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On the Coexistence of Deformed and Spherical States in Odd-Mass Nuclei*

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The problem of coexisting deformed and spherical states in Odd-Mass Nuclei is discussed within the framework of the Unified Model. The emphasis is made on the coupling between such states. Numerical calculations were performed for ^{115}In. The basic spherical states are either ^{116}Sn plus a (proton) hole or ^{114}Cd plus a (proton) particle. The agreement between calculated and experimental quantities is rather good.

I. Introduction

A comparison of the spectra for the odd-mass indium isotopes (see Fig. 1) shows a very similar pattern for all of them. Therefore one is justified to treat them from the same theoretical point of view. Up to now different interpretations have been reported. For the sake of completeness and for future reference, a resumé of them is presented below.

a) Ground State and the First Two Excited States with Negative Parity

The ground state and the isomeric state are well described by the spherical shell model. A hole in the $1g_{9/2}$ and $2p_{1/2}$ states explain the spin and parity assignments for these two levels, respectively.

Silverberg [1] has studied $1g_{9/2}$ and $2p_{1/2}$ energy difference occuring in the odd-mass indium isotopes ($A=109-119$). He found that after the

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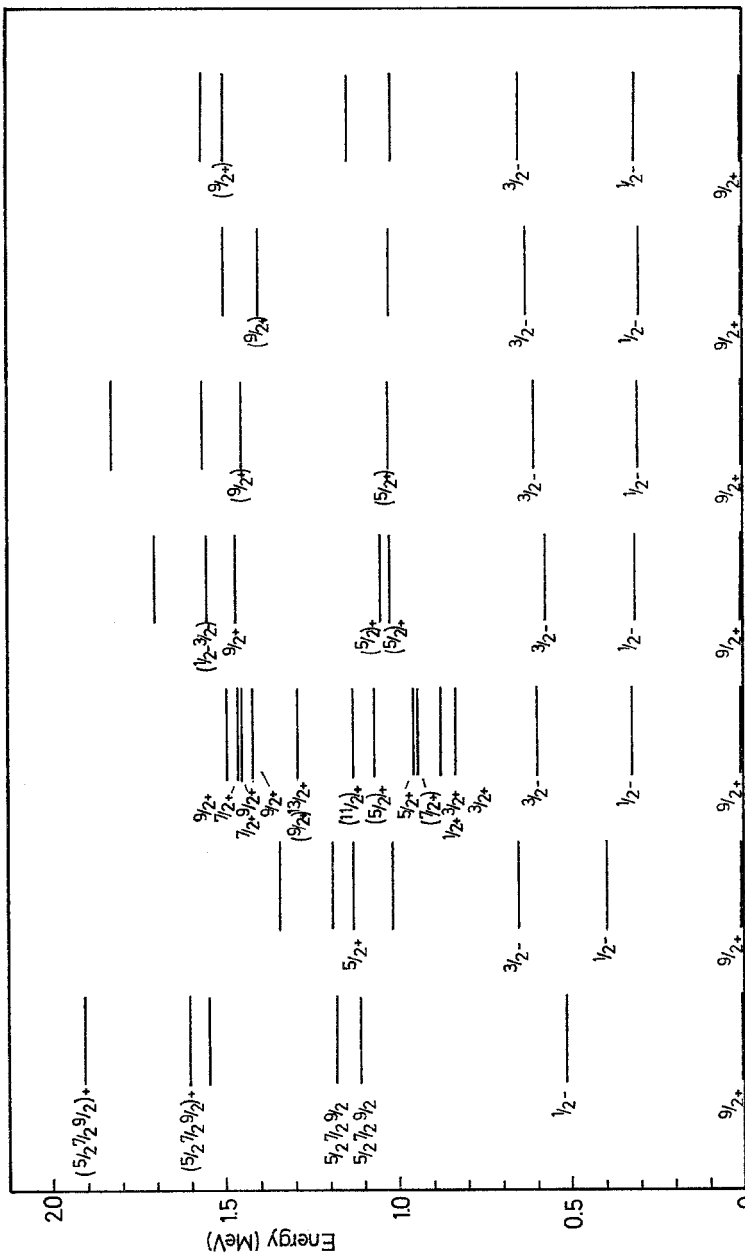

Fig. 1. Experimental nuclear level schemes for indium nuclei in the mass region $A=111$–113. Data ^{111}In Ref. [32]; ^{113}In Ref. [12]; ^{115}In Refs. [5, 7, 11, 33–35]; ^{117}In — ^{123}In Ref. [5]

filling of the $1g_{7/2}$ state – which occur in In^{113} – the $1h$, $2d$ and $3s$ states are filling simultaneously. In this way, the energy difference between the $1g_{9/2}$ and $2p_{1/2}$ states is practically constant in all odd indium isotopes.

A 3/2-state appears above the $p_{1/2}$ isomeric state. Pandharipande *et al.* [2] have calculated the excitation of a proton hole in the $p_{3/2}$ orbital which can give rise to a 3/2-state. They found that this level would lie at 1.2 MeV above the isomeric state. On the other hand, this level can also be due to the coupling of a proton hole in the $p_{1/2}$ state with a phonon of the neighbouring tin isotopes. In this case, the 3/2-level can be expected in the energy region of 1.2 MeV too (which is the first phonon energy of the even-even tin isotopes). Since both 3/2-levels lie practically at the same energy, they may interact strongly. So that, one of these levels is pushed down in energy. Silverberg [3] following the calculation of Kisslinger and Sorensen [4] has suggested that the lowest 3/2-state is of a single-particle nature whereas the other one is too far off in energy. Recently, Weiffenbach and Tickle [5] studied the $Sn(d, {}^3He)$ pick up reaction on a series of even tin targets. These experiments indicate that the second excited states are composed largely of the configuration $2p_{3/2}^{-1}$, confirming the suggestion by Silverberg [3].

b) Levels as Members of a Proposed $K=1/2+$ Rotational Band

The Nilsson model [6] can well account for energies and transition rates of some states in In-isotopes if these states are members of a $K=1/2+$ rotational band, which is based on the state $1/2+$ [431] [7]. Following this suggestion, Pandharipande *et al.* [8] were able to reproduce the energies of those states using as values for the inertial and decoupling parameters, $\hbar^2/2J=28$ keV and $a=-2.2$, respectively, which in turn are in close agreement with those originally deduced by Bäcklin *et al.* [7].

c) Hole-Vibration Coupling

The coupling of the $g_{9/2}$ hole with the one phonon state (2^+) of the neighbouring even-even tin isotopes form a multiplet of five positive parity levels with spins ranging from $5/2+$ up to $13/2+$. Calculations of this kind were originally performed by Silverberg [3] but the agreement with experiment was fairly poor. Later on, Pandharipande [9] recalculated the energies of these states using a rather simple residual interaction within the shell model. The coupling of the $g_{9/2}$ proton hole with various states of the neutron core was considered. The agreement with experiment as far as energies and spins are concerned is a little bit more satisfactory than in the previous case, at least for ^{117}In [10], but is still

rather poor for the other indium isotopes [10, 11]. Moreover, this model cannot account for the position of the center of mass of the multiplet. It also fails in explaining the observed $B(E2)$ values [11, 12].

Recently, Dietrich *et al.* [11] applied the intermediate coupling approach within the unified model [13] to explain their experimental data of ^{115}In. In this model, the $1g_{9/2}$ proton hole is considered to be coupled to the oscillatory motion of ^{116}Sn. Vibrational states of the core up to two phonons were taken into account. When three phonons were also included the agreement with the experimental level positions and $B(E2)$ values was worse than in the simpler calculation. The calculated values for the level positions, $B(E2)$ and $B(M1)$ transition rates, ground-state quadrupole moment and spectroscopic factors are however in reasonably good agreement with their experimental data for ^{115}In. Since they did not observe the states at 828 and 864 keV found by Bäcklin *et al.* [7], the existence of the proposed rotational band in ^{115}In was out of the scope of their analysis.

With the same approach, Atalay and Chiao-Yap [14] have analyzed the structure of the odd-mass indium isotopes considering the coupling of quadrupole vibrations of the tin core to a proton hole which has available the $1g_{9/2}$, $2p_{1/2}$ and $2p_{3/2}$ orbits. Their results show an overall fair agreement with experiment except for some levels and properties which in turn could be explained by the simple rotational model.

Based on the same type of approach [11, 14], Covello *et al.* [15] have recently analyzed the nuclear properties of ^{115}In by coupling four proton-hole states to quadrupole and octupole core vibrations. Most of the experimental levels were well predicted but others cannot be described by the model.

The aim of the present work is to discuss the structure of the In-isotopes under the assumption that deformed and spherical states coexist in these nuclei, an approach which has proven its usefulness in the treatment of oxygen [16] and calcium [17, 18] isotopes.

During the course of this work an additional investigation on this subject was performed by Sen [19] with his extended hole-coupling model. In this model, the mixing of the core-plus-hole states with the deformed states was considered and the interaction matrix elements were treated as adjustable parameters.

In our work, the mixing of the hole-core and particle-core states as well as deformed states is considered on the same footing and the matrix elements of the various interactions are treated in an exact way except in the case of two-phonon vibrations in which they are neglected. Moreover, we have taken $E2$ transitions between different parts of the wave functions into account.

II. The Model Hamiltonian

For the odd-mass nuclei we assume a system of nuclear quadrupole surface oscillation with coordinates α_μ coupled to the single-particle motion with coordinates x. The model Hamiltonian can be written in the usual form [20]

$$H = H_{\text{coll}}(\alpha_\mu) + H_P(x) + H_{\text{int}}(x, \alpha_\mu), \tag{II.1}$$

where

$$H_{\text{coll}}(\alpha_\mu) = \tfrac{1}{2}\sum_\mu B\,|\dot{\alpha}_\mu|^2 + C\,|\alpha_\mu|^2 = T(\dot{\alpha}_\mu) + V(\alpha_\mu), \tag{II.2}$$

with T and V being the kinetic and potential energy of the surface vibration, and $H_P(x)$ is a single-particle Hamiltonian with a spherical potential. The coordinates α_μ are the expansion parameters of the nuclear surface defined – for the quadrupole case – by

$$R(\theta, \phi) = R_0\left[1 + \sum_\mu \alpha_\mu\, Y_2^\mu(\theta, \phi)\right],$$

where R_0 is the equilibrium radius and Y_2^μ, the normalized second-order spherical harmonic.

The coupling term H_{int} represents the interaction between a particle (or a hole) and the quadrupole vibration of the core. In first approximation it is given by [20]

$$H_{\text{int}}(x, \alpha_\mu) = k(r)\sum_\mu \alpha_\mu\, Y_2^\mu \tag{II.3}$$

where $k(r)$ is the radial part of H_{int}. The dependence of the matrix element on the particle quantum numbers n and l is sufficiently weak so that it can be treated as a constant, as it is customarily done. For example, the matrix element of the interaction between a one-phonon state and the ground state can be written as

$$\langle j00J{=}jM\,|\,H_{\text{int}}\,|\,j'12JM\rangle = k\left(\frac{\hbar\omega}{2c}\right)^{\frac{1}{2}}\langle j\|\,Y_2\,\|j'\rangle \tag{II.4}$$

where $\langle j\,\|\,Y_2\|j'\rangle$ is the reduced matrix element of a second order spherical harmonic between states with angular momenta j and j'. We use in this paper the convention adopted by de-Shalit and Talmi [21].

In order to find approximate solutions of the Schrödinger equation corresponding to the Hamiltonian (II.1)

$$H\,|\psi\rangle = E\,|\psi\rangle, \tag{II.5}$$

two different possibilities can be considered. They are described in the following paragraphs.

a) Spherical Solution

It is well known from the weak coupling scheme in the unified model [22] that the Hamiltonian H can be diagonalized within a Hilbert space spanned by the solutions of

$$(H_{\text{coll}}+H_P)\,|\phi_{N,\nu}\rangle=(\varepsilon_N^{\text{coll}}+\varepsilon_\nu^P)\,|\phi_{N,\nu}\rangle, \tag{II.6a}$$

with

$$|\phi_{N,\nu}\rangle=|\varphi_N(\alpha_\mu)\psi_\nu(x)\rangle. \tag{II.6b}$$

The states $\psi_\nu(x)$ are the spherical single-particle states whereas the collective states $\varphi_N(\alpha_\mu)$ are classified according to the number of phonons N present in a particular state.

These solutions are characterized by small average values of the collective expansion parameters α_μ. This fact suggests to search for a second type of solutions for which the average values of these coefficients need not be small.

b) Deformed Solution

In this case one assumes a permanent and sufficiently large nuclear deformation. Then it is convenient to introduce an intrinsic coordinate system in such a way that the kinetic energy of the surface vibration separates into a vibrational and rotational part. The new system of coordinates is related to the original coordinate system by the relations [23]

$$\begin{aligned} &a_m=\sum_{\mu=-2}^{2} D_{\mu m}\alpha_\mu, \qquad a_2=a_{-2}=\frac{1}{\sqrt{2}}\beta\sin\gamma, \\ &a_1=a_{-1}=0, \qquad\qquad a_0=\beta\cos\gamma. \end{aligned} \tag{II.7}$$

The Hamiltonian (II.1) in the new coordinate system reads [24]

$$H=T_{\text{rot}}+T_{\text{vib}}+H_N+H', \tag{II.8}$$

where

$$H_N=H_P+\rho r^2 Y_2^0 \tag{II.9}$$

is a single-particle Hamiltonian with a deformed potential as was originally discussed by Nilsson [6]. The last term in Eq. (II.8) stands for all the remaining interactions. It is discussed below.

The constant g which appears in Eq. (II.9) is the coupling strength of the particle-vibrator interaction.

The solutions of the single-particle Schrödinger equation corresponding to the Hamiltonian (II.9) can be defined by

$$H_N\,|\chi_K\rangle=\varepsilon_K\,|\chi_K\rangle \tag{II.10}$$

which of course depend on the coordinates a_0 and a_2 in the following way

$$|\chi_K\rangle = \sum_{j,m} C^K_{j,m}(a_0, a_2)\,|\psi^m_j\rangle. \tag{II.11}$$

The wavefunctions $|\psi^m_j\rangle$ are solutions of the spherical single-particle Hamiltonian H_P

$$H_P\,|\psi^m_j\rangle = \varepsilon_j\,|\psi^m_j\rangle. \tag{II.12}$$

Therefore the wavefunction (cf. Eq. (II.5)) takes the form

$$|\psi\rangle = \sum_{K,\tau} |D^J_{M,K+\tau}\varphi_{K,\tau}(a_0, a_2)\chi_K\rangle \tag{II.13}$$

where $\varphi_{K,\tau}$ describes the vibration of the nucleus around its equilibrium shape. The solution of the Schrödinger equation (II.5) yields a set of coupled equations for the vibrational wavefunctions $\varphi_{K,\tau}$ [24, 25]. We shall not write down this set of equations but only discuss the nature of the various coupling terms which form the Hamiltonian H'(Eq. (II.8)). The kinetic energy of the rotation and vibration gives rise to different interactions: rotation-vibration-coupling, rotation-particle-coupling, vibration-rotation-particle-coupling. They are customarily labelled by RVC, RPC (or Coriolis coupling) and VRPC [25]. As a first approximation, these terms will be omitted in the present treatment unless otherwise stated.

We are interested in the fact that the potential energy $W(a_0, a_2)$ can now be expressed as a sum of two terms $V(a_0, a_2)$ and $\varepsilon_K(a_0, a_2)$ which represent the collective potential and single-particle energies respectively,

$$W(a_0, a_2) = V(a_0, a_2) + \varepsilon_K(a_0, a_2). \tag{II.14a}$$

For the sake of simplicity, only a_0(β-vibration) is taken into account. Thus, Eq. (II.14a) can be written as

$$W(a_0) = V(a_0) + \varepsilon_K(a_0) = \tfrac{1}{2} C a_0^2 + \varepsilon_K(a_0). \tag{II.14b}$$

This potential function when plotted for the Nilsson state corresponding to the low-lying deformed state in the even-cadmium isotopes [26], as a function of the deformation a_0 allows one to conclude that although the collective potential energy V has its minimum at $a_0 = 0$, the total potential energy of the coupled system may have its minimum at $a_0 = a_0^0 \neq 0$. Moreover, the equilibrium position a_0^0 may correspond to a sizable deformation.

c) Coexistence of Spherical and Deformed Solutions

At this point one arrives to the conclusion that in a odd-mass nuclei, low-lying deformed states can exist even though the neighbouring even-

mass nuclei do not posses such states at low energy. Provided that the equilibrium deformation a_0^0 is reasonably large to reach the strong coupling approximation [20] one can now write down the solutions of the Schrödinger equation under the usual assumptions that the above mentioned couplings (cf. Eq. (II.8)) can be neglected [20, 24]. The solutions are denoted by $|JM, \mathrm{Rot}\rangle$ (Rot = rotational),

$$|JM, \mathrm{Rot}\rangle = \left(\frac{2J+1}{16\pi^2}\right)^{\frac{1}{2}} \cdot \{D^J_{MK}\chi_K + (-)^{J-K} D^J_{M-K}\chi_{-K}\}\,\tilde{\varphi}(a_0, a_2). \tag{II.15}$$

Let us emphasize that χ_K describes the motion of the particle with respect to the deformed nucleus; $\tilde{\varphi}$ represents the collective vibrational motion in a_0 and a_2 coordinates while D^J_{MK} are the eigenfunctions of the symmetric top which describe the nuclear rotation. The vibrational wave functions are explicitly given in Ref. [24]. Here we consider only the case of zero phonons:

$$|\tilde{\varphi}\rangle = \left(\frac{2J+1}{8\pi^2}\right)^{\frac{1}{2}} D^J_{M0}\sqrt{\lambda}\sqrt{|a_2|}\exp\left(-\frac{\lambda}{2}a_2^2\right)\varphi_0(a_0),$$
$$\varphi_0(a_0) = \left(\frac{\tau}{\pi}\right)^{\frac{1}{4}} \exp\left[-\frac{\tau}{2}(a_0 - a_0^0)^2\right]. \tag{II.16}$$

With the help of Eqs. (II.15) and (II.16) one is able to calculate the matrix elements of any operator between deformed and spherical states.

The following quantities enter into a numerical solution of the Schrödinger equation.

(i) The wavefunction of a particle or hole coupled to vibration around a spherical equilibrium shape

$$|JM, \mathrm{Vib}\rangle = \sqrt{2J+1}\sum_K D^J_{MK}(-)^{R-j+K} \cdot \sum_m \begin{pmatrix} R & j & J \\ m & K-m & -K \end{pmatrix} |\varphi^N_m(a_0, a_2)\chi^{K-m}_j(x)\rangle. \tag{II.17}$$

Here the coordinates α_μ have been transformed to intrinsic coordinates a_0, a_2 and Euler angles θ_i, $i = 1, 2, 3$.

The functions $\varphi^N_m(a_0, a_2)$ are given by:

$$\varphi^0_0 = \frac{1}{2}\left(\frac{\kappa}{\pi}\right)^{\frac{5}{4}} \exp\left[-\frac{\kappa}{2}(a_0^2 + 2a_2^2)\right] \tag{II.18}$$

for the ground state and

$$\varphi^1_{(\frac{0}{2})} = \frac{1}{\sqrt{2}}\left(\frac{\kappa}{\pi}\right)^{\frac{7}{4}} \begin{pmatrix} a_0 \\ a_2 \end{pmatrix} \exp\left[-\frac{\kappa}{2}(a_0^2 + 2a_2^2)\right] \tag{II.19}$$

for the one-phonon state.

The functions χ_j^{K-m} are the single-particle wavefunctions.

(ii) Overlap between spherical and deformed collective vibrational functions

$$I_m^N \equiv \langle \varphi_m^N | \tilde{\varphi}(a_0, a_2) \rangle \tag{II.20}$$

where $\tilde{\varphi}(a_0, a_2)$ is defined by Eq. (II.16).

We assume that

$$I_2^1 = \langle \varphi_2^1 | \tilde{\varphi}(a_0, a_2) \rangle \equiv 0. \tag{II.21}$$

(iii) Overlap between spherical and deformed nuclear states

$$\mathcal{N}_J^N \equiv \langle JM; \text{Vib} | JM; \text{Rot} \rangle$$

$$= 4\pi(-)^{R-j+K} \begin{pmatrix} R & j & J \\ 0 & K & -K \end{pmatrix} \langle \varphi_m^N \chi_j^K | \tilde{\varphi}(a_0, a_2) \chi_K \rangle \tag{II.22}$$

or equivalently

$$\mathcal{N}_J^N = 4\pi(-)^{R-j+K} \begin{pmatrix} R & j & J \\ 0 & K & -K \end{pmatrix} I_m^N C_j^K \tag{II.23}$$

where C_j^K are Nilsson coefficients for a given deformation and a given set of single-particle energies defined by

$$|\chi_K\rangle = \sum_j C_j^K |\chi_j^K\rangle. \tag{II.24}$$

(iv) Interaction between spherical and deformed collective vibrational functions

$$V_{(2)}^N \equiv \langle \varphi_m^N | \begin{pmatrix} a_0 \\ a_2 \end{pmatrix} | \tilde{\varphi}(a_0, a_2) \rangle \tag{II.25}$$

and

$$V_0^0 = \langle \varphi_0^0 | a_0 | \tilde{\varphi}(a_0, a_2) \rangle = \frac{1}{\sqrt{2\pi}} I_0^1 \tag{II.26}$$

where I_0^1 is given by Eq. (II.20) for the case $N=1$ and $m=0$.

(v) Interaction between spherical and deformed nuclear states

$$H_{SD} \equiv \langle JM; \text{Vib} | k \sum_{m=-2} a_m Y_2^m | JM; \text{Rot} \rangle$$

$$= \langle k \rangle \sqrt{\frac{5\pi}{2}} \sqrt{2j+1} \sum_{m=-2} V_{|m|}^N \begin{pmatrix} R & j & J \\ m & \frac{1}{2}-m & -\frac{1}{2} \end{pmatrix} (-)^{R-j+\frac{1}{2}} \tag{II.27}$$

$$\cdot \sum_n \begin{pmatrix} j'(n) & 2 & j \\ -\frac{1}{2} & m & \frac{1}{2}-m \end{pmatrix} \begin{pmatrix} j'(n) & 2 & j \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$\cdot \sqrt{2j'(n)+1} \left[1+(-)^{l+l'(n)}\right] C_{j'(n)}^K.$$

d) Diagonalization of the Model Hamiltonian

In order to account for the mixing of spherical and deformed states, as a first step the energy matrix is calculated in the basis of the non-orthogonal deformed and spherical states.

The diagonal matrix element of the total Hamiltonian in the strong coupling limit for the special case $K=1/2$ is given by [26]

$$\begin{aligned}\langle JM;\,\mathrm{Rot}|\,H\,|JM;\,\mathrm{Rot}\rangle &= E_{\mathrm{rot}}\\ &= \varepsilon_{\frac{1}{2}}+\left[J(J+1)-\tfrac{1}{4}+a^{J+\frac{1}{2}}(J+\tfrac{1}{2})\right],\end{aligned} \tag{II.28}$$

where $\varepsilon_{\frac{1}{2}}$ is the Nilsson eigenvalue. In Eq. (II.28) the rotation-particle coupling or Coriolis term was included. The matrix elements of the Coriolis interaction H_C for symmetric nuclei and $K=1/2$ are given by [26]

$$\langle JM;\,\mathrm{Rot}|\,H_C\,|JM;\,\mathrm{Rot}\rangle=(-)^{J+\frac{1}{2}}\frac{\hbar^2}{2\mathscr{J}}(J+\tfrac{1}{2})\,a \tag{II.29}$$

where the decoupling parameter a is defined by

$$a\equiv\sum_j(-)^{j-\frac{1}{2}}(j+\tfrac{1}{2})[C_j^{K=\frac{1}{2}}]^2. \tag{II.30}$$

For the spherical case, the diagonal and off-diagonal matrix elements are given in an obvious notation by [22]

$$\begin{aligned}&\langle JM;\,\mathrm{Vib}|\,H\,|JM;\,\mathrm{Vib}\rangle\\ &=\langle(n'l'j',N'R')J'M'\,|\,H\,|(nlj,NR)JM\rangle\\ &=[\varepsilon(nlj)+N\hbar\omega]\,\delta_{nn'}\delta_{ll'}\delta_{jj'}\delta_{NN'}\delta_{RR'}\\ &\quad+\langle k\rangle(-)^{J+R'+j+1}\begin{Bmatrix}j' & R' & J\\ R & j & 2\end{Bmatrix}\langle j'\|\,Y_2\,\|j\rangle\\ &\quad\cdot\frac{1}{\sqrt{2\kappa}}\left[\langle N'R'\|\,b_2\,\|NR\rangle\,\delta_{NN'+1}+\langle N'R'\|\,b_2\,\|NR\rangle\,\delta_{N'N+1}\right].\end{aligned} \tag{II.31}$$

Finally, the other off-diagonal elements of the total model Hamiltonian come from the interaction of spherical and deformed states. They are given by

$$\begin{aligned}&\langle JM;\,\mathrm{Vib}|\,H\,|JM;\,\mathrm{Rot}\rangle\\ &\quad=[\varepsilon(nlj)+N\hbar\omega]\,\mathscr{N}_J^N+H_{SD}.\end{aligned} \tag{II.32}$$

As a second step in our calculation, the orthonormalization of the deformed and spherical basic states was performed according to the following procedure. Let us denote $\bar{x}_1$ and $\bar{x}_n$ with $n\geqq 2$, the deformed and spherical states respectively, obtained up to now. The following

relations hold

$$\langle \bar{x}_n, \bar{x}_m \rangle = \delta_{n,m} \quad \text{for } n, m \geqq 2 \tag{II.33}$$

for the spherical states and

$$\langle \bar{x}_1, \bar{x}_n \rangle = \mathcal{N}_J^N(n) \tag{II.34}$$

for the nonorthogonal deformed and spherical states according to Eq. (II.22).

The new deformed state $|x_1\rangle$ can be expanded in terms of these state vectors,

$$|x_1\rangle = a_1 |\bar{x}_1\rangle + \sum_{n=2} a_n |\bar{x}_n\rangle. \tag{II.35}$$

Taking into account the orthogonalization relations for the new vectors,

$$\langle x_1, x_1 \rangle = 1 \quad \text{and} \quad \langle x_1, x_n \rangle = 0 \tag{II.36}$$

it follows that

$$a_1 = \frac{1}{\sqrt{1-S}} \quad \text{and} \quad a_n = -\frac{\mathcal{N}_J^N(n)}{\sqrt{1-S}} \tag{II.37a}$$

with

$$S = \sum_{n=2} \mathcal{N}_J^N(n). \tag{II.37b}$$

Therefore, the matrix elements for the new deformed state become

$$\langle x_1 | H | x_1 \rangle = \sum_{n,m=1} a_n a_m \langle \bar{x}_m | H | \bar{x}_n \rangle, \tag{II.38}$$

and for the new spherical states, they read

$$\langle x_1 | H | x_n \rangle = \sum_{m=1} a_m \langle \bar{x}_m | H | \bar{x}_n \rangle \quad \text{for } n \geqq 2 \tag{II.39}$$

As a third and last step, the diagonalization of the energy matrix was performed.

e) Energy Levels and Electric Properties

The diagonalization of the total Hamiltonian described above yields the wave functions that describe the nuclear energy levels of our model. From the choice of our basis, it is evident that the wave functions satisfy the relation

$$|{}^{\rho}J\rangle = b_1 |x_1\rangle + \sum_{n \geqq 2} b_n |x_n\rangle \tag{II.40}$$

where the symbol ρ acts as an ordering number for states of same angular momentum and b_i $(i=1, \ldots, n)$ are the expansion coefficients.

In order to have an explicit expression for the electric quadrupole operator acting between initial and final states – which can be of different

nature – the wave functions (II.40) should be expanded in terms of the original non-orthogonalized rotational states $|\bar{x}_1\rangle$ and the vibrational states $|\bar{x}_n\rangle$, $n \geqq 2$. According to Eq. (II.35) the wave functions take the form

$$|{}^{\rho}J\rangle = c_1 |\bar{x}_1\rangle + \sum_{n \geqq 2} c_n |\bar{x}_n\rangle \tag{II.41}$$

where the new expansion coefficients c_i are expressed as

$$c_1 = a_1 b_1 \tag{II.42a}$$

and

$$c_n = a_n b_1 + b_n, \quad \text{for } n \geqq 2. \tag{II.42b}$$

Then, the reduced matrix elements for the electric quadrupole transition operator between the eigenstates (II.41) reads

$$\begin{aligned}
&\langle {}^{\rho'}J' \| \mathcal{M}(E2) \| {}^{\rho}J \rangle \\
&\quad = c_1({}^{\rho}J)\, c_1({}^{\rho'}J') \langle {}^{\rho'}J' \| \mathcal{M}(E2) \| {}^{\rho}J \rangle_{\text{rot}} \\
&\qquad + \sum_{n \geqq 2} \sum_{n' \geqq 2} c_n({}^{\rho}J)\, c_{n'}({}^{\rho'}J') \langle {}^{\rho'}J' \| \mathcal{M}(E2) \| {}^{\rho}J \rangle_{\text{vib}} \\
&\qquad + c_1({}^{\rho'}J') \sum_{n \geqq 2} c_n({}^{\rho}J) \langle {}^{\rho'}J' \| \mathcal{M}(E2) \| {}^{\rho}J \rangle_{\text{rot-vib}} \\
&\qquad + c_1({}^{\rho}J) \sum_{n' \geqq 2} c_{n'}({}^{\rho'}J') \langle {}^{\rho'}J' \| \mathcal{M}(E2) \| {}^{\rho}J \rangle_{\text{rot-vib}},
\end{aligned} \tag{II.43}$$

where the subscripts rot, vib and rot-vib labell the pure rotational, the pure vibrational and the mixing of the rotational-vibrational wave functions.

III. Calculations and Results

Among the odd-mass indium isotopes $^{109-123}$In we have tried to fit the theoretical energy spectra to the experimental spectra of a typical nuclei, namely ^{115}In. Even in this most favourable case, the available experimental information does not allow the use of a least-squares procedure to search for the best values of the parameters which enter into the calculation. Therefore, estimates of some of them had to be made from the knowledge of nuclear properties. For the others, the results obtained from a variation between reasonable limits show at which value they have to be fixed.

First of all we calculated the matrix elements corresponding to the single-particle Hamiltonian (II.9) [6]

$$\begin{aligned}
\langle j l n \tfrac{1}{2} | H_N | j' l' n' \tfrac{1}{2} \rangle &= \varepsilon(j l n) \delta_{j j'} \delta_{l l'} \delta_{n n'} \\
&\quad + \rho \langle j l n \tfrac{1}{2} | Y_2^0 | j' l' n' \tfrac{1}{2} \rangle \langle n l | r^2 | n' l' \rangle,
\end{aligned} \tag{III.1}$$

where

$$\rho = -\delta \hbar\omega_0 \frac{4}{3}\sqrt{\frac{\pi}{5}} \tag{III.2}$$

with

$$\hbar\omega_0 = \hbar\mathring{\omega}_0 \left(1 - \frac{4}{3}\delta^2 - \frac{16}{27}\delta^3\right)^{-\frac{1}{6}} \tag{III.3}$$

and

$$\hbar\mathring{\omega}_0 = 41\, A^{-\frac{1}{3}}. \tag{III.4}$$

The matrix elements for the operator $Y_2^{m-m'}$ are given in the general case [21] by

$$\begin{aligned}\langle j l n m | Y_2^{m-m'} | j' l' n' m' \rangle &= (-)^{m-\frac{1}{2}} \left[\frac{1+(-)^{l+l'}}{2}\right] \\ &\cdot \sqrt{\frac{5(2j+1)(2j'+1)}{4\pi}} \begin{pmatrix} j & 2 & j' \\ -m & m-m' & m' \end{pmatrix} \begin{pmatrix} j & 2 & j \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.\end{aligned} \tag{III.5}$$

The matrix elements of the radial part r^2 are taken from Nilsson [6],

$$\begin{aligned}\langle n l | r^2 | n l \rangle &= 2n + l + \tfrac{3}{2}, \\ \langle n+1\, l-2 | r^2 | n l \rangle &= 2\sqrt{(n+1)(n+l+\tfrac{1}{2})},\end{aligned} \tag{III.6}$$

and

$$\langle n-1\, l+2 | r^2 | n l \rangle = 2\sqrt{n(n-l+\tfrac{3}{2})}.$$

These calculations were performed for different sets of shell-model single-particle energies and different deformations. Then the level structure of the rotational band $K=1/2$ was computed using Eqs. (II.29, 30) adopting the value for the inertial parameter $\hbar^2/2\mathscr{I}$ already found by Bäcklin *et al.* [7] .We choose such a set of single particle energies for which the decoupling parameter a is sufficiently large and negative to yield an anomalous behaviour of the rotational band, i.e., its $J=3/2$ level will appear below the $J=1/2$ level. For this to be the case, the $g_{7/2}$ stste should be below the $d_{5/2}$ state. This fact conflicts with the single-particle positions used by Lopac [27] in her calculations of ^{119}Sb.

As far as the vibrational case is concerned estimates had to be made of the values of the different parameters which enter into the wavefunctions. The values for the parameters κ_{Sn} and κ_{Cd} were deduced from the measured reduced electromagnetic transition probability from the ground to the first excited 2+state. Actually they can be expressed – in the usual notation – by

$$B(E2; 0+_{\mathrm{gr.\,st.}} \rightarrow {}^{1}2+) = 5\left(\frac{3}{4\pi} Z R_0^2\right)\frac{\hbar\omega}{2C} \tag{III.7}$$

with

$$\frac{\hbar\omega}{2C}=\frac{1}{2}\frac{\hbar}{\sqrt{BC}}=\frac{1}{2\kappa}. \tag{III.8}$$

Therefore, the parameters κ are given by

$$\kappa=\frac{45}{32}\frac{Z^2R_0^4}{B(E2;\,0+_{\text{gr. st.}}\rightarrow{}^{2}2+)}. \tag{III.9}$$

The experimental values for $B(E2)$ were taken from Ref. [28]. For the nuclear radius we have used $R_0=1.2A^{+1/3}$ fm.

The γ vibrational parameter λ for deformed states was estimated as lying in between the vibrational parameters for the spherical case. With respect to the β vibrational parameter τ for deformed states it was considered to be the half of λ. When performing the calculations it was found that these parameters do not affect seriously the final results, within a wide range of variation.

As usual, the core deformation parameter a_0^0 was considered to be related to the oscillator deformation parameter δ by

$$a_0^0=\frac{2}{3}\sqrt{\frac{4\pi}{5}}\delta(1+\tfrac{2}{3}\delta). \tag{III.10}$$

The values for the overlap and interaction between spherical and deformed collective vibrational functions were calculated by integrals of the type

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|a_2|\exp\left(-\frac{\lambda}{2}a_2^2\right)\varphi_m^N\tilde{\varphi}(a_0,a_2)\sqrt{3a_0^2-2a_2^2}\,da_0\,da_2. \tag{III.11}$$

In order to do it Femenia [29] has extended the Romberg method to two variables. The values obtained for I_0^0, I_0, V_0^0, V_0 and V_2 for tin and cadmium as well as the quantities aforementioned are presented in Table 1.

For the states built upon the two-phonon states of the vibrating core the overlap and the interactions mentioned above were considered to be nulle. This is justified by their small values for one-phonon states (cf. Table 1).

The one-phonon energy $\hbar\omega$ was taken from the neighbouring tin or cadmium isotopes depending upon whether a hole or a particle respectively was being considered. Following the suggestion by Dietrich *et al.* [11] we took different energies for the two-phonon states of ^{116}Sn

Table 1. Values for the parameters used in the present calculation for level sequence and quadrupole electric transitions in ^{115}In. Calculated overlap and interactions between spherical and deformed stated are also listed

1. Set of single-particle energies (in MeV)

$$(g_{9/2})-(g_{7/2})=-3.070$$
$$(d_{5/2})-(g_{7/2})=\ 0.750$$
$$(d_{3/2})-(g_{7/2})=\ 2.850$$
$$(s_{1/2})-(g_{7/2})=\ 3.150.$$

2. Oscillator deformation $\delta=0.20$
   The core deformation becomes $a_0^0=0.2396$.

3. Inertial parameter $\hbar^2/2J=25$ keV.
   From the adopted values indicated in steps 1–3, the decoupling parameter is $a=-2.595$.

4. β vibrational parameter for spherical states for ^{116}Sn vibrating core $\kappa_{Sn}=266$.

5. γ vibrational parameter for spherical states for ^{114}Cd vibrating core $\kappa_{Cd}=90$.

6. γ vibrational parameter for deformed states $\lambda=134$.

7. γ vibrational parameter for deformed states $\tau=67$.

8. $I_0^0(\mathrm{Sn})=0.03349$ $\quad I_0^0(\mathrm{Cd})=0.07327$
   $I_0(\mathrm{Sn})=0.05471$ $\quad I_0(\mathrm{Cd})=0.1427$
   $V_0^0(\mathrm{Sn})=0.002372$ $\quad V_0^0(\mathrm{Cd})=0.01063$
   $V_0(\mathrm{Sn})=0.006973$ $\quad V_0(\mathrm{Cd})=0.02737$
   $V_2(\mathrm{Sn})=0.002378$ $\quad V_2(\mathrm{Cd})=0.006445$.

9. $^12+$ phonon energy of ^{114}Cd, $\hbar\omega_{Cd}=0.5581$ MeV.

10. $^12+$ phonon energy of ^{116}Sn, $\hbar\omega_{Sn}=1.293$ MeV
    $^10+$ phonon energy of ^{116}Sn, $2\hbar\omega_{Sn}(^10)=1.716$ MeV
    $^22+$ phonon energy of ^{116}Sn, $2\hbar\omega_{Sn}(^22)=2.146$ MeV
    $^14+$ phonon energy of ^{116}Sn, $2\hbar\omega_{Sn}(^14)=2.586$ MeV.

11. Strength of the particle (hole)-vibrator interaction: $k=50$ MeV.

depending upon which of the levels $^10+$, $^22+$ or $^14+$ is involved in the coupling with a particle to built states in ^{115}In.

The coupling strength of the particle-(hole) vibrator interaction k was varied between reasonable limits. Then the value of 50 MeV was adopted which is in agreement with that quoted by Dietrich *et al.* [11].

The calculated positive-parity levels of ^{115}In are compared in Fig. 2 with the experimental data. In Table 2 the components of the wavefunc-

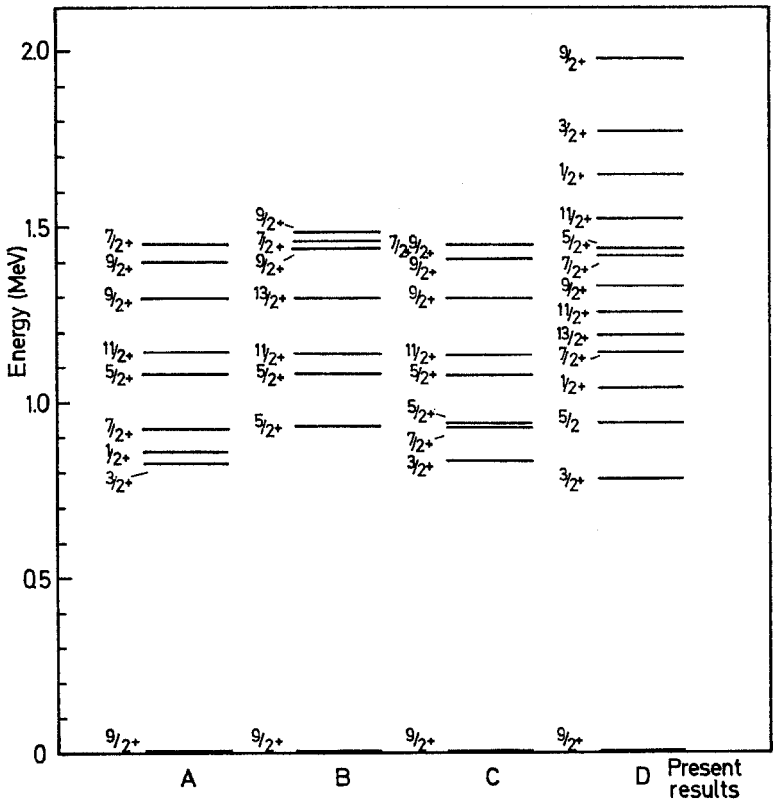

Fig. 2. Positive-parity level scheme of ^{115}In observed via: *A* The decay of ^{115m,115g}Cd (Refs. [7, 33] and [34]); *B* the reactions ^{115}In(^{16}O, ^{16}O′γ) Ref. [11]; *C* the reaction ^{115}In(n, $n'\gamma$) Ref. [35]; compared with the present prediction

tions of the low-lying states are presented. Only those components are listed which contribute more than 5%. The states with $^{n}I=^{1}1/2$, $^{1}3/2$, $^{2}5/2$, $^{1}7/2$ and $^{2}11/2$ can be in some extent identified as the members of the $K=1/2+$ rotational band. The states with $^{n}I=^{1}5/2$, $^{2}7/2$, $^{0}9/2$, $^{1}9/2$, $^{1}11/2$ and $^{1}13/2$ are mainly originated from the coupling of the $g_{9/2}$ proton hole to quadrupole excitations in ^{116}Sn core. The remaining states $^{2}1/2$ and $^{2}3/2$ are mixed with those yielded by the coupling of a proton particle coupled to quadrupole vibrations of the ^{114}Cd core.

A general acceptable agreement is obtained between the energies and spins found experimentally and those calculated by us. However, some discrepancies still occur such as the displacement to higher energies of

Table 2. Components of the calculated wave-functions for the low-lying states

$|(1/2)_1; 1038\rangle = 0.97\,|\mathrm{rot}\rangle.$

$|(1/2)_2; 1655\rangle = -0.27\,|s_{1/2}, 00\rangle + 0.61\,|d_{5/2}, 12\rangle + 0.27|d_{3/2}, 12\rangle$
$+0.25\,|d_{5/2}, 12\rangle + 0.57|_{g_{9/2},\mathcal{S}}\, 24\rangle.$

$|(3/2)_1; 784\rangle = 0.94\,|\mathrm{rot}\rangle + 0.27\,|d_{3/2}, 00\rangle.$

$|(3/2)_2; 1766\rangle = -0.32\,|d_{3/2}, 00\rangle + 0.62\,|g_{7/2}, 12\rangle + 0.30\,|d_{5/2}, 12\rangle$
$-0.23\,|d_{3/2}, 12\rangle + 0.32\,|g_{7/2}, 22\rangle + 0.33\,|g_{9/2}, 24\rangle.$

$|(5/2)_1; 947\rangle = 0.311\,|d_{5/2}, 00\rangle + 0.85\,|g_{9/2}, 12\rangle - 0.32\,|g_{9/2}, 22\rangle.$

$|(5/2)_2; 1435\rangle = 0.99\,|\mathrm{rot}\rangle.$
$|(7/2)_1; 1141\rangle = 0.99\,|\mathrm{rot}\rangle.$

$|(7/2)_2; 1420\rangle = 0.84\,|g_{9/2}, 12\rangle - 0.26\,|g_{9/2}, 24\rangle.$

$|(9/2)_0; 0\rangle \quad = 0.88\,|g_{9/2}, 00\rangle - 0.42\,|g_{9/2}, 12\rangle.$

$|(9/2)_1; 1337\rangle = 0.38\,|g_{9/2}, 00\rangle + 0.72\,|g_{9/2}, 12\rangle$
$-0.25\,|g_{9/2}, 20\rangle + 0.35\,|g_{9/2}, 22\rangle - 0.23\,|g_{9/2}, 24\rangle.$

$|(9/2)_2; 1974\rangle = 0.89\,|\mathrm{rot}\rangle - 0.38\,|g_{9/2}, 20\rangle.$

$|(11/2)_1; 1256\rangle = 0.88\,|g_{9/2}, 12\rangle + 0.29\,|g_{9/2}, 22\rangle - 0.34\,|g_{9/2}, 24\rangle.$

$|(11/2)_2; 1524\rangle = 0.997\,|\mathrm{rot}\rangle.$

$|(13/2)_1; 1198\rangle = 0.90\,|g_{9/2}, 12\rangle - 0.24\,|g_{9/2}, 22\rangle - 0.29\,|g_{9/2}, 24\rangle.$

Each state is indicated by its spin, ordinal number and energy in keV.

Table 3. Calculated transition probabilities $B(E2\uparrow)$ in units of $e^2 \cdot 10^{-50}$ cm^4 for ^{115}In

Final state $^nI+$	$B(E2\uparrow)$ in $e^2 \cdot 10^{-50}$ cm^4
$^2 9/2$	4.62
$^3 9/2$	0.06
$^1 5/2$	5.41
$^2 5/2$	0.005
$^1 7/2$	0.016
$^2 7/2$	3.63
$^1 11/2$	8.00
$^2 11/2$	0.02
$^1 13/2$	9.84
$^2 13/2$	0.03

pure rotational states when they are mixed with hole-core and particle-core coupled states.

In Table 3 are presented the calculated electric quadrupole transitions. One immediately observes that the transitions from the quasi-rotational states to the spherical ground state are very small as we expect for shape-forbidden transitions. Some of the remaining transitions agree with those experimentally determined by Dietrich *et al.* [11] while others differ noticeably from them.

The Quadrupole moment of the ground state and the $^{1}3/2$ state are calculated to be 0.60 b and -0.60 b which are in good agreement with their respective experimental value of 0.861 ± 0.045 b Ref. [30] and 0.60 ± 0.08 b Ref. [31]. From this value the intrinsic quadrupole moment Q_0 for $K=1/2+$ rotational band is calculated to be 3.0 b which in turn agrees with 2.67 b obtained from $B(E2)$ values [7] as well as with a deformation $\delta=0.20$.

IV. Concluding Remarks

The present work permits an overall description o fthe positive-parity low-lying states of odd-mass indium isotopes with a reasonable success. The model consists of the coupling of a proton hole and a proton particle to the quadrupole vibrations of the tin and cadmium cores together with the mixing of deformed sttaes. The interaction matrix elements are treated exactly and consequently the $B(E2)$ transition rates between initial and final states of similar or different nature could be calculated.

As mentioned before some discrepancies still occur that perhaps would be overcome with a more sophisticated treatment. In spite of them, we feel inclined to conclude that our model is a good starting point to analyze the coexistence of deformed and spherical states in odd-mass nuclei.

It is hoped that some experiments will be encouraged to establish definitely the nature of the different states predicted by us.

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