Algebraic functions in quasiprimal algebras

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Abstract

A function is *algebraic* on an algebra $\mathbf{A}$ if it can be implicitly defined by a system of equations on $\mathbf{A}$. In this note we give a semantic characterization for algebraic functions on quasiprimal algebras. This characterization is applied to obtain necessary and sufficient conditions for a quasiprimal algebra $\mathbf{A}$ to have every one of its algebraic functions be a term function. We also apply our results to particular algebras such as finite fields and monadic algebras.

1 Introduction

Let $\mathbf{A}$ be an algebra and let $t_i(x_1, \ldots, x_n, z_1, \ldots, z_m), s_i(x_1, \ldots, x_n, z_1, \ldots, z_m)$ be terms, for $i = 1, \ldots, k$. Suppose that for every $(a_1, \ldots, a_n) \in A^n$ the system of equations

$$\begin{aligned} t_1(\overline{a}, \overline{z}) &= s_1(\overline{a}, \overline{z}) \\ &\vdots \\ t_k(\overline{a}, \overline{z}) &= s_k(\overline{a}, \overline{z}) \end{aligned}$$

has a unique solution $\overline{b} \in A^m$. One such system on $\mathbf{A}$ implicitly defines m functions $f_1, \ldots, f_m : A^n \to A$ by letting $(f_1(\overline{a}), \ldots, f_m(\overline{a}))$ be the unique $\overline{b} \in A^m$ such that

$$t_i(\overline{a}, \overline{b}) = s_i(\overline{a}, \overline{b}), \text{ for } i = 1, \ldots, k.$$

We call a function $f : A^n \to A$ *algebraic* on $\mathbf{A}$ if it can be implicitly defined by a system in the above described manner. There are many natural examples of algebraic functions. For instance, let $\mathbf{D} = (D, \wedge, \vee, 0, 1)$ be a complemented distributive lattice, and consider the complementation operation $c : D \to D$. This function is implicitly defined by the system

$$\begin{aligned} x \wedge z &= 0 \\ x \vee z &= 1, \end{aligned}$$

and thus is algebraic on $\mathbf{D}$.

Algebraic functions have some appealing properties, and are an interesting choice as expanding operations. The authors have investigated the subject of algebraic functions for general algebras (see [3]) as a natural development of their study of axiomatizability by sentences of the form $\forall\exists! \wedge p = q$ (see [2]).

A finite algebra $\mathbf{A}$ is called *quasiprimal* if the *ternary discriminator*, $\mathrm{d}^A : A^3 \to A$ defined by

$$\mathrm{d}^A(a,b,c) = \begin{cases} a & \text{if } a \neq b \\ c & \text{if } a = b, \end{cases}$$

is a term operation of $\mathbf{A}$.

In the current note we provide a complete characterization for algebraic functions on quasiprimal algebras (Theorem 6). We apply this result to several particular cases such as finite fields and monadic algebras, and to describe those quasiprimal algebras in which every algebraic function is a term-function.

2 Preliminaries

Throughout this note an *algebra* is a model of a first order language without relation symbols. Let $\mathbf{A}$ be an algebra. We write $\mathrm{Clo}\,\mathbf{A}$ to denote the set of term-functions of $\mathbf{A}$.

Let γ be an *inner isomorphism* of $\mathbf{A}$, that is, an isomorphism between subalgebras of $\mathbf{A}$. We say that a function $f : A^n \to A$ *preserves* γ if $\mathrm{Dom}(\gamma)$ is closed under f, and for any $a_1, \ldots, a_n \in \mathrm{Dom}(\gamma)$ we have $\gamma f(a_1, \ldots, a_n) = f(\gamma(a_1), \ldots, \gamma(a_n))$.

EFD-sentences and algebraic functions

An *equational function definition sentence* (EFD-sentence for brevity) in the language $\mathcal{L}$ is a sentence of the form

$$\forall x_1 \ldots x_n \exists! z_1 \ldots z_m \bigwedge_{i=1}^{k} s_i(\overline{x}, \overline{z}) = t_i(\overline{x}, \overline{z}),$$

where s_i, t_i are $\mathcal{L}$-terms, $n, m \geq 0$. In this note we are only concerned with EFD-sentences having $m \geq 1$, so in the sequel we assume that EFD-sentences have at least one existential quantifier. Let φ be as in the display above and suppose $\mathbf{A} \vDash \varphi$, the *function defined by φ in* $\mathbf{A}$ is the map

$$\begin{array}{lcl} A^n & \to & A^m \\ \overline{a} & \mapsto & \text{the only } \overline{b} \in A^m \text{ such that } \bigwedge_{i=1}^{k} s_i(\overline{a}, \overline{b}) = t_i(\overline{a}, \overline{b}). \end{array}$$

We write $[\varphi]^{\mathbf{A}}$ to denote this function, and if $\pi_j : A^m \to A$ is the jth canonical projection, let $[\varphi]_j^{\mathbf{A}} = \pi_j \circ [\varphi]^{\mathbf{A}}$, for $j = 1, \ldots, m$. After introducing this notation we have an easy way to identify algebraic functions, in fact:

Remark 1 *A function is algebraic on $\mathbf{A}$ if and only if it is of the form $[\varphi]_j^{\mathbf{A}}$, for some EFD-sentence φ that holds in $\mathbf{A}$.*

For a more thorough account on the basic properties of algebraic functions we refer the reader to [3]. One elementary fact we do need to mention is:

Lemma 2 *Let φ be an EFD-sentence such that $\mathbf{A} \vDash \varphi$, and let $\mathbf{B} \in \mathbb{S}(\mathbf{A})$. Then $\mathbf{B} \vDash \varphi$ iff $[\varphi]^{\mathbf{A}}(B^n) \subseteq B^m$.*

Recall that a *clone* on a set A is a set of operations of finite arity on A which contains all projections and is closed under composition. Of course, the term-functions on an algebra $\mathbf{A}$ form a clone on A.

As shown in [3], the set of algebraic functions on $\mathbf{A}$ is a clone, which we denote by $\mathrm{Clo}_{alg}\,\mathbf{A}$. It is easily seen that $\mathrm{Clo}\,\mathbf{A} \subseteq \mathrm{Clo}_{alg}\,\mathbf{A}$.

We shall need the following:

Lemma 3 *Let $\mathbf{A}$ be any algebra and let $f_1, \dots, f_k \in \mathrm{Clo}_{alg}\,\mathbf{A}$.*

(1) There is an EFD-sentence φ such that $\mathbf{A} \vDash \varphi$ and

$$f_1, \dots, f_k \in \mathrm{Clo}(\mathbf{A}, [\varphi]_1^{\mathbf{A}}, \dots, [\varphi]_m^{\mathbf{A}}).$$

(2) $\mathrm{Clo}_{alg}\,(\mathbf{A}, f_1, \dots, f_k) = \mathrm{Clo}_{alg}\,\mathbf{A}$.

Proof. (1). Let $\varphi_i = \forall x_1 \dots x_{n_i} \exists! z_1 \dots z_{m_i}\ \varepsilon_i(\overline{x}, \overline{z})$ be such that $[\varphi_i]_1^{\mathbf{A}} = f_i$, for $i = 1, \dots, k$, and let $n = \max\{n_i : i = 1, \dots, k\}$. If we take

$$\varphi = \forall x_1 \dots x_n \exists! z_1^1 \dots z_{m_1}^1 \dots z_1^k \dots z_{m_k}^k\ \varepsilon_1(\overline{x}, z_1^1, \dots, z_{m_1}^1) \wedge \dots \wedge \varepsilon_k(\overline{x}, z_1^k, \dots, z_{m_k}^k)$$

it is clear that φ has the desired property.

(2). First note that $\mathrm{Clo}_{alg}\,(\mathbf{A}, f_1, \dots, f_k) = \mathrm{Clo}_{alg}((\mathbf{A}, f_1, \dots, f_{k-1}), f_k)$, thus it suffices to prove the case $k = 1$. Let $f \in \mathrm{Clo}_{alg}\,(\mathbf{A}, f_1)$ and let $\varphi = \forall \overline{x} \exists! \overline{z}\ \varepsilon(\overline{x}, \overline{z})$ be an EFD-sentence of the expanded language such that $f = [\varphi]_1^{(\mathbf{A}, f_1)}$. By means of a routine inductive argument we can suppose w.l.o.g. that each term occurring in φ is either a variable or a basic operation applied to variables. Take $\psi = \forall \overline{y} \exists! \overline{w}\ \delta(\overline{y}, \overline{w})$ such that $f_1 = [\psi]_1^{\mathbf{A}}$. Let $\varepsilon'(\overline{x}, \overline{z}, u_1)$ be the formula obtained by taking the first occurrence of a term of the form $f_1(v_1, \dots, v_n)$ in $\varepsilon(\overline{x}, \overline{z})$ and replacing it by a new variable u_1. Next, take

$$\varphi' = \forall \overline{x} \exists! \overline{z} u_1 \dots u_l\ \varepsilon'(\overline{x}, \overline{z}, u_1) \wedge \delta(v_1, \dots, v_n, \overline{u}),$$

where $u_2, \dots, u_l$ are new variables. Observe that $[\varphi']_1^{(\mathbf{A}, f_1)} = f$ and φ' has one less occurrence of f_1 than φ. Thus, it can be proved by induction that there is an EFD-sentence λ in the language of $\mathbf{A}$ such that $[\lambda]_1^{\mathbf{A}} = f$. ■

Quasiprimal algebras

Recall from the Introduction that a *quasiprimal* algebra is a finite algebra having the ternary discriminator as a term-operation. We shall need the following characterization of quasiprimal algebras by A. Pixley:

Theorem 4 ([7]) *For a finite algebra $\mathbf{A}$ the following are equivalent:*

(1) $\mathbf{A}$ is quasiprimal.

(2) If $f : A^n \to A$, with $n \geq 1$, preserves all inner isomorphisms of $\mathbf{A}$ then $f \in \mathrm{Clo}\,\mathbf{A}$.

3 Main results

A *fixed-point subalgebra* of $\mathbf{A}$ is a subalgebra of $\mathbf{A}$ having as its universe the fixed-point set of an automorphism of $\mathbf{A}$. A class of models $\mathcal{C}$ has the *finite intersection property* if given $\mathbf{A} \in \mathcal{C}$ and $\mathbf{A} \supseteq \mathbf{B}, \mathbf{C} \in \mathcal{C}$, such that $B \cap C \neq \emptyset$, then $\mathbf{B} \cap \mathbf{C} \in \mathcal{C}$. Before we can state our main result we need to introduce a closure operator on $\mathbb{S}(\mathbf{A})$. Given $\mathcal{F} \subseteq \mathbb{S}(\mathbf{A})$ we define $\mathbb{K}(\mathcal{F})$ to be the smallest subset of $\mathbb{S}(\mathbf{A})$ such that:

- $\mathcal{F} \subseteq \mathbb{K}(\mathcal{F})$
- all trivial subalgebras of $\mathbf{A}$ are in $\mathbb{K}(\mathcal{F})$
- $\mathbb{K}(\mathcal{F})$ is closed under fixed-point subalgebras
- $\mathbb{K}(\mathcal{F})$ has the finite intersection property
- $\mathbb{K}(\mathcal{F})$ is closed under isomorphisms relative to $\mathbb{S}(\mathbf{A})$.

Lemma 5 *Let $\mathbf{A}$ be a quasiprimal algebra, and let $\mathcal{F} \subseteq \mathbb{S}(\mathbf{A})$. Then, there is an EFD-sentence φ such that $\mathcal{F} = \mathrm{Mod}(\varphi) \cap \mathbb{S}(\mathbf{A})$ if and only if $\mathbb{K}(\mathcal{F}) = \mathcal{F}$.*

Proof. We prove the nontrivial direction. Since $\mathbb{K}(\mathcal{F}) = \mathcal{F}$, Theorem 1 of [4] says that there is a finite set Σ of $\forall\exists!$-sentences such that $\mathcal{F} = \mathrm{Mod}(\Sigma) \cap \mathbb{S}(\mathbf{A})$. A detailed inspection of the way the sentences of Σ are built in the proof of the aforementioned theorem shows that the matrix of each one of these sentences is satisfiable by trivial algebras. (For this to be true it is a key fact that all trivial subalgebras of $\mathbf{A}$ are in $\mathcal{F}$.) Now, by the well known translation result for quasiprimal algebras [8], we can obtain a set Σ' of EFD-sentences such that $\mathrm{Mod}(\Sigma') \cap \mathbb{S}(\mathbf{A}) = \mathrm{Mod}(\Sigma) \cap \mathbb{S}(\mathbf{A}) = \mathcal{F}$. Finally, it is a routine exercise to check that the conjunction of EFD-sentences is logically equivalent to an EFD-sentence. ■

We are now ready to present our characterization of algebraic functions in quasiprimal algebras.

Theorem 6 *Let $\mathbf{A}$ be quasiprimal, and let $f : A^n \to A$ with $n \geq 1$. T.f.a.e.:*

(1) f is algebraic on $\mathbf{A}$.

(2) There are $f_1, \ldots, f_{m-1} : A^n \to A$ such that:

(a) every trivial subalgebra of $\mathbf{A}$ is closed under $f, f_1, \ldots, f_{m-1}$, and

(b) if $\mathbf{B}, \mathbf{C}$ are subalgebras of $\mathbf{A}$ such that $f(\bar{b}), f_1(\bar{b}), \ldots, f_{m-1}(\bar{b}) \in B$ for every $\bar{b} \in B^n$, and $\gamma : \mathbf{B} \to \mathbf{C}$ is an isomorphism, then $f, f_1, \ldots, f_{m-1}$ preserve γ.

(3) f preserves isomorphisms between members of $\mathbb{K}(\mathbf{A})$.

(4) $f \in \mathrm{Clo}(\mathbf{A}, [\varphi]_1^{\mathbf{A}}, \ldots, [\varphi]_k^{\mathbf{A}})$ *for some EFD-sentence* φ *such that* $\mathrm{Mod}(\varphi) \cap \mathbb{S}(\mathbf{A}) = \mathbb{K}(\mathbf{A})$.

Furthermore, if φ *is any EFD-sentence satisfying* $\mathrm{Mod}(\varphi) \cap \mathbb{S}(\mathbf{A}) = \mathbb{K}(\mathbf{A})$ *then*

$$\mathrm{Clo}_{alg}\,\mathbf{A} = \mathrm{Clo}(\mathbf{A}, [\varphi]_1^{\mathbf{A}}, \ldots, [\varphi]_k^{\mathbf{A}}).$$

Proof. (1)⇒(2). This is easy.

(2)⇒(3). Let $\mathcal{F} = \{\mathbf{B} \in \mathbb{S}(\mathbf{A}) : f(\bar{b}), f_1(\bar{b}), \ldots, f_{m-1}(\bar{b}) \in B$, for every $\bar{b} \in B^n\}$. It is straightforward to check that $\mathbb{K}(\mathcal{F}) = \mathcal{F}$, and since $\mathbf{A} \in \mathcal{F}$ we have $\mathbb{K}(\mathbf{A}) \subseteq \mathcal{F}$. Thus (3) follows.

(3)⇒(4). Lemma 5 produces an EFD-sentence φ such that $\mathrm{Mod}(\varphi) \cap \mathbb{S}(\mathbf{A}) = \mathbb{K}(\mathbf{A})$. Let $\mathbf{A}_e = (\mathbf{A}, [\varphi]_1^{\mathbf{A}}, \ldots, [\varphi]_k^{\mathbf{A}})$. Observe that from Lemma 2 we have $\mathbb{S}(\mathbf{A}_e) = \mathbb{K}(\mathbf{A})$, and so f preserves inner isomorphisms of $\mathbf{A}_e$. Thus, by Theorem 4, f is a term-function of $\mathbf{A}_e$, and we have proven (4).

(4)⇒(1). Direct from Lemma 3.

Finally, observe that the only fact used about φ in (3)⇒(4) is that $\mathrm{Mod}(\varphi) \cap \mathbb{S}(\mathbf{A}) = \mathbb{K}(\mathbf{A})$, thus (1) and (4) are equivalent for any such φ, proving the furthermore part. ■

Interestingly, a result almost identical to the above holds for a class of algebras broader than that of quasiprimal ones.

Corollary 7 *Suppose* $\mathbf{A}$ *is a finite algebra having the discriminator as an algebraic function. Then (1)-(3) of Theorem 6 are equivalent for* $\mathbf{A}$*, and these three conditions are equivalent to*

(4′) $f \in \mathrm{Clo}(\mathbf{A}, [\varphi]_1^{\mathbf{A}}, \ldots, [\varphi]_k^{\mathbf{A}})$ *for some EFD-sentence* φ *such that* $\mathrm{Mod}(\varphi) \cap \mathbb{S}(\mathbf{A}) = \mathbb{K}(\mathbf{A})$ *and* $\mathrm{d}^A \in \mathrm{Clo}(\mathbf{A}, [\varphi]_1^{\mathbf{A}}, \ldots, [\varphi]_k^{\mathbf{A}})$.

Furthermore, if φ *is any EFD-sentence satisfying* $\mathrm{Mod}(\varphi) \cap \mathbb{S}(\mathbf{A}) = \mathbb{K}(\mathbf{A})$ *and* $\mathrm{d}^A \in \mathrm{Clo}(\mathbf{A}, [\varphi]_1^{\mathbf{A}}, \ldots, [\varphi]_k^{\mathbf{A}})$*, then*

$$\mathrm{Clo}_{alg}\,\mathbf{A} = \mathrm{Clo}(\mathbf{A}, [\varphi]_1^{\mathbf{A}}, \ldots, [\varphi]_k^{\mathbf{A}}).$$

Proof. Observe that the proofs of (4)⇒(1)⇒(2)⇒(3) of Theorem 6 are valid for any algebra $\mathbf{A}$. We prove (3)⇒(4′). Let $\mathbf{A}^*$ be the expansion obtained from $\mathbf{A}$ by adding the ternary discriminator as a basic operation. Since $\mathbf{A}$ and $\mathbf{A}^*$ have the same subalgebras and inner isomorphims it is clear that $\mathbb{K}(\mathbf{A}^*) = \mathbb{K}(\mathbf{A})$. So, Theorem 6 says that there is an EFD-sentence ψ such that $\mathrm{Mod}(\psi) \cap \mathbb{S}(\mathbf{A}^*) = \mathbb{K}(\mathbf{A}^*)$, and $f \in \mathrm{Clo}(\mathbf{A}^*, [\psi]_1^{\mathbf{A}^*}, \ldots, [\psi]_k^{\mathbf{A}^*})$. By (2) of Lemma 3, the functions $[\psi]_1^{\mathbf{A}^*}, \ldots, [\psi]_k^{\mathbf{A}^*}$ are algebraic on $\mathbf{A}$, and now by (1) of the same lemma, there is an EFD-sentence φ such that

$$f \in \mathrm{Clo}(\mathbf{A}^*, [\psi]_1^{\mathbf{A}^*}, \ldots, [\psi]_k^{\mathbf{A}^*}) \subseteq \mathrm{Clo}(\mathbf{A}, [\varphi]_1^{\mathbf{A}}, \ldots, [\varphi]_m^{\mathbf{A}}).$$

It remains to check that $\mathrm{Mod}(\varphi) \cap \mathbb{S}(\mathbf{A}) = \mathbb{K}(\mathbf{A})$. Clearly, $\mathbb{K}(\mathbf{A}) \subseteq \mathrm{Mod}(\varphi) \cap \mathbb{S}(\mathbf{A})$, since $\mathbf{A} \vDash \varphi$. Finally, if $\mathbf{B} \in \mathrm{Mod}(\varphi) \cap \mathbb{S}(\mathbf{A})$, from the inclusion in the display above it follows that B is closed under $[\psi]_1^{\mathbf{A}^*}, \ldots, [\psi]_k^{\mathbf{A}^*}$. Thus, by Lemma 2,

$$\mathbf{B}^* \in \mathrm{Mod}(\psi) \cap \mathbb{S}(\mathbf{A}^*) = \mathbb{K}(\mathbf{A}^*).$$

and hence $\mathbf{B} \in \mathbb{K}(\mathbf{A})$.

The furthermore part is proved exactly as in Theorem 6. ■

Recall that the quaternary discriminator on a set A is the function

$$\mathrm{s}^A(x, y, z, w) = \begin{cases} z & \text{if } x = y, \\ w & \text{if } x \neq y. \end{cases}$$

Each discriminator is obtainable from the other as a term-operation [1]. Thus for any algebra $\mathbf{A}$ we have

$$\mathrm{s}^A \in \mathrm{Clo}_{alg}\, \mathbf{A} \iff \mathrm{d}^A \in \mathrm{Clo}_{alg}\, \mathbf{A}.$$

As shown in [5], it is possible to describe those algebras in which the quaternary discriminator is *mono-algebraic*, i.e., algebraic and definable by an EFD-sentence with only one existential quantifier.

Theorem 8 ([5]) *Let $\mathbf{A}$ be a finite algebra and let $\mathcal{Q}$ be the quasivariety generated by $\mathbf{A}$. The following are equivalent:*

(1) The quaternary discriminator is mono-algebraic on $\mathbf{A}$.

(2) $\mathcal{Q}$ has equationally definable relative principal congruences, and every non-trivial subalgebra of $\mathbf{A}$ is a relatively simple member of $\mathcal{Q}$.

So Corollary 7 applies to all algebras satisfying (2) of Theorem 8.

To give a quick example, let $\mathbf{3}$ be the simple 3-element de Morgan algebra. It is easy to see that the quasivariety generated by $\mathbf{3}$ coincides with the variety generated by $\mathbf{3}$, and it is well known that this variety has equationally definable principal congruences. Now, $\mathbf{3}$ has no proper automorphisms, so $\mathbb{K}(\mathbf{3}) = \{\mathbf{3}\}$ and it follows that every finitary operation is an algebraic function on $\mathbf{3}$ (cf. Proposition 11 below).

4 Applications and examples

Quasiprimal algebras without new algebraic functions An interesting application of Theorem 6 is the following:

Theorem 9 *Let $\mathbf{A}$ be a quasiprimal algebra. The following are equivalent:*

(1) If $f : A^n \to A$, with $n \geq 1$, is algebraic on $\mathbf{A}$ then $f \in \mathrm{Clo}\, \mathbf{A}$.

(2) $\mathbb{S}(\mathbf{A}) = \mathbb{K}(\mathbf{A})$.

Proof. (1)$\Rightarrow$(2) For the sake of contradiction suppose there is $\mathbf{B} \in \mathbb{S}(\mathbf{A}) \setminus \mathbb{K}(\mathbf{A})$. As $\mathbb{K}(\mathbf{A})$ is closed under non-empty intersections, there is a smallest $\mathbf{C} \in \mathbb{K}(\mathbf{A})$ such that $B \subseteq C$. Let $b_1, \dots, b_n$ be an enumeration of B, and let $a \in C \setminus B$. Define $f : A^n \to A$ by

$$f(\overline{x}) = \begin{cases} \gamma(a) & \text{if } \gamma(\overline{b}) = \overline{x} \text{ and } \gamma \text{ is an isom. between members of } \mathbb{K}(\mathbf{A}), \\ x_1 & \text{otherwise.} \end{cases}$$

To see that f is well defined suppose $\gamma(\overline{b}) = \delta(\overline{b})$. Observe that by the minimality of $\mathbf{C}$ we have that $C \subseteq \mathrm{Dom}(\gamma) \cap \mathrm{Dom}(\delta)$, thus we may actually assume $C = \mathrm{Dom}(\gamma) = \mathrm{Dom}(\delta)$. Now $B \subseteq \mathrm{Fix}(\delta^{-1} \circ \gamma) \in \mathbb{K}(\mathbf{A})$, and so $C \subseteq \mathrm{Fix}(\delta^{-1} \circ \gamma)$. Clearly this yields $\gamma(a) = \delta(a)$.

Next we prove that f satisfies (3) of Theorem 6. We only show that f preserves the universes of algebras in $\mathbb{K}(\mathbf{A})$. Take $\mathbf{D} \in \mathbb{K}(\mathbf{A})$ and suppose γ is an isomorphism between members of $\mathbb{K}(\mathbf{A})$ such that $\gamma(\overline{b}) \in D$. Observe that $\gamma(\mathbf{C})$ is the smallest member of $\mathbb{K}(\mathbf{A})$ containing $\gamma(B)$. Thus $f(\gamma(\overline{b})) = \gamma(a) \in \gamma(C) \subseteq D$. We conclude that f is algebraic on $\mathbf{A}$, but since $f(\overline{b}) = a \notin B$, f cannot be a term-function.

(2)$\Rightarrow$(1) This follows directly from Theorems 4 and 6. ∎

For example, let A be a finite set and take $\mathbf{A} = (A, \mathrm{d}^A)$. Since every permutation of A is an isomorphism of $\mathbf{A}$ it follows that $\mathbb{K}(\mathbf{A}) = \mathcal{P}(A) = \mathbb{S}(\mathbf{A})$, where $\mathcal{P}(A)$ stands for the power set of A. Thus, $\mathrm{Clo}_{alg}\,\mathbf{A} = \mathrm{Clo}\,\mathbf{A}$.

Monadic algebras Recall that every finite and simple *monadic algebra* is of the form $\mathbf{M} = (M, \wedge, \vee, ^c, c_0, 0, 1)$ where $(M, \wedge, \vee, ^c, 0, 1)$ is a finite boolean algebra and c_0 is a unary operation on A satisfying

$$c_0(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0. \end{cases}$$

Finite and simple monadic algebra are quasiprimal. For proofs of these facts and a more detailed account on monadic algebras see [1]. It is easy to check that given $\mathbf{S}$ a subalgebra of $\mathbf{M}$, there is an automorphism γ of $\mathbf{M}$ such that $\mathrm{Fix}(\gamma) = S$. Thus, applying Theorem 9 yields

$$\mathrm{Clo}_{alg}\,\mathbf{M} = \mathrm{Clo}\,\mathbf{M}.$$

It is worth mentioning that this fact was obtained through different means in [3].

Finite fields Let $\mathbf{GF}(p^n)$ be the finite field of order p^n, considered as a model of the language $\{+, -, \cdot, 0, 1\}$. Note that the term

$$x(x-y)^{p^n-1} + z(1-(x-y)^{p^n-1})$$

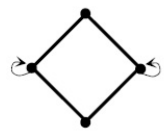
represents the ternary discriminator in $\mathbf{GF}(p^n)$. It is well known that the automorphism group of $\mathbf{GF}(p^n)$ is cyclic and consists of the successive powers of the

Frobenius automorphism $\gamma(x) = x^p$. Also, for each divisor d of n, $\mathbf{GF}(p^n)$ has exactly one subalgebra $\mathbf{S}_d$ of order p^d, which is isomorphic with $\mathbf{GF}(p^d)$, and these are all the subalgebras of $\mathbf{GF}(p^n)$. Now, given d a divisor of n, observe that $\mathrm{Fix}(\gamma^d) = S_d$ and thus

$$\mathrm{Clo}_{alg}\, \mathbf{GF}(p^n) = \mathrm{Clo}\, \mathbf{GF}(p^n).$$

It is interesting to note that in the case of infinite fields the situation changes as the multiplicative inverse (with $0^{-1} = 0$) is algebraic but not a term-operation. Moreover, in the case of algebraically closed fields of characteristic 0, the multiplicative inverse together with the basic operations of the field generate the clone of algebraic functions (see [6]).

Two simple De Morgan algebras Let $\mathbf{D}$ denote the De Morgan algebra in the picture.



Of course, the language of $\mathbf{D}$ is $\{\vee, \wedge, -, 0, 1\}$. Let $\mathbf{2}$ denote the subalgebra of $\mathbf{D}$ with universe $\{0, 1\}$, and let ^{c} be the complement operation on D.

Proposition 10 *For a function $f : D^n \to D$ the following are equivalent.*

(1) $f \in \mathrm{Clo}_{alg}\, \mathbf{D}$

(2) $f \in \mathrm{Clo}\,(\mathbf{D},^c)$

(3) $f(\{0,1\}^n) \subseteq \{0,1\}$ *and f preserves the only nontrivial automorphism of* $\mathbf{D}$.

Proof. We show first that the ternary discriminator is algebraic on $\mathbf{D}$. Observe that the complement operation $x \mapsto x^c$ is algebraic on $\mathbf{D}$ since

$$\varphi_c = \forall x \exists! z\ x \wedge z = 0 \ \&\ x \vee z = 1$$

holds in $\mathbf{D}$ and $x \mapsto x^c = [\varphi_c]^{\mathbf{D}}$. Further observe that the term

$$t(x,y) = (\overline{x}^c \vee \overline{y}) \wedge (\overline{x} \vee \overline{y}^c) \wedge (x^c \vee y) \wedge (x \vee y^c)$$

represents the equality test on $(\mathbf{D},^c)$. That is, $t^{(\mathbf{D},^c)}(x,y)$ is 1 when $x = y$ and 0 otherwise. So, the term

$$(x \wedge t(x,y)^c) \vee (z \wedge t(x,y))$$

represents d^D, the ternary discriminator on $(\mathbf{D},^c)$. Thus $\mathrm{d}^D \in \mathrm{Clo}\,(\mathbf{D},^c) \subseteq \mathrm{Clo}_{alg}\, \mathbf{D}$.

So, we can apply Corollary 7 and the fact that $\mathbb{K}(\mathbf{D}) = \{\mathbf{D}, \mathbf{2}\} = \mathrm{Mod}(\varphi_c) \cap \mathbb{S}(\mathbf{D})$ to conclude the proof. ■

Let $\mathbf{3} = (\{0, \frac{1}{2}, 1\}, \vee, \wedge, -, 0, 1)$ be the three-element De Morgan algebra. For $x \in \{0, \frac{1}{2}, 1\}$ define

$$\lfloor x \rfloor = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1. \end{cases}$$

Proposition 11 *Every function of finite arity on $\{0, \frac{1}{2}, 1\}$ is algebraic on $\mathbf{3}$. Furthermore,*

$$\mathrm{Clo}_{alg}\, \mathbf{3} = \mathrm{Clo}\left(\mathbf{3}, \lfloor . \rfloor, \frac{1}{2}\right).$$

Proof. The EFD-sentence

$$\varphi_{\lfloor . \rfloor} = \forall x \exists ! z \; (x = z \vee (x \wedge \overline{x}) \; \& \; z \vee \overline{z} = 1)$$

witnesses the fact that the function $x \mapsto \lfloor x \rfloor$ is algebraic on $\mathbf{D}$. Next, note that the term

$$t(x, y) = \lfloor x \wedge y \rfloor \vee \lfloor \overline{x} \wedge \overline{y} \rfloor \vee \overline{\lfloor x \vee \overline{x} \vee y \vee \overline{y} \rfloor}$$

represents the equality test on $(\mathbf{3}, \lfloor . \rfloor)$, and as in the proof of Proposition 10 we obtain $\mathrm{d}^{\{0, \frac{1}{2}, 1\}} \in \mathrm{Clo}\,(\mathbf{3}, \lfloor . \rfloor) \subseteq \mathrm{Clo}_{alg}\, \mathbf{3}$.

Clearly $\mathbb{K}(\mathbf{3}) = \{\mathbf{3}\}$, and since the only automorphism of $\mathbf{3}$ is the identity, by (3)⇔(1) of Corollary 7 every $f : \{0, \frac{1}{2}, 1\}^n \to \{0, \frac{1}{2}, 1\}$ is algebraic on $\mathbf{3}$.

It is a routine exercise to check that if the discriminator and all constant functions are in the clone of a finite algebra then every finite operation is in that clone. Since this is true of $\mathrm{Clo}\left(\mathbf{3}, \lfloor . \rfloor, \frac{1}{2}\right)$ the furthermore part is proved. ■

References

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