On the linear stability of the extreme Kerr black hole under axially symmetric perturbations

Sergio Dain and Ivan Gentile de Austria
Facultad de Matemática, Astronomía y Física, FaMAF,
Universidad Nacional de Córdoba,
Instituto de Física Enrique Gaviola, IFEG, CONICET,
Ciudad Universitaria, (5000) Córdoba, Argentina.

September 22, 2014

Abstract

We prove that for axially symmetric linear gravitational perturbations of the extreme Kerr black hole there exists a positive definite and conserved energy. This provides a basic criteria for linear stability in axial symmetry. In the particular case of Minkowski, using this energy we also prove pointwise boundedness of the perturbation in a remarkable simple way.

1 Introduction

Recently there has been considerable progress on the long standing and central open problem of black hole stability in General Relativity (see the review articles [15] [14] and reference therein). The following three aspects of this problem motivated the present work.

(i) Non-modal stability of linear gravitational perturbations: the non-modal stability of linear gravitational perturbations for the Kerr black hole still remains unsolved. The works of Regge, Wheeler [39], Zerilli [47] [46] and Moncrief [36] determined the modal linear stability of gravitational perturbations for the Schwarzschild black hole by ruling out exponential growth in time for every individual mode. The modal stability for the Kerr black hole was proved by Whiting [45] using the Teukolsky equation. However, modal stability is not enough to exclude that general linear perturbations grow unbounded in time (see, for example, the discussion in [43] and [15]). The study of black hole non-modal stability was initiated by Kay and Wald in [43] [32]. They prove that solutions of the linear wave equation on a Schwarzschild black hole background remain bounded by a constant for all time. An important ingredient in this proof is the use of conserved energies to control the norm of the solution. The analog of the Kay-Wald theorem on a large class of backgrounds which includes the slow rotating Kerr black hole was first proved by Dafermos and Rodniaski [16] and then, independently, in the special case of slow rotating Kerr by Andersson and Blue [1]. In [15] Dafermos and Rodniaski provide the essential elements of the proof of this theorem for the general subextremal Kerr black hole. Recently, this problem was finally solved in [17]. For a complete list of references with important related works on this subject see the review articles [15] [14] [28]. All these results concern the wave equation. For gravitational perturbations the only non-modal stability result was given very recently by Dotti [26] for the Schwarzschild black hole. There are, so far, no results regarding the non-modal stability of the Kerr black hole under linear gravitational perturbations.

(ii) Stability and instability of extreme black holes: extreme black holes are relevant because they lie on the boundary between black holes and naked singularities and hence it is expected that their study shed light on the cosmic censorship conjecture. Recently, Aretakis discovered certain instabilities for extreme black holes [3] [4]. These instabilities concern transverse derivatives of the field at the horizon: a conservation law ensures that the first transverse derivative of the field on the event horizon generically does not decay, this implies that the second transverse derivative of the field generically grows with time on the horizon. These instabilities were discovered first for the scalar wave equation on the extreme Reissner-Nordström black hole, a similar result also holds for the extreme Kerr black hole [6] [5]. These works were extended in several directions: for generic extreme black holes and linear gravitational perturbations [35], for certain higher dimensional extreme vacuum black holes [37]; for massive scalar field and for coupled linearized gravitational and electromagnetic perturbations [34], for a test scalar field with a nonlinear self-interaction in the extreme Kerr geometry [7]. An interesting relation between these instabilities and the Newman-Penrose constants was pointed out by Bizon and Friedrich [9]. This relation was also independently observed by Lucietti, Murata, Reall and Tanahashi [34]. Finally, a numerical study of nonlinear evolution of this instability for spherically symmetric perturbations of an extreme Reissner-Nordström black hole was performed by Murata, H. S. Reall, and N. Tanahashi in [38].

An important question regarding the dynamical behaviour of extremal black holes is whether a non-extremal black hole can evolve to an extremal one at late times. In [40] Reiris proved that there exists arbitrary small perturbations of the extreme black hole initial data that can not decay in time into any extreme black hole. On the other hand, in [38] fine tuned initial data are numerically constructed which settle to an extreme Reissner-Nordström black hole. There is no contradiction between these two results since they apply to different kind of data. It is interesting to note that the construction in [40] relies on geometrical inequalities between area and charges on trapped surfaces (see [21] and reference therein), in contrast in the spacetime considered in [38] there are no trapped surfaces.

The discussion above concern instability of extreme black holes. However, there are also stability results for this class of black holes. The most relevant of them is that the solutions of the wave equations remain pointwise bounded in the black hole exterior region [3] (see also [22]).

(iii) Non-linear stability: the problem of the black hole non-linear stability remains largely open (see the discussion in [15] and reference therein). The linear studies previously discussed are expected to provide insight into the non-linear problem. However, this will be possible only if they rely on techniques that can be suitable extended to the non-linear regime. One of the most important of these techniques are the energy estimates.

The main result of this article is the following:

> *For axially symmetric linear gravitational perturbation of the extreme Kerr black hole there exists an energy which is positive definite and conserved.*

A precise version of this statement is given in Theorem 4.1. In the following we discuss the relation of this result with the points (i), (ii) and (iii) discussed above.

(i) The conserved energy for the linear perturbation has a similar structure as the energy of the wave equation: it is an integral over an spacelike surface of terms that involves squares of first derivatives of the perturbations. This energy is related with the second order expansion of the ADM mass. However it is important to stress that the positiveness of this energy can not be easily deduced from the positiveness of the ADM mass. In fact, as we will see, this result is proved as a consequence of highly non-trivial identities. It is also important to emphasize that this energy is positive also inside the ergosphere.

The energy expression and its conservation do not require any mode expansion of the fields. The existence of this conserved quantity provides a basic non-modal stability criteria for axially symmetric linear perturbation of the extreme Kerr black hole. Since the equation are linear and the coefficients of them do not depend on time, it is possible to construct an infinitely number of higher order conserved energies. We expect that these higher order energies can be used to prove pointwise boundedness of the solution, in a similar fashion as in [22]. In that reference the pointwise boundedness of solutions of the wave equation on the extreme Reissner-Nordström black hole was proved using only higher order energies estimates. But, up to now, we were not able to extend this result to the present context. However, in the particular case of the Minkowski background we prove a pointwise bound for the linear perturbations in a remarkable simple way. Comparing with the Minkowski case, the main difficulties to obtain pointwise estimates from the energy in the Kerr case are two: first, the equations for the norm and the twist are coupled and hence it is not possible to separate them as in the Minkowski case. Second, the coefficient of the equations are singular at the horizon and hence we can not use standard Sobolev estimates.

This conserved energy is closely related with the energy studied by Hollands and Wald [31] (see also [33]). We expect that the techniques used here to prove positiveness should also be useful in that context. Also, the boundary conditions at the horizon proposed in [31] are likely to be useful to generalize our results to the non-extreme case.

(ii) The existence of this conserved energy and its related stability criteria are not in contradiction with Aretakis instabilities. The situation is very similar as the one discussed in [22] for the case of the wave equation: the energy is only defined in the black hole exterior region and it does not control any transverse derivative at the horizon.

(iii) As we pointed out above, the energy used here is related with the ADM mass which is also conserved in the non-linear regime (see the discussion in [24]) . That is, the energy estimates used here are very likely to be useful in the non-linear case.

The plan of the article is the following. The expression of the conserved energy arises naturally in a particular gauge for the Einstein equation: the maximal–isothermal gauge. We review this gauge in section 2. In that section we also present the linearized equations on a class of stationary backgrounds. In section 3 we study the particular case of the Minkowski background, where we prove that the solutions are pointwise bounded in terms of a constant that depends only on the conserved energy, see theorem 3.1. In section 4 we study the extreme Kerr background and we prove the main result of this article given by theorem 4.1. Finally, in the appendices we write the Kerr solution in the maximal–isothermal gauge and we also prove a Sobolev like estimate needed in the proof of theorem 3.1.

2 Axisymmetric Einstein equations in the maximal–isothermal gauge

In axial symmetry, the maximal-isothermal gauge has the important property that the total ADM mass can be written as a positive definite integral on the spacelike hypersurfaces of the foliation and the integral is constant along the evolution [19]. The conserved energy for the linear perturbations will be obtained as an appropriate second order expansion of this integral. In this section we first review the full Einstein equations in this gauge in subsection 2.1 and then in subsection 2.2 we perform the linearization on a class of stationary backgrounds that include the Kerr black hole. On this class of backgrounds the linearized equations in this gauge have a remarkably simply form.

2.1 Einstein equations

Einstein equations in the maximal-isothermal gauge were studied, with slight variations, in several works [11], [41], [29], [24]. In this section we review these equations, we closely follow [24].

In axial symmetry, it possible to perform a symmetry reduction of Einstein equations to obtain a set of geometrical equations in the 3-dimensional quotient manifold in terms a Lorenzian 3-dimensional metric. See [24] for the details. In appendix A we explicitly perform this reduction for the Kerr metric.

On the 3-dimensional quotient manifold we take a foliation of spacelike surfaces. The intrinsic metric on the slices of the foliation is denoted by q_{AB} and the extrinsic curvature by χ_{AB}. Here the indices $A, B \cdots$ are 2-dimensional.

The maximal-isothermal gauge and its associated cylindrical coordinates (t, ρ, z) are defined by the following two conditions. For the the lapse, denoted by α, we impose the maximal condition on the 2-surfaces $t = constant$. That is, the trace χ of the extrinsic curvature vanishes

$$\chi = q^{AB}\chi_{AB} = 0. \tag{1}$$

The shift, denoted by β^A, is fixed by the requirement that the intrinsic metric q_{AB} has the following form

$$q_{AB} = e^{2u}\delta_{AB}, \tag{2}$$

where δ_{AB} is the fixed flat metric

$$\delta = d\rho^2 + dz^2. \tag{3}$$

For our purposes, the relevant geometries for the 2-dimensional spacelike surfaces are the half plane $\mathbb{R}^2_+$ (defined by $-\infty < z < \infty$, $0 \leq \rho < \infty$) for the Minkowski case or $\mathbb{R}^2_+ \setminus \{0\}$ for the black hole case. In that case the origin will represent an extra asymptotic end. For both cases the axis of symmetry is defined by $\rho = 0$.

The dynamical degree of freedom of the gravitational field are encoded in two geometrical scalars η and ω, the square of the norm and the twist of the axial Killing vector respectively. Due to the behaviour at the axis, instead of η, α and u it is often convenient to work with the auxiliary function σ, $\bar{\alpha}$ and q defined by

$$\eta = \rho^2 e^{\sigma}, \quad \alpha = \rho\bar{\alpha}, \quad u = \ln\rho + \sigma + q. \tag{4}$$

To write the equations we will make use of the following differential operators. The 2-dimensional Laplacian Δ defined by

$$\Delta q = \partial_\rho^2 q + \partial_z^2 q, \tag{5}$$

and the operator ${}^{(3)}\Delta$ defined as

$${}^{(3)}\Delta\sigma = \Delta\sigma + \frac{\partial_\rho\sigma}{\rho}. \tag{6}$$

This operator, which appears frequently in the rest of the article, is the flat Laplace operator in 3-dimensions written in cylindrical coordinates and acting on axially symmetric functions. The conformal Killing operator $\mathcal{L}$ acting on a vector β_A is defined by

$$(\mathcal{L}\beta)_{AB} = \partial_A\beta_B + \partial_B\beta_A - \delta_{AB}\partial_C\beta^C. \tag{7}$$

In these equations ∂ denotes partial derivatives with respect to the space coordinates (ρ, z) and all the indices are moved with the flat metric δ_{AB}. We denote by a dot the partial derivative with respect to t and we define the prime operator as

$$\eta' = \frac{1}{\alpha}\left(\dot{\eta} - \beta^A\partial_A\eta\right). \tag{8}$$

Einstein equations in the maximal-isothermal gauge are divided into three groups: evolution equations, constraint equations and gauge equations. The evolution equations are further divided into two groups, evolution equations for the dynamical degree of freedom (σ, ω) and evolution equations for the metric q_{AB} and second fundamental form χ_{AB}. Due to the axial symmetry, these equations are not independent (see the discussion in [41]). For example, the constraint equations are essentially equivalent to the evolution equations for the metric and second fundamental form. In particular, in this article we will not make use of the evolution equations for the metric and second fundamental form, we will always use instead a time derivative of the constraint equations.

Bellow we write the equations, for the deduction of them see [24]. We divide them in the three groups discussed above. In the next sections the linearization of these equations on different background is performed, for the sake of clarity we will always group them in the same way.

Evolution equations:

The evolution equations for σ and ω are given by[1]

$$-e^{2u}\sigma'' +^{(3)} \Delta\sigma + \partial_A\sigma\frac{\partial^A\bar{\alpha}}{\bar{\alpha}} - 2e^{2u}(\log\rho)'' + 2\frac{\partial_\rho\bar{\alpha}}{\bar{\alpha}\rho} = \frac{(e^{2u}\omega'^2 - |\partial\omega|^2)}{\eta^2}, \quad (9)$$

$$-e^{2u}\omega'' +^{(3)} \Delta\omega + \partial_A\omega\frac{\partial^A\bar{\alpha}}{\bar{\alpha}} = \frac{2\left(\partial_A\omega\partial^A\eta - e^{2u}\omega'\eta'\right)}{\eta}. \quad (10)$$

The evolution equation for the metric q_{AB} (by equation (2) this is only one equation for the conformal factor u) and the second fundamental form χ_{AB} are given by

$$2\dot{u} = \partial_A\beta^A + 2\beta^A\partial_A u, \quad (11)$$

$$\dot{\chi}_{AB} = \pounds_\beta\chi_{AB} - F_{AB} - \alpha G_{AB} - 2\alpha\chi_{AC}\chi^C_B, \quad (12)$$

where $\pounds$ denotes Lie derivative and we have defined

$$F_{AB} = \partial_A\partial_B\alpha - \frac{1}{2}\delta_{AB}\Delta\alpha - 2\partial_{(A}\alpha\partial_{B)}u + \partial_C\alpha\partial^C u\delta_{AB}, \quad (13)$$

and

$$G_{AB} = {}^{(3)}R_{AB} - \frac{1}{2}\delta_{AB}\,{}^{(3)}R_{CD}\delta^{CD}, \quad (14)$$

$${}^{(3)}R_{AB} = \frac{1}{2\eta^2}(\partial_A\eta\partial_B\eta + \partial_A\omega\partial_B\omega). \quad (15)$$

Constraint equations:

The momentum and Hamiltonian constraints are given by

$$\partial^B\chi_{AB} = -\frac{e^{2u}}{2\eta^2}\left(\eta'\partial_A\eta + \omega'\partial_A\omega\right), \quad (16)$$

$${}^{(3)}\Delta\sigma + \Delta q = -\frac{\varepsilon}{4\rho}, \quad (17)$$

[1] There were a misprint in equation (63) in [24], a minus sign is missing on the right hand side of this equation. We have corrected that in equation (9).

where we have defined the energy density ε by

$$\varepsilon = \left(\frac{e^{2u}}{\eta^2} \left(\eta'^2 + \omega'^2 \right) + |\partial \sigma|^2 + \frac{|\partial \omega|^2}{\eta^2} + 2e^{-2u} \chi^{AB} \chi_{AB} \right) \rho. \tag{18}$$

It is important to emphasize that ε is positive definite.

Gauge equations:

The gauge equations for lapse and shift are given by

$$\Delta \alpha = \alpha \left(e^{-2u} \chi^{AB} \chi_{AB} + e^{2u} \bar{\mu} \right), \tag{19}$$

$$(\mathcal{L}\beta)_{AB} = 2\alpha e^{-2u} \chi_{AB}, \tag{20}$$

where we have defined $\bar{\mu}$ by

$$\bar{\mu} = \frac{1}{2\eta^2} \left(\eta'^2 + \omega'^2 \right). \tag{21}$$

As we mentioned above, the most important property of this gauge is that the total ADM mass of the spacetime is given by the following integral on the half plane $\mathbb{R}^2_+$ of the positive definite energy density ε

$$m = \frac{1}{16} \int_{\mathbb{R}^2_+} \varepsilon \, d\rho dz. \tag{22}$$

Moreover, this quantity is conserved along the evolution in this gauge (see [19]). We emphasize that the domain of integration in (22) is $\mathbb{R}^2_+$ even in the case of a black hole (see the discussion in [20]).

We have introduced two slight changes of notation with respect to [24]. First we have suppressed the hat symbol over tensors like $\hat{\chi}^{AB}$ introduced in [24] to distinguish between indices moved with the flat metric δ_{AB} and with the metric q_{AB}. In this article there is no danger of confusion since all the indices are moved with the flat metric δ_{AB}. Second, we have defined the energy density ε in (18) with an extra factor ρ. This is convenient for the calculations presented in the next section since the integral in the mass (22) has then the flat volume element in $\mathbb{R}^2_+$ (in [24] the ρ factor appears in the volume element). The only disadvantage of this notation is that in the right hand side of the Hamiltonian constraint (17) an extra ρ appears in the denominator.

Boundary conditions:

At spacelike infinity we assume the following standard asymptotically flat fall off condition in the limit $r \to \infty$

$$\sigma, \beta^A, \chi_{AB}, \dot{\sigma}, \dot{\beta}^A, \dot{\chi}_{AB} = o_1(r^{-1/2}), \quad \bar{\alpha} - 1 = o_1(1), \tag{23}$$

where we write $f = o_j(r^k)$ if f satisfies $\partial^\alpha f = o(r^{k-|\alpha|})$, for $|\alpha| \leq j$, where α is a multi-index and the spherical radius r is defined by $r = \sqrt{\rho^2 + z^2}$. In the following we will also make use of a similar notation for $f = O_j(r^k)$.

At the axis the functions must satisfy the following parity conditions

$$\eta,\ \omega,\ \bar{\alpha},\ u,\ q,\ \sigma,\ \chi_{\rho\rho},\ \beta^z \text{ are even functions of } \rho, \tag{24}$$

and

$$\alpha,\ \chi_{\rho z},\ \beta^\rho \text{ are odd functions of } \rho. \tag{25}$$

Note that odd functions vanish at the axis and ρ derivative of even functions vanishes at the axis.

In the case of extreme Kerr black hole we have an extra asymptotic end, which in these coordinates is located at the origin. For that case we will assume the following behaviour in the limit $r \to 0$

$$\sigma, \beta^A, \chi_{AB}, \dot{\sigma}, \dot{\beta}^A, \dot{\chi}_{AB} = o_1(r^{-1/2}), \quad \bar{\alpha} - 1 = o_1(1). \tag{26}$$

These conditions encompass the asymptotically cylindrical behaviour typical of extreme black hole at this end (see the discussion in [20] and [23]).

The behaviour of the twist ω is more subtle because it contains the information of the angular momentum. It will be discussed in the next sections.

2.2 Linearization

Denote by ψ any of the unknowns of the previous equations. Consider a one-parameter family of exact solutions $\psi(\lambda)$. To linearize the equations with respect to the family $\psi(\lambda)$ means to take a derivative with respect to λ to the equations and evaluate them at $\lambda = 0$. We will use the following notation for the background and the first order linearization

$$\psi_0 = \psi(\lambda)|_{\lambda=0}, \quad \psi_1 = \left.\frac{d\psi(\lambda)}{d\lambda}\right|_{\lambda=0}. \tag{27}$$

We will assume that the background solution is stationary in this gauge, that is

$$\dot{\psi}_0 = 0. \tag{28}$$

Moreover, we will also assume that the background shift and second fundamental form vanished

$$\beta_0^A = 0, \quad \chi_{0AB} = 0. \tag{29}$$

The condition (29) is satisfied by the Kerr solution for any choice of the mass and angular momentum parameters, see appendix A. This condition simplifies considerably the equations. In particular, from (28) and (29) we deduce

$$\psi_0' = 0. \tag{30}$$

The first important consequence of the background assumptions (29) is that the first order expansion of the lapse is trivial. Namely, the right hand side of equation (19) is second order in λ, hence we obtain

$$\Delta\alpha_0 = 0, \quad \Delta\alpha_1 = 0. \tag{31}$$

Since the boundary condition for for α are independent of λ, it follows that the first order perturbation α_1 satisfies homogeneous boundary condition both at the axis and at infinity, and hence from equation (31) we obtain that

$$\alpha_1 = 0. \tag{32}$$

In contrast, the zero order lapse α_0 satisfies non-trivial boundary conditions. The specific value of α_0 will depend, of course, on the choice of background. Remarkably, for Minkowski and extreme Kerr we have $\alpha_0 = \rho$, as we will see in the next sections. But for non-extreme Kerr it has a different value (see appendix A). In this section we keep α_0 arbitrary in order to obtain general equations that can be used in future works for non-extreme black holes.

Using (32), (29) and (28) we find the following useful formulas

$$\psi_1' = \frac{1}{\alpha_0}\left(\dot{\psi}_1 - \beta_1^A\partial_A\psi_0\right), \tag{33}$$

$$\psi_1'' = \frac{1}{\alpha_0^2}\left(\ddot{\psi}_1 - \dot{\beta}_1^A\partial_A\psi_0\right). \tag{34}$$

Also, as consequence of the definition (4) we have the following relations between η and σ

$$\eta_0 = \rho^2 e^{\sigma_0}, \quad \eta_1 = \eta_0\sigma_1. \tag{35}$$

Using these assumptions it is straightforward to obtain the linearization of the equations presented in section 2.1. The result is the following.

Evolution equations:

The evolution equation for σ_1 and ω_1 are given by

$$-\frac{e^{2u_0}}{\alpha_0^2}\dot{p} +^{(3)}\Delta\sigma_1 + \frac{\partial_A\sigma_1\partial^A\bar{\alpha}_0}{\bar{\alpha}_0} = \frac{2}{\eta_0^2}\left(\sigma_1|\partial\omega_0|^2 - \partial_A\omega_1\partial^A\omega_0\right), \tag{36}$$

$$-\frac{e^{2u_0}}{\alpha_0^2}\dot{d} +^{(3)}\Delta\omega_1 + \frac{\partial_A\omega_1\partial^A\bar{\alpha}_0}{\bar{\alpha}_0} = 4\frac{\partial_\rho\omega_1}{\rho} + 2\partial_A\omega_1\partial^A\sigma_0 + 2\partial_A\omega_0\partial^A\sigma_1, \tag{37}$$

where we have defined the following two useful auxiliary variables

$$p = \dot{\sigma}_1 - \beta_1^A \partial_A \sigma_0 - 2\frac{\beta^\rho}{\rho}, \tag{38}$$

$$d = \dot{\omega}_1 - \beta_1^A \partial_A \omega_0. \tag{39}$$

The evolution equation for the metric and second fundamental form are given by

$$2\dot{u}_1 = \partial_A \beta_1^A + 2\beta_1^A \partial_A u_0, \tag{40}$$

$$\dot{\chi}_{1AB} = -\left(F_{1AB} + \alpha_0 G_{1AB}\right), \tag{41}$$

where

$$F_{1AB} = -2\partial_{(A}\alpha_0 \partial_{B)} u_1 + \delta_{AB}\partial_C \alpha_0 \partial^C u_1, \tag{42}$$

and

$$\begin{aligned} G_{1AB} = \frac{1}{2\eta_0^2}&\left(\partial_A \eta_1 \partial_B \eta_0 + \partial_A \eta_0 \partial_B \eta_1 + \partial_A \omega_1 \partial_B \omega_0 + \partial_A \omega_0 \partial_B \omega_1\right) \\ &- \frac{\sigma_1}{\eta_0^2}\left(\partial_A \eta_0 \partial_B \eta_0 + \partial_A \omega_0 \partial_B \omega_0\right) \\ &- \frac{\delta_{AB}}{2}\left[\frac{1}{\eta_0^2}\left(\partial_C \eta_0 \partial^C \eta_1 + \partial_C \omega_0 \partial^C \omega_1\right) - \frac{\sigma_1}{\eta_0^2}\left(|\partial \eta_0|^2 + |\partial \omega_0|^2\right)\right]. \end{aligned} \tag{43}$$

Constraint equations:
The momentum constraint and Hamiltonian constraints are given by

$$\partial^B \chi_{1AB} = -\frac{e^{2u_0}}{2\alpha_0}\left(p\left(\partial_A \sigma_0 + 2\frac{\partial_A \rho}{\rho}\right) + \frac{\partial_A \omega_0}{\eta_0^2} d\right), \tag{44}$$

$${}^{(3)}\Delta \sigma_1 + \Delta q_1 = -\frac{\varepsilon_1}{4\rho}, \tag{45}$$

where ε_1 is the first order term of the energy density (18), that is

$$\varepsilon_1 = \left(2\partial_A \sigma_0 \partial^A \sigma_1 + \frac{2\partial_A \omega_0 \partial^A \omega_1}{\eta_0^2} - \frac{2\sigma_1 |\partial \omega_0|^2}{\eta_0^2}\right)\rho. \tag{46}$$

Gauge equations:
We have seen that the first order lapse is zero. For the shift we have

$$\left(\mathcal{L}\beta_1\right)^{AB} = 2e^{-2u_0}\alpha_0 \chi_1^{AB}. \tag{47}$$

We have presented above the complete set of axially symmetric linear equations in the maximal–isothermal gauge. The conserved energy for this system of equation is calculated from the second variation of the energy density (18) as follows. Assume that $\psi(\lambda)$ has the following form

$$\psi(\lambda) = \psi_0 + \lambda\psi_1. \tag{48}$$

That it, we assume that the second order derivative with respect to λ of $\psi(\lambda)$ vanishes at $\lambda = 0$. For this kind of linear perturbations we define the second variation of ε as

$$\varepsilon_2 = \left.\frac{d^2\varepsilon(\lambda)}{d\lambda^2}\right|_{\lambda=0}. \tag{49}$$

Using (18) we obtain

$$\varepsilon_2 = \left(\frac{2e^{2u_0}}{\alpha_0^2}\left(p^2 + \frac{d^2}{\eta_0^2}\right) - 8\frac{\sigma_1\partial_A\omega_0\partial^A\omega_1}{\eta_0^2} + \right.$$
$$\left. +2|\partial\sigma_1|^2 + 4e^{-2u_0}\chi_1^{AB}\chi_{1AB} + 2\frac{|\partial\omega_1|^2}{\eta_0^2} + 4\frac{|\partial\omega_0|^2}{\eta_0^2}\sigma_1^2\right)\rho. \tag{50}$$

Note that ε_2, in contrast with ε, is not positive definite.

For further reference we write also the zero order expression for the energy density

$$\varepsilon_0 = \left(|\partial\sigma_0|^2 + \frac{|\partial\omega_0|^2}{\eta_0^2}\right)\rho, \tag{51}$$

and the masses associated with the different orders of the energy density

$$m_0 = \frac{1}{16}\int_{\mathbb{R}^2_+} \varepsilon_0 \, d\rho dz, \tag{52}$$

$$m_1 = \frac{1}{16}\int_{\mathbb{R}^2_+} \varepsilon_1 \, d\rho dz, \tag{53}$$

$$m_2 = \frac{1}{16}\int_{\mathbb{R}^2_+} \varepsilon_2 \, d\rho dz. \tag{54}$$

Recall that ε_1 has been calculated in (46).

We will prove that m_1 vanished and that m_2 is conserved and positive definite. Since we are interested in the study of linear stability, it is important for our present purpose (and also for future works on this subject) to prove these statements using only the linear equations, without referring to the original non-linear system. In the next sections we will perform these proofs. However, from the conceptual point of view and for further possible applications to the non-linear stability problem, it is important also to deduce these properties from the full equations. We discuss this point bellow.

Consider a general one-parameter family of exact solutions $\psi(\lambda)$ (i.e. we are not assuming the particular linear form (48)). For this family we compute the exact mass $m(\lambda)$ given by equation (22). This quantity is conserved, that is

$$\frac{dm(\lambda)}{dt} = 0, \tag{55}$$

This equation is valid for all λ. Taking derivatives with respect to λ of equation (55) and then evaluating them in $\lambda = 0$ we obtain that

$$\frac{d}{dt} m|_{\lambda=0} = 0, \tag{56}$$

$$\frac{d}{dt} \frac{dm}{d\lambda}\bigg|_{\lambda=0} = 0, \tag{57}$$

$$\frac{d}{dt} \frac{d^2m}{d\lambda^2}\bigg|_{\lambda=0} = 0. \tag{58}$$

We can, of course, take more derivatives with respect to λ, but this will not provide any useful conserved quantity for the linear equations.

It is clear that equations (56) and (57) are precisely

$$\frac{dm_0}{dt} = 0, \tag{59}$$

$$\frac{dm_1}{dt} = 0, \tag{60}$$

where m_0 and m_1 are given by (52) and (53) respectively.

The first equation (59) asserts that the mass of the background metric is conserved. This is of course valid even when the background solution is not stationary. In our case, since the background metric is stationary, not only m_0 is conserved but also the integrand ε_0, given by equation (51), is time independent, and hence the conservation (59) is trivial.

Since m_1 depends only on the background solution ψ_0 and the first order perturbation ψ_1 (recall that ψ_0 and ψ_1 are defined by (27) for a general family $\psi(\lambda)$) then equation (60) asserts that m_1 is a conserved quantity for the linear equations. That is, from the exact conservation law (55) we have deduced the conservation of m_1 for the linear equations.

For a general background, m_1 will be non-zero. However, using the Hamiltonian formulation of General Relativity, it is possible to show that the first variation of the ADM mass vanishes on stationary solutions (see [8] and reference therein). In section 4 we explicitly perform this computation adapted to our settings.

For the third equation (58) the situation is different. This equation asserts that the quantity

$$\hat{m}_2 = \left.\frac{d^2 m}{d\lambda^2}\right|_{\lambda=0}, \tag{61}$$

is conserved

$$\frac{d\hat{m}_2}{dt} = 0. \tag{62}$$

However, $\hat{m}_2$ depends on the background solution ψ_0, the linear perturbation ψ_1 but also on the second order perturbation

$$\psi_2 = \left.\frac{d^2\psi(\lambda)}{d\lambda^2}\right|_{\lambda=0}. \tag{63}$$

Then $\hat{m}_2$ is not a quantity that can be computed purely in terms of the background solution ψ_0 and the linear perturbation ψ_1 and hence it can not be used for the linearized equations.

Note that the mass m_2 defined in (54) is computed only using first order perturbations (since we have assumed (48) to compute it). In principle, m_2 and $\hat{m}_2$ are different quantities. Hence the conservation law

$$\frac{dm_2}{dt} = 0, \tag{64}$$

can not be deduced directly from (62). But, as we will prove bellow, it turns out that if the background is stationary and hence the first variation m_1 vanishes, then we have $\hat{m}_2 = m_2$.

Let us compute explicitly $\hat{m}_2$. We define

$$\hat{\varepsilon}_2 = \left.\frac{d^2\varepsilon(\lambda)}{d\lambda^2}\right|_{\lambda=0}. \tag{65}$$

We emphasize that in (65) we are not assuming (48) and hence this is different from (49). The difference between ε_2 and $\hat{\varepsilon}_2$ is given by

$$\hat{\varepsilon}_2 - \varepsilon_2 = \left(2\partial_A\sigma_0\partial^A\sigma_2 + \frac{2\partial_A\omega_0\partial^A\omega_2}{\eta_0^2} - \frac{2\sigma_2|\partial\omega_0|^2}{\eta_0^2}\right)\rho. \tag{66}$$

In this calculation we have assumed that the background is stationary in this gauge (namely, we have assumed (28) and (29)). The difference between ε_2 and $\hat{\varepsilon}_2$ involves, of course, the second order perturbation σ_2 and ω_2. However, remarkably, the right hand side of (66) has exactly the same for as the first variation ε_1 if we replace σ_1 and ω_1 in ε_1 (given by (46)) by σ_2 and ω_2. Hence, if m_1 vanishes on stationary solutions then $\hat{m}_2 = m_2$ (that is, the integral of the right hand side of (66) vanishes). In fact, this result is general and well known in the calculus of variations with non-linear variations (see, for example, [30] p. 267).

Finally, let us discuss the sign of the second variation m_2. On Minkowski, the positive mass theorem clearly implies that the second variation of the mass should be positive since flat space is a global minimum of the mass. In the extreme Kerr case there is no obvious connection between the positivity of the mass and the second variation. However, it has been proved that the mass has a minimum at extreme Kerr under variations with fixed angular momentum [18][20]. To prove the positivity of the second variation m_2 on extreme Kerr in section 4 we will use similar techniques as in those references. As we pointed out above, for our purpose, it is important to prove this in terms only of the linearized equations.

3 Minkowski perturbations

The natural first application of the linear equations obtained in section 2.2 is to study the linear stability of Minkowski in axial symmetry. The problem of linear stability of Minkowski, without any symmetry assumptions, was solved in [12] and the non-linear stability of Minkowski was finally proved in [13]. The purpose of this section in to provide an alternative proof of the linear stability of Minkowski in axial symmetry using the gauge presented in the previous section. This is given in theorem 3.1 which constitutes the main result of this section.

In comparison with the results in [12], theorem 3.1 has the obvious disadvantage that it only applies to axially symmetric perturbation. Moreover in this theorem only pointwise boundedness of the solution is proved and not precise decay rates as in [12]. However, the advantage of this result is that it make use only of energy estimates that can be generalized to the black hole case as we will see in section 4.

This system of linear equations was studied numerically in [24] and analytically in [25]. The main difficulty is that the system is formally singular at the axis where $\rho = 0$. Theorem 3.1 generalize those works by including the twist and, more important, by obtaining a pointwise estimate of the solution in terms of conserved energies. We explain in more detail this point bellow.

The Minkowski background satisfies the assumptions (29). The value of the other background quantities are the following

$$\omega_0 = q_0 = \sigma_0 = 0, \tag{67}$$

and

$$u_0 = \ln \rho, \quad \eta_0 = \rho^2, \quad \alpha_0 = \rho. \tag{68}$$

Introducing the background quantities (67)–(68) on the linearized equations obtained in section 2.2 we arrive at the following set of equations for the linear axially symmetric perturbations of Minkowski.

Evolution equations:

The evolution equations for σ_1 and ω_1 are given by

$$-\dot{p} + {}^{(3)}\Delta\sigma_1 = 0, \tag{69}$$

$$-\ddot{\omega}_1 + {}^{(3)}\Delta\omega_1 = 4\frac{\partial_\rho \omega_1}{\rho}, \tag{70}$$

where we defined the auxiliary function p by

$$p = \dot{\sigma}_1 - \frac{2\beta_1^\rho}{\rho}. \tag{71}$$

The evolution equations for the metric and the extrinsic curvature are given by

$$2\dot{u}_1 = \partial_A \beta_1^A + 2\frac{\beta_1^\rho}{\rho}, \tag{72}$$

$$\dot{\chi}_{1AB} = 2\partial_{(A} q_1 \partial_{B)} \rho - \delta_{AB} \partial_\rho q_1. \tag{73}$$

Constraint equations:

The momentum and the Hamiltonian constraints takes the following form

$$\partial^A \chi_{1AB} = -p\partial_B \rho, \tag{74}$$

$$\Delta q_1 +^{(3)} \Delta\sigma_1 = 0. \tag{75}$$

Gauge equations for lapse and shift:

We have proved in section 2.2 that the first order lapse is zero. The equation for the shift is given by

$$(\mathcal{L}\beta_1)^{AB} = \frac{2}{\rho}\chi_1^{AB}. \tag{76}$$

For the mass density we have that

$$\varepsilon_0 = \varepsilon_1 = 0, \tag{77}$$

and hence we have

$$m_0 = m_1 = 0. \tag{78}$$

The second order mass density is given by

$$\varepsilon_2 = \left(2p^2 + 2\frac{\dot{\omega}_1^2}{\rho^4} + 2|\partial\sigma_1|^2 + 2\frac{|\partial\omega_1|^2}{\rho^4} + 4\frac{\chi_1^{AB}\chi_{1AB}}{\rho^2}\right)\rho. \tag{79}$$

It is important to note that ε_2, in the particular case of the Minkowski background, is positive definite.

Before presenting the main result, let us first discuss two simple but important properties of this set of equations. The first one (which only holds for the Minkowski background) is that the equation for the twist ω_1 (70) decouples completely from the other equations [2]. Then, it is useful to split the density ε_2 in two terms

$$\varepsilon_2 = \varepsilon_\sigma + \varepsilon_\omega, \tag{80}$$

where

$$\varepsilon_\sigma = \left(2p^2 + 2|\partial\sigma_1|^2 + 4\frac{\chi_1^{AB}\chi_{1AB}}{\rho^2}\right)\rho, \tag{81}$$

$$\varepsilon_\omega = 2\frac{\dot{\omega}_1^2}{\rho^3} + 2\frac{|\partial\omega_1|^2}{\rho^3}, \tag{82}$$

and the corresponding masses

$$m_2 = m_\sigma + m_\omega, \tag{83}$$

where

$$m_\sigma = \int_{\mathbb{R}^2_+} \varepsilon_\sigma \, d\rho dz, \quad m_\omega = \int_{\mathbb{R}^2_+} \varepsilon_\omega \, d\rho dz. \tag{84}$$

Note that all the densities are positive definite.

Equation (70) is equivalent to the following homogeneous wave equation

$$-\ddot{\bar{\omega}}_1 +^{(7)} \Delta\bar{\omega}_1 = 0, \tag{85}$$

where ${}^{(7)}\Delta$ is the Laplacian in 7-dimensions acting on axially symmetric functions [3], namely

$${}^{(7)}\Delta\bar{\omega}_1 = \Delta\bar{\omega}_1 + 5\frac{\partial_\rho\bar{\omega}_1}{\rho}, \tag{86}$$

[2] We thank O. Rinne for pointing this out to us before this work was started.

[3] The trick of writing the 2-dimensional equations that appears in axially symmetric (which are formally singular at the axis) as regular equations in higher dimensions has provided to be very useful. It has been used, in a similar context, in [44] and [2].

and we have defined

$$\bar{\omega}_1 = \frac{\omega_1}{\rho^4}. \tag{87}$$

That is, the dynamic of the twist potential is determined by a wave equation and hence it is clear how to obtain decay estimates for the solution. In contrast, the equations for σ_1 are coupled and non-standard due to the formal singular behaviour at the axis (see the discussion in [24] and [25]). The wave equation (85) has associated the canonical energy density

$$\epsilon_{\bar{\omega}} = 2\left(\dot{\bar{\omega}}_1^2 + |\partial\bar{\omega}_1|^2\right)\rho^5, \tag{88}$$

and corresponding energy

$$m_{\bar{\omega}} = \int_{\mathbb{R}^3} \epsilon_{\bar{\omega}}\, d\rho dz. \tag{89}$$

The factor ρ^5 in (88) comes from the expression of the volume element in 7-dimensions in terms of the cylindrical coordinates $dx^7 = \rho^5 d\rho dz$. The two densities $\varepsilon_{\bar{\omega}}$ and ε_{ω} are related by a boundary term

$$\epsilon_{\bar{\omega}} - \varepsilon_{\omega} = -4\partial_\rho\left(\frac{\omega_1^2}{\rho^4}\right), \tag{90}$$

and hence $m_{\bar{\omega}} = m_\omega$ provided ω_1 satisfies appropriate boundary conditions. Note that equation (85) suggests that $\bar{\omega}_1$ and not ω_1 is the most convenient variable to impose the boundary conditions.

The second property (which will be also satisfied for the Kerr background and in general for any stationary background) is the following. The coefficients of the equations do not depend on time, hence if we take a time derivative to all equations we get a new set of equations for the time derivatives of the unknowns which are formally identical to the original ones. That is, the variables $\sigma_1, \omega_1, u_1, \beta_1, \chi_1$ satisfy the same equations as the time derivatives $\dot{\sigma}_1, \dot{\omega}_1, \dot{u}_1, \dot{\beta}_1, \dot{\chi}_1$. And the same is of course true for any number of time derivatives. In particular, if m is a conserved quantity, then we automatically get an infinity number of conserved quantities which has the same form as m but in terms of the n-th time derivatives of $\sigma_1, \omega_1, u_1, \beta_1, \chi_1$. For example, let us consider the mass m_σ defined by (81) and (84). It depends on the the functions p, σ_1 and χ_1, to emphasize this dependence we use the notation $m_\sigma[p, \sigma_1, \chi_1]$. Then we define $m_\sigma[\dot{p}, \dot{\sigma}_1, \dot{\chi}_1]$ as

$$m_\sigma[\dot{p}, \dot{\sigma}_1, \dot{\chi}_1] = \int_{\mathbb{R}^2_+}\left(2\dot{p}^2 + 2|\partial\dot{\sigma}_1|^2 + 4\frac{\dot{\chi}_1^{AB}\dot{\chi}_{1AB}}{\rho^2}\right)\rho d\rho dz. \tag{91}$$

If $m_\sigma[p, \sigma_1, \chi_1]$ is conserved along the evolution then also $m_\sigma[\dot{p}, \dot{\sigma}_1, \dot{\chi}_1]$ is conserved. The same applies for $m_\omega[\omega_1]$ and $m_{\bar{\omega}}[\bar{\omega}_1]$, for example we have

$$m_{\bar{\omega}}[\dot{\bar{\omega}}_1] = \int_{\mathbb{R}^2_+} \left(\ddot{\bar{\omega}}_1^2 + |\partial \dot{\bar{\omega}}_1|^2\right) \rho^5 d\rho dz. \tag{92}$$

We will make use also of the higher order masses $m_{\bar{\omega}}[\ddot{\bar{\omega}}_1]$ and $m_{\bar{\omega}}[\dddot{\bar{\omega}}_1]$.

Theorem 3.1. *Consider a smooth solution of the linearized equations presented above that satisfies the fall off conditions at infinity (23) and the regularity conditions at the axis (24), (25). Assume also that*

$$\dot{\bar{\omega}}_1, \bar{\omega}_1 = O_1(1), \tag{93}$$

at the axis and

$$\dot{\bar{\omega}}_1, \bar{\omega}_1 = o_1(r^{-5/2}), \tag{94}$$

at infinity, where we have defined

$$\bar{\omega}_1 = \frac{\omega_1}{\rho^4}. \tag{95}$$

Then, we have:

(i) The masses m_σ, m_ω and $m_{\bar{\omega}}$ defined by (84) and (89) are conserved along de evolution and $m_\omega = m_{\bar{\omega}}$. And hence, all higher order masses are also conserved.

(ii) The solution σ_1, ω_1 satisfy the following (time independent) bounds

$$C|\sigma_1| \leq m_\sigma[p, \sigma_1, \chi_1] + m_\sigma[\dot{p}, \dot{\sigma}_1, \dot{\chi}_1], \tag{96}$$

$$C\frac{|\omega_1|}{\rho^4} \leq m_\omega[\ddot{\omega}_1] + m_\omega[\dddot{\omega}_1], \tag{97}$$

where $C > 0$ is a numerical constant.

The value of ω at the axis determines the angular momentum (see, for example, [20]). Hence, the physical interpretation of the boundary conditions (93) is that the perturbations do not change the angular momentum of the background (which is zero in the case of Minkowski).

The conservation of m_σ in point (i) was proved in [24] . For completeness we review this proof and also we perform it in different variables which are the appropriate ones for the extreme Kerr black hole case treated in the next section.

We have already shown that the equation for ω_1 is decoupled and it can be converted into an standard wave equation in higher dimensions. Hence the dynamics of ω_1 is well known. In particular one has the classical pointwise estimates for solutions of the wave equation in 7-dimensions $|\bar{\omega}_1| \leq t^{-3}C$, where the constant C depends only on the initial data (see, for example, [42]). We present the weaker estimate (97) because it can be proved using only the conserved energies and is likely to be useful in the more complex case of the Kerr black hole, where the pure wave equations estimates are not available.

The most important part of theorem 3.1 is the estimate (96). In a previous work [25] the existence of solution of this set of equations was proved using an explicit (but rather complicated) representation in terms of integral transforms. In contrast, the a priori estimate (96) is proved in terms of only the conserved masses in a remarkably simple way. This estimate is expected to be useful in the non-linear regime.

Proof. (i) Since the equations are decoupled, we can treat the conservation for m_σ and m_ω separately. We begin with m_σ. Taking the time derivative of ε_σ we obtain

$$\dot{\varepsilon}_\sigma = 4\rho p\dot{p} + 4\rho\partial_A\sigma_1\partial^A\dot{\sigma}_1 + 8\frac{\chi_1^{AB}\dot{\chi}_{1AB}}{\rho}. \tag{98}$$

The strategy is to prove (using the linearized equations) that the right hand side of (98) is a total divergence and hence it integrates to zero (under appropriate boundary conditions). We calculate each terms individually.

For the first term we just use the definition of p given in equation (71) to obtain

$$4\rho p\dot{p} = 4\rho\dot{\sigma}_1\dot{p} - 8\beta_1^\rho\dot{p}. \tag{99}$$

For the second term we obtain

$$\begin{aligned} 4\rho\partial_A\sigma_1\partial^A\dot{\sigma}_1 &= 4\partial^A\left(\rho\dot{\sigma}_1\partial_A\sigma_1\right) - 4\dot{\sigma}_1\partial^A\left(\rho\partial_A\sigma_1\right), && (100)\\ &= 4\partial^A\left(\rho\dot{\sigma}_1\partial_A\sigma_1\right) - 4\rho\dot{\sigma}_1\,{}^{(3)}\Delta\sigma_1, && (101)\\ &= 4\partial^A\left(\rho\dot{\sigma}_1\partial_A\sigma_1\right) - 4\rho\dot{\sigma}_1\dot{p}, && (102)\end{aligned}$$

where in line (101) we have used the definition of the operator ${}^{(3)}\Delta$ given in equation (6) and in line (102) we have used equation (69).

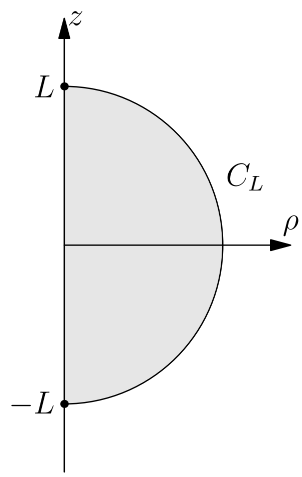

Figure 1: Domain of integration in $\mathbb{R}^2_+$.

Finally, for the third term we have

$$8\frac{\chi_1^{AB}\dot{\chi}_{1AB}}{\rho} = 4(\mathcal{L}\beta_1)^{AB}\dot{\chi}_{1AB}, \tag{103}$$

$$= 8\partial^A\beta_1^B\dot{\chi}_{1AB}, \tag{104}$$

$$= 8\partial^A\left(\beta_1^B\dot{\chi}_{1AB}\right) - 8\beta_1^B\partial^A\dot{\chi}_{1AB}, \tag{105}$$

$$= 8\partial^A\left(\beta_1^B\dot{\chi}_{1AB}\right) + 8\dot{p}\beta_1^\rho, \tag{106}$$

where in line (103) we have used the gauge equation (76), in line (104) the fact that χ_{1AB} is trace-free and in line (105) we have used the time derivative of equation (74).

Summing these results we see that only the total divergence terms remain. That is

$$\dot{\varepsilon}_\sigma = \partial_A t^A, \tag{107}$$

where

$$t_A = 4\rho\dot{\sigma}_1\partial_A\sigma_1 + 8\beta_1^B\dot{\chi}_{1AB}. \tag{108}$$

We integrate (107) in the half disk D_L of radius L in $\mathbb{R}^2_+$, where C_L denote the semi-circle of radius L, see figure 1. Using the divergence theorem in

2-dimensions we obtain

$$\int_{D_L} \dot{\varepsilon}_\sigma \, d\rho dz = \int_{D_L} \partial_A t^A \, d\rho dz, \tag{109}$$

$$= \int_{\partial D_L} t^A n_A \, ds, \tag{110}$$

$$= -\int_{-L}^{L} t^\rho|_{\rho=0} \, dz + \int_{C_L} t^A n_A \, ds. \tag{111}$$

where n^A is the outwards unit normal vector and ds the line element of C_L.

The integrated of the first term in line (111) is given by

$$t_\rho = 4\rho\dot{\sigma}_1\partial_\rho\sigma_1 + 8\beta_1^\rho\dot{\chi}_{1\rho\rho} + 8\beta_1^z\dot{\chi}_{1\rho z}. \tag{112}$$

The first term clearly vanished at the axis $\rho = 0$. The second and third term also vanish at the axis since the regularity conditions (25) implies that β_1^ρ and $\chi_{1\rho z}$ are zero at the axis. Hence we obtain

$$\int_{D_L} \dot{\varepsilon}_\sigma \, d\rho dz = \int_{C_L} t^A n_A \, ds. \tag{113}$$

Taking the limit $L \to \infty$ and using the fall off conditions (23) we obtain that the integral vanished and hence $\dot{m}_\sigma = 0$ (recall that on C_L we have $ds = r d\theta$ where $\tan\theta = z/\rho$).

The conservation of m_ω is similar. We take the time derivative of the mass density ε_ω

$$\dot{\varepsilon}_\omega = 4\frac{\dot{\omega}_1\ddot{\omega}_1}{\rho^3} + 4\frac{\partial_A\omega_1\partial^A\dot{\omega}_1}{\rho^3}. \tag{114}$$

For the first term we have

$$4\frac{\dot{\omega}_1\ddot{\omega}_1}{\rho^3} = 4\frac{\dot{\omega}_1}{\rho^3}\left(\partial^A\partial_A\omega_1 - \frac{3\partial_\rho\omega_1}{\rho}\right), \tag{115}$$

where we have used equation (70).

For the second term we have

$$4\frac{\partial_A\omega_1\partial^A\dot{\omega}_1}{\rho^3} = 4\partial^A\left(\frac{\dot{\omega}_1\partial_A\omega_1}{\rho^3}\right) - 4\frac{\dot{\omega}_1}{\rho^3}\left(\partial^A\partial_A\omega_1 - \frac{3\partial_\rho\omega_1}{\rho}\right). \tag{116}$$

Hence we obtain

$$\dot{\varepsilon}_\omega = \partial_A t^A, \tag{117}$$

with

$$t_A = 4\frac{\dot{\omega}_1\partial_A\omega_1}{\rho^3}. \tag{118}$$

Integrating in the same domain as above and using the behavior at the axis (93) and the fall off conditions (94) at infinity we obtain that $\dot{m}_\omega = 0$. Finally, the equality $m_\omega = m_{\bar{\omega}}$ is deduced from (90) and the assumption (93).

(ii) To prove the estimate (96) note that we have the following bounds

$$m_\sigma[p, \sigma_1, \chi_1] \geq \int_{\mathbb{R}^2_+} |\partial\sigma_1|^2 \rho \, d\rho dz, \tag{119}$$

$$m_\sigma[\dot{p}, \dot{\sigma}_1, \dot{\chi}_1] \geq 2\int_{\mathbb{R}^2_+} \dot{p}^2 \rho \, d\rho dz = 2\int_{\mathbb{R}^2_+} \left({}^{(3)}\Delta\sigma_1\right)^2 \rho \, d\rho dz, \tag{120}$$

where in the last equality of line (120) we have used equation (69). The right hand side of (120) can be written in the following form

$$\int_{\mathbb{R}^2_+} \left({}^{(3)}\Delta\sigma_1\right)^2 \rho \, d\rho dz = \int_{\mathbb{R}^3} ({}^{(3)}\Delta\sigma_1)^2 \, dx^3, \tag{121}$$

$$= \int_{\mathbb{R}^3} |\partial^2\sigma_1|^2 \, dx^3. \tag{122}$$

where in the right hand side of line (121) we have changed from cylindrical coordinates (ρ, z) to Cartesian coordinates (x, y, z) in $\mathbb{R}^3$, with $x = \rho\cos\phi$, $y = \rho\sin\phi$. For axially symmetric functions (i.e. functions in $\mathbb{R}^3$ that do not depends on ϕ) we have that $dx^3 = \rho \, d\rho dz$. In Cartesian coordinates the Laplacian ${}^{(3)}\Delta$ is given by

$${}^{(3)}\Delta\sigma_1 = \partial_x^2\sigma_1 + \partial_y^2\sigma_1 + \partial_z^2\sigma_1. \tag{123}$$

And in line (122) we have integrated by parts, due to the fall off assumptions on σ_1 the boundary terms vanishes. In this equation $|\partial^2\sigma_1|^2$ denote the sum of the squares of all second derivatives in terms of the Cartesian coordinates in $\mathbb{R}^3$, that is

$$|\partial^2\sigma_1|^2 = (\partial_x^2\sigma_1)^2 + (\partial_y^2\sigma_1)^2 + (\partial_z^2\sigma_1)^2 + (\partial_x\partial_y\sigma_1)^2 + (\partial_x\partial_z\sigma_1)^2 + (\partial_y\partial_z\sigma_1)^2. \tag{124}$$

From (122), (120) and (119) we obtain the following crucial estimate

$$m_\sigma[p, \sigma_1, \chi_1] + m_\sigma[\dot{p}, \dot{\sigma}_1, \dot{\chi}_1] \geq \int_{\mathbb{R}^3} \left(|\partial^2\sigma_1|^2 + |\partial\sigma_1|^2\right) dx^3. \tag{125}$$

Note that on the right hand side of (125) there are no terms with σ_1^2 and hence we can not use directly the standard Sobolev estimate to control pointwise the solution σ_1. However, using the estimate given by lemma B.1 with $n = 3$ and $k = 2$ we obtain the desired result (96).

To obtain the estimate (97) for $\bar{\omega}_1$, we proceed in a similar manner. From the definition of $m_{\bar{\omega}}$ we obtain

$$m_{\bar{\omega}} \geq \int_{\mathbb{R}^2_+} |\partial\bar{\omega}_1|^2 \rho^5 \, d\rho dz = \int_{\mathbb{R}^7} |\partial\bar{\omega}_1|^2 \, dx^7, \tag{126}$$

where we have used that $dx^7 = \rho^5 d\rho dz$. For the higher order masses we have

$$m_{\bar{\omega}}[\dot{\bar{\omega}}_1] \geq \int_{\mathbb{R}^2_+} \ddot{\bar{\omega}}_1^2 \rho^5 \, d\rho dz, \tag{127}$$

$$= \int_{\mathbb{R}^7} ({}^{(7)}\Delta\bar{\omega}_1)^2 \, dx^7, \tag{128}$$

$$= \int_{\mathbb{R}^7} |\partial^2\bar{\omega}_1|^2 \, dx^7, \tag{129}$$

where in line (128) we have used the wave equation (85) and in line (129) we have integrated by part and used that $\bar{\omega}_1$ decay at infinity. In a similar way, we obtain that energies with n-time derivatives control $n+1$ spatial derivatives, in particular

$$m_{\bar{\omega}}[\ddot{\bar{\omega}}_1] \geq \int_{\mathbb{R}^7} |\partial^3\bar{\omega}_1|^2 \, dx^7, \tag{130}$$

$$m_{\bar{\omega}}[\dddot{\bar{\omega}}_1] \geq \int_{\mathbb{R}^7} |\partial^4\bar{\omega}_1|^2 \, dx^7. \tag{131}$$

Using the bound (130), (131) and Lemma B.1 with $n = 7$ and $k = 4$ the estimate (97) follows.

□

We finally remark that in the proof of the conservation of m_2 we have used only the evolution equations for σ_1 and ω_1, the time derivative of the momentum constraint and the gauge equation for the shift.

4 Extreme Kerr perturbations

In this section we study the linearized equation obtained in section 2.2 for the case of extreme Kerr background. The main difference with respect to the previous case of Minkowski is that the background quantities q_0, σ_0, ω_0 are not zero. However, we still have that (see appendix A)

$$\alpha_0 = \rho. \tag{132}$$

This is the main remarkably simplification of the extreme Kerr case compared with the non-extreme Kerr black hole.

For the explicit form of $(q_0, \sigma_0, \omega_0)$ see the appendix A. These functions depend on one parameter, the mass m_0 of the black hole. This mass is given by (52). The only properties of these functions that we will are the following. They satisfy the stationary equations

$$^{(3)}\Delta\sigma_0 = \frac{|\partial\omega_0|^2}{\eta_0^2}, \tag{133}$$

$$\partial^A\left(\frac{\rho\partial_A\omega_0}{\eta_0^2}\right) = 0. \tag{134}$$

They satisfy the fall off conditions (23), (26). They satisfy the following inequality in $\mathbb{R}^2_+$ (i.e. including both the origin and infinity) (see [18])

$$\frac{|\partial\omega_0|^2}{\eta_0^2} \leq \frac{C}{r^2}, \quad |\partial\sigma_0|^2 \leq \frac{C}{r^2}, \tag{135}$$

where C is a constant that depends only on m_0. Finally, near the axis we have

$$\frac{\partial_\rho\omega_0}{\eta_0} = O(\rho). \tag{136}$$

The complete set of linearized equations, in axial symmetry, for the extreme Kerr black hole is the following.

Evolution equations:

The evolution equations for σ_1 and ω_1 are given by

$$-\frac{e^{2u_0}}{\rho^2}\dot{p} +^{(3)} \Delta\sigma_1 = \frac{2}{\eta_0^2}\left(\sigma_1|\partial\omega_0|^2 - \partial_A\omega_1\partial^A\omega_0\right), \tag{137}$$

$$-\frac{e^{2u_0}}{\rho^2}\dot{d} +^{(3)} \Delta\omega_1 = 4\frac{\partial_\rho\omega_1}{\rho} + 2\partial_A\omega_1\partial^A\sigma_0 + 2\partial_A\omega_0\partial^A\sigma_1, \tag{138}$$

with

$$p = \dot{\sigma}_1 - 2\frac{\beta_1^\rho}{\rho} - \beta_1^A\partial_A\sigma_0, \tag{139}$$

$$d = \dot{\omega}_1 - \beta_1^A\partial_A\omega_0. \tag{140}$$

The evolution equation for the metric and the second fundamental are obtained replacing (132) in equations (40) and (41). No relevant simplification occur in these equations compared with the general expressions (40) and (41), and hence we do not write them again in this section. Also, we will not make use of these equations in the proof of theorem 4.1.

Constraint equations:
The momentum constraint and Hamiltonian constraint are given by

$$\partial^B \chi_{1AB} = -\frac{e^{2u_0}}{2\rho}\left(p\left(2\frac{\partial_A \rho}{\rho} + \partial_A \sigma_0\right) + \frac{\partial_A \omega_0}{\eta_0^2} d\right), \tag{141}$$

$$^{(3)}\Delta \sigma_1 + \Delta q_1 = -\frac{\varepsilon_1}{4\rho}, \tag{142}$$

where ε_1 is given by

$$\varepsilon_1 = \left(2\partial_A \sigma_0 \partial^A \sigma_1 + \frac{2\partial_A \omega_0 \partial^A \omega_1}{\eta_0^2} - \frac{2\sigma_1 |\partial \omega_0|^2}{\eta_0^2}\right)\rho. \tag{143}$$

Gauge equations:
For the shift we have

$$(\mathcal{L}\beta_1)^{AB} = 2e^{-2u_0}\rho \chi_1^{AB}. \tag{144}$$

The energy density ε_2 defined previously in equation (49) is given by

$$\varepsilon_2 = \left(2\frac{e^{2u_0}}{\rho^2}p^2 + 2\frac{e^{2u_0}}{\rho^2 \eta_0^2}d^2 + 4e^{-2u_0}\chi_1^{AB}\chi_{1AB} + \right. \\ \left. +2|\partial \sigma_1|^2 + 2\frac{|\partial \omega_1|^2}{\eta_0^2} + 4\frac{|\partial \omega_0|^2}{\eta_0^2}\sigma_1^2 - 8\frac{\partial_A \omega_0 \partial^A \omega_1 \sigma_1}{\eta_0^2}\right)\rho \tag{145}$$

Note that the energy density (145) is not positive definite and hence it is by no means obvious that the energy m_2 is positive.

Theorem 4.1. *Consider a smooth solution of the linearized equations presented above, such that it satisfies the fall off decay conditions at infinity* (23)*, the decay conditions at the extra asymptotic end at the origin* (26) *and the regularity conditions* (24), (25) *at the axis. Assume also that* ω_1 *satisfies the following conditions. At the axis we have*

$$\dot{\bar{\omega}}_1, \bar{\omega}_1 = O_1(1), \tag{146}$$

and both at infinity and at the origin we impose

$$\dot{\bar{\omega}}_1, \bar{\omega}_1 = o_1(r^{-5/2}), \tag{147}$$

where we have defined

$$\bar{\omega}_1 = \frac{\omega_1}{\eta_0^2}. \tag{148}$$

Then, we have:

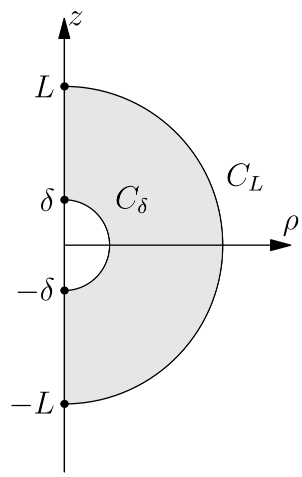

Figure 2: Domain of integration in $\mathbb{R}^2_+$ for the extreme Kerr black hole.

(i) The first order mass m_1 defined by (53) with ε_1 given by (143) vanishes $m_1 = 0$. The second order mass m_2 defined by (54) with ε_2 given by (145) is equal to the following expression, which is explicitly definite positive

$$m_2 = \frac{1}{16}\int_{\mathbb{R}^2_+} \bar{\epsilon}_2\, \rho d\rho dz, \tag{149}$$

where

$$\begin{aligned}\bar{\epsilon}_2 = \Bigg(& 2\frac{e^{2u_0}}{\rho^2}p^2 + 2\frac{e^{2u_0}}{\rho^2\eta_0^2}d^2 + 4e^{-2u_0}\chi_1^{AB}\chi_{1AB} + \\ & + \left(\partial\sigma_1 + \omega_1\eta_0^{-2}\partial\omega_0\right)^2 + \left(\partial\left(\omega_1\eta_0^{-1}\right) - \eta_0^{-1}\sigma_1\partial\omega_0\right)^2 + \\ & + \left(\eta_0^{-1}\sigma_1\partial\omega_0 - \omega_1\eta_0^{-2}\partial\eta_0\right)^2\Bigg)\rho.\end{aligned} \tag{150}$$

(ii) The mass m_2 is conserved along the evolution.

Note that the boundary condition (146) at the axis (outside the origin) is identical to the one used in Minkowski in section 3, since η_0 behaves like ρ^2 at the axis.

Proof. (i) We first prove that $m_1 = 0$. Take the density ε_1 given by (143), for the first term we have

$$2\rho\partial_A\sigma_0\partial^A\sigma_1 = 2\partial^A\left(\rho\sigma_1\partial_A\sigma_0\right) - 2\sigma_1\partial^A\left(\rho\partial_A\sigma_0\right), \tag{151}$$

$$= 2\partial^A\left(\rho\sigma_1\partial_A\sigma_0\right) - 2\rho\sigma_1{}^{(3)}\Delta\sigma_0, \tag{152}$$

$$= 2\partial^A\left(\rho\sigma_1\partial_A\sigma_0\right) - 2\rho\sigma_1\frac{|\partial\omega_0|^2}{\eta_0^2}, \tag{153}$$

where in line (152) we have used the definition of ${}^{(3)}\Delta$ given by equation (6) and in line (153) we have used the stationary equation (133).

For the second term we have

$$\frac{2\rho\partial_A\omega_0\partial^A\omega_1}{\eta_0^2} = 2\partial^A\left(\frac{\rho\omega_1\partial_A\omega_0}{\eta_0^2}\right) - 2\omega_1\partial^A\left(\frac{\rho\partial_A\omega_0}{\eta_0^2}\right), \tag{154}$$

$$= 2\partial^A\left(\frac{\rho\omega_1\partial_A\omega_0}{\eta_0^2}\right), \tag{155}$$

where in line (155) we have used the stationary equation (134). Summing up these terms we find

$$\varepsilon_1 = \partial_A t^A, \tag{156}$$

where

$$t_A = 2\rho\sigma_1\partial_A\sigma_0 + 2\rho\omega_1\frac{\partial_A\omega_0}{\eta_0^2}. \tag{157}$$

We integrate equation (156) in the domain showed in figure 2 for some finite δ and L with $0 < \delta < L$. At the axis the axis the first term in (157) clearly vanished. The second term also vanishes by the assumption (146) and the behavior (136) of the background quantities. Hence, the integral of (156) contains only the two boundary terms C_δ and C_L. Then, we take the limit $\delta \to 0$ and $L \to \infty$. Using the assumptions (147) on ω_1, the assumptions (23) and (26) on σ_1 and the background fall off (135) we obtain that these two boundary integrals vanish. Hence, it follows that $m_1 = 0$.

We prove now the positivity of m_2. The proof is identical to the proof of positivity presented in section 3 of [18], which is based on the Carter identity [10]. The last four terms in (145) are identical to the integrand of equation (24) in [18] (in that reference a different notation is used, namely $\sigma_1 = \alpha$, $\omega_1 = y$, $\eta_0 = X$ and $\omega_0 = Y$). Then, Carter identity given by equation (57) in [18] in the notation of this article can be written as

$$\bar{\epsilon}_2 - \epsilon_2 = \partial_A t^A, \tag{158}$$

where

$$t_A = 2\rho\left(2\sigma_1\partial_A\sigma_1 + \omega_1\frac{\partial_A\omega_1}{\eta_0^2} - 2\sigma_1\omega_1\frac{\partial_A\omega_0}{\eta_0^2} + \frac{\omega_1}{\eta_0}\partial_A\left(\frac{\omega_1}{\eta_0}\right)\right), \tag{159}$$

and $\bar{\epsilon}_2$ is given by (150). Recall that the divergence term in the right hand side of equation (158) has two contributions, one is the right hand side of equation (57) in [18] and the other comes from the integration by parts in equation (63) in [18]. Also note that in [18] Cartesian coordinates in $\mathbb{R}^3$ are used for the integration, and here we use cylindrical coordinates, and hence the factor ρ appears in (159). Integrating equation (158) and using the fall off conditions at infinity and at the axis it follows that m_2 is given by (149) and hence it is positive.

(ii) To prove the conservation of m_2 we take a time derivative of the mass density (145), we obtain

$$
\dot{\varepsilon}_2 = 4\frac{e^{2u_0}}{\rho}p\dot{p} + 4\frac{e^{2u_0}}{\rho\eta_0^2}d\dot{d} + 8e^{-2u_0}\rho\chi_1^{AB}\dot{\chi}_{1AB} +
$$

$$
+4\rho\partial_A\sigma_1\partial^A\dot{\sigma}_1 + 4\rho\frac{\partial_A\omega_1\partial^A\dot{\omega}_1}{\eta_0^2} - 8\rho\sigma_1\frac{\partial_A\omega_0}{\eta_0^2}\partial^A\dot{\omega}_1 + 8\rho\frac{|\partial\omega_0|^2\sigma_1\dot{\sigma}_1}{\eta_0^2} - 8\rho\dot{\sigma}_1\frac{\partial_A\omega_0}{\eta_0^2}\partial^A\omega_1. \tag{160}
$$

The strategy is very similar (but the calculations are more lengthy) than in the Minkowski case discussed in section 3: using the linearized equations we will write the right hand side of (160) as a total divergence. We proceed analyzing term by term.

For the first two terms we just use the definition of p and d given in equations (139) and (140) respectively. We obtain

$$
4\frac{e^{2u_0}}{\rho}p\dot{p} = 4\frac{e^{2u_0}}{\rho}\dot{p}\left(\dot{\sigma}_1 - \frac{2\beta_1^\rho}{\rho} - \beta_1^A\partial_A\sigma_0\right), \tag{161}
$$

$$
4\frac{e^{2u_0}}{\rho\eta_0^2}d\dot{d} = 4\frac{e^{2u_0}}{\rho\eta_0^2}\dot{d}\left(\dot{\omega}_1 - \beta_1^A\partial_A\omega_0\right). \tag{162}
$$

For the third term we have

$$
8e^{-2u_0}\rho\chi_1^{AB}\dot{\chi}_{1AB} = 8\dot{\chi}_1^{AB}\partial_A\beta_{1B}, \tag{163}
$$

$$
= 8\partial_A(\beta_{1B}\dot{\chi}_1^{AB}) - 8\beta_{1B}\partial_A\dot{\chi}_1^{AB}, \tag{164}
$$

$$
= 8\partial_A(\beta_{1B}\dot{\chi}_1^{AB}) + 4\frac{e^{2u_0}}{\rho}\left(\dot{p}\left(2\frac{\beta_1^\rho}{\rho} + \beta_1^A\partial_A\sigma_0\right) + \dot{d}\frac{\beta_1^A\partial_A\omega_0}{\eta_0^2}\right), \tag{165}
$$

where in the line (163) we have used equation (144) and the fact that χ_1^{AB} is trace free and in the line (165) we have used the time derivative of equation (141).

For the fourth term we have

$$
4\rho\partial_A\sigma_1\partial^A\dot{\sigma}_1 = 4\partial_A(\rho\dot{\sigma}_1\partial^A\sigma_1) - 4\dot{\sigma}_1\partial^A(\rho\partial_A\sigma_1), \tag{166}
$$

$$
= 4\partial_A(\rho\dot{\sigma}_1\partial^A\sigma_1) - 4\dot{\sigma}_1\rho^{(3)}\Delta\sigma_1, \tag{167}
$$

$$
= 4\partial_A(\rho\dot{\sigma}_1\partial^A\sigma_1) - 4\frac{\dot{\sigma}_1\dot{p}e^{2u_0}}{\rho} + 8\frac{\dot{\sigma}_1}{\eta_0^2}\rho\partial_A\omega_1\partial^A\omega_0 - 8\frac{\dot{\sigma}_1\sigma_1\rho|\partial\omega_0|^2}{\eta_0^2}, \tag{168}
$$

where in line (167) we used the definition of the operator ${}^{(3)}\Delta$ given by equation (6), in line (168) we used equation (137).

For the fifth term we obtain

$$
\begin{aligned}
4\rho\frac{\partial_A\dot\omega_1\partial^A\omega_1}{\eta_0^2} &= 4\partial^A\left(\frac{\rho\dot\omega_1\partial_A\omega_1}{\eta_0^2}\right) - 4\dot\omega_1\partial^A\left(\frac{\rho\partial_A\omega_1}{\eta_0^2}\right), && (169)\\
&= 4\partial^A\left(\frac{\rho\dot\omega_1\partial_A\omega_1}{\eta_0^2}\right) - 4\frac{\rho\dot\omega_1}{\eta_0^2}\left({}^{(3)}\Delta\omega_1 - \frac{4\partial_\rho\omega_1}{\rho} - 2\partial_A\omega_1\partial^A\sigma_0\right), && (170)\\
&= 4\partial^A\left(\frac{\rho\dot\omega_1\partial_A\omega_1}{\eta_0^2}\right) - 4\frac{e^{2u_0}}{\rho\eta_0^2}\dot d\dot\omega_1 - 8\frac{\rho\dot\omega_1\partial_A\omega_0\partial^A\sigma_1}{\eta_0^2}, && (171)
\end{aligned}
$$

where in line (170) we have used the definition of the operator ${}^{(3)}\Delta$ given in equation (6) and the definition of η_0 given in equation (35). In the line (171) we have used the evolution equation (138).

For the sixth term we obtain

$$
\begin{aligned}
-8\rho\sigma_1\frac{\partial_A\omega_0\partial^A\dot\omega_1}{\eta_0^2} &= -8\partial^A\left(\rho\dot\omega_1\sigma_1\frac{\partial_A\omega_0}{\eta_0^2}\right) + 8\dot\omega_1\partial^A\left(\rho\sigma_1\frac{\partial_A\omega_0}{\eta_0^2}\right), && (172)\\
&= -8\partial^A\left(\dot\omega_1\sigma_1\frac{\partial_A\omega_0}{\eta_0^2}\right) + 8\rho\dot\omega_1\frac{\partial_A\omega_0\partial^A\sigma_1}{\eta_0^2} + 8\dot\omega_1\sigma_1\partial^A\left(\frac{\rho\partial_A\omega_0}{\eta_0^2}\right), && (173)\\
&= -8\partial^A\left(\dot\omega_1\sigma_1\frac{\partial_A\omega_0}{\eta_0^2}\right) + 8\rho\dot\omega_1\frac{\partial_A\omega_0\partial^A\sigma_1}{\eta_0^2}, && (174)
\end{aligned}
$$

where in line (174) we have used the stationary equation (134).

We sum the six terms obtained above plus the two last terms in (160), only the divergence terms survive, we obtain

$$
\dot\varepsilon_2 = \partial_A t^A, \tag{175}
$$

where

$$
t^A = 4\rho\dot\sigma_1\partial^A\sigma_1 + 8\beta_{1B}\dot\chi_1^{AB} + 4\frac{\rho\dot\omega_1\partial^A\omega_1}{\eta_0^2} - 8\dot\omega_1\sigma_1\frac{\partial^A\omega_0}{\eta_0^2}. \tag{176}
$$

Remarkably we get only one extra term compared with the Minkowski case (compare (176) with the sum of (108) and (118)).

We integrate equation (175) in the domain showed in figure 2. The boundary term at the axis vanished by the hypothesis (25). Then, we take the limit $\delta \to 0$ and $L \to \infty$, and the other two boundary integrals also vanished by the fall off conditions (23), (26) and (146), (147).

□

Acknowledgments

This work was supported in by grant PICT-2010-1387 of CONICET (Argentina) and grant Secyt-UNC (Argentina).

A Kerr black hole in the maximal-isothermal gauge

In this appendix we explicitly write the Kerr black hole metric in the maximal – isothermal gauge described in section 2. In particular, we show that in this gauge the metric satisfies the conditions (29).

The Kerr metric, with parameters (m, a), in Boyer-Lindquist coordinates $(t, \tilde{r}, \theta, \phi)$ is given by

$$g = -V dt^2 + 2W dt d\phi + \frac{\Sigma}{\Xi} d\tilde{r}^2 + \Sigma d\theta^2 + \eta d\phi^2, \tag{177}$$

where

$$\Xi = \tilde{r}^2 + a^2 - 2m\tilde{r}, \quad \Sigma = \tilde{r}^2 + a^2 \cos^2\theta, \tag{178}$$

and

$$V = \frac{\Xi - a^2 \sin^2\theta}{\Sigma}, \tag{179}$$

$$W = -\frac{2ma\tilde{r}\sin^2\theta}{\Sigma}, \tag{180}$$

$$\eta = \left(\frac{(\tilde{r}^2 + a^2)^2 - \Xi a^2 \sin^2\theta}{\Sigma} \right) \sin^2\theta. \tag{181}$$

The angular momentum is given by

$$J = ma. \tag{182}$$

The metric (177) is stationary and axially symmetric because it has the following two Killing vectors

$$\xi^\mu = \left(\frac{\partial}{\partial t}\right)^\mu, \quad \eta^\nu = \left(\frac{\partial}{\partial \phi}\right)^\nu, \tag{183}$$

where ξ^μ is timelike near infinity (i.e. outside the ergosphere) and η^μ is spacelike and it vanished at the axis. The scalars (179), (180) and (181) are written in terms of the Killing vectors as follows

$$V = -\xi^\mu \xi^\nu g_{\mu\nu}, \quad \eta = \eta^\mu \eta^\nu g_{\mu\nu}, \quad W = \eta^\mu \xi^\nu g_{\mu\nu}. \tag{184}$$

In particular, η is the square norm of the axial Killing vector η^μ. In these equations we are using 4-dimensional indices $\mu, \nu \cdots$.

The twist potential ω of the axial Killing vector η^μ is given by

$$\omega = 2ma(\cos^3\theta - 3\cos\theta) - \frac{2ma^3\cos\theta\sin^4\theta}{\Sigma}. \tag{185}$$

The 3-dimensional Lorenzian metric h on the quotient manifold (see equation (26) on [24], we follow the notation of that article) is defined by

$$\eta g_{\mu\nu} = h_{\mu\nu} + \eta_\mu\eta_\nu. \tag{186}$$

Using the explicit form of the Kerr metric (177) and the Killing vector η^μ we obtain that h is given by

$$h = -(V\eta + W^2)dt^2 + \frac{\eta\Sigma}{\Xi}d\tilde{r}^2 + \eta\Sigma d\theta^2. \tag{187}$$

For the Kerr metric, the following remarkably relation holds

$$V\eta + W^2 = \Xi\sin^2\theta. \tag{188}$$

Using (188) we further simplify the expression for the metric h

$$h = -\Xi\sin^2\theta dt^2 + \frac{\eta\Sigma}{\Xi}d\tilde{r}^2 + \eta\Sigma d\theta^2. \tag{189}$$

This metric is static. The foliation $t = \textit{constant}$ has zero extrinsic curvature and hence it is a maximal foliation. The shift of this foliation also vanished, then the condition (29) is satisfied. However, the coordinates $(\tilde{r}, \theta)$ are not isothermal because they do not satisfy the condition (2).

To introduce isothermal coordinates we will assume that $m \geq |a|$ (i.e. the Kerr metric (177) describe a black hole). Let r be defined as the positive root of the equation

$$\tilde{r} = r + m + \frac{m^2 - a^2}{4r}, \tag{190}$$

that is

$$r = \frac{1}{2}\left(\tilde{r} - m + \sqrt{\Xi}\right). \tag{191}$$

We have

$$d\tilde{r} = \frac{\sqrt{\Xi}}{r}dr. \tag{192}$$

We define the cylindrical coordinates (ρ, z) in terms of the spherical coordinates (r, θ) by the standard formula

$$\rho = r\sin\theta, \quad z = r\cos\theta. \tag{193}$$

Then the metric h in the new coordinate system (t, ρ, z) is given by

$$h = -\alpha^2 dt^2 + e^{2u}(d\rho^2 + dz^2), \tag{194}$$

where

$$\alpha = \sqrt{\Xi} \sin\theta = \rho \left(1 - \frac{(m^2 - a^2)}{4r^2}\right), \tag{195}$$

and

$$e^{2u} = \frac{\eta\Sigma}{r^2}. \tag{196}$$

The intrinsic metric of the $t = \textit{constant}$ of the slices is

$$q = e^{2u}\left(d\rho^2 + dz^2\right). \tag{197}$$

That is, the coordinates system is isothermal.

The function σ is defined in terms of the norm η by

$$e^{\sigma} = \frac{\eta}{\rho^2}, \tag{198}$$

The function q is given by

$$e^{2q} = \frac{\sin^2\theta\Sigma}{\eta}, \tag{199}$$

We have the relation

$$u = q + \sigma + \log\rho. \tag{200}$$

Note that the lapse satisfies the maximal gauge condition

$$\Delta\alpha = 0. \tag{201}$$

In the extreme case $m = |a|$ and hence we have

$$\alpha = \rho. \tag{202}$$

B A Sobolev like estimate

In this appendix we prove the following Sobolev type estimate.

Lemma B.1. *There exists a constant $C > 0$ such that for all $u \in C_0^\infty(\mathbb{R}^n)$, with $n \geq 3$, the following inequality holds*

$$C\left(\int_{\mathbb{R}^n}\left(|\partial^k u|^2 + |\partial^{k-1}u|^2\right)\, dx^n\right)^{1/2} \geq \sup_{x\in\mathbb{R}^n}|u(x)|, \tag{203}$$

where $k > n/2$.

Proof. The estimate (203) will be a consequence of the following two classical estimates. The first one is the Gagliardo-Nirenberg-Sobolev inequality: assume that $1 \leq p < n$, then exists a constant C, depending only on p and n, such that

$$||u||_{L^q(\mathbb{R}^n)} \leq C||\partial u||_{L^p(\mathbb{R}^n)}, \tag{204}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$, where

$$q = \frac{pn}{n-p}. \tag{205}$$

The second estimate is the Morrey's inequality: assume $n < p \leq \infty$, then there exists a constant depending only on p and n, such that

$$\sup_{x\in\mathbb{R}^n} |u(x)| \leq C||u||_{W^{1,p}(\mathbb{R}^n)}. \tag{206}$$

See [27] for an elementary and clear presentation of these inequalities and the functional spaces $L^p(\mathbb{R}^n)$, $W^{1,p}(\mathbb{R}^n)$ involved in them.

We first observe that the estimate (204) can be iterated as follows:

$$||u||_{L^{p_k}(\mathbb{R}^n)} \leq C||\partial^k u||_{L^p(\mathbb{R}^n)}, \tag{207}$$

where $1 \leq k \leq n/p$, $1 < p$ and p_k is given by

$$p_k = \frac{pn}{n-pk}. \tag{208}$$

To prove (207) we use induction in k. For $k = 1$ the inequality (207) reduces to (204). Assume that (207) is valid for k. If $\partial^{k+1}u \in L^p(\mathbb{R}^n)$, then by (204) we obtain that $\partial^k u \in L^q(\mathbb{R}^n)$ with q given by

$$q = \frac{pn}{n-p}. \tag{209}$$

By the inductive hypothesis we obtain that $u \in L^{q_k}(\mathbb{R}^n)$ with

$$q_k = \frac{qn}{n-qk}. \tag{210}$$

We substitute (209) in (34) to obtain

$$q_k = \frac{pn}{n-(k+1)p}. \tag{211}$$

And then the desired result is proved.

To prove (203), note that the left hand side of (203) implies that $\partial^{k-1}w$, $\partial^{k-1}u \in L^2(\mathbb{R}^n)$ where $w = \partial u$. Then, we apply the inequality (207) for both w and u, to obtain that $w, u \in L^p(\mathbb{R}^n)$, with p given by

$$p = \frac{2n}{n - 2k + 2}. \tag{212}$$

By hypothesis $k > n/2$, then we obtain that $p > n$. Hence, we have proved that $u \in W^{1,p}(\mathbb{R}^n)$ with $p > n$. We use the Morrey inequality (206) and the desired result follows.

□

References

[1] L. Andersson and P. Blue. Hidden symmetries and decay for the wave equation on the Kerr spacetime. *ArXiv e-prints*, Aug. 2009, 0908.2265.

[2] H. Andreasson, M. Kunze, and G. Rein. Rotating, stationary, axially symmetric spacetimes with collisionless matter. *Commun.Math.Phys.*, 329:787–808, 2014, 1212.5028.

[3] S. Aretakis. Stability and Instability of Extreme Reissner-Nordström Black Hole Spacetimes for Linear Scalar Perturbations I. *Commun.Math.Phys.*, 307:17–63, 2011, 1110.2007.

[4] S. Aretakis. Stability and Instability of Extreme Reissner-Nordström Black Hole Spacetimes for Linear Scalar Perturbations II. *Annales Henri Poincare*, 12:1491–1538, 2011, 1110.2009.

[5] S. Aretakis. Decay of Axisymmetric Solutions of the Wave Equation on Extreme Kerr Backgrounds. *J.Funct.Anal.*, 263:2770–2831, 2012, 1110.2006.

[6] S. Aretakis. Horizon Instability of Extremal Black Holes, 2012, 1206.6598.

[7] S. Aretakis. Nonlinear instability of scalar fields on extremal black holes. *Phys.Rev.*, D87:084052, 2013, 1304.4616.

[8] R. Bartnik. Phase space for the Einstein equations. *Comm. Anal. Geom.*, 13(5):845–885, 2005, gr-qc/0402070.

[9] P. Bizon and H. Friedrich. A remark about wave equations on the extreme Reissner-Nordstróm black hole exterior. *Class.Quant.Grav.*, 30:065001, 2013, 1212.0729.

[10] B. Carter. Axisymmetric black hole has only two degrees of freedom. *Phys. Rev. Lett.*, 26(6):331–333, 1971.

[11] M. W. Choptuik, E. W. Hirschmann, S. L. Liebling, and F. Pretorius. An Axisymmetric Gravitational Collapse Code. *Class. Quant. Grav.*, 20:1857–1878, 2003, gr-qc/0301006.

[12] D. Christodoulou and S. Klainerman. Asymptotic properties of linear field equations in Minkowski space. *Comm. Pure Appl. Math.*, 43(2):137–199, 1990.

[13] D. Christodoulou and S. Klainerman. *The Global Nonlinear Stability of the Minkowski Space*, volume 41 of *Princeton Mathematical series.* Princeton University Press, Princeton, NJ, 1993.

[14] M. Dafermos and I. Rodnianski. Lectures on black holes and linear waves, 2008, 0811.0354.

[15] M. Dafermos and I. Rodnianski. The black hole stability problem for linear scalar perturbations, 2010, 1010.5137.

[16] M. Dafermos and I. Rodnianski. A Proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds. *Invent.Math.*, 185:467–559, 2011, 0805.4309.

[17] M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman. Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case $|a| < M$, 2014, 1402.7034.

[18] S. Dain. Proof of the (local) angular momemtum-mass inequality for axisymmetric black holes. *Class. Quantum. Grav.*, 23:6845–6855, 2006, gr-qc/0511087.

[19] S. Dain. Axisymmetric evolution of Einstein equations and mass conservation. *Class. Quantum. Grav.*, 25:145021, 2008, 0804.2679.

[20] S. Dain. Proof of the angular momentum-mass inequality for axisymmetric black holes. *J. Differential Geometry*, 79(1):33–67, 2008, gr-qc/0606105.

[21] S. Dain. Geometric inequalities for axially symmetric black holes. *Classical and Quantum Gravity*, 29(7):073001, 2012, 1111.3615.

[22] S. Dain and G. Dotti. The wave equation on the extreme Reissner-Nordstróm black hole. *Class.Quant.Grav.*, 30:055011, 2013, 1209.0213.

[23] S. Dain and M. E. Gabach Clément. Small deformations of extreme Kerr black hole initial data. *Class. Quant. Grav.*, 28:075003, 2011, 1001.0178.

[24] S. Dain and O. E. Ortiz. Well-posedness, linear perturbations, and mass conservation for the axisymmetric einstein equations. *Phys. Rev. D*, 81(4):044040, Feb 2010.

[25] S. Dain and M. Reiris. Linear perturbations for the vacuum axisymmetric Einstein equations. *Annales Henri Poincare*, 12:49–65, 2011, 1005.5347.

[26] G. Dotti. Non-modal linear stability of the Schwarzschild black hole. *Phys.Rev.Lett.*, 112:191101, 2014, 1307.3340.

[27] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[28] F. Finster, N. Kamran, J. Smoller, and S.-T. Yau. Linear waves in the Kerr geometry: a mathematical voyage to black hole physics. *Bull. Amer. Math. Soc. (N.S.)*, 46(4):635–659, 2009, 0801.1423.

[29] D. Garfinkle and G. C. Duncan. Numerical evolution of Brill waves. *Phys. Rev.*, D63:044011, 2001, gr-qc/0006073.

[30] M. Giaquinta and S. Hildebrandt. *Calculus of variations. I*, volume 310 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1996. The Lagrangian formalism.

[31] S. Hollands and R. M. Wald. Stability of Black Holes and Black Branes. *Commun.Math.Phys.*, 321:629–680, 2013, 1201.0463.

[32] B. S. Kay and R. M. Wald. Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation two sphere. *Class.Quant.Grav.*, 4:893–898, 1987.

[33] J. Keir. Stability, Instability, Canonical Energy and Charged Black Holes, 2013, 1306.6087.

[34] J. Lucietti, K. Murata, H. S. Reall, and N. Tanahashi. On the horizon instability of an extreme Reissner-Nordstróm black hole. *JHEP*, 1303:035, 2013, 1212.2557.

[35] J. Lucietti and H. S. Reall. Gravitational instability of an extreme Kerr black hole. *Phys.Rev.*, D86:104030, 2012, 1208.1437.

[36] V. Moncrief. Gauge-invariant perturbations of Reissner-Nordstrom black holes. *Phys.Rev.*, D12:1526–1537, 1975.

[37] K. Murata. Instability of higher dimensional extreme black holes. *Class.Quant.Grav.*, 30:075002, 2013, 1211.6903.

[38] K. Murata, H. S. Reall, and N. Tanahashi. What happens at the horizon(s) of an extreme black hole? *Class.Quant.Grav.*, 30:235007, 2013, 1307.6800.

[39] T. Regge and J. A. Wheeler. Stability of a Schwarzschild singularity. *Phys.Rev.*, 108:1063–1069, 1957.

[40] M. Reiris. Instability of the extreme Kerr-Newman black-holes, 2013, 1311.3156.

[41] O. Rinne. *Axisymmetric Numerical Relativity.* PhD thesis, University of Cambridge, 2005, gr-qc/0601064.

[42] J. Shatah and M. Struwe. *Geometric wave equations*, volume 2 of *Courant Lecture Notes in Mathematics.* New York University Courant Institute of Mathematical Sciences, New York, 1998.

[43] R. M. Wald. Note on the stability of the schwarzschild metric. *Journal of Mathematical Physics*, 20(6):1056–1058, 1979.

[44] G. Weinstein. On rotating black holes in equilibrium in general relativity. *Comm. Pure App. Math.*, 43(7):903–948, 1990.

[45] B. F. Whiting. Mode stability of the Kerr black hole. *J. Math. Phys.*, 30(6):1301–1305, 1989.

[46] F. Zerilli. Perturbation analysis for gravitational and electromagnetic radiation in a Reissner-Nordström geometry. *Phys.Rev.*, D9:860–868, 1974.

[47] F. J. Zerilli. Effective potential for even parity Regge-Wheeler gravitational perturbation equations. *Phys.Rev.Lett.*, 24:737–738, 1970.