LINEAR AND MULTILINEAR ISOMETRIES IN A NONCOMPACT FRAMEWORK

MALIHEH HOSSEINI AND JUAN J. FONT

Abstract. Both classical linear and multilinear isometries defined between subalgebras of bounded continuous functions on (complete) metric spaces are studied. Particularly, we prove that certain such subalgebras, including the subalgebras of uniformly continuous, Lipschitz or locally Lipschitz functions, determine the topology of (complete) metric spaces. As consequence, it is proved that the subalgebra of Lipschitz functions determines the Lipschitz in the small structure of a complete metric space.

Furthermore, we provide a weighted composition representation for multilinear isometries from similar subalgebras on (not necessarily complete) metric spaces. We apply this general representation to obtain more specific ones for subalgebras of uniformly continuous and Lipschitz functions.

1. Introduction

According to the classical Banach-Stone theorem, $C(X)$ and $C(Y)$ are linearly isometric if and only if the underlying compact spaces X and Y are homeomorphic, which is to say that the geometric structure of $C(X)$ determines the compact space X. Furthermore, any such linear isometry is given by a composition with that homeomorphism, followed by a multiplication by a unimodular function. Generalizations of the Banach-Stone theorem have been intensely investigated; we now know that similar results are valid for various other function and operator spaces and algebras. Even in the nonsurjective case, which is clearly more involved and, consequently, less studied, the weighted composition representation is available, at least, on certain nonempty subspaces of the underlying spaces (see [2]).

Almost all of these results assume that the spaces X, Y are either compact or, at least, locally compact. The reason might be that if no conditions on compactness are required, the existence of a linear isometry between the spaces $C_b(X)$ and $C_b(Y)$ of bounded continuous functions (endowed with the supremum norm) does not imply that the Tychonoff spaces X and Y are homeomorphic

2010 *Mathematics Subject Classification.* Primary 47B38; Secondary 46E25.

Key words and phrases: spaces of bounded continuous functions, uniformly continuous functions, Lipschitz functions, isometry, multilinear isometry, Stone-Čech compactification.

J.J. Font is supported by Spanish Government grant MTM2016-77143-P (AEI/FEDER, UE) and by Generalitat Valenciana grant GV/2018/110.

This work was partially supported by a grant from the IMU-CDC.

but only that their Stone-Čech compactifications are homeomorphic, that is, the geometric structure of $C_b(X)$ determines the topology of the Stone-Čech compactifiction of X but not the one of X. Indeed it is known that the algebra, the lattice, and the geometric structures of $C_b(X)$ altogether are still not enough to determine X. We should remark here that, surprisingly, the vector-valued setting is different from that scalar-valued one in the sense that $C_b(X, E)$, E a Banach space, determines X in certain general cases, as was shown in [1].

However, if X is a metric space, then the geometric structure of $C_b(X)$ does determine X and a question arises naturally: is it required to consider the whole $C_b(X)$? That is, can we recover the topology of a metric space X from a (proper) subalgebra of $C_b(X)$? This does not seem an easy question since, for example, the subalgebra of $C_b(X)$ consisting of the uniformly continuous functions on X coincides with the same subalgebra on its completion.

The literature on linear isometries in a noncompact framework is reduced compared to the plethora of results treating the compact (or locally compact) case. Indeed, as far as we know, non-surjective linear isometries and multilinear isometries have not been addressed in this noncompact context yet. So, in this paper we address these questions by dealing with isometries, both classical linear (1-linear) and multilinear, defined between subalgebras of $C_b(X)$ being X a (complete) metric space.

In Section 3, we deal with classical 1-linear isometries and we prove that certain subalgebras of $C_b(X)$ determine the topology of a (complete) metric space X. The subalgebras of uniformly continuous, Lipschitz and locally Lipschitz functions belong to this class of subalgebras. As a consequence, it is shown that the subalgebra of Lipschitz functions determines the Lipschitz in the small structure of a complete metric space.

In Section 4, we provide a weighted composition representation for multilinear isometries from certain subalgebras of $C_b(X)$-spaces, where X is a (not necessarily complete) metric space. We apply this general representation to obtain more specific ones for subalgebras of uniformly continuous and Lipschitz functions.

2. Preliminaries

Let X be a metric space and let $C_b(X)$ stand for the space of scalar-valued bounded continuous functions defined on X. Let $\hat{f}$ denote the unique extension of $f \in C_b(X)$ to the Stone-Čech compactification, βX, of X.

Given a subalgebra A of $C_b(X)$, we denote $\hat{A} := \{\hat{f} : f \in A\}$, which turns out to be a subalgebra of $C(\beta X)$ too. For such A, we can define an equivalence relation on βX as follows: given $x_1, x_2 \in \beta X$, $x_1 \sim x_2$ if $\hat{f}(x_1) = \hat{f}(x_2)$ for all $f \in A$. It is apparent that the quotient $\gamma X := \frac{\beta X}{\sim}$ is a compactification of X.

Let $\mathcal{A}$ be a subalgebra of continuous functions defined on a compact Hausdorff space $\mathcal{X}$. It is said that $\mathcal{A}$ separates (resp. strongly) the points of $\mathcal{X}$ if given two distinct points $x, y \in \mathcal{X}$, then there exists $f \in \mathcal{A}$ with $f(x) \neq f(y)$ (resp. $|f(x)| \neq |f(y)|$). We denote the uniform closure of $\mathcal{A}$ by $\overline{\mathcal{A}}$. The unique minimal closed subset of $\mathcal{X}$ with the property that every function in $\mathcal{A}$ assumes its maximum modulus on this set, which exists by [4], is called the *Šilov boundary* for $\mathcal{A}$ and is denoted by $\partial\mathcal{A}$. The *Choquet boundary* $Ch(\mathcal{A})$ of $\mathcal{A}$ is the set of all $x \in \mathcal{X}$ for which δ_x, the evaluation functional at the point x, is an extreme point of the unit ball of the dual space of $(\mathcal{A}, \|\cdot\|)$, where $\|\cdot\|$ denotes the uniform norm. So it is apparent that $Ch(\mathcal{A}) = Ch(\overline{\mathcal{A}})$. Besides, note that for each point-separating algebra $\mathcal{A}$, $\partial\mathcal{A}$ is the closure of $Ch(\mathcal{A})$ [4, Theorem 1].

We say that a subalgebra A of $C_b(X)$ is *normal* if given two subsets, M and N, of X with $d(M, N) > 0$, then there exists $f \in A$ such that $|f_{|M}| > \frac{3}{4}$ and $|f_{|N}| < \frac{1}{4}$.

We say that a subalgebra A of $C_b(X)$ *peaks at* X is for every $x \in X$, there is a function $f \in A$ such that $\hat{f}$ peaks at x, that is, $|\hat{f}(x)| > |\hat{f}(y)|$ for all $y \in \beta X$.

The subalgebras of bounded uniformly continuous, $UC(X)$, and bounded Lipschitz functions, $Lip(X)$, on X are examples of normal subalgebras of $C_b(X)$ peaking at X (see, e.g. [13, Section 7.3]). Let us also point out that if X is a closed unit ball of a normed space, then each uniformly continuous function is bounded.

3. Linear isometries on subalgebras of $C_b(X)$

Despite we are working in a noncompact context, classical 1-linear isometries still produce interesting results. Indeed, for a complete metric space X, the metric structure of normal subalgebras of $C_b(X)$ determines the topology of X. First, we need the following result:

Lemma 3.1. *Let A be a subalgebra of $C_b(X)$.*

(a) Every $x \in X$ is a G_δ-point in γX.

(b) If X is complete and A is normal, then no point in $\gamma X \setminus X$ is a G_δ-point.

Proof. (a) As X is dense in γX and each $x \in X$ has a countable local base $\{U_n\}$ in X, then it is known (see, e.g., [8, 9.7]) that $\{cl_{\gamma X} U_n\}$ is a countable local base at x in γX.

(b) Let us suppose that there exists $x_0 \in \gamma X \setminus X$ which has a countable local base. Then, since X is dense in γX, there exists a sequence (x_n) in X converging to x_0. Hence, since X is complete, no subsequence of (x_n) is Cauchy. Consequently, there is $\epsilon > 0$ and a subsequence (x_{n_j}) of (x_n) such that $d(x_{n_j}, x_{n_k}) > \epsilon$ for $j \neq k$. The normality of A allows us to find $f \in A$ such that $f(x_{n_{2j}}) > \frac{3}{4}$ and $f(x_{n_{2j+1}}) < \frac{1}{4}$. It is apparent that such f cannot be extended continuously to γX, which is a contradiction. $\square$

Theorem 3.2. *Let A and B be unital normal subalgebras of $C_b(X)$ and $C_b(Y)$, for X and Y complete metric spaces, respectively.*

(1) *If there exists $T : A \longrightarrow C_b(Y)$ a linear isometry whose range has finite codimension in $C_b(Y)$, then there exist a nonempty subset Y_{00} of Y, a continuous surjective map $\varphi : Y_{00} \longrightarrow X$ and a unimodular continuous function $a : Y_{00} \longrightarrow \mathbb{T}$ such that $T(f)(y) = a(y)f(\varphi(y))$ for all $f \in A$ and $y \in Y$.*

(2) *If there exists $T : A \longrightarrow B$ a surjective linear isometry, then X and Y are homeomorphic.*

Proof. (1) Let us first check that $\hat{A}$ separates strongly the points of γX. To this end, let x and y be two distinct points in γX. It is clear that there is a function $f \in A$ such that $\hat{f}(x) \neq \hat{f}(y)$. If $|\hat{f}(x)| \neq |\hat{f}(y)|$, then we are done. Otherwise, choose $\alpha \in \mathbb{C}$ such that $|\alpha \hat{f}(x) + \hat{f}(x)^2| \neq |\alpha \hat{f}(y) + \hat{f}(y)^2|$, which implies that $\hat{A}$ separates strongly the points of γX.

It is apparent that the range of the induced mapping $\hat{T} : \hat{A} \longrightarrow C(\beta Y)$ has also finite codimension in $C(\beta Y)$. Let n be this finite codimension.

Claim. A subset B of βY which cannot be separated strongly with functions of the range of $\hat{T}$ has, at most, $n+1$ elements.

Assume, contrary to what we claim, that there exist $n+2$ elements of βY, say $y_1, ..., y_{n+2}$, which cannot be separated strongly with functions of the range of $\hat{T}$. Let us take $n+1$ functions in $C(\beta Y)$ such that

$$f_l(y_j) = \begin{cases} 1 & l = j \\ 0 & l \neq j \end{cases}$$

for $l \in \{1, 2, ..., n+1\}$ and $j \in \{1, 2, ..., n+2\}$. From our assumption, it is clear that $\{f_1, ..., f_{n+1}\} \cap \hat{T}(\hat{A}) = \emptyset$. It is also apparent that such $n+1$ functions are linearly independent. Hence, since the codimension of the range of $\hat{T}$ is n, we can find $n+1$ constants (not all zero), $\alpha_1, ..., \alpha_{n+1}$ such that the function

$$F := \alpha_1 f_1 + ... + \alpha_{n+1} f_{n+1}$$

belongs to $\hat{T}(\hat{A})$. Again from our assumption, we infer $|F(y_1)| = ... = |F(y_{n+2})|$, which yields $\alpha_1 = ... = \alpha_{n+1} = 0$, a contradiction.

By [2, Theorem 3.1], there exist a nonempty subset Y_0 of βY, a continuous surjective map $\varphi : Y_0 \longrightarrow \partial_0 \hat{A}$ and a continuous map $a : Y_0 \longrightarrow \mathbb{T}$ such that $\hat{T}\hat{f}(y) = a(y)\hat{f}(\varphi(y))$ for all $f \in A$ and $y \in Y_0$, where $\partial_0 \hat{A} = \partial \hat{A} \cap \{x \in \gamma X : \exists f \in A \text{ with } \hat{f}(x) \neq 0\}$. Since A is unital, we have $\partial_0 \hat{A} = \partial \hat{A}$.

Fix $x_0 \in X$ and a neighborhood U of x_0 in γX. Since $d(\{x_0\}, X \setminus U) > 0$ and A is normal, there exists $f \in A$ such that $|f(x_0)| > \frac{3}{4}$ and $|f| < \frac{1}{4}$ on $X \setminus U$. From the density of X in γX, we infer that $|\hat{f}| \leq \frac{1}{4}$ on $\gamma X \setminus U$. Hence, by [2, Lemma 2.1], we infer that X is a subset of the Šilov boundary, $\partial \hat{A} \subseteq \gamma X$, of $\hat{A}$. Moreover, since $\partial \hat{A}$ is a closed subset of γX and X is dense in γX, we conclude that $\partial \hat{A} = \gamma X$.

By the above representation of $\hat{T}$ and the Claim, we deduce that $\varphi^{-1}(x)$ has, at most, $n+1$ elements for each $x \in \gamma X$, in particular for each $x \in X$. By Lemma 3.1, we know that if we take $x \in X$, then it is a G_δ-point in γX. Hence, since φ is continuous, $\varphi^{-1}(x)$ is a finite G_δ-set in βY, which implies that each element of $\varphi^{-1}(x)$ is a G_δ-point in βY. This fact yields $\varphi^{-1}(x) \subseteq Y$ since the only points in βY which are G_δ are those in Y ([8, 9.7]). As a consequence, we infer that $Y_{00} := Y_0 \cap \varphi^{-1}(X)$ is a nonempty subset of Y.

(2) Let $\hat{T} : \hat{A} \longrightarrow \hat{B}$ be the surjective linear isometry induced by $T : A \longrightarrow B$. Hence, by [2, Theorem 4.1], there exist a homeomorphism $\varphi : \partial_0 \hat{B} \longrightarrow \partial_0 \hat{A}$ and a continuous map $a : \partial_0 \hat{B} \longrightarrow \mathbb{T}$ such that $\hat{T}\hat{f}(y) = a(y)\hat{f}(\varphi(y))$ for all $f \in A$ and $y \in \partial_0 \hat{B}$.

Hence, from the reasonings in (1), we infer that $\varphi : \gamma Y \longrightarrow \gamma X$ is a homeomorphism and that its restriction to Y is a homeomorphism from Y onto X since homeomorphisms preserve G_δ-sets. □

We can compare Theorem 3.2 with Corollary 3.1 in [5], where similar results for the surjective case are obtained in a noncompact framework for certain function spaces which satisfy several technical conditions.

Since the uniformly continuous functions of a metric space and its completion are the same, we infer that the above theorem is not true for noncomplete metric spaces. Indeed, it is not true either if we do not assume both X and Y to be metric spaces as the following example shows:

Example 3.3. *([8, 4M]) Let $X = \mathbb{N}$, the natural numbers, and let $Y = \mathbb{N} \cup \{\sigma\}$ ($\sigma \in \beta\mathbb{N} \setminus \mathbb{N}$). It is clear that any $f \in C_b(X)$ can be extended uniquely to a function $f^\sigma \in C_b(Y)$. Then, we can define a bijective isometry, $T(f) = f^\sigma$, between $C_b(X)$ and $C_b(Y)$ but X and Y are not homeomorphic.*

Remark 3.4. Note that if $\hat{A}$ (resp. $\hat{B}$) separates the points of βX (resp. βY), taking into account the fact that no point in $\beta X \setminus X$ and $\beta Y \setminus Y$ is G_δ, with a proof similar to above, we can deduce that the completeness of metric spaces can be removed from Theorem 3.2. In particular, one can obtain this known result that metric spaces X and Y are homeomorphic if and only if $C_b(X)$ and $C_b(Y)$ are isometrically isomorphic.

Meantime, the above situation happens in the context of locally Lipschitz function spaces. First, let us recall that a map $f : X \longrightarrow Y$ is called *locally Lipschitz* if each point of X has a neighborhood on which f is Lipschitz. Meantime, if f is a bijection and f and f^{-1} are locally Lipschitz, then X and Y are said to be *locally Lipschitz homeomorphic*.

Furthermore, let $Lip_{loc}(X)$ denote the space of all bounded scalar-valued functions on X which are locally Lipschitz. It is known that a bounded scalar-valued function f belongs to $Lip_{loc}(X)$ if and only if f is Lipschitz on each compact subset of X (see [12, Corollary 2.2]). Now, we can state our next result on isometries of $Lip_{loc}(X)$-spaces (compare with [6, Theorem 3.16]).

Corollary 3.5. *Two metric spaces X and Y are locally Lipschitz homeomorphic if and only if $Lip_{loc}(X)$ and $Lip_{loc}(Y)$ are isometrically isomorphic.*

Proof. If $h : Y \longrightarrow X$ is a locally Lipschitz homeomorphism, it is easy to check that $Tf = f \circ h$ is a linear isometry from $Lip_{loc}(X)$ onto $Lip_{loc}(Y)$, which yields the sufficiency. To see the necessity, assume that $T : Lip_{loc}(X) \longrightarrow Lip_{loc}(Y)$ is a surjective linear isometry. Taking into account that $Lip_{loc}(X)$ and $Lip_{loc}(Y)$ are uniformly dense in $C_b(X)$ and $C_b(Y)$, respectively (see, e.g., [6]), from Theorem 3.2 and Remark 3.4 we conclude that there exists a homeomorphism $h : Y \longrightarrow X$ such that $Tf = f \circ h$ for every $f \in Lip_{loc}(X)$. Hence, by [6, Lemma 3.15], h is a locally Lipschitz homeomorphism. □

Next, we concentrate on the algebra of bounded Lipschitz functions. First we state the concept of Lipschitz in the small functions which is due to Luukkainen [11].

Definition 3.6. *A function $f : (X_1, d_1) \longrightarrow (X_2, d_2)$ is said to be Lipschitz in the small (abbreviated LS) if there exist $r > 0$ and $k > 0$ such that $d_2(f(x), f(x')) \leq k d_1(x, x')$ for any $x, x' \in X_1$ with $d_1(x, x') \leq r$. Moreover, X_1 and X_2 are called LS-homeomorphic if f is a bijection and f and f^{-1} are Lipschitz in the small.*

It is worth mentioning that a bounded function is Lipschitz if and only if it is Lipschitz in the small.

The space of all scalar-valued functions defined on a metric space X which are Lipschitz in the small is denoted by $LS(X)$. It is clear that the space of all bounded functions in $LS(X)$ coincides with $Lip(X)$. We also note that, in general, the space of all scalar-valued Lipschitz functions on X is a subset of $LS(X)$, but when X is quasi-convex or precompact these spaces coincide (for more details, see [7]).

The next result says that $Lip(X)$ determines the LS-structure of complete metric spaces as follows (see also [7, Theorem 2] and [11, Corollary 3.9]):

Corollary 3.7. *Let X and Y be complete metric spaces. Then the following are equivalent:*

(a) X and Y are LS-homeomorphic.

(b) $Lip(X)$ and $Lip(Y)$ are isometrically isomorphic.

Proof. (*a*) ⇒ (*b*) Let $h : Y \longrightarrow X$ be an LS-homeomorphism. Define

$$Tf = f \circ h \quad (f \in Lip(X)).$$

Let $f \in Lip(X)$. Since f is Lipschitz, there is a constant $k > 0$ such that $d(f(x), f(x')) \leq kd(x, x')$ for all $x, x' \in X$. On the other hand, since h is Lipschitz in the small, there exist $r > 0$ and $m > 0$ such that $d(h(y), h(y')) \leq md(y, y')$ for any $y, y' \in Y$ with $d(y, y') \leq r$. Hence

$$d(f(h(y)), f(h(y'))) \leq kmd(y, y'),$$

which, taking into account that $f \circ h$ is bounded, shows that $f \circ h \in Lip(Y)$. Now, it is easy to see that T is a surjective linear isometry, as desired.

(*b*) ⇒ (*a*) Let $T : Lip(X) \longrightarrow Lip(Y)$ be a surjective linear isometry. From Theorem 3.2, there is a homeomorphism $h : Y \longrightarrow X$ such that $Tf = f \circ h$ for every $f \in Lip(X)$. Now the result follows immediately from [7, Lemma 1]. □

It is worth pointing out we cannot replace LS-homeomorphic with Lipschitz-homeomorphic in the sense that there is a bijection h such that h and h^{-1} are Lipschitz. Let us state a simple example borrowed from [6]. Suppose that (X, d) is a metric space with infinite diameter and $d' = \min\{1, d\}$ (note that (X, d') is complete whenever (X, d) is). Hence (X, d) and (X, d') are LS-homeomorphic but, despite $Lip(X, d) = Lip(X, d')$, they are not Lipschitz-homeomorphic.

Taking into account [7] and Corollary 3.7, we may obtain easily the next result.

Corollary 3.8. *Let X and Y be complete quasi-convex metric spaces or complete precompact metric spaces. Then X and Y are Lipschitz-homeomorphic if and only if $Lip(X)$ and $Lip(Y)$ are isometrically isomorphic.*

4. Multilinear isometries on subalgebras of $C_b(X)$

Let $X_1, ..., X_k$ and Z be metric spaces. Let $A_1, ..., A_k$ be subalgebras of $C_b(X_1), ..., C_b(X_k)$, respectively. A k-linear map $T : A_1 \times ... \times A_k \longrightarrow C_b(Z)$ is called a *multilinear (or k-linear) isometry* if

$$\|T(f_1, ..., f_k)\| = \prod_{i=1}^{k} \|f_i\| \qquad ((f_1, ..., f_k) \in A_1 \times ... \times A_k).$$

First note that it is not difficult to extend $T : A_1 \times ... \times A_k \longrightarrow C_b(Z)$ to a k-linear isometry $T : \overline{A_1} \times ... \times \overline{A_k} \longrightarrow C_b(Z)$, where $\overline{A_i}$ is the uniform closure of A_i $(i = 1, ..., k)$. So, without loss of generality, we can assume each A_i $(i = 1, ..., k)$ is uniformly closed.

Theorem 4.1. *Assume that A_i peaks at X_i for $i = 1, ..., k$. Let $T : A_1 \times ... \times A_k \longrightarrow C_b(Z)$ be a k-linear isometry such that the linear span of its range has finite codimension in $C_b(Z)$. Then there exist a nonempty subset Z_{00} of Z, a continuous surjective map $\varphi : Z_{00} \longrightarrow X_1 \times ... \times X_k$ and a unimodular continuous function $a : Z_{00} \longrightarrow \mathbb{T}$ such that $T(f_1, ..., f_k)(z) = a(z) \prod_{i=1}^{k} f_i(\pi_i(\varphi(z)))$ for all $(f_1, ..., f_k) \in A_1 \times ... \times A_k$ and $z \in Z_{00}$, where π_i is the ith projection map.*

Proof. It is apparent that the linear span of the range of the induced mapping $\hat{T} : \hat{A}_1 \times ... \times \hat{A}_k \longrightarrow C(\beta Z)$ has also finite codimension in $C(\beta Z)$. Let n be this finite codimension. By an argument similar to the proof of Theorem 3.2, we can obtain the following claim:

Claim A subset B of βZ which cannot be separated strongly with functions of the range of $\hat{T}$ has, at most, $n + 1$ elements.

By [9, Theorem 4.5], there exist a nonempty subset Z_0 of βZ, a continuous surjective map $\varphi : Z_0 \longrightarrow Ch(\hat{A}_1) \times ... \times Ch(\hat{A}_k)$, a unimodular continuous function $a : Z_0 \longrightarrow \mathbb{T}$, such that $\hat{T}(\hat{f})(z) = a(z) \prod_{i=1}^{k} \hat{f}_i(\pi_i(\varphi(z)))$ for all $(f_1, ..., f_k) \in A_1 \times ... \times A_k$ and $z \in Z_0$, where π_i is the ith projection map.

By the above representation of $\hat{T}$ and the Claim, we deduce that $\varphi^{-1}(x_1, ..., x_k)$ has, at most, $n + 1$ elements for each $(x_1, ..., x_k) \in Ch(\hat{A}_1) \times ... \times Ch(\hat{A}_k)$.

Since, for each $i = 1, ..., k$, A_i peaks at X_i and $Ch(\hat{A}_i)$ is a boundary for $\hat{A}_i$, it is apparent that $X_i \subseteq Ch(\hat{A}_i)$. By Lemma 3.1, we know that if we take $x_i \in X_i \subseteq Ch(\hat{A}_i)$, then it is a G_δ-point in γX_i. Hence, since φ is continuous, $\varphi^{-1}(x_1, ..., x_k)$ is a finite G_δ-set in βZ, which implies that each element of $\varphi^{-1}(x_1, ..., x_k)$ is a G_δ-point in βZ. This fact yields $\varphi^{-1}(x_1, ..., x_k) \subseteq Z$ since the only points in βZ which are G_δ are those in Z ([8, 9.7]). As a consequence, we infer that $Z_{00} := Z_0 \cap \varphi^{-1}(X_1 \times ... \times X_k)$ is a nonempty subset of Z. □

If we focus on the context of uniformly continuous functions, Theorem 4.1 can be sharpened as follows. Let $X_1, ..., X_k$ and Z be metric spaces. Also let $A_1, ..., A_k$ be normal subalgebras of $UC(X_1), ..., UC(X_k)$ which peak at $X_1, ..., X_k$, respectively.

Theorem 4.2. *Let $T : A_1 \times ... \times A_k \longrightarrow UC(Z)$ be a k-linear isometry such that the linear span of its range has finite codimension in $UC(Z)$. Then there exist a nonempty subset Z_{00} of Z, a continuous surjective map $\varphi : Z_{00} \longrightarrow X_1 \times ... \times X_k$ and a unimodular uniformly continuous function $a : Z_{00} \longrightarrow \mathbb{T}$ such that $T(f_1, ..., f_k)(z) = a(z) \prod_{i=1}^{k} f_i(\pi_i(\varphi(z)))$ for all $(f_1, ..., f_k) \in A_1 \times ... \times A_k$ and $z \in Z_{00}$, and each $\pi_i \circ \varphi$ is uniformly continuous $(i = 1, ..., k)$.*

Proof. The representation of T follows from Theorem 4.1 having into account that we can mimic the proof of the Claim to prove that a subset of γZ which cannot be separated strongly with functions of the linear span of the range of $\hat{T}$ has, at most, $n+1$ elements.

Let us next prove that the weight function $a(z)$ is uniformly continuous. Consider, contrary to what we claim, that there exist two sequences, (z_n) and (z'_n), in Z_{00} with $\lim_{n\to\infty} d(z_n, z'_n) = 0$ such that $|a(z_n) - a(z'_n)| \geq \epsilon$ for a certain $\epsilon > 0$ and for every $n \in \mathbb{N}$. Since $A_1, ..., A_k$ are normal, we can choose $(f_1, ..., f_k) \in A_1 \times ... \times A_k$ such that

$$|f_i(\pi_i(\varphi(z_n)))| < 1/4 \quad \text{and} \quad |f_i(\pi_i(\varphi(z'_n)))| \geq 3/4 \;\; (i = 1, ..., k). \quad (1)$$

Since T preserves uniformly continuous functions, we infer that

$$\lim_{n\to\infty} \left(a(z_n) \prod_{i=1}^{k} f_i(\pi_i(\varphi(z_n))) - a(z'_n) \prod_{i=1}^{k} f_i(\pi_i(\varphi(z'_n))) \right) = 0. \quad (2)$$

Hence, multiplying by $f_1(\pi_1(\varphi(z_n)))$, we get

$$\lim_{n\to\infty} \left(a(z_n) f_1^2(\pi_1(\varphi(z_n))) \prod_{i=2}^{k} f_i(\pi_i(\varphi(z_n))) - a(z'_n) f_1(\pi_1(\varphi(z_n))) \prod_{i=1}^{k} f_i(\pi_i(\varphi(z'_n))) \right) = 0.$$

On the other hand, we also have

$$\lim_{n\to\infty} \left(a(z_n) f_1^2(\pi_1(\varphi(z_n))) \prod_{i=2}^{k} f_i(\pi_i(\varphi(z_n))) - a(z'_n) f_1^2(\pi_1(\varphi(z'_n))) \prod_{i=2}^{k} f_i(\pi_i(\varphi(z'_n))) \right) = 0.$$

Consequently, we infer

$$\lim_{n\to\infty} \left(a(z'_n) \prod_{i=1}^{k} f_i(\pi_i(\varphi(z'_n)))(f_1(\pi_1(\varphi(z_n))) - f_1(\pi_1(\varphi(z'_n))) \right) = 0$$

or equivalently by (1),

$$\lim_{n\to\infty} (f_1(\pi_1(\varphi(z_n))) - f_1(\pi_1(\varphi(z'_n)))) = 0.$$

Similarly, for every $i = 2, ..., k$ we can claim

$$\lim_{n\to\infty} (f_i(\pi_1(\varphi(z_n))) - f_i(\pi_1(\varphi(z'_n)))) = 0.$$

Hence

$$\lim_{n\to\infty} \left(a(z_n) \prod_{i=1}^{k} f_i(\pi_i(\varphi(z_n))) - a(z_n) f_1(\pi_1(\varphi(z'_n))) \prod_{i=2}^{k} f_i(\pi_i(\varphi(z_n))) \right) = 0,$$

$$\lim_{n\to\infty} \left(a(z_n) f_1(\pi_1(\varphi(z'_n))) \prod_{i=2}^{k} f_i(\pi_i(\varphi(z_n))) - a(z_n) f_1(\pi_1(\varphi(z'_n))) f_1(\pi_2(\varphi(z'_n))) \prod_{i=3}^{k} f_i(\pi_i(\varphi(z_n))) \right) = 0$$

$$\vdots$$

$$\lim_{n\to\infty} \left(a(z_n) f_k(\pi_1(\varphi(z_n))) \prod_{i=1}^{k-1} f_i(\pi_i(\varphi(z_n))) - a(z_n) \prod_{i=1}^{k} f_i(\pi_i(\varphi(z'_n))) \right) = 0.$$

Summing up,

$$\lim_{n\to\infty}\left(a(z_n)\prod_{i=1}^{k} f_i(\pi_i(\varphi(z_n))) - a(z_n)\prod_{i=1}^{k} f_i(\pi_i(\varphi(z_n')))\right) = 0.$$

From (2) above,

$$\lim_{n\to\infty}\left(a(z_n)\prod_{i=1}^{k} f_i(\pi_i(\varphi(z_n'))) - a(z_n')\prod_{i=1}^{k} f_i(\pi_i(\varphi(z_n')))\right) = 0,$$

which is to say,

$$\lim_{n\to\infty}\left((a(z_n) - a(z_n'))\prod_{i=1}^{k} f_i(\pi_i(\varphi(z_n')))\right) = 0.$$

From (1), we obtain

$$\lim_{n\to\infty}((a(z_n) - a(z_n')) = 0,$$

which contradicts our assumption.

In order to prove that $\frac{1}{a(z)}$ is also uniformly continuous, suppose that $\lim_{n\to\infty} d(z_n, z_n') = 0$. Then

$$\lim_{n\to\infty}\left|\frac{1}{a(z_n)} - \frac{1}{a(z_n')}\right| = \lim_{n\to\infty}\left|\frac{a(z_n) - a(z_n')}{a(z_n)a(z_n')}\right| = \lim_{n\to\infty}|a(z_n) - a(z_n')| = 0$$

since $a(z)$ is uniformly continuous.

If we assume that T is 1-linear, then

$$\frac{1}{a(z)}Tf(z) = \frac{1}{a(z)}a(z)f(\varphi(z)) = f(\varphi(z))$$

is uniformly continuous for every $f \in A_1$, so φ is uniformly continuous by an argument similar to [3, Lemma 4.1] and [10, Theorem 2.3 and its Remark].

If we assume that T is k-linear, $k \geq 2$, then we can choose $(f_2, ..., f_k) \in A_2 \times ... \times A_k$ such that $f_i(\pi_i \circ \varphi(z)) = 1$ for $i = 2, ..., k$. Then

$$\frac{1}{a(z)}T(f_1, ..., f_k)(z) = \frac{1}{a(z)}a(z)f_1(\pi_1 \circ \varphi(z)) \cdot 1 \cdot ... \cdot 1 = f_1(\pi_1 \circ \varphi(z))$$

is uniformly continuous for every $f_1 \in A_1$, so $\pi_1 \circ \varphi$ is uniformly continuous as in the preceding paragraph. Similarly we infer $\pi_i \circ \varphi$ is uniformly continuous for every $i = 2, ..., k$. □

Using Theorems 4.2 and 3.2, we easily obtain the following result which is a known result in the context of uniformly continuous function spaces.

Corollary 4.3. *Let X and Y be complete metric spaces. Then X and Y are uniformly homeomorphic if and only if $UC(X)$ and $UC(Y)$ are isometrically isomorphic.*

Similarly to the case of uniformly continuous functions, we can obtain an improved version of Theorem 4.1 for algebras of bounded Lipschitz functions.

Theorem 4.4. *Let $T : Lip(X_1) \times ... \times Lip(X_k) \longrightarrow Lip(Z)$ be a k-linear isometry such that the linear span of its range has finite codimension in $Lip(Z)$. Then there exist a nonempty subset Z_{00} of Z, a continuous surjective map $\varphi : Z_{00} \longrightarrow X_1 \times ... \times X_k$ and a Lipschitz function $a : Z_{00} \longrightarrow \mathbb{T}$ such that $\pi_i \circ \varphi$ is Lipschitz in the small ($i = 1, ..., k$), and $T(f_1, ..., f_k)(z) = a(z) \prod_{i=1}^{k} f_i(\pi_i(\varphi(z)))$ for all $(f_1, ..., f_k) \in Lip(X_1) \times ... \times Lip(X_k)$ and $z \in Z_{00}$.*

Proof. Taking into account that $Lip(X)$-spaces are normal subalgebras peaking at X, as mentioned in the proof of Theorem 4.2, we can obtain the representation of T. Moreover, it is evident that a is a Lipschitz function on Z_{00} because $a = T(1, ..., 1)|_{Z_{00}}$.

Now, we show that $\pi_i \circ \varphi$ is Lipschitz in the small ($i = 1, ..., k$). We assume, without loss of generality, that $i = 1$. Let $f \in Lip(X_1)$. From the representation of T we have

$$(f_1 \circ (\pi_1 \circ \varphi))(z) = \frac{T(f_1, 1, ..., 1)(z)}{a(z)} \quad (z \in Z_{00}),$$

which easily shows that $f_1 \circ (\pi_1 \circ \varphi)$ belongs to $Lip(Z_{00})$. Next, taking into account [7, Lemma 1], we deduce that $\pi_1 \circ \varphi$ is Lipschitz in the small. □

Remark 4.5. Although in most of the results provided in this manuscript, surjectivity is not assumed, a question arises naturally: is it possible to prove a Holsztynski theorem in a noncompact framework? That is, can we obtain a representation of the isometry on a nonempty subset of the underlying spaces with no assumption on its range?

References

[1] J. Araujo, *The noncompact Banach-Stone theorem*, J. Operator Theory 55 (2006), 285-294.

[2] J. Araujo and J.J. Font, *Linear isometries between subspaces of continuous functions*, Trans. Amer. Math. Soc. 349 (1997), 413-428.

[3] J. Araujo and J.J. Font, *Linear isometries on subalgebras of uniformly continuous functions*, Proc. Edinburgh Math. Soc. (2) 43 (2000), no. 1, 139-147.

[4] J. Araujo and J.J. Font, *On Šilov boundaries for subspaces of continuous functions*, Topology Appl. 77 (1997), 79-85.

[5] M. Bachir, *An extension of the Banach-Stone theorem*, J. Aust. Math. Soc. 105 (2018), no. 1, 1-23.

[6] M.I. Garrido and J.A. Jaramillo, *Homomorphisms on function lattices*, Monatsh. Math. 141 (2004), no. 2, 127-146.

[7] M.I. Garrido and J.A. Jaramillo, *Lipschitz-type functions on metric spaces*, J. Math. Anal. Appl. 340 (2008), no. 1, 282-290.

[8] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, 1960.

[9] M. Hosseini, J.J. Font and M. Sanchis, *Multilinear isometries on function algebras*, Linear Multilinear Algebra 63 (2015), no. 7, 1448-1457.

[10] M. Lacruz and J.G. Llavona, *Composition operators between algebras of uniformly continuous functions*, Arch. Math. 69 (1997), 52-56.

[11] J. Luukkainen, *Rings of functions in Lipschitz topology*, Ann. Acad. Sci. Fenn. Ser. A I Math. 4 (1979), no. 1, 119-135.

[12] C.H. Scanlon, *Rings of functions with certain Lipschitz properties*, Pacific J. Math. 32 (1970), 197-201.

[13] N. Weaver, *Lipschitz algebras*, 2nd ed., World Scientific Publishing, 2018.

Faculty of Mathematics, K. N. Toosi University of Technology, Tehran, 16315-1618, Iran

E-mail address: m.hosseini@kntu.ac.ir

Departamento de Matemáticas, Universitat Jaume I, Campus Riu Sec, 8029 AP, Castellón, Spain

E-mail address: font@mat.uji.es