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Small-amplitude solitons in a nonlocal sine-Gordon model

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Abstract

It is shown that small amplitude solitons of a nonlocal sine-Gordon model corresponding to different frequencies of the carrier wave can create coupled states. The effect is due to a change of the dispersion originated by a nonlocal nonlinearity. Within the framework of the multiscale expansion such pulses are described by a system of nonlinear Schrödinger equations which possesses coupled mode solutions in the form of running localized waves (breathers). Such breathers consist of modes with different frequencies and are characterized by two internal frequencies.

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Recently much attention has been paid to various nonlinear nonlocal models due to their prominent role in the description of the lattice dynamics [1,2], superconductivity [3], and magnetic systems [4]. Those models take into account long-range interactions which naturally result in new physical phenomena. One of such systems, the so-called nonlocal sine-Gordon (NSG) equation

$$u_{xx} - u_{tt} = 2\cos[\tfrac{1}{2}u(x,t)] \int dy\, f(x-y) \sin[\tfrac{1}{2}u(y,t)], \quad (1)$$

has been introduced in Ref. [5], and can describe a 1D linear chain of torsionally coupled permanent electric dipoles positioned in a narrow gap between two parallel infinite earthed conducting planes. The authors have studied numerically the existence and dynamical stability of static solutions of (1). It has been found that due to the nonlocality kink solutions possessing nonzero topological charge can create zero-topological-charge localized excitations of odd and even parity. These static solutions are shown to be stable at $\sigma > \sigma_{cr}$, where σ is a parameter characterizing the range of interactions (see (6) below) and σ_{cr} is its critical value: in terms of the present paper $\sigma_{cr} = 1.77/\sqrt{2} \approx 1.25$. Being static the mentioned solutions differ from another known type of localized excitations which are called breathers. Taking this fact into account and recalling that the small-amplitude breathers of the local sine-Gordon model are governed by the nonlinear Schrödinger (NLS) equation, see e.g. Ref. [6], it is natural to analyze solutions of the NSG model in the limit of a small amplitude. This is the main purpose of this work.

The NLS equation being obtained by means of the multiscale expansion represents the general property

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of small amplitude solutions of the nonlinear Klein-Gordon models. Differences appear in the coefficients related to both the group velocity dispersion of the linear waves and nonlinearity. More precisely, the effects discussed below are caused by nontrivial dependences of the mentioned coefficients on the carrier wave number. In particular, it will be shown that the nonlocality introduces qualitative changes in small amplitude soliton dynamics. The most interesting of them is the possibility of creating coupled states of modes. Such excitations are localized in space, move with finite velocity, and are characterized by internal frequencies (because of this last property we call them breathers). Though we particularize the consideration to the NSG Eq. (1), the results are naturally generalized to any nonlocal model generated by the action as follows,

$$A = \frac{1}{2}\int dt \int dx \Big(u_t^2 - u_x^2 - 4\Phi[u(x,t)] \int dy\, f(x-y)\, \Phi[u(y,t)] \Big), \quad (2)$$

where $\Phi[u]$ is a function on u.

Being interested in the small amplitude limit we make use of the method of multiple scales [6]. It implies that the wave field is represented in the form of the series $u = \epsilon u_1 + \epsilon^2 u_2 + \ldots$, where the small parameter ϵ is introduced in explicit form. The functions u_n are assumed to be varying on different space ($x_n = \epsilon^n x$) and time ($t_n = \epsilon^n t$) scales, where x_n and t_n are considered as independent variables.

The behavior of the nonlocal system depends essentially on the characteristic scale of the kernel $f(x)$. So, for instance, we have the exactly integrable sine-Gordon equation in the limit $f(x) \to \delta(x)$ ($\delta(x)$ being the Dirac delta). Bearing in mind the findings of Ref. [5] we restrict the consideration to the case when the characteristic scale of $f(x)$ is of the order of one. In other words, it will be assumed that in terms of the scaled variables the function $f(x)$ depends only on the "rapid" spatial coordinate x_0. This allows us to develop $u(x,t)$ in the integrand of (1) in a series with respect to the slow independent variables $\{x_n\}$ ($n \geqslant 1$).

Then proceeding in the usual manner, i.e., collecting the terms containing equal powers of ϵ, we find the set of equations of the form $Lu_n = F_n$ where the linear integro-differential operator L is given by

$$Lu = \frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial t_0^2} - \int dy\, f(y)\, u(x_0 - y, t_0) \quad (3)$$

(hereafter in the arguments of functions we indicate only the most "rapid" variable) and F_n for the second and third orders are given in the appendix. It is not difficult to find the dispersion relation associated with the linear operator L. It reads

$$\omega^2(k) = k^2 + \hat{f}(k), \quad (4)$$

where $\hat{f}(k)$ is the Fourier transform of $f(x)$.

First, let us concentrate on the conventional single-mode solution of the first order system ($F_1 = 0$), which can be represented in the form $u_1 = A(x_1, \ldots; t_1, \ldots)e^{i\theta} + \text{c.c.}$, where $\theta = kx_0 - \omega(k)\, t_0$. As is customary the condition of the absence of secular terms in the second order leads to the conclusion that A depends on x_1 and t_1 only through the combination $\zeta = x_1 - v_{gr}(k)\, t_1$ where $v_{gr}(k) = d\omega/dk$ is the group velocity of the carrier wave. Finally, considering the solutions independent of x_2 in the third order of the small parameter ϵ we obtain the NLS equation

$$2i\frac{\partial A}{\partial t_2} + \omega'' \frac{\partial^2 A}{\partial \zeta^2} + \chi |A|^2 A = 0, \quad (5)$$

where $\omega'' = d^2\omega/dk^2$ and $\chi = (1/2\omega)\hat{f}(k)$. Let us now analyze the features of the problem using as an example the non-local kernel introduced in Ref. [5],

$$f(x) = \frac{1}{2\sqrt{\pi}\sigma} e^{-x^2/4\sigma^2}, \quad (6)$$

where σ is a parameter characterizing the range of interactions. The first characteristic feature of the problem follows from (5), (6). The nonlinearity coefficient χ goes to zero with k and has a maximum at $k = 0$. This means that localized (i.e. soliton) solutions are available only in the limited region $|k| < 1/\sigma$ of the wave numbers. Then, with the kernel (6), we have the group velocity in the form

$$v_{gr}(k) = k\frac{1 - \sigma^2 e^{-\sigma^2 k^2}}{\sqrt{k^2 + e^{-\sigma^2 k^2}}}. \quad (7)$$

Its dependence on the wave number is sketched in Fig. 1. The peculiarity of the case at hand is the N-shape form of the curve $v_{gr}(k)$ for the interaction of a sufficiently long range, more precisely, when $\sigma >$

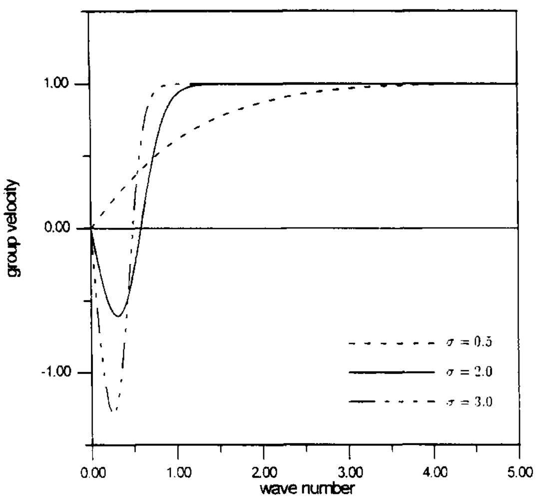

Fig. 1. Dependence of the group velocity on the carrier wave number. The curves correspond to three different situations: $\sigma < \sigma_{cr}^{(1)}$, $\sigma_{cr}^{(1)} < \sigma < \sigma_{cr}^{(2)}$ and $\sigma_{cr}^{(2)} < \sigma$.

$\sigma_{cr}^{(1)} = 1$. First of all this means the existence of solitons of different types at different wave numbers. In particular, in the vicinity of $k = 0$, where the group velocity dispersion is negative, there can exist a dark soliton, while at $|k| > k_0$ (k_0 being the wave number corresponding to the minimum of $v_{gr}(k)$) the group velocity dispersion is positive, which means the existence of bright soliton solutions. It is to be mentioned here that the similar phenomenon of co-existence of bright and dark solitons in nonlinear lattices with long range interactions has been reported in Ref. [2]. Further increasing σ leads to linear waves appearing with velocity more than one (i.e. than the "light" velocity). This happens when the range of interactions exceeds the second critical value, $\sigma > \sigma_{cr}^{(2)}$, where $\sigma_{cr}^{(2)} = 2e^{1/4}$ (≈ 2.568). One more peculiarity displayed in Fig. 1 is the existence of waves with different frequencies and group velocity dispersions but with the same group velocity. Such waves can create travelling bound states due to the nonlinearity.

In order to illustrate the last effect let us consider the propagation of two modes having the same velocity $v_{gr}(k)$. The respective solution can be represented as follows,

$$u_1 = A_1(x_1, \ldots; t_1, \ldots)e^{i\theta_1} + A_2(x_1, \ldots; t_1, \ldots)e^{i\theta_2} + \text{c.c.}, \tag{8}$$

hereafter we use the designation $\theta_n = k_n x_0 - \omega(k_n)\, t_0$. Thus it is assumed that $v_{gr}(k_1) = v_{gr}(k_2) = v_{gr}$ (this can be called a matching condition) though the frequencies of both modes are different, $\omega_1 \neq \omega_2$. Therefore equations for both mode amplitudes A_1 and A_2 in the second order coincide. This means that $A_{1,2}$ depends only on the running variable ζ. The explicit form (8) of u_1 in the third order leads to

$$2\omega_1 i\frac{\partial A_1}{\partial t_2} + \omega_1\omega_1''\frac{\partial^2 A_1}{\partial \zeta^2} + (\chi_{11}|A_1|^2 + \chi_{12}|A_2|^2)A_1 = 0,$$

$$2\omega_2 i\frac{\partial A_2}{\partial t_2} + \omega_2\omega_2''\frac{\partial^2 A_2}{\partial \zeta^2} + (\chi_{21}|A_1|^2 + \chi_{22}|A_2|^2)A_2 = 0, \tag{9}$$

where $\chi_{jj} = \frac{1}{2}\hat{f}(k_j)$, $\chi_{ij} = \frac{1}{2}[\hat{f}(k_i) + \hat{f}(k_j)]$ $(i \neq j)$, $\omega_n = \omega(k_n)$, and $\omega_n'' = \omega''(k_n)$.

It is to be mentioned here that in fact the matching condition explored above is not a necessary requirement for the derivation of the system (9). Indeed, let us consider a wave with wave number $\tilde{k}_2 = k_2 + \epsilon\kappa$ (κ being of the order of k_2). This results in the change of the frequency of the second mode by $\epsilon\kappa v_{gr} + \frac{1}{2}\epsilon^2\kappa^2\omega_2''$ and therefore in the change of the running phase: $\tilde{\theta}_2 = \theta_2 + \kappa\zeta - \frac{1}{2}\kappa^2\omega_2'' t_2$ (the tilde stands for the functions in the point $\tilde{k}_2$). The above modification can be taken into account by simple renormalization of A_2 and therefore leads to the renormalization of both the velocity and the phase of the coupled mode solution (if any). In this way we found that the solutions represented below are structurally stable with respect to a small (or order ϵ) deviation from the matching condition.

The system of equations (9) is well known and can be solved in a number of particular cases. Moreover, it has already been studied in the context of coupled mode dynamics (see e.g. Refs. [7,8]). However, the coupling considered here is essentially different from the situation treated elsewhere. So, for example, in Ref. [7] the tied states of two modes with close wave vectors have been described and in Ref. [8] it has appeared at the same frequency in the system having two branches of the spectrum. In our case $k_2 - k_1 = O(k_{1,2})$ and both values of the wave numbers belong to the same branch and the modes are characterized by different frequencies and wave numbers. Also it is to be emphasized that the effect is a direct consequence of the nonlocality and disappears at $\sigma \to 0$, which is explicitly seen from Fig. 1. Our model, where nonlocality originates "material" dispersion, allows coupling of modes in the form of a running bright pulse, which is demonstrated below.

Now we concentrate on two particular cases. First, we analyze the situation when the group velocity dispersion of one of the modes is zero. More precisely, we consider the points k_1, k_2 such that $\omega_2'' = 0$, $\omega_1'' > 0$ (i.e. $k_2 = \pm k_0$). This requirement can be fulfilled in a finite range of the nonlocality parameter σ. Namely, there must be $\sigma_{cr}^{(1)} < \sigma < \sigma_{cr}^{(2)}$. If $\sigma < \sigma_{cr}^{(1)}$ the group velocity is a monotonic function of the wave number, while at $\sigma > \sigma_{cr}^{(2)}$ the group velocity at the point when $\omega'' = 0$ exceeds one (see Fig. 1). Then a solitary wave solution (we call it breather) of the system (9) can be written as follows,

$$A_1 = \alpha e^{i\Omega t_2} \operatorname{sech}(\beta\zeta),$$
$$A_2 = \alpha e^{i\phi(\zeta)t_2} \operatorname{sech}(\beta\zeta), \tag{10}$$

where

$$\Omega = \tfrac{1}{2}\omega_1''\beta^2, \quad \phi(\zeta) = \frac{\chi_{11} + 2\chi_{22}}{2\omega_2} \frac{\alpha^2}{\cosh^2 \beta\zeta},$$
$$\beta = \sqrt{\frac{2\chi_{11} + \chi_{22}}{2\omega_1\omega_1''}}\,\alpha, \tag{11}$$

and α is the constant amplitude of the breather. The ansatz (10) is based on the solution of the NLS equation since in the case at hand the second of Eqs. (9) does not contain the derivative with respect to ζ and is resolved directly subject to the supposition that the modula of $A_{1,2}$ depend only on ζ. The solution (10), (11) exists in a range of parameters which is intermediate between regions where there exist stable pairs of coupled bright solitons and unstable coupled pairs of bright and dark solitons [10]. In the respective region one of the modes (the second one in the case at hand) displays dispersionless propagation and being alone it should require taking into account higher dispersion. The coupling, however, introduces dispersion in the system of the two linked modes which now cannot be considered independently. Looking for an explicit form of u_1, we get

$$u_1 = \frac{2\alpha}{\cosh \beta\zeta}\{\cos[k_1 x + (\Omega - \omega_1)t]$$
$$+ \cos[k_2 x + (\phi(\zeta) - \omega_2)t]\}. \tag{12}$$

Thus the breather, moving with group velocity v_{gr}, is characterized by the amplitude and two internal frequencies depending on the spatial coordinate. Such a solution has been numerically obtained in Ref. [9] where a stable structure was observed for quite a long time.

Another interesting coupled mode solution appears when $\sigma > \sigma_{cr}^{(2)}$. Then one can consider the coupling of two modes having unit velocity. If one of them corresponds to $\omega_1'' > 0$ and another is characterized by rather large $|k_2|$, for which $\omega_2'' = \hat{f}(k_2) = O(\epsilon)$, then the system (9) is degenerated,

$$2\omega_1 i\frac{\partial A_1}{\partial t_2} + \omega_1\omega_1''\frac{\partial^2 A_1}{\partial \zeta^2} + \chi_{11}(|A_1|^2 + I_2)A_1 = 0. \tag{13}$$

Here $I_2 = |A_2|^2$ is the intensity of the second mode. Since now the mode 2 is associated with linear dispersionless propagation, $A_2 = \sqrt{I_2}\exp[i(\chi_{11}/2\omega_2)|A_1|^2 t_2]$, the intensity I_2 may depend on ζ in an arbitrary way. Thus we have dynamics when the first mode is driven by the second one. In particular, if the mode 2 is a monochromatic wave of a constant amplitude (i.e. I_2 does not depend on t_2) then there exists a breather of the NSG which is exactly the NLS soliton with the frequency determined by $(\chi_{11}/2\omega_1)I_2$.

The effects of coupling discussed above are caused by a nonmonotonic dependence of the group velocity on the wave number. This is a rather general characteristic feature of nonlocal models rather than a peculiarity of the problem at hand. In order to illustrate this, in Fig. 2 we depict the group velocity for our model with a different kernel [5], $f(x) = 1/(x^4 + \sigma^4)$, and for another NSG model [11],

$$u_{tt} + \sin u = \int dy\, f_x(x-y)\, u_y(y,t), \tag{14}$$

which describes the propagation of signals in certain nonlinear transmission lines. Also, as it has been found recently [12], all the above effects are observed in the dynamics of nonlinear chains with long-range interactions of Kac–Baker type.

To conclude we have shown that in the limit of small amplitudes there exist coupled localized states originated by long range interactions. In the case of the NSG (1), (6) the properties of the system are

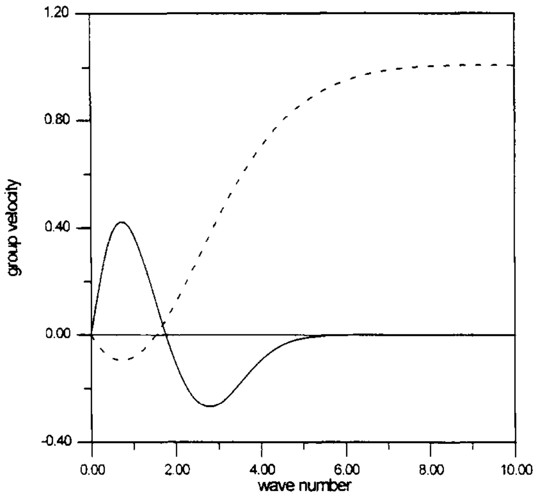

Fig. 2. The group velocity versus wave number for the model (1) at $f(x) = 1/(x^4+\sigma^4)$ (dashed line) and for the model (18) at $f(x) = (1/2\sqrt{\pi}\sigma)\exp(-x^2/4\sigma^2)$ (solid line). Both curves correspond to $\sigma \approx 0.565$.

characterized by two critical values of the nonlocality parameter σ. One of them, $\sigma_{cr}^{(1)}$, is close to the critical value found numerically in Ref. [5] for the excitations of large amplitudes. As is evident the coupled mode dynamics is much richer and is not exhausted by the phenomena described here. In particular, it is possible to have coupling of three running modes, as well as more sophisticated multimode dynamics, which will be reported elsewhere.

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Appendix

The multiscale analysis for our model differs from the conventional approach in the analysis of the nonlocal kernel. Expansion of the last depends on the scale associated to the kernel. In our case, the scale of the kernel is taken to be of order one. This allows one to develop u_n (in the operator L given by (3)) as a function of the slow variables as follows,

$$f(x-y)\,u_n = f(z)\left(u_n - \epsilon z\frac{\partial u_n}{\partial x_1} - \epsilon^2 z\frac{\partial u_n}{\partial x_2} + \tfrac{1}{2}\epsilon^2 z^2\frac{\partial^2 u_n}{\partial x_1^2} + 2\epsilon^3 z^2\frac{\partial^2 u_n}{\partial x_1\partial x_2} + \epsilon^4 z^2\frac{\partial^2 u_n}{\partial x_2^2}\right), \tag{A.1}$$

where $z = x - y$. Then the functions F_n used in the calculations take the form

$$F_2 = -2\frac{\partial^2 u_1}{\partial x_0\partial x_1} + 2\frac{\partial^2 u_1}{\partial t_0\partial t_1} - \int \mathrm{d}x_0\, f(x_0)\, x_0\frac{\partial u_1}{\partial x_1}, \tag{A.2}$$

$$F_3 = -\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial t_1^2} + \tfrac{1}{2}\int \mathrm{d}x_0\, f(x_0)\, x_0^2\frac{\partial^2 u_1}{\partial x_1^2} - 2\frac{\partial^2 u_1}{\partial x_0\partial x_2} + 2\frac{\partial^2 u_1}{\partial t_0\partial t_2} - \int \mathrm{d}x_0\, f(x_0)\, x_0\frac{\partial u_1}{\partial x_2} - \tfrac{1}{24}\int \mathrm{d}x_0\, f(x_0)\, u_1^3 - \tfrac{1}{8}u_1^2\int \mathrm{d}x_0\, f(x_0)\, u_1. \tag{A.3}$$

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