Lambert Function in the Determining of the Noise Equilibrium Frequency for Radiation Detectors

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***Abstract*—We analyze several cases of combined noise in sensors for radiation and in sensor-amplifier circuits. Solutions are shown to involve the Lambert function of various variables when the unknown is a frequency limit of the bandwidth. The coverage of the paper is essentially theoretical and its two aims are to exemplify the use of Lambert function in noise analysis and to help improving the design of pre-amplifiers, especially for radiation detectors.**

Keywords—Lambert function, radiation detector, amplifier, optimal frequency bandwidth, equilibrium noise frequency, optimization.

I. Introduction

The optimization of applications based on low noise amplifiers (LNA) and sensors coupled to their input remains a challenge, among others because of the optimization of the balance between the noise of the sensors, that of the LNA and the bandwidth required in the application. A special case is that of photodetectors, including those for nuclear radiation, and the related signal to noise ratio [1], [2,] [3] [4], [5], [6]. Consequently, better designs for the circuits including various sensors and especially photodetectors are of interest in medical fields, for example in medical electronics and industrial testing. Most designs concentrate on conditions related to amplifiers and sensor parameters; we focus on the bandwidth aspect, asking for conditions that the frequency limits satisfy specific conditions.

We propose an analysis of the noise in LNA-sensor circuits based on the notion of noise equilibrium frequency, which we apply to the problem in hand first for detectors alone and then for detectors with the amplifier circuit. The notion of noise equilibrium frequency is similar to that of (noise-wise) optimal amplifier for a specified signal generator. A basic optimality condition for the amplifier is that it produces the same noise as the generator internal resistance, in the given bandwidth, $e_{n,a}^2(\Delta f) = 4ktR_g\Delta f$, where $e_{n,a}^2$ is the noise power of the amplifier, Δf is the bandwidth, R_g is the electrical resistance of the signal generator.

The notions of corner frequency and of noise figure are useful when approximatively assessing the noise quality of an amplifier in a specified bandwidth. However, it does not provide a clear understanding of the contributions of the white and $1/f$ noises in the specified bandwidth. This limit is removed by the concept of noise equilibrium frequency, as discussed subsequently.

The noise equilibrium frequency for an amplifier was defined as a bandwidth where white noise and $1/f$ noise have the same power [7]. When the lower limit f_1 of the bandwidth is specified and the noise generators are known, determining the noise equilibrium frequency condition is equivalent with finding the upper frequency, f_2 , of the bandwidth. For amplifier noise, for the case of white and $1/f$ noise, it was shown in [7] that the problem leads to a solution involving the Lambert function. As far as I know, no previous study has applied Lambert function to the radiation sensor plus amplifier circuits. In addition, the use of Lambert functions applied to the noise of amplifiers (to computing the "equilibrium frequency" for amplifiers) was never stated or approached, at my best knowledge, before the paper [7].

The problem can be restated as: Find the general condition for ideal bandpass filters such that in the selected band the contributions of the white noise and of the $1/f$ noise are equal. One can easily imagine several other problems leading to the same type of solution, based on the Lambert function. Similar problems can easily be stated.

A general form of equation leading to a Lambert function-based solution is [8]

$$a^x = x + b. \tag{1}$$

This leads to, after some basic manipulations and assuming $a > 0$,

$$x \ln a = \ln(x + b) \rightarrow (u - b) \ln a =$$

$$= \ln u \Rightarrow Au + B = \ln u, \tag{2}$$

where $A = \ln a\,, B = -b \ln a$, and $u = x + b$, under the condition $x + b > 0$ ($x > -b$). Recall that we assume throughout the paper that the condition $x + b > 0$ is satisfied; this may not be true in all applications discussed. In addition, this requires to find the value of x and then to decide if it is admissible.

The above equation (1) has the solution (see eq. (5) in [8])

$$x = -b - \frac{W(-a^{-b} \ln a)}{\ln a}, \tag{3}$$

where $W()$ is Lambert function. Reversing the notations above, $a = e^A$, $b = -B/\ln a = -B/A$, the equation $Au + B = \ln u$ has the solution derived from (3)

$$u = x + b = -\frac{1}{A} W\left(-A \cdot (e^A)^{\frac{B}{A}}\right) = -\frac{1}{A} W(-Ae^B). \tag{4}$$

Other typical forms of the equation with Lambert function solution are, [8], $W(z)e^{W(z)} = z$, $z \in \boldsymbol{C}$, which is often considered the standard form, and [8]

$$x^{x^a} = b. \tag{5}$$

Throughout the paper we will show how equations in the form $a^x = x + b$ occur when dealing with noise problems, without further explicating the details of the solutions. If otherwise not stated, we will assume that the noise of the detectors and the amplifiers can be represented as a sum of white and $1/f$ noises. Notice that as long as the imposed conditions are reasonable, the actual form of the above Lambert equation has a (real) solution. The reasonable conditions consist in requiring that the bandwidth of interest (see next Section) includes the corner frequency of the noise of the overall circuit.

The remaining part of the study addresses some of the presented issues and points to design considerations.

II. Noise Equilibrium Frequency of Sensor Circuits

The applicative importance of the discussion can be stated as follows: Considering a sensor and its circuitry (preamplifier, filters, amplifier), an assuming the expected signals determine a specific lower frequency of the bandwidth, f_1, where f_1 is low enough such that the $1/f$ noise of the sensor and amplifier is important, further assuming that it is desirable to extend the bandwidth as much as possible without adding too much (white) noise, then the question is how to choose the upper frequency of the circuit. Here, "not adding too much noise" is interpreted as limiting the white noise contribution to the total noise such that the white noise power does not become dominant.

Applications illustrating the above problem setting include signals with spectra extending toward higher frequencies (compared with f_1), yet the higher frequencies being less informative than the lower frequency components. Examples are the electrocardiographic signals and other biological signals.

A. A Basic Case of Sensor-Amplifier Circuit

We start with a simple case, partly discussed in [7]. According to the definition, in the bandwidth $[f_1, f_x]$, the white and $1/f$ noise have the same power [7]. This is not always possible when f_1 is predetermined; thus, the solution is not guaranteed for any f_1, moreover the solution is of little use when the result is a too narrow bandwidth for the application considered. The same remarks apply for a choice of the bandwidth $[f_x, f_2]$.

Consider a sensor, such as a radiation sensor. Consider a sensor with both white and $1/f$ noise voltage generators and with no noise current generators, with the total noise with the power spectral density

$$e_s^2(f) = a_s + b_s/f \tag{6}$$

Also consider that the preamplifier has only voltage noise generators,

$$e_{am}^2(f) = a_{am} + b_{am}/f. \tag{7}$$

Assuming a bandwidth $[f_1, f_x]$, with the lower frequency fixed, we are interested in the bandwidth condition that the sensor and the amplifier contribute equal noises,

$$\int_{f_1}^{f_x}\left(a_s + \frac{b_s}{f}\right)df = \int_{f_1}^{x}\left(a_{am} + \frac{b_{am}}{f}\right)df. \tag{8}$$

One obtains from (8) that

$$(a_s - a_{am})(f_x - f_1) = (b_s - b_{am})\ln\frac{f_x}{f_1}. \tag{9}$$

The above can be written as

$$(a_s - a_{am})f_x = (b_s - b_{am})\ln f_x + (a_s - a_{am})f_1 - (b_s - b_{am})\ln f_1 . \tag{10}$$

Denoting $A = \frac{(b_s - b_{am})}{(a_s - a_{am})}$, $B = \frac{(a_s - a_{am})f_1 - (b_s - b_{am})\ln f_1}{(a_s - a_{am})}$, one obtains from (10)

$$f_x = A\ln f_x + B,\ \ln f_x = \frac{1}{A}f_x - \frac{B}{A} \tag{11}$$

which is again the typical Lambert equation, see [7], [8]. Notice that both A and B can have positive or negative values. The solution is

$$f_x = -A \cdot W\left(\frac{1}{A} \cdot e^{-\frac{B}{A}}\right), \tag{12}$$

and the variable in the Lambert function can be either positive or negative, depending on the sign of A. Notice that the joint noise (sensor plus amplifier) has a corner frequency f_{cj} given by

$$a_s + a_{am} = \frac{b_s}{f} + \frac{b_{am}}{f} \tag{13}$$

or $f_{cj} = \frac{b_s + b_{am}}{a_s + a_{am}}$. The corner frequency is important because when $f_1 > f_{cj}$ and when $f_2 < f_{cj}$, there is no possible solution to the problem.

For physical reasons, the solution f_x of (12) satisfies the condition $f_x > f_{cj}$, as discussed previously. Because $f_x > 0$, the solution (12) is positive only when $A < 0$ (because the Lambert function should be negative); therefore, the solution (12) is meaningful only when $\frac{(b_s - b_{am})}{(a_s - a_{am})} < 0$, that is when $(b_s - b_{am}) \cdot (a_s - a_{am}) < 0$. In addition, the solution should satisfy the condition $x + b > 0$, see eq. (2).

When the sensor has only white noise and its noise is thermic, due to its physical resistance R_s, $e_s^2(f) = 4kTR_s$, the equation for f_x becomes

$$(4kTR_s - a_{am})f_x + b_{am}\ln f_x = C, \tag{14}$$

where $C = (a_s - a_{am})f_1 - (b_s - b_{am})\ln f_1$. This is again a Lambert equation.

The same type of equation is expected when the sensor has shot noise, as all photosensors have; recall that shot noise is also white, but is best modelled by a noise current generator.

B. Case of Amplifier with Current Noise Only

Amplifiers with FETs/MOS input stage have negligible current noise, but bipolar input stages may have the current noise contribution higher than the voltage noise. Next, we neglect the voltage noise of the amplifier. For a sensor with both thermal and $1/f$ noise e_s, the combined noise of the amplifier and sensor is, in a bandwidth $[f_1, f_x]$,

$$4kTR + \frac{a_s}{f} + R^2 i_{am}^2(f). \tag{15}$$

Assuming the amplifier current noise has both a white component and a $1/f$ component, the combined power density of the noise is

$$4kTR + \frac{a_s}{f} + R^2\left(a_{am} + \frac{b_{am}}{f}\right) \quad (16)$$

and the total noise in the bandwidth is

$$4kTR(f_x - f_1) + a_s \ln\frac{f_x}{f_1} + R^2 a_{am}(f_x - f_1) + b_{am}\ln\frac{f_x}{f_2}. \quad (17)$$

The condition of equal contribution to the combined noise of the amplifier and the sensor becomes

$$4kTR(f_x - f_1) + a_s \ln\frac{f_x}{f_2} = R^2 a_{am}(f_x - f_1) + b_{am}\ln\frac{f_x}{f_2}, \quad (18)$$

or

$$(-R^2 a_{am} + 4kTR)f_x + R^2 a_{am} f_1 - 4kTRf_1 + (b_{am} - a_s)\ln f_1 = (b_{am} - a_s)\ln f_x. \quad (19)$$

Denoting $C_1 = -R^2 a_{am} + 4kTR$, $C_2 = R^2 a_{am} f_1 - 4kTRf_1 + (b_{am} - a_s)\ln f_1$, and $C_3 = b_{am} - a_s$, one obtains $C_1 f_x + C_2 = C_3 \ln f_x$, again a Lambert equation.

C. *Case of Amplifier with Current Noise only and Sensor with Shot Noise*

Denote by R the resistance seen by the noise current generators in the typical transimpedance circuit, see for example [9]. Because amplifiers have higher noise at low frequencies, which are of interest in some applications such as astronomy, assume that the higher frequency f_2 of the desired bandwidth is fixed and the problem is to determine the lower frequency $f_1 = f_x$ such that the total equivalent noise of the sensor and the preamplifier is below a specified threshold, n_{th}. The condition is equivalent with:

$$\int_{f_x}^{f_2}\left(\left(2qI_\phi + a_{am} + \frac{b_{am}}{f}\right)R^2 + 4kTR\right)df \le n_{th}^2 \quad (20)$$

See [10], [11], Figure 28 in [12], [13], [14], [15] for details on the terms in the sum of the noise. The above (20) leads to

$$\left((2qI_\phi + a_{am})R^2 + 4kTR\right)(f_2 - f_x) + b_{am}R^2 + b_{am}R^2(\ln f_2 - \ln f_x) \le n_{th}^2 \quad (21)$$

or, denoting $A = \left((2qI_\phi + a_{am})R^2 + 4kTR\right)$, $B = b_{am}R^2(1 + \ln f_2)$, $C = b_{am}R^2$ and considering the limit equality:

$$Af_2 - Af_x + B - C\ln f_x = n_{th}^2 \quad (22)$$

Finally, with $D = Af_2 + B - n_{th}^2$, the equation is

$$Af_x + C\ln f_x = D \quad (23)$$

which again is a Lambert equation.

III. A Problem of SNR

Assume that the system involves both 1/f and white noise, $e^2(f) = a + b/f$, a known, and that the signal has a power spectrum $S^2(f)$ in the band $[f_1, f_2]$. We ask for the condition that b should satisfy such that the SNR is the same below and above f_c, where f_c is the corner frequency. Denote $x = f_c$.

The same SNR means that

$$SNR_L = \frac{\int_{f_1}^{x} S^2(f)df}{\int_{f_1}^{x}\left(a+\frac{b}{f}\right)df} = SNR_H = \frac{\int_{x}^{f_2} S^2(f)df}{\int_{x}^{f_2}\left(a+\frac{b}{f}\right)df}. \quad (24)$$

Denoting $\frac{\int_{x}^{f_2} S^2(f)df}{\int_{f_1}^{x} S^2(f)df} = \gamma(x)$, (24) can be written as

$$\left(a(f_2 - x) + b(\ln f_2 - \ln x)\right) = \gamma(x)\left(a(x - f_1) + b(\ln x - \ln f_1)\right). \quad (25)$$

Then, after basic manipulations,

$$-ax - b\ln x + af_2 + b\ln f_2 = \gamma(x)(ax - af_1 + b\ln x - b\ln f_1) \quad (26)$$

and finally

$$a\left(1 + \gamma(x)\right)x + b\left(1 + \gamma(x)\right)\ln x = b\ln f_2 + af_2 + \gamma(x)(af_1 + b\ln f_1). \quad (27)$$

Dividing in (27) by $1 + \gamma(x)$, one obtains

$$ax + b\ln x = \frac{b\ln f_2 + af_2 + \gamma(x)(af_1 + b\ln f_1)}{1 + \gamma(x)}. \quad (28)$$

The above can be written as

$$x + c\ln x = g(x), c = \frac{b}{a}, \quad (28)$$

where

$$g(x) = \frac{b\ln f_2 + af_2 + \gamma(x)(af_1 + b\ln f_1)}{1 + \gamma(x)}, \quad (29)$$

which is the condition that noise affects the same way the low and high frequency range of the signal.

Next, we ask for the condition that b should satisfy such that the signal is as uniformly as possible affected in the frequency band by the noise (in the sense given below). The uniform effect of the noise means that there is a constant, k, and a given error limit ϵ such that the error between the $S^2(f)$ curve and $k \times e^2(f)$, satisfies

$$\varepsilon^2(f) = \left(S^2(f) - k \times e^2(f)\right)^2 < \epsilon^2 \;\; \forall f.$$

The condition is equivalent with saying that the two curves, the signal power spectrum and the noise power spectrum are as similar as possibly made by a good choice of the sensor and the amplifier. The condition asks that any spectral component in the signal is affected by a limited amount of noise comparable to the noise affecting any other component.

IV. Discussion and Conclusions

The treatment present is limited to some basic situations. Amplifiers bandwidth is limited and the low-pass effect sometimes occurs in the frequency band of interest; we have not analyzed this case. The issue of noise in radiation detectors is much affected by the collimation process [16] and should be dealt with specifically for that case, with some collimators more potent than others in reducing unavoidable background radiation noise [17].

For simplifying the discussion and focusing on the main topic of the paper, we have not considered the roles of the capacitors in parallel with the sensor and at the output of the amplifier. For a detailed treatment considering the capacitors in the circuit, see [18]. Taking into account the low pass filtering effect of these capacitors adds in the equation a $1/f$ term filtering the white noise at higher frequencies, which combines with the $1/f$ term from the noise of the amplifier; in addition, a term $2\pi f e_{am} C$, see [18] which ads by integration a term in f_x^2 . The last term complicates the equation, which may be no long Lambert-type. Also, we have not considered the effect of the temperature on the noise, an effect known to be strong (e.g., [19]). The solution of the Lambert equation, expressing the optimal bandwidth in the problems discussed, is actually dependent on the temperature.

We need to stress that the Lambert function solution may not exist. In addition, the restrictions stated in the text explaining eq. (2) have to be carefully checked, because only under those conditions the solution was derived.

Concluding, we presented the computations necessary for an optimized design with radiation sensors, the related amplifiers, and radiation shields / screens, for several types of noises in the detectors and the amplifiers. We have shown that these design problems require the use of the Lambert function when the bandwidth is not pre-determined, except one of its limits. Methods to approximate the solutions (of the Lambert function) can be found in the literature, see for example [7], [20].

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