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Research article

Generalized (edge-)connectivity of join, corona and cluster graphs*

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Abstract: The generalized k-connectivity $\kappa_k(G)$ of a graph G, introduced by Hager in 1985, is a natural generalization of the classical connectivity. As a natural counterpart, Li et al. proposed the concept of generalized k-edge-connectivity $\lambda_k(G)$. In this paper, we obtain exact values or sharp upper and lower bounds of $\kappa_k(G)$ and $\lambda_k(G)$ for join, corona and cluster graphs.

Keywords: Steiner tree; generalized connectivity; join; corona; cluster
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1. Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to the book [3] for graph theoretical notation and terminology not described here. For a graph G, let $V(G)$, $E(G)$ and $\delta(G)$ denote the set of vertices, the set of edges and the minimum degree of G, respectively. For $S \subseteq V(G)$, we denote by $G - S$ the subgraph obtained by deleting from G the vertices of S together with the edges incident with them. To show the properties of these generalizations clearly, we hope to state from the connectivity in graph theory. We divide our introduction into the following three subsections to state the motivations and our results of this paper.

*In Summer 2020, this paper was accepted for publication in "Ars Combinatoria". As of Dec 15, 2021, the editorial board and managing editors of this journal have resigned. So we contacted the publisher and withdrawn the paper from "Ars Combinatoria" and re-submitted here.

1.1. Connectivity and generalized connectivity

Connectivity and edge-connectivity are two of the most basic concepts of graph-theoretic subjects, both in a combinatorial sense and an algorithmic sense. As we know, the classical connectivity has two equivalent definitions. The *connectivity* of a graph G, written $\kappa(G)$, is the minimum size of a set $S \subseteq V(G)$ such that $G - S$ is disconnected or has only one vertex. If $G - S$ is disconnected, then we call such a set S a *vertex cut-set* for G. We call this definition the 'cut' version definition of connectivity. The well-known Menger's theorem provides an equivalent definition of connectivity, which can be called the 'path' version definition of connectivity. For any two distinct vertices x and y in G, the *local connectivity* $\kappa_G(x, y)$ is the maximum number of internally disjoint paths connecting x and y. Then $\kappa(G) = \min\{\kappa_G(x, y) \mid x, y \in V(G), x \neq y\}$ is defined to be the *connectivity* of G.

The generalized connectivity of a graph G, introduced by Hager [17], is a natural and nice generalization of the 'path' version definition of connectivity. For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, *an S-Steiner tree* or *a Steiner tree connecting S* (or simply, *an S-tree*) is a subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Two Steiner trees T and T' connecting S are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local connectivity* $\kappa(S)$ is the maximum number of internally disjoint Steiner trees connecting S in G. Note that when $|S| = 2$, a minimal Steiner tree connecting S is just a path connecting the two vertices of S. For an integer k with $2 \leq k \leq n$, *generalized k-connectivity* (or *k-tree-connectivity*) is defined as $\kappa_k(G) = \min\{\kappa(S) \mid S \subseteq V(G), |S| = k\}$. Clearly, when $|S| = 2$, $\kappa_2(G)$ is nothing new but the connectivity $\kappa(G)$ of G, that is, $\kappa_2(G) = \kappa(G)$, which is the reason why one addresses $\kappa_k(G)$ as the generalized connectivity of G. By convention, for a connected graph G with less than k vertices, we set $\kappa_k(G) = 1$. Set $\kappa_k(G) = 0$ when G is disconnected. This concept appears to have been introduced by Hager in [17].

The following Table 1 shows how the generalization proceeds.

Table 1. Classical connectivity and generalized connectivity.

	Classical connectivity	Generalized connectivity
Vertex subset	$S = \{x, y\} \subseteq V(G)$ ($\lvert S\rvert = 2$)	$S \subseteq V(G)$ ($\lvert S\rvert \geq 2$)
Set of Steiner trees	$\begin{cases} \mathscr{P}_{x,y} = \{P_1, P_2, \cdots, P_\ell\} \\ \{x, y\} \subseteq V(P_i), \\ E(P_i) \cap E(P_j) = \emptyset \\ V(P_i) \cap V(P_j) = \{x, y\} \end{cases}$	$\begin{cases} \mathscr{T}_S = \{T_1, T_2, \cdots, T_\ell\} \\ S \subseteq V(T_i), \\ E(T_i) \cap E(T_j) = \emptyset, \\ V(T_i) \cap V(T_j) = S \end{cases}$
Local parameter	$\kappa(x, y) = \max \lvert\mathscr{P}_{x,y}\rvert$	$\kappa(S) = \max \lvert\mathscr{T}_S\rvert$
Global parameter	$\kappa(G) = \min\limits_{x,y \in V(G)} \kappa(x, y)$	$\kappa_k(G) = \min\limits_{S \subseteq V(G), \lvert S\rvert = k} \kappa(S)$

As mentioned above, $\kappa_2(G) = \kappa(G)$ is just the connectivity of a graph G when $k = |S| = 2$. Another extreme of $\kappa_k(G)$ is when $k = n$. For $k = n$, one can see that $S = V(G)$ and $\kappa_n(G)$ is just the maximum number of edge-disjoint spanning trees in G (For $k = n$, each Steiner tree connecting S is a spanning tree of G). Then $\kappa_n(G)$ is called the *spanning tree packing number* of G. For the spanning tree packing number, we refer to [35, 36]. For a given graph G, the problem of finding out the spanning tree packing number of G is called the *Spanning tree packing problem*. Note that Spanning tree packing problem is a special case of the generalized k-connectivity.

1.2. Edge-connectivity and generalized edge-connectivity

The classical edge-connectivity also has two equivalent definitions. The *edge-connectivity* of G, written $\lambda(G)$, is the minimum size of an edge set $M \subseteq E(G)$ such that $G - M$ is disconnected or has only one vertex. We call this definition the 'cut' version definition of edge-connectivity. Menger's theorem also provides an equivalent definition of edge-connectivity, which can be called the 'path' version definition. For any two distinct vertices x and y in G, the *local edge-connectivity* $\lambda_G(x, y)$ is the maximum number of edge-disjoint paths connecting x and y. Then $\lambda(G) = \min\{\lambda_G(x, y) \,|\, x, y \in V(G), x \neq y\}$ is defined to be the *edge-connectivity* of G. For connectivity and edge-connectivity, Oellermann gave a survey paper on this subject, see [31].

As a natural counterpart of the generalized connectivity, Li et al. [29] introduced the concept of generalized edge-connectivity, which is a generalization of the 'path' version definition of edge-connectivity. For $S \subseteq V(G)$ and $|S| \geq 2$, the *generalized local edge-connectivity* $\lambda(S)$ is the maximum number of edge-disjoint Steiner trees connecting S in G. For an integer k with $2 \leq k \leq n$, the *generalized k-edge-connectivity* $\lambda_k(G)$ of G is then defined as $\lambda_k(G) = \min\{\lambda(S) \,|\, S \subseteq V(G) \textit{ and } |S| = k\}$. It is also clear that when $|S| = 2$, $\lambda_2(G)$ is nothing new but the standard edge-connectivity $\lambda(G)$ of G, that is, $\lambda_2(G) = \lambda(G)$, which is the reason why we address $\lambda_k(G)$ as the generalized edge-connectivity of G. Also set $\lambda_k(G) = 0$ when G is disconnected.

There are many results on the generalized connectivity, tree-connectivity and strong connectivity, we refer to the recent book [28] and the papers [7–9, 20, 26, 27, 34, 43].

1.3. Application background of generalized connectivity and product networks

In addition to being a natural combinatorial measure, the generalized connectivity can be motivated by its interesting interpretation in practice. For example, suppose that G represents a network. If one considers to connect a pair of vertices of G, then a path is used to connect them. However, if one wants to connect a set S of vertices of G with $|S| \geq 3$, then a tree has to be used to connect them. This kind of tree for connecting a set of vertices is usually called a *Steiner tree*, and popularly used in the physical design of VLSI circuits (see [15, 16, 38]). In this application, a Steiner tree is needed to share an electric signal by a set of terminal nodes. Steiner tree is also used in computer communication networks (see [10]) and optical wireless communication networks (see [6]). Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then, the number of totally independent ways to connect them is a measure for this purpose. The generalized k-connectivity can serve for measuring the capability of a network G to connect any k vertices in G.

Join, corona and cluster [42] are all graph product operations and can be defined as follows.

(1) The *join* or *complete product* of two disjoint graphs G and H, denoted by $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{gh \,|\, g \in V(G), h \in V(H)\}$.
(2) The *corona* $G * H$ is obtained by taking one copy of G and $|V(G)|$ copies of H, and by joining each vertex of the i-th copy of H with the i-th vertex of G, where $i = 1, 2, \ldots, |V(G)|$.
(3) The *cluster* $G \odot H$ is obtained by taking one copy of G and $|V(G)|$ copies of a rooted graph H, and by identifying the root of the i-th copy of H with the i-th vertex of G, where $i = 1, 2, \ldots, |V(G)|$.

Product networks were proposed based upon the idea of using the cross product as a tool for "combining" two known graphs with established properties to obtain a new one that inherits properties from both [11]. Inspired by product networks and the application of generalized connectivity on networks, H. Li et al. [26] studied the generalized 3-connectivity of cartesian product and lexicographic product of graphs, and S. Li et al. [27] investigated the generalized 3-connectivity of star graphs and bubble-sort graphs. In this paper, we focus on the generalized k-connectivity and generalized k-edge-connectivity of three other graph product operations, i.e., join, corona and cluster. In the following three sections, we obtain exact values or sharp upper and lower bounds of $\kappa_k(G)$ and $\lambda_k(G)$ for these three graph product operations respectively.

2. Cluster

In this section, let G and H be two graphs with $V(G) = \{g_1, g_2, \ldots, g_n\}$ and $V(H) = \{h_1, h_2, \ldots, h_m\}$, respectively. From the definition of cluster, $V(G \odot H) = \{(g_i, h_j) \mid 1 \le i \le n,\ 1 \le j \le m\}$, where $\odot$ denotes the cluster product operation. For $g \in V(G)$, we use $H(g)$ to denote the subgraph of $G \odot H$ induced by the vertex set $\{(g, h_j) \mid 1 \le j \le m\}$. Without loss of generality, we assume (g_i, h_1) is the root of $H(g_i)$ for each $g_i \in V(G)$. Let $G(h_1)$ be the graph induced by the vertices in $\{(g_i, h_1) \mid 1 \le i \le n\}$. Clearly, $G(h_1) \cong G$, and $V(G \odot H) = V(H(g_1)) \cup V(H(g_2)) \cup \ldots \cup V(H(g_n))$.

For generalized edge-connectivity, we have the following result.

Theorem 2.1. *Let k, n, m be three integers with $2 \le k \le nm$, and let G, H be two connected graphs with n, m vertices, respectively. Then*

$$\lambda_k(G \odot H) = \begin{cases} \min\{\lambda_k(G), \lambda_k(H)\} & \text{if } 2 \le k < \min\{m, n\}, \\ \min\{\lambda_n(G), \lambda_m(H)\} & \text{if } \max\{m, n\} < k \le mn, \\ \min\{\lambda_k(G), \lambda_m(H)\} & \text{if } m \le k \le n, \\ \min\{\lambda_n(G), \lambda_k(H)\} & \text{if } n \le k \le m. \end{cases}$$

Proof. We only give the proof of cases $2 \le k \le \min\{m, n\}$ and $k > \max\{m, n\}$, and the other two cases can be proved similarly. For $2 \le k \le \min\{m, n\}$, we first show that $\lambda_k(G \odot H) \le \min\{\lambda_k(G), \lambda_k(H)\}$. Choose $S \subseteq V(G(h_1))$ and $|S| = k$. From the structure of $G \odot H$, there are at most $\lambda_k(G)$ edge-disjoint S-Steiner trees in $G \odot H$, and hence $\lambda_k(G \odot H) \le \lambda(S) \le \lambda_k(G)$. Choose $S' \subseteq V(H(g_1))$ and $|S'| = k$. From the structure of $G \odot H$, there are at most $\lambda_k(H)$ edge-disjoint S'-Steiner trees in $G \odot H$, and hence $\lambda_k(G \odot H) \le \lambda(S') \le \lambda_k(H)$. Therefore, we have $\lambda_k(G \odot H) \le \min\{\lambda_k(G), \lambda_k(H)\}$. Next, we show that $\lambda_k(G \odot H) \ge \min\{\lambda_k(G), \lambda_k(H)\}$. For any $S \subseteq V(G \odot H)$ and $|S| = k$, we assume that $S \cap V(H(g_i)) \ne \emptyset$ for $1 \le i \le r$; $S \cap V(H(g_i)) = \emptyset$ for $r+1 \le i \le n$. Let $\lambda_k(G) = x$, $\lambda_k(H) = y$, and $S_G = \{g_1, g_2, \ldots, g_r\}$, where $1 \le r \le k$. If $r = 1$, then $S \subseteq V(H(g_1))$, and there are at least y edge-disjoint S-Steiner trees in $H(g_1)$, and hence $\lambda(S) \ge y = \lambda_k(H)$. From now on, we assume $r \ge 2$. It follows that $1 \le |S \cap V(H(g_i))| \le k - 1$ for each i ($1 \le i \le r$). For each i ($1 \le i \le r$), we let

$$S_i = \begin{cases} S \cap V(H(g_i)) & \text{if } (g_i, h_1) \in S, \\ (S \cap V(H(g_i))) \cup \{(g_i, h_1)\} & \text{if } (g_i, h_1) \notin S. \end{cases}$$

Since $|S_i| \le k$, it follows that there exist at least y edge-disjoint S_i-Steiner trees in $H(g_i)$, say $T_{i,1}, T_{i,2}, \ldots, T_{i,y}$, where $1 \le i \le r$. Since $\lambda_k(G) = x$ and $S_G = \{g_1, g_2, \ldots, g_r\}$, there are x edge-disjoint S_G-Steiner trees in $G(h_1)$, say $T'_1, T'_2, \ldots, T'_x$. If $x \ge y$, then the tree induced by the edges in $E(T'_j) \cup E(T_{1,j}) \cup E(T_{2,j}) \cup \ldots \cup E(T_{r,j})$ $(1 \leq j \leq y)$ are edge-disjoint S-Steiner trees in $G \odot H$, and hence $\lambda(S) \geq y = \lambda_k(H)$. If $x < y$, then the tree induced by the edges in $E(T'_j) \cup E(T_{1,j}) \cup E(T_{2,j}) \cup \ldots \cup E(T_{r,j})$ $(1 \leq j \leq x)$ are edge-disjoint S-Steiner trees in $G \odot H$, and hence $\lambda(S) \geq x = \lambda_k(G)$. From the above argument, we have $\lambda_k(G \odot H) = \min\{\lambda_k(G), \lambda_k(H)\}$.

For $k > \max\{m, n\}$, we first show that $\lambda_k(G \odot H) \leq \min\{\lambda_n(G), \lambda_m(H)\}$. Since $k > n$, we choose $S \subseteq V(G(h_1))$ and $|S| = k$ such that $V(G(h_1)) \subseteq S$. From the structure of $G \odot H$, there exist at most $\lambda_n(G)$ edge-disjoint S-Steiner trees in $G \odot H$, and hence $\lambda_k(G \odot H) \leq \lambda(S) \leq \lambda_n(G)$. Since $k > m$, we choose $S \subseteq V(H(g_1))$ and $|S| = k$ such that $V(H(g_1)) \subseteq S$. From the structure of $G \odot H$, there exist at most $\lambda_m(H)$ edge-disjoint S-Steiner trees in $G \odot H$, and hence $\lambda_k(G \odot H) \leq \lambda(S) \leq \lambda_m(H)$. Therefore, we have $\lambda_k(G \odot H) \leq \min\{\lambda_n(G), \lambda_m(H)\}$. Next, we show that $\lambda_k(G \odot H) \geq \min\{\lambda_n(G), \lambda_m(H)\}$. Let $\lambda_n(G) = x$ and $\lambda_m(H) = y$. Clearly, there exist at least x edge-disjoint S-Steiner trees in $G(h_1)$, say $T'_1, T'_2, \ldots, T'_x$, and there exist at least y edge-disjoint S-Steiner trees in $H(g_i)$, say $T_{i,1}, T_{i,2}, \ldots, T_{i,y}$, where $1 \leq i \leq n$. For any $S \subseteq V(G \odot H)$ and $|S| = k$, if $x \geq y$, then the tree induced by the edges in $E(T'_j) \cup E(T_{1,j}) \cup E(T_{2,j}) \cup \ldots \cup E(T_{n,j})$ $(1 \leq j \leq y)$ are edge-disjoint spanning trees of $G \odot H$, and they are edge-disjoint S-Steiner trees in $G \odot H$, and so $\lambda(S) \geq y = \lambda_m(H)$. If $x < y$, then the tree induced by the edges in $E(T'_j) \cup E(T_{1,j}) \cup E(T_{2,j}) \cup \ldots \cup E(T_{n,j})$ $(1 \leq j \leq x)$ are edge-disjoint S-Steiner trees in $G \odot H$, and so $\lambda(S) \geq x = \lambda_n(G)$. From the arbitrariness of S, we have $\lambda_k(G \odot H) \geq \min\{x, y\} = \min\{\lambda_n(G), \lambda_m(H)\}$, as desired. □

For generalized connectivity, we have the following result.

Proposition 2.1. *Let k, n, m be three integers with $2 \leq k \leq nm$, and let G, H be two connected graphs with n, m vertices, respectively. Then*

$$\kappa_k(G \odot H) = \begin{cases} 1 & \text{if } 2 \leq k \leq mn - 1, \\ \min\{\kappa_n(G), \kappa_m(H)\} & \text{if } k = mn. \end{cases}$$

Proof. For $2 \leq k \leq mn - 1$, since (g_1, h_1) is a cut vertex of $G \odot H$, it follows that $\kappa_k(G \odot H) \leq 1$. Since $G \odot H$ is connected, it follows that $G \odot H$ contains a spanning tree T, For any $S \subseteq V(G \odot H)$ and $|S| = k$, T is an S-Steiner tree, and hence $\kappa(S) \geq 1$. From the arbitrariness of S, we have $\kappa_k(G \odot H) \geq 1$, and hence $\kappa_k(G \odot H) = 1$.

For $k = mn$, we let $\kappa_n(G) = x$ and $\kappa_m(H) = y$. Clearly, there exist at least x edge-disjoint S-Steiner trees in $G(h_1)$, say $T'_1, T'_2, \ldots, T'_x$, and there exist at least y S-Steiner trees in $H(g_i)$, say $T_{i,1}, T_{i,2}, \ldots, T_{i,y}$, for each i $(1 \leq i \leq n)$. For any $S \subseteq V(G \odot H)$ and $|S| = k$, if $x \geq y$, then the tree induced by the edges in $E(T'_j) \cup E(T_{1,j}) \cup E(T_{2,j}) \cup \ldots \cup E(T_{n,j})$ $(1 \leq j \leq y)$ are edge-disjoint spanning trees of $G \odot H$, and they are edge-disjoint S-Steiner trees in $G \odot H$, and so $\kappa(S) \geq y = \kappa_m(H)$. If $x < y$, then the tree induced by the edges in $E(T'_j) \cup E(T_{1,j}) \cup E(T_{2,j}) \cup \ldots \cup E(T_{n,j})$ $(1 \leq j \leq x)$ are edge-disjoint S-Steiner trees in $G \odot H$, and so $\kappa(S) \geq x = \kappa_n(G)$. Therefore, we have $\kappa_k(G \odot H) \geq \min\{\kappa_n(G), \kappa_m(H)\}$. On the other hand, there are at most $\min\{\kappa_n(G), \kappa_m(H)\}$ edge-disjoint spanning trees in $G \odot H$, which implies that $\kappa_k(G \odot H) \leq \min\{\kappa_n(G), \kappa_m(H)\}$ for $k = mn = |V(G \odot H)|$. □

3. Corona

In this section, let G and H be two graphs with $V(G) = \{g_1, g_2, \ldots, g_n\}$ and $V(H) = \{h_1, h_2, \ldots, h_m\}$, respectively. From the definition of corona graphs, $V(G * H) = V(G) \cup \{(g_i, h_j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$, where $*$ denotes the corona product operation. For $g \in V(G)$, we use $H(g)$ to denote the subgraph of $G*H$ induced by the vertex set $\{(g, h_j) \mid 1 \leq j \leq m\}$. For fixed i $(1 \leq i \leq n)$, we have $g_i(g_i, h_j) \in E(G*H)$ for each j $(1 \leq j \leq m)$. Then $V(G * H) = V(G) \cup V(H(g_1)) \cup V(H(g_2)) \cup \ldots \cup V(H(g_n))$.

For $\kappa_k(G * H)$, we have the following result.

Proposition 3.1. *Let k, n, m be three integers with $2 \leq k \leq (m + 1)n$, and let G, H be two connected graphs with n, m vertices, respectively.*

(1) *If $2 \leq k \leq (m + 1)n - 1$, then $\kappa_k(G * H) = 1$.*

(2) *If $k = (m + 1)n$, then $\min\{\kappa_m(H), \kappa_n(G)\} \leq \kappa_k(G * H) \leq \min\{\kappa_n(G), \delta(H) + 1\}$.*

Proof. For $2 \leq k \leq (m + 1)n - 1$, since g_1 is a cut vertex of $G * H$, it follows that $\kappa_k(G * H) \leq 1$. Since $G * H$ is connected, it follows that $G * H$ contains a spanning tree T. For any $S \subseteq V(G * H)$ and $|S| = k$, T is an S-Steiner tree, and hence $\kappa(S) \geq 1$. From the arbitrariness of S, we have $\kappa_k(G * H) \geq 1$, and hence $\kappa_k(G * H) = 1$.

For $k = (m+1)n$, we have $G*H = G\odot(H\vee K_1)$. From Proposition 2.1, $\kappa_k(G*H) = \kappa_k(G\odot(H\vee K_1)) = \min\{\kappa_n(G), \kappa_{m+1}(H \vee K_1)\}$. Since $\kappa_{m+1}(H \vee K_1) \leq \delta(H \vee K_1) = \delta(H) + 1$ and $\kappa_{m+1}(H \vee K_1) \geq \kappa_m(H)$, it follows that $\min\{\kappa_m(H), \kappa_n(G)\} \leq \kappa_k(G * H) \leq \min\{\kappa_n(G), \delta(H) + 1\}$. □

Proposition 3.2. *Let k, n, m be three integers with $2 \leq k \leq (m + 1)n$, and let G, H be two connected graphs with n, m vertices, respectively. Then*

$$\lambda_k(G * H) \leq \begin{cases} \min\{\lambda_k(G), \delta(H) + 1\} & \text{if } 2 \leq k < n, \\ \min\{\lambda_n(G), \delta(H) + 1\} & \text{if } n \leq k \leq (m + 1)n \end{cases}$$

and

$$\lambda_k(G * H) \geq \begin{cases} \min\{\lambda_k(G), \lambda_k(H)\} & \text{if } 2 \leq k < \min\{m + 1, n\}, \\ \min\{\lambda_n(G), \lambda_m(H)\} & \text{if } k \geq \max\{m + 1, n\}, \\ \min\{\lambda_k(G), \lambda_m(H)\} & \text{if } m + 1 \leq k < n, \\ \min\{\lambda_n(G), \lambda_k(H)\} & \text{if } n \leq k < m + 1. \end{cases}$$

Proof. It is clear that $G * H = G \odot (H \vee K_1)$. From Theorem 2.1, we have

$$\lambda_k(G * H) = \lambda_k(G \odot (H \vee K_1)) = \begin{cases} \min\{\lambda_k(G), \lambda_k(H \vee K_1)\} & \text{if } 2 \leq k < \min\{m + 1, n\}, \\ \min\{\lambda_n(G), \lambda_{m+1}(H \vee K_1)\} & \text{if } k \geq \max\{m + 1, n\}, \\ \min\{\lambda_k(G), \lambda_{m+1}(H \vee K_1)\} & \text{if } m + 1 \leq k < n, \\ \min\{\lambda_n(G), \lambda_k(H \vee K_1)\} & \text{if } n \leq k < m + 1. \end{cases}$$

Since $\lambda_k(H \vee K_1) \leq \delta(H \vee K_1) = \delta(H) + 1$ and $\lambda_k(H \vee K_1) \geq \lambda_k(H)$, the result follows. □

4. Join

Chartrand et al. [5] got the exact value of the generalized k-connectivity for the complete graph K_n.

Lemma 4.1. [5] *For every two integers n and k with $2 \leq k \leq n$, $\kappa_k(K_n) = n - \lceil k/2 \rceil$.*

In [29] we obtained some results on the generalized k-edge-connectivity.

Lemma 4.2. [29] *For every two integers n and k with $2 \le k \le n$, $\lambda_k(K_n) = n - \lceil k/2 \rceil$.*

Theorem 4.1. *Let G and H be two graphs, respectively. Then*

$$\kappa(G \vee H) = \min\{\kappa(G) + |V(H)|, \kappa(H) + |V(G)|\}.$$

For general k, we can obtain the following upper bounds.

Proposition 4.1. *Let G, H be two graphs of order n, m, respectively. Then*

$$\kappa_k(G \vee H) \le \begin{cases} \min\{\kappa_k(G) + m, \kappa_k(H) + n\} & \text{if } 2 \le k \le \min\{n, m\}, \\ m + n - \lceil k/2 \rceil & \text{if } \max\{n, m\} < k \le nm, \\ \kappa_k(G) + m & \text{if } m \le k \le n, \\ \kappa_k(H) + n & \text{if } n \le k \le m. \end{cases}$$

Moreover, the bounds are sharp.

Proof. For $k \le \min\{n, m\}$, we choose $S \subseteq V(G)$ and $|S| = k$. Then there are at most $\kappa_k(G)$ S-Steiner trees in G. For any other S-Steiner tree, it contains at least one vertex of H, and so there are at most $|V(H)|$ such S-Steiner trees in $G \vee H$. So, we have $\kappa_k(G \vee H) \le \kappa(S) \le \kappa_k(G) + m$. By symmetry, $\kappa_k(G \vee H) \le \min\{\kappa_k(G) + m, \kappa_k(H) + n\}$. For $\max\{n, m\} < k \le nm$, from Lemma 4.1, we have $\kappa_k(G \vee H) \le m + n - \lceil k/2 \rceil$. Similarly, we can prove that $\kappa_k(G \vee H) \le \kappa_k(G) + m$ if $m \le k \le n$; $\kappa_k(G \vee H) \le \kappa_k(H) + n$ if $n \le k \le m$. □

To show the sharpness of the upper bounds, we consider the following example.

Example 1. Let $G = K_n$ and $H = K_m$. If $2 \le k \le \min\{n, m\}$, then $\kappa_k(G \vee H) = \kappa_k(K_{m+n}) = n + m - \lceil k/2 \rceil = \min\{n - \lceil k/2 \rceil + m, m - \lceil k/2 \rceil + n\} = \min\{\kappa_k(G) + m, \kappa_k(H) + n\}$. If $\max\{n, m\} < k \le nm$, then it follows from Lemma 4.1 that $\kappa_k(G \vee H) = \kappa_k(K_{m+n}) = n + m - \lceil k/2 \rceil$. For $m \le k \le n$, we let $G = K_n$, $m = 1$, and $H = K_1$. Then $G \vee H = K_{n+1}$. From Lemma 4.1, we have $\kappa_k(G \vee H) = \kappa_k(K_{n+1}) = n + 1 - \lceil k/2 \rceil = \kappa_k(K_n) + 1$.

Proposition 4.2. *Let G, H be two graphs of order n, m, respectively. Let k be an integer with $2 \le k \le n + m$. Then*

$$\lambda_k(G \vee H) \le \min\{\delta(G) + m, \delta(H) + n\}.$$

Moreover, the bound is sharp for $k = 2$.

Proof. For $2 \le k \le n + m$, we have $\lambda_k(G \vee H) \le \delta(G \vee H) = \min\{\delta(G) + m, \delta(H) + n\}$. Let $G = K_n$ and $H = K_m$. Then $G \vee H = K_{m+n}$ and $\min\{\delta(G) + m, \delta(H) + n\} = m + n - 1$. From Lemma 4.2, if $k = 2$, then $\lambda_k(G \vee H) = \lambda_k(K_{m+n}) = m + n - \lceil k/2 \rceil = \min\{\delta(G) + m, \delta(H) + n\}$. This implies that the upper bound is sharp. □

Palmer [36] gave the spanning tree packing number of some special graph classes.

Lemma 4.3. [36] *For a complete bipartite graph $K_{a,b}$, $\kappa_{a+b}(K_{a,b}) = \lfloor \frac{ab}{a+b-1} \rfloor$.*

We now give an lower bound of $\kappa_k(G \vee H)$.

Proposition 4.3. *Let G and H be two graphs. Then*

$$\kappa_k(G \vee H) \geq \min_{r+s=k} \left\{ \left\lfloor \frac{rs}{r+s-1} \right\rfloor + \min\{|V(G)| - r, |V(H)| - s\} \right\}.$$

Proof. For any $S \subseteq V(G \vee H)$ and $|S| = k$, we assume that $|S \cap V(G)| = r$ and $|S \cap V(H)| = s$. From Lemma 4.3, there are $\lfloor \frac{rs}{r+s-1} \rfloor$ spanning trees of $G[S]$, and they are all internally disjoint S-Steiner trees in $G \vee H$. Without loss of generality, let $S \cap V(G) = \{g_1, g_2, \ldots, g_r\}$ and $S \cap V(H) = \{h_1, h_2, \ldots, h_s\}$. Then $V(G) \setminus S = \{g_{r+1}, g_{r+2}, \ldots, g_n\}$ and $V(H) \setminus S = \{h_{s+1}, h_{s+2}, \ldots, h_m\}$. Without loss of generality, let $n - r \leq m - s$. Clearly, the trees induced by the edges in $\{g_i h_1, g_i h_2, \ldots, g_i h_s\} \cup \{h_j g_1, h_j g_2, \ldots, h_j g_r\} \cup \{g_i h_j\}$ $(r + 1 \leq i \leq n,\ s + 1 \leq j \leq s + (n - r) \leq m)$ are $n - r$ internally disjoint S-Steiner trees in $G \vee H$, say $T_1, T_2, \ldots, T_{n-r}$. These trees together with the $\lfloor \frac{rs}{r+s-1} \rfloor$ S-Steiner trees in $G \vee H[S]$ are internally disjoint S-Steiner trees in $G \vee H$, and hence $\kappa(S) \geq \left\lfloor \frac{rs}{r+s-1} \right\rfloor + \min\{|V(G)| - r, |V(H)| - s\}$. From the arbitrariness of S, we have

$$\kappa_k(G \vee H) \geq \min_{r+s=k} \left\{ \left\lfloor \frac{rs}{r+s-1} \right\rfloor + \min\{|V(G)| - r, |V(H)| - s\} \right\}.$$

To show the sharpness of the lower bound, we consider the following example.

Example 2. Let G, H be two trees of order n, m, respectively, such that $\frac{nm}{n+m-1}$ is an integer. If $k = n + m$, then $r = n$, $s = m$. From Proposition 4.3, we have

$$\begin{aligned} \kappa_{m+n}(G \vee H) &\geq \min_{r=n,\ s=m} \left\{ \left\lfloor \frac{rs}{r+s-1} \right\rfloor + \min\{n - r, m - s\} \right\} \\ &= \frac{nm}{n+m-1}. \end{aligned}$$

Since $e(G \vee H) = mn + n + m - 2$, it follows that there are at most $\lfloor \frac{mn+n+m-2}{n+m-1} \rfloor = \frac{nm}{n+m-1} + \lfloor \frac{n+m-2}{n+m-1} \rfloor = \frac{nm}{n+m-1}$ spanning trees of $G \vee H$, that is, $\kappa_{m+n}(G \vee H) \leq \frac{nm}{n+m-1}$. So $\kappa_{m+n}(G \vee H) = \frac{nm}{n+m-1}$ which implies that the lower bound is sharp for $k = |V(G)| + |V(H)|$.

For $\lambda_k(G \vee H)$, we have the following lower bound.

Proposition 4.4. *Let G and H be two connected graphs of order n, m, respectively. Then*

$$\lambda_k(G \vee H) \geq \min_{r+s=k} \left\{ x + \left(\left\lfloor \frac{rs}{r+s-1} \right\rfloor - \left\lceil \frac{x}{r+s-1} \right\rceil \right) + \min\{n - r, m - s\} \right\},$$

where $x = \min\{\lambda_r(G), \lambda_s(H), rs\}$.

Proof. For any $S \subseteq V(G \vee H)$ and $|S| = k$, we assume that $|S \cap V(G)| = r$ and $|S \cap V(H)| = s$. Let $\lambda_r(G) = a$ and $\lambda_s(H) = b$. Clearly, there are a edge-disjoint $S \cap V(G)$-Steiner trees in G, say $T_1, T_2, \ldots, T_a$, and there are b edge-disjoint $S \cap V(H)$-Steiner trees in H, say $T'_1, T'_2, \ldots, T'_b$. From Lemma 4.3, there are $\lfloor \frac{rs}{r+s-1} \rfloor$ spanning trees of $(G \vee H)[S]$, say $T^*_1, T^*_2, \ldots, T^*_c$, and they are all edge-disjoint S-Steiner trees in $G \vee H$, where $c = \lfloor \frac{rs}{r+s-1} \rfloor$. We choose $T^*_1, T^*_2, \ldots, T^*_d$, and these trees have $d(r + s - 1) \geq x$ edges between $V(G)$ and $V(H)$, where $d = \left\lceil \frac{x}{r+s-1} \right\rceil$. Choose x such edges, say $e_1, e_2, \ldots, e_x$. Note that the union of one tree in $\{T_1, T_2, \ldots, T_a\}$, one tree in $\{T'_1, T'_2, \ldots, T'_b\}$, and one edge in $\{e_1, e_2, \ldots, e_x\}$ is an S-Steiner tree in $G \vee H$. Since $x = \min\{\lambda_r(G), \lambda_s(H), rs\}$, we can get x edge-disjoint S-Steiner trees in

$G \vee H$, say $T_1^{**}, T_2^{**}, \ldots, T_x^{**}$. Without loss of generality, let $n-r \leq m-s$. Let $S \cap V(G) = \{g_1, g_2, \ldots, g_r\}$ and $S \cap V(H) = \{h_1, h_2, \ldots, h_s\}$. Then $V(G) \backslash S = \{g_{r+1}, g_{r+2}, \ldots, g_n\}$ and $V(H) \backslash S = \{h_{s+1}, h_{s+2}, \ldots, h_m\}$. Clearly, the trees induced by the edges in $\{g_i h_1, g_i h_2, \ldots, g_i h_s\} \cup \{h_j g_1, h_j g_2, \ldots, h_j g_r\} \cup \{g_i h_j\}$ $(r+1 \leq i \leq n,\ s+1 \leq j \leq s+(n-r) \leq m)$ are $n-r$ edge-disjoint S-Steiner trees in $G \vee H$, say $T_1^{***}, T_2^{***}, \ldots, T_{n-r}^{***}$. Since $T_1^{**}, T_2^{**}, \ldots, T_x^{**}$, and $T_1^{***}, T_2^{***}, \ldots, T_{n-r}^{***}$, and $T_{d+1}^{*}, T_{d+2}^{*}, \ldots, T_c^{*}$ are

$$\left\{x + \left(\left\lfloor \frac{rs}{r+s-1} \right\rfloor - \left\lceil \frac{x}{r+s-1} \right\rceil\right) + \min\{n-r, m-s\}\right\}$$

edge-disjoint S-Steiner trees in $G \vee H$, and hence

$$\lambda(S) \geq \left\{x + \left(\left\lfloor \frac{rs}{r+s-1} \right\rfloor - \left\lceil \frac{x}{r+s-1} \right\rceil\right) + \min\{n-r, m-s\}\right\}.$$

From the arbitrariness of S, we have

$$\lambda_k(G \vee H) \geq \min_{r+s=k} \left\{x + \left(\left\lfloor \frac{rs}{r+s-1} \right\rfloor - \left\lceil \frac{x}{r+s-1} \right\rceil\right) + \min\{n-r, m-s\}\right\},$$

as desired. □

To show the sharpness of the lower bound, we consider the following example.

Example 3. Let G, H be two trees of order n, m, respectively, such that $\frac{nm}{n+m-1}$ is an integer. If $k = n+m$, then $r = n$, $s = m$, $\lambda_r(G) = \lambda_n(G) = 1$, $\lambda_s(H) = \lambda_m(H) = 1$, and $x = 1$. From Proposition 4.4, we have

$$\begin{aligned}
\lambda_{m+n}(G \vee H) &\geq \min_{r=n,\ s=m} \left\{x + \left(\left\lfloor \frac{rs}{r+s-1} \right\rfloor - \left\lceil \frac{x}{r+s-1} \right\rceil\right) + \min\{n-r, m-s\}\right\} \\
&\geq \frac{nm}{n+m-1}.
\end{aligned}$$

Since $e(G \vee H) = mn+n+m-2$, it follows that there are at most $\lfloor \frac{mn+n+m-2}{n+m-1} \rfloor = \frac{nm}{n+m-1} + \lfloor \frac{n+m-2}{n+m-1} \rfloor = \frac{nm}{n+m-1}$ spanning trees of $G \vee H$, that is, $\lambda_{m+n}(G \vee H) \leq \frac{nm}{n+m-1}$. So $\lambda_{m+n}(G \vee H) = \frac{nm}{n+m-1}$ and the lower bound is sharp for $k = n+m$.

5. Conclusions

In this paper, we focus on the generalized k-connectivity and the generalized k-edge-connectivity of join, corona and cluster graphs. We determine exact values of $\kappa_k(G \odot H)$, $\lambda_k(G \odot H)$ and sharp upper and lower bounds of $\kappa_k(G * H)$, $\lambda_k(G * H)$, $\kappa_k(G \vee H)$ and $\lambda_k(G \vee H)$. In addition, we give some examples to show the sharpness of these bounds. And as a future work, it's interesting to think about whether it is possible to characterize all the extremal graphs arriving at these sharp upper and lower bounds.

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Conflict of interest

The authors declare no conflict of interest.

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