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Research article

A novel analytical Aboodh residual power series method for solving linear and nonlinear time-fractional partial differential equations with variable coefficients

Muhammad Imran Liaqat[1], Sina Etemad[2], Shahram Rezapour[2,3,*] and Choonkil Park[4,*]

1 National College of Business Administration & Economics, Lahore, Pakistan
2 Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
3 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan
4 Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea

* **Correspondence:** Emails: sh.rezapour@azaruniv.ac.ir; baak@hanyang.ac.kr.

Abstract: The goal of this research is to develop a novel analytic technique for obtaining the approximate and exact solutions of the Caputo time-fractional partial differential equations (PDEs) with variable coefficients. We call this technique as the Aboodh residual power series method (ARPSM), because it apply the Aboodh transform along with the residual power series method (RPSM). It is based on a new version of Taylor's series that generates a convergent series as a solution. Establishing the coefficients for a series, like the RPSM, necessitates the computation of the fractional derivatives each time. As ARPSM just requires the idea of an infinite limit, we simply need a few computations to get the coefficients. This technique solves nonlinear problems without the He's polynomials and Adomian polynomials, so the small size of computation of this technique is the strength of the scheme, which is an advantage over the homotopy perturbation method and the Adomian decomposition method. The absolute and relative errors of five linear and non-linear problems are numerically examined to determine the efficacy and accuracy of ARPSM for time-fractional PDEs with variable coefficients. In addition, numerical results are also compared with other methods such as the RPSM and the natural transform decomposition method (NTDM). Some graphs are also plotted for various values of fractional orders. The results show that our technique is easy to use, accurate, and effective. Mathematica software is used to calculate the numerical and symbolic quantities in the paper.

Keywords: Aboodh transform; residual power series method; Caputo fractional derivative; approximate solution; exact solution; partial differential equations
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1. Introduction

Fractional calculus (FC) in the field of mathematical analysis studies the derivative and integral of arbitrary real or even complex orders. FC is also known as Non-Newtonian calculus and generalized calculus. In a famous letter, L'Hospital asked Leibniz what would happen if the order of the derivative were to be $\frac{1}{2}$, and the reply of Leibniz on September 30, 1695, is considered the birth of Non-Newtonian calculus [1–4]. FC has become a valuable tool in various disciplines of engineering, physics, image processing, biology, chemistry, control theory, viscoelasticity, solid-state, stochastic designed finance, economics, signal, and fiber optics [5–10]. There are several techniques to define fractional derivatives, but not all of them are generally used. The most commonly used fractional derivatives are Riemann-Liouville (R-L), Caputo fractional derivative (CFD), Caputo-Fabrizio, Atangana-Baleanu, and conformable operators [11–23]. In some circumstances, fractional derivatives are superior to integer-order derivatives for modeling, and they can simulate and analyze complicated systems with higher-order dynamics and sophisticated non-linear processes. This is due to two main factors: first, we can choose arbitrary orders for the derivative operator rather than being restricted to an integer order. Second, non-integer order derivatives rely on the past as well as local conditions, which is advantageous when the system has a long-term memory.

In the disciplines of science and engineering, we find natural and physical events that, when characterized by mathematical models, happen to be differential equations (DEs). For instance, equation of motion, simple harmonic motion, deflection of the beam, and so on are characterized by DEs. Consequently, the solutions of DEs are important and applicable. Many DEs that arise in applications are sufficiently complex that close-form solutions are sometimes impracticable. Under the specified preliminary conditions, numerical methods provide a powerful alternative tool for solving DEs. In recent years, various approaches for solving fractional-order DEs have been described, including the Laplace decomposition method [24], the differential transform method [25–27], the variational iteration method [28], the operational matrix method [29], the homotopy analysis method [30], the Chebyshev polynomials method [31], the Aboodh transform decomposition method [32], the Elzaki transform-variational iteration method [33], the Shehu transform iterative method [34] and the residual power series method [35–38].

The Jordanian mathematician Omar Abu Arqub created the RPSM in 2013 [39]. The RPSM is a semi-analytical method; it is a combination of Taylor's series and the residual error function. It provides series solutions of linear and nonlinear DEs in the form of convergence series. In 2013, RPSM was implemented for the first time to find solutions to fuzzy DEs. Arqub et al. constructed a new set of algorithms for RPSM to derive rapid power series solutions for ordinary DEs [40]. Also, Arqub et al. [41] established a new interesting RPSM algorithm for the solution of fractional order non-linear boundary value problems. El-Ajou et al. formed a new iterative algorithm of RPSM to find approximate solutions of KdV-burgers equations of fractional order [42]. Xu et al. provided a new algorithm for the fractional power series solutions of the second and fourth-order Boussinesq DEs [43]. Zhang et al. presented an effective numerical method which is a combination of RPSM and least square methods [44]. For more details about RPSM, see [45–47].

Researchers combined two powerful methods to develop a new method for solving fractional-order differential equations (FODEs). Some of these groups are described as a combination of the homotopy perturbation approach and the Sumudu transform [48], as well as the homotopy analysis method and the natural transform [49], the Shehu transformation and the Adomian decomposition method [50], and the Laplace transform with RPSM [51–53]. For more details about combining the two methods, see [54–56]. In this study, we applied the novel combined technique known as the ATRPSM to provide approximate and exact solutions for time-fractional PDEs with variable coefficients. The Aboodh transform method and the RPSM are combined in this novel technique. To assess the efficiency and consistency of the proposed method, the relative and absolute errors of the five problems are examined. In addition, numerical results are also compared with other methods such as the NTDM and RPSM. The results obtained by the proposed method show excellent agreement with these methods, which indicates the effectiveness and reliability of the proposed method. Graphical significance is also found for various values of fractional-order derivatives. As a result, the technique is precise, simple to use, not affected by computational rounds of errors, and does not require large computer memory and extensive time.

The set of rules for this new technique depends on transforming the given equation into the Aboodh transform space, establishing a series of solutions to the new form of the equation, and then acquiring the solution to the actual equation by applying the inverse Aboodh transform. Without linearization, perturbation, or discretization, the new technique can be utilized to create power series expansion solutions for linear and nonlinear PDEs. Unlike the traditional power series method, this method does not require matching the coefficients of the corresponding terms, nor does it necessitate the use of a recursion relation. The proposed technique, which is based on the limit concept, finds the series coefficients but not the fractional derivatives like the RPSM does. In contrast to RPSM, which requires many times to calculate various fractional derivatives in the solution phases, only a few calculations are necessary to determine the coefficients. The recommended technique can yield both closed-form and accurate approximate solutions by including a rapid convergence series.

Finding the solutions of time-fractional PDEs with variable coefficients is an interesting and important field for researchers [57–62]. This paper introduces a new semi-analytical technique for solving time-fractional PDEs with variable coefficients that is both simple and efficient. The obtained results by using the recommended technique are the same as those previously published in the literature. But these methods require many computational work and long running times. The ARPSM, which is a combination of the Aboodh transform and RPSM, is our suggested technique.

We chose the most common types of time-fractional PDEs with variable coefficients to highlight the key principles of our recommended technique, such as its reliability, capability, and applicability. In 1822, Joseph Fourier proposed the heat equation, which states how a certain quantity of heat diffuses over a region. Consider the time-space fractional PDE with variable coefficients in the following general form [57]:

$$D_{\tau}^{q\alpha}\Phi(x,\tau) + \vartheta(x)\aleph(\Phi) = \xi(x,\Phi), \tag{1.1}$$

subject to the initial condition:

$$D_{\tau}^{w\alpha}\Phi(x,0) = \Omega_w,$$

where

$$w = 0,1,2,3\ldots,q-1,\ x = (x_1,x_2,\cdots,x_p) \in \mathbb{R}^p,\ \alpha \in (\frac{q-1}{q}],\ q \in \mathbb{N},$$

and

$$\aleph(\Phi) = \aleph(\Phi, D_\tau^\alpha\Phi, D_\tau^{2\alpha}\Phi, \cdots, D_\tau^{(q-1)\alpha}\Phi, D_{x_1}^{\beta_{11}}\Phi, D_{x_2}^{\beta_{12}}\Phi, \cdots, D_{x_p}^{\beta_{1p}}\Phi, \cdots, D_{x_1}^{\beta_{c1}}\Phi, D_{x_2}^{\beta_{c2}}\Phi, \cdots, D_{x_p}^{\beta_{cp}}\Phi), \text{ with } g-1 < \beta_{gf} \le g, \ g = 1, 2, \cdots, c; \ f = 1, 2, \cdots, p.$$

Here $D_\tau^{w\alpha}$ and $D_{x_f}^{\beta_{gf}}$ mean the CFD w.r.t. τ of order $w\alpha$ and x_f of order β_{gf}, respectively. This type of PDEs provide precise descriptions of a variety of physical phenomena in electrodynamics, elastic mechanics, and fluid dynamics [58,59].

The framework of this study is as follows. Firstly, we employ significant definitions and conclusions from FC theory in Section 2. Furthermore, some new results are established, which is the basis of the new technique in the same Section 2. Next, in Section 3, we obtain the solutions with the ARPSM for time-fractional PDEs with variable coefficients. In Section 4, some problems are solved with the help of ARPSM. Section 5 explains our findings, which are given in the form of figures and tables. Finally, in the conclusion, we summarize our findings.

2. Non-Newtonian calculus

This section includes several definitions and characteristics as well as some useful results that serve as the basis for the new technique. The classical Fourier integral is used to derive Aboodh transform. Khalid Aboodh founded the Aboodh transform in 2013 to facilitate the approach to solve ordinary DEs and PDEs in the time intervals [63]. This integral transform has the inmost interrelation with the Elzaki and Laplace transforms. Some important notations, a basic definition, and a few characteristics of the Aboodh transform are discussed below.

Definition 2.1. *[63] Assume that the function $\Phi(x,\tau)$ is of exponential order and piecewise continuous. Then the Aboodh transform of $\Phi(x,\tau)$ for $\tau \ge 0$ is formulated as:*

$$A[\Phi(x,\tau)] = \Psi(x,\upsilon) = \frac{1}{\upsilon}\int_0^\infty \Phi(x,\tau)e^{-\tau\upsilon}d\tau, \ \gamma_1 \le \upsilon \le \gamma_2,$$

and the inverse Aboodh transform is defined by:

$$A^{-1}[\Psi(x,\upsilon)] = \Phi(x,\tau) = \frac{1}{2\pi\iota}\int_{u-\iota\infty}^{u+\iota\infty} \upsilon e^{\upsilon\tau}\Psi(x,\upsilon)d\upsilon,$$

where $x = (x_1, x_2, \cdots, x_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$.

Lemma 2.2. *[64,65] Let $\Phi_1(x,\tau)$ and $\Phi_2(x,\tau)$ be piecewise continuous on $[0,\infty[$ and be of exponential order. Assume that $A[\Phi_1(x,\tau)] = \Psi_1(x,\upsilon)$, $A[\Phi_2(x,\tau)] = \Psi_2(x,\upsilon)$ and λ_1, λ_2 are constants. Then the properties mentioned below are valid:*

(i) $A[\lambda_1\Phi_1(x,\tau) + \lambda_2\Phi_2(x,\tau)] = \lambda_1\Psi_1(x,\upsilon) + \lambda_2\Psi_2(x,\upsilon)$,

(ii) $A^{-1}[\lambda_1\Psi_1(x,\upsilon) + \lambda_2\Psi_2(x,\upsilon)] = \lambda_1\Phi_1(x,\tau) + \lambda_2\Phi_2(x,\tau)$,

(iii) $A[J_\tau^\alpha\Phi(x,\tau)] = \dfrac{\Psi(x,\upsilon)}{\upsilon^\alpha}$,

(iv) $A[D_\tau^\alpha\Phi(x,\tau)] = \upsilon^\alpha\Psi(x,\upsilon) - \displaystyle\sum_{\kappa=0}^{r-1}\frac{\Phi^{(\kappa)}(x,0)}{\upsilon^{\kappa-\alpha+2}}, \ r-1 < \alpha \le r, \ r \in \mathbb{N}$.

Definition 2.3. *[66] The fractional derivative of $\Phi(x,\tau)$ of order α in the Caputo sense is defined as follows:*

$$D_{\tau}^{\alpha}\Phi(x,\tau) = J_{\tau}^{m-\alpha}\Phi^{(m)}(x,\tau),\ \tau \geq 0,\ m-1 < \alpha \leq m,$$

where $x = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p$ and $m, p \in R$, $J_{\tau}^{m-\alpha}$ is the R-L integral of $\Phi(x,\tau)$.

Definition 2.4. *[67] The power series representation is in the following form*

$$\sum_{r=0}^{\infty} \hbar_r(x)(\tau-\tau_0)^{r\alpha} = \hbar_0(\tau-\tau_0)^0 + \hbar_1(\tau-\tau_0)^{\alpha} + \hbar_2(\tau-\tau_0)^{2\alpha} + \cdots,$$

where $x = (x_1, x_2, \ldots, x_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. It is known as multiple fractional power series (MFPS) about τ_0, where τ denotes a variable and $\hbar_r(x)$'s are the series coefficients.

Lemma 2.5. *Assume that $\Phi(x,\tau)$ is an exponential order function. Then the Aboodh transform is as $A[\Phi(x,\tau)] = \Psi(x,\upsilon)$. Hence,*

$$A[D_{\tau}^{r\alpha}\Phi(x,\tau)] = \upsilon^{r\alpha}\Psi(x,\upsilon) - \sum_{j=0}^{r-1}\upsilon^{\alpha(r-j)-2}D_{\tau}^{j\alpha}\Phi(x,0),\ 0 < \alpha \leq 1, \tag{2.1}$$

where, $x = (x_1, x_2, \ldots x_p) \in \mathbb{R}^p$, $p \in \mathbb{N}$ and $D_{\tau}^{r\alpha} = D_{\tau}^{\alpha}.D_{\tau}^{\alpha}.\cdots D_{\tau}^{\alpha}(r - times)$.

Proof. Let us prove Eq (2.1) by induction. We obtain as follows when we choose r=1 in Eq (2.1):

$$A[D_{\tau}^{\alpha}\Phi(x,\tau)] = \upsilon^{\alpha}\Psi(x,\upsilon) - \upsilon^{\alpha-2}\Phi(x,0).$$

For r=1, Eq (2.1) is valid based on part (iv) of Lemma 2.2. Using $r = 2$ in Eq (2.1), we have

$$A[D_{\tau}^{2\alpha}\Phi(x,\tau)] = \upsilon^{2\alpha}\Psi(x,\upsilon) - \upsilon^{2\alpha-2}\Phi(x,0) - \upsilon^{\alpha-2}D_{\tau}^{\alpha}\Phi(x,0). \tag{2.2}$$

In view of the L.H.S. of Eq (2.2), we obtain

$$L.H.S = A[D_{\tau}^{2\alpha}\Phi(x,\tau)]. \tag{2.3}$$

The Eq (2.3) can be written as

$$L.H.S = A[D_{\tau}^{\alpha}(D_{\tau}^{\alpha}\Phi(x,\tau))]. \tag{2.4}$$

Let

$$z(x,\tau) = D_{\tau}^{\alpha}\Phi(x,\tau). \tag{2.5}$$

As a result Eq (2.4) becomes

$$L.H.S = A[D_{\tau}^{\alpha}z(x,\tau)]. \tag{2.6}$$

By using the fractional derivative of the Caputo type, Eq (2.6) becomes

$$L.H.S = A[J^{1-\alpha}z'(x,\tau)]. \tag{2.7}$$

By using the R-L fractional integral formula of the Aboodh transform in Eq (2.7), we get

$$L.H.S = \frac{A[z'(x,\tau)]}{\upsilon^{1-\alpha}}. \tag{2.8}$$

By using the differential property of the Aboodh transform, Eq (2.8) becomes as

$$L.H.S = \upsilon^{\alpha} Z(x,\upsilon) - \frac{z(x,0)}{\upsilon^{2-\alpha}}. \tag{2.9}$$

From Eq (2.5), we get

$$Z(x,\upsilon) = \upsilon^{\alpha}\Psi(x,\upsilon) - \frac{\Phi(x,0)}{\upsilon^{2-\alpha}},$$

where, $A[z(x,\tau)] = Z(x,\upsilon)$. Therefore, Eq (2.9) is converted to

$$L.H.S = \upsilon^{2\alpha}\Psi(x,\upsilon) - \frac{\Phi(x,0)}{\upsilon^{2-2\alpha}} - \frac{D_{\tau}^{\alpha}\Phi(x,0)}{\upsilon^{2-\alpha}}. \tag{2.10}$$

when $r = \kappa$. Eq (2.10) is compatible with Eq (2.1).

Now, assume that Eq (2.1) is true for $r = \kappa$. Thus, put $r = \kappa$ in Eq (2.1):

$$A[D_{\tau}^{\kappa\alpha}\Phi(x,\tau)] = \upsilon^{\kappa\alpha}\Psi(x,\upsilon) - \sum_{j=0}^{\kappa-1}\upsilon^{\alpha(\kappa-j)-2}D_{\tau}^{j\alpha}\Phi(x,0),\ 0<\alpha\le 1. \tag{2.11}$$

We will now prove Eq (2.1) for $r = \kappa + 1$. From Eq (2.1), we write

$$A[D_{\tau}^{(\kappa+1)\alpha}\Phi(x,\tau)] = \upsilon^{(\kappa+1)\alpha}\Psi(x,\upsilon) - \sum_{j=0}^{\kappa}\upsilon^{\alpha((\kappa+1)-j)-2}D_{\tau}^{j\alpha}\Phi(x,0). \tag{2.12}$$

By considering the L.H.S. of Eq (2.12), we get

$$L.H.S = A[D_{\tau}^{\alpha}(D_{\tau}^{\kappa\alpha}\Phi(x,\tau))]. \tag{2.13}$$

Let

$$D_{\tau}^{\kappa\alpha}\Phi(x,\tau) = g(x,\tau).$$

By Eq (2.13), we get

$$L.H.S = A[D_{\tau}^{\alpha} g(x,\tau)]. \tag{2.14}$$

By utilizing the Caputo fractional derivative and R-L integral formulas, Eq (2.14) becomes

$$L.H.S = \upsilon^{\alpha}A[D_{\tau}^{\kappa\alpha}\Phi(x,\tau)] - \frac{g(x,0)}{\upsilon^{2-\alpha}}. \tag{2.15}$$

By utilizing Eq (2.11), Eq (2.15) is transformed into

$$L.H.S = \upsilon^{r\alpha}\Psi(x,\upsilon) - \sum_{j=0}^{r-1}\upsilon^{\alpha(r-j)-2}D_{\tau}^{j\alpha}\Phi(x,0), \tag{2.16}$$

and from Eq (2.16), we have the following result

$$L.H.S = A[D_{\tau}^{r\alpha}\Phi(x,0)].$$

Therefore, the formula Eq (2.1) is to be held for $r = \kappa + 1$. Consequently, by using the mathematical induction method, we proved that the formula Eq (2.1) is true for all positive integers. □

In the next lemma, we provide a new form of multiple fractional Taylor's formula, which will be helpful for the ARPSM.

Lemma 2.6. *Suppose that $\Phi(x,\tau)$ is a function of exponential order. Then the Aboodh transform of $\Phi(x,\tau)$, given by $A[\Phi(x,\tau)] = \Psi(x,\upsilon)$, has multiple fractional Taylor's series representation as follows:*

$$\Psi(x,\upsilon) = \sum_{r=0}^{\infty} \frac{\hbar_r(x)}{\upsilon^{r\alpha+2}},\ \upsilon > 0, \tag{2.17}$$

where, $x = (x_1, x_2, \dots x_p) \in \mathbb{R}^p,\ p \in \mathbb{N}$.

Proof. Consider the Taylor's series in the fractional order as

$$\Phi(x,\tau) = \hbar_0(x) + \hbar_1(x)\frac{\tau^{\alpha}}{\Gamma[\alpha+1]} + \hbar_2(x)\frac{\tau^{2\alpha}}{\Gamma[2\alpha+1]} + \cdots . \tag{2.18}$$

We obtain the following equality by applying the Aboodh transform on Eq (2.18):

$$A\Big[\Phi(x,\tau)\Big] = A\Big[\hbar_0(x)\Big] + A\left[\hbar_1(x)\frac{\tau^{\alpha}}{\Gamma[\alpha+1]}\right] + A\left[\hbar_2(x)\frac{\tau^{2\alpha}}{\Gamma[2\alpha+1]}\right] + \dots .$$

Therefore by using the properties of the Aboodh transform, we get

$$A\Big[\Phi(x,\tau)\Big] = \hbar_0(x)\frac{1}{\upsilon^2} + \hbar_1(x)\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\frac{1}{\upsilon^{\alpha+2}} + \hbar_2(x)\frac{\Gamma(2\alpha+1)}{\Gamma(2\alpha+1)}\frac{1}{\upsilon^{2\alpha+2}} + \dots .$$

So (2.17) is obtained (In the Aboodh transform, this is a new form of Taylor's series). □

Lemma 2.7. *Assume that the function $A[\Phi(x,\tau)] = \Psi(x,\upsilon)$ has MFPS representation in the new form of Taylor'series* (2.17). *Then we have*

$$\hbar_0(x) = \lim_{\upsilon\to\infty} \upsilon^2\Psi(x,\upsilon) = \Phi(x,0). \tag{2.19}$$

Proof. The preceding is derived from the new form of Taylor's series:

$$\hbar_0(x) = \upsilon^2\Psi(x,\upsilon) - \frac{\hbar_1(x)}{\upsilon^{\alpha}} - \frac{\hbar_2(x)}{\upsilon^{2\alpha}} - \dots . \tag{2.20}$$

Applying $\lim_{\upsilon\to\infty}$ to the Eq (2.20), and by making a simple calculation, we get the required result represented by (2.19). □

Theorem 2.8. *Assume that the MFPS representation for the function $A[\Phi(x,\tau)] = \Psi(x,\upsilon)$ is given by*

$$\Psi(x,\upsilon) = \sum_{0}^{\infty} \frac{\hbar_r(x)}{\upsilon^{r\alpha+2}},\ \upsilon > 0,$$

where $x = (x_1, x_2, \dots x_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. Then we have

$$\hbar_r(x) = D_{\tau}^{r\alpha}\Phi(x,0),$$

where, $D_{\tau}^{r\alpha} = D_{\tau}^{\alpha}.D_{\tau}^{\alpha}\dots D_{\tau}^{\alpha}(r - times)$.

Proof. From the new form of Taylor's series we have

$$\hbar_1(x) = \upsilon^{\alpha+2}\Psi(x,\upsilon) - \upsilon^{\alpha}\hbar_0(x) - \frac{\hbar_2(x)}{\upsilon^{\alpha}} - \frac{\hbar_3(x)}{\upsilon^{2\alpha}} - \dots. \tag{2.21}$$

Applying $lim_{\upsilon\to\infty}$ to the Eq (2.21), we get

$$\hbar_1(x) = lim_{\upsilon\to\infty}(\upsilon^{\alpha+2}\Psi(x,\upsilon) - \upsilon^{\alpha}\hbar_0(x)) - lim_{\upsilon\to\infty}\frac{\hbar_2(x)}{\upsilon^{\alpha}} - lim_{\upsilon\to\infty}\frac{\hbar_3(x)}{\upsilon^{\alpha}} - \dots.$$

We obtain the following equality after taking the limit as

$$\hbar_1(x) = lim_{\upsilon\to\infty}(\upsilon^{\alpha+2}\Psi(x,\upsilon) - \upsilon^{\alpha}\hbar_0(x)). \tag{2.22}$$

By employing Lemma 2.5, to Eq (2.22), it becomes

$$\hbar_1(x) = lim_{\upsilon\to\infty}(\upsilon^2 A[D_\tau^{\alpha}\Phi(x,\tau)](\upsilon)). \tag{2.23}$$

Further, by employing Lemma 2.7 to Eq (2.23), it becomes

$$\hbar_1(x) = D_\tau^{\alpha}\Phi(x,0).$$

Again, by considering the new form of Taylor's series and as $\upsilon \to \infty$, we have

$$\hbar_2(x) = \upsilon^{2\alpha+2}\Psi(x,\upsilon) - \upsilon^{2\alpha}\hbar_0(x) - \upsilon^{\alpha}\hbar_1(x) - \frac{\hbar_3(x)}{\upsilon^{\alpha}} - \dots.$$

From Lemma 2.7, we get

$$\hbar_2(x) = lim_{\upsilon\to\infty}\upsilon^2(\upsilon^{2\alpha}\Psi(x,\upsilon) - \upsilon^{2\alpha-2}\hbar_0(x) - \upsilon^{\alpha-2}\hbar_1(x)). \tag{2.24}$$

Again, by using Lemmas 2.5 and 2.7, Eq (2.24) becomes

$$\hbar_2(x) = D_\tau^{2\alpha}\Phi(x,0).$$

By repeating the same process on the new Taylor's series, we have

$$\hbar_3(x) = lim_{\upsilon\to\infty}\upsilon^2(A[D_\tau^{3\alpha}\Phi(x,\alpha)](\upsilon)).$$

The last equation is obtained when Lemma 2.7 is used,

$$\hbar_3(x) = D_\tau^{3\alpha}\Phi(x,0).$$

In the general case, we get

$$\hbar_r(x) = D_\tau^{r\alpha}\Phi(x,0).$$

This ends the proof. □

The conditions for the convergence of the new form of Taylor's formula are explained and determined in the following theorem.

Theorem 2.9. *Let* $A\ [\Phi(x,\tau)] = \Psi(x,\upsilon)$ *can be denoted as the new form of multiple fractional Taylor's formula given in Lemma 2.6. If* $|\upsilon^2 A[D_\tau^{(\kappa+1)\alpha}\Phi(x,\tau)]| \leq T$*, on* $0 < \upsilon \leq s$ *with* $0 < \alpha \leq 1$*, then the remainder* $R_\kappa(x,\upsilon)$ *of the new form of multiple fractional Taylor's formula satisfies the following inequality:*

$$|R_\kappa(x,\upsilon)| \leq \frac{T}{\upsilon^{(\kappa+1)\alpha+2}},\ 0 < \upsilon \leq s.$$

Proof. To begin the proof, we assume that $A[D_\tau^{r\alpha}\Phi(x,\tau)](\upsilon)$ is defined on $0 < \upsilon \leq s$ for $r = 0, 1, 2, \ldots, \kappa+1$. As given, assume that $\left|\upsilon^2 A[D_\tau^{(\kappa+1)\alpha}\Phi(x,\tau)]\right| \leq T$, on $0 < \upsilon \leq s$. Consider the following relation from the new form of Taylor's series:

$$R_\kappa(x,\upsilon) = \Psi(x,\upsilon) - \sum_{r=0}^{\kappa} \frac{\hbar_r(x)}{\upsilon^{r\alpha+2}}. \tag{2.25}$$

By applying Theorem 2.8, Eq (2.25) becomes

$$R_\kappa(x,\upsilon) = \Psi(x,\upsilon) - \sum_{r=0}^{\kappa} \frac{D_\tau^{r\alpha}\Phi(x,0)}{\upsilon^{r\alpha+2}}. \tag{2.26}$$

Multiply by $\upsilon^{(\kappa+1)\alpha+2}$ on both sides of the Eq (2.26). We have

$$\upsilon^{(\kappa+1)\alpha+2}R_\kappa(x,\upsilon) = \upsilon^2(\upsilon^{(\kappa+1)\alpha}\Psi(x,\upsilon) - \sum_{r=0}^{\kappa} \upsilon^{(\kappa+1-r)\alpha-2}D_\tau^{r\alpha}\Phi(x,0)). \tag{2.27}$$

Lemma 2.5 is utilized to Eq (2.27), and we get

$$\upsilon^{(\kappa+1)\alpha+2}R_\kappa(x,\upsilon) = \upsilon^2 A[D_\tau^{(\kappa+1)\alpha}\Phi(x,\tau)]. \tag{2.28}$$

Using the absolute sign on Eq (2.28), we get

$$\left|\upsilon^{(\kappa+1)\alpha+2}R_\kappa(x,\upsilon)\right| = \left|\upsilon^2 A[D_\tau^{(\kappa+1)\alpha}\Phi(x,\tau)]\right|. \tag{2.29}$$

We get the following conclusion by employing the given condition in Eq (2.29), and so

$$\frac{-T}{\upsilon^{(\kappa+1)\alpha+2}} \leq R_\kappa(x,\upsilon) \leq \frac{T}{\upsilon^{(\kappa+1)\alpha+2}}. \tag{2.30}$$

From Eq (2.30), we have the required result

$$|R_\kappa(x,\upsilon)| \leq \frac{T}{\upsilon^{(\kappa+1)\alpha+2}}.$$

As a result, the new series convergence condition is established. □

3. ARPSM technique for solving time-fractional PDEs with variable coefficients

We use our new ARPSM to derive the solutions of the linear and nonlinear PDEs with variable coefficients. The following steps can be used to create a set of rules for this technique to solve time-fractional PDEs. The solution equation is then introduced into the new space using the new form of Taylor's series. In the most recent step, the coefficients of this series are determined using a novel approach. Finally, we use the inverse Aboodh transform to find the solution of the problem in real space.

3.1. The ARPSM algorithm for linear and nonlinear PDEs

We explain the set of rules of the ARPSM for solving Eq (1.1).

Step 1: Rewrite Eq (1.1). We have

$$D_{\tau}^{q\alpha}\Phi(x,\tau)+\vartheta(x)\aleph(\Phi)-\xi(x,\Phi)=0. \tag{3.1}$$

Step 2: By applying the Aboodh transform on both sides of Eq (3.1), we get

$$A\big[D_{\tau}^{q\alpha}\Phi(x,\tau)+\vartheta(x)\aleph(\Phi)-\xi(x,\Phi)\big]=0. \tag{3.2}$$

By utilizing Lemma 2.5, Eq (3.2) becomes

$$\Psi(x,\upsilon)=\sum_{j=0}^{q-1}\frac{D_{\tau}^{j}\Phi(x,0)}{\upsilon^{q\alpha+2}}-\frac{\vartheta(x)Y(\upsilon)}{\upsilon^{q\alpha}}+\frac{F(x,\upsilon)}{\upsilon^{q\alpha}}, \tag{3.3}$$

where, $A[\xi(x,\Phi)]=F(x,\upsilon), A[\aleph(\Phi)]=Y(\upsilon)$.

Step 3: Consider the solution of Eq (3.3), which has the following form:

$$\Psi(x,\upsilon)=\sum_{r=0}^{\infty}\frac{\hbar_r(x)}{\upsilon^{r\alpha+2}},\ \upsilon>0.$$

Step 4: Follow the following procedure:

$$\hbar_0(x)=lim_{\upsilon\to\infty}\upsilon^2\Psi(x,\upsilon)=\Phi(x,0),$$

and by using Theorem 2.9, we have the following

$$\hbar_1(x)=D_{\tau}^{\alpha}\Phi(x,0),$$

$$\hbar_2(x)=D_{\tau}^{2\alpha}\Phi(x,0),$$

$$\hbar_w(x)=D_{\tau}^{w\alpha}\Phi(x,0).$$

Step 5: Obtain the $\kappa-$th-truncated series of $\Psi(x,\upsilon)$ as:

$$\Psi_{\kappa}(x,\upsilon)=\sum_{r=0}^{\kappa}\frac{\hbar_r(x)}{\upsilon^{r\alpha+2}},\ \upsilon>0,$$

$$\Psi_{\kappa}(x,\upsilon)=\frac{\hbar_0(x)}{\upsilon^2}+\frac{\hbar_1(x)}{\hbar^{\alpha+2}}+\cdots+\frac{\hbar_w(x)}{\upsilon^{w\alpha+2}}+\sum_{r=w+1}^{\kappa}\frac{\hbar_r(x)}{\upsilon^{r\alpha+2}}.$$

Step 6: Consider separately the Aboodh residual function (ARF) of Eq (3.3) and the κth-truncated Aboodh residual function, so that

$$ARes(x,\upsilon)=\Psi(x,\upsilon)-\sum_{j=0}^{q-1}\frac{D_{\tau}^{j}\Phi(x,0)}{\upsilon^{j\alpha+2}}+\frac{\vartheta(x)Y(\upsilon)}{\upsilon^{j\alpha}}-\frac{F(x,\upsilon)}{\upsilon^{j\alpha}},$$

and

$$ARes_\kappa(x,\upsilon) = \Psi_\kappa(x,\upsilon) - \sum_{j=0}^{q-1} \frac{D_\tau^j \Phi(x,0)}{\upsilon^{j\alpha+2}} + \frac{\vartheta(x)Y(\upsilon)}{\upsilon^{j\alpha}} - \frac{F(x,\upsilon)}{\upsilon^{j\alpha}}. \tag{3.4}$$

Step 7: Replace the expansion form of $\Psi_\kappa(x,\upsilon)$ into Eq (3.4).

$$\begin{aligned} ARes_\kappa(x,\upsilon) =& \Big(\frac{\hbar_0(x)}{\upsilon^2} + \frac{\hbar_1(x)}{\hbar^{\alpha+2}} + \cdots + \frac{\hbar_w(x)}{\upsilon^{w\alpha+2}} + \sum_{r=w+1}^{\kappa} \frac{\hbar_r(x)}{\upsilon^{r\alpha+2}}\Big) - \\ & \sum_{j=0}^{q-1} \frac{D_\tau^j \Phi(x,0)}{\upsilon^{j\alpha+2}} + \frac{\vartheta(x)Y(\upsilon)}{\upsilon^{j\alpha}} - \frac{F(x,\upsilon)}{\upsilon^{j\alpha}}. \end{aligned} \tag{3.5}$$

Step 8: Multiply by $\upsilon^{\kappa\alpha+2}$ on both sides of Eq (3.5):

$$\begin{aligned} \upsilon^{\kappa\alpha+2} ARes_\kappa(x,\upsilon) =& \upsilon^{\kappa\alpha+2}\Big(\frac{\hbar_0(x)}{\upsilon^2} + \frac{\hbar_1(x)}{\hbar^{\alpha+2}} + \cdots + \frac{\hbar_w(x)}{\upsilon^{w\alpha+2}} + \sum_{r=w+1}^{\kappa} \frac{\hbar_r(x)}{\upsilon^{r\alpha+2}} - \\ & \sum_{j=0}^{q-1} \frac{D_\tau^j \Phi(x,0)}{\upsilon^{j\alpha+2}} + \frac{\vartheta(x)Y(\upsilon)}{\upsilon^{j\alpha}} - \frac{F(x,\upsilon)}{\upsilon^{j\alpha}}\Big). \end{aligned} \tag{3.6}$$

Step 9: Taking $lim_{\upsilon\to\infty}$ on both sides of Eq (3.6):

$$\begin{aligned} lim_{\upsilon\to\infty} \upsilon^{\kappa\alpha+2} ARes_\kappa(x,\upsilon) =& lim_{\upsilon\to\infty} \upsilon^{\kappa\alpha+2}\Big(\frac{\hbar_0(x)}{\upsilon^2} + \frac{\hbar_1(x)}{\hbar^{\alpha+2}} + \cdots + \frac{\hbar_w(x)}{\upsilon^{w\alpha+2}} \\ & + \sum_{r=w+1}^{\kappa} \frac{\hbar_r(x)}{\upsilon^{r\alpha+2}} - \sum_{j=0}^{q-1} \frac{D_\tau^j \Phi(x,0)}{\upsilon^{j\alpha+2}} + \frac{\vartheta(x)Y(\upsilon)}{\upsilon^{j\alpha}} - \frac{F(x,\upsilon)}{\upsilon^{j\alpha}}\Big). \end{aligned}$$

Step 10: Solve the following equation for $\hbar_\kappa(x)$,

$$lim_{\upsilon\to\infty}(\upsilon^{\kappa\alpha+2} ARes_\kappa(x,\upsilon)) = 0,$$

where, $\kappa = w+1, w+2, \cdots$.
Step 11: Replace the obtained values of $\hbar_\kappa(x)$ into κ-truncated series of $\Psi(x,\upsilon)$ to derive the κ-approximate solution of Eq (3.3).
Step 12: Use the inverse Aboodh transform on $\Psi_\kappa(x,\upsilon)$ to obtain the κ-approximate solution $\Phi_\kappa(x,\tau)$.

4. Applications to non-linear and linear PDEs with variable coefficients

To demonstrate the performance and applicability of ARPSM, we consider three well-known and important problems for PDEs with variable coefficients.
Problem 1. Consider the following nonlinear (1+1) wave-like equation with variable coefficients [58]:

$$D_\tau^{2\alpha}\Phi(x,\tau) = x^2\frac{\partial}{\partial x}\Phi(x,\tau)\frac{\partial^2}{\partial x^2}\Phi(x,\tau) - x^2\Big(\frac{\partial^2}{\partial x^2}\Phi(x,\tau)\Big)^2 - \Phi(x,\tau), \tag{4.1}$$

where, $0 < \alpha \le 1, x \in \mathbb{R}$ and $\tau \ge 0$, with the initial conditions:

$$\Phi(x,0) = 0,\ D_\tau^\alpha \Phi(x,0) = x^2.$$

Applying the Aboodh transform on Eq (4.1), we get

$$A[D_\tau^{2\alpha}\Phi(x,\tau)] = A[x^2\frac{\partial}{\partial x}\Phi(x,\tau)\frac{\partial^2}{\partial x^2}\Phi(x,\tau) - x^2\Big(\frac{\partial^2}{\partial x^2}\Phi(x,\tau)\Big)^2 - \Phi(x,\tau)]. \tag{4.2}$$

Using the approach mentioned in Section 3, we obtain the following results from Eq (4.2) as

$$\begin{aligned}\Psi(x,\upsilon) =&\frac{x^2}{\upsilon^{\alpha+2}} + \frac{x^2}{\upsilon^{2\alpha}}\frac{\partial}{\partial x}A\Big[\frac{\partial}{\partial x}A^{-1}[\Psi(x,\upsilon)]\frac{\partial^2}{\partial x^2}A^{-1}[\Psi(x,\upsilon)]\Big]-\\ &\frac{1}{\upsilon^{2\alpha}}x^2\Big(\frac{\partial^2}{\partial x^2}A^{-1}[\Psi(x,\upsilon)]\Big)^2 - \frac{1}{2\alpha}\Psi(x,\upsilon).\end{aligned} \tag{4.3}$$

Assume that Eq (4.3) has a series solution in the following form:

$$\Psi(x,\upsilon) = \sum_{r=0}^{\infty}\frac{\hbar_r(x)}{\upsilon^{r\alpha+2}},\ \upsilon > 0.$$

The κ-truncated expansion is as

$$\Psi_\kappa(x,\upsilon) = \sum_{r=0}^{\kappa}\frac{\hbar_r(x)}{\upsilon^{r\alpha+2}},\ \upsilon > 0.$$

By using Lemma 2.7 and Theorem 2.9, we get

$$lim_{\upsilon\to\infty}(\upsilon^2\Psi(x,\upsilon)) = \Phi(x,0) = \hbar_0(x) = 0,\ \hbar_1(x) = D_\tau^\alpha\Phi(x,0) = x^2.$$

Therefore, κ-truncated expansion becomes as

$$\Psi_\kappa(x,\upsilon) = \frac{x^2}{\upsilon^{\alpha+2}} + \sum_{r=2}^{\kappa}\frac{\hbar_r(x)}{\upsilon^{r\alpha+2}},\ \upsilon > 0. \tag{4.4}$$

The ARF is formulated as

$$\begin{aligned}ARes(x,\upsilon) =&\Psi(x,\upsilon) - \frac{x^2}{\upsilon^{\alpha+2}} - \frac{x^2}{\upsilon^{2\alpha}}\frac{\partial}{\partial x}A\Big[\frac{\partial}{\partial x}A^{-1}[\Psi(x,\upsilon)]\frac{\partial^2}{\partial x^2}A^{-1}[\Psi(x,\upsilon)]\Big]+\\ &\frac{1}{\upsilon^{2\alpha}}x^2\Big(\frac{\partial^2}{\partial x^2}A^{-1}[\Psi(x,\upsilon)]\Big)^2 + \frac{1}{\upsilon^{2\alpha}}\Psi(x,\upsilon).\end{aligned}$$

The κth-truncated ARF takes the following form

$$\begin{aligned}ARes_\kappa(x,\upsilon) =&\Psi_\kappa(x,\upsilon) - \frac{x^2}{\upsilon^{\alpha+2}} - \frac{x^2}{\upsilon^{2\alpha}}\frac{\partial}{\partial x}A\Big[\frac{\partial}{\partial x}A^{-1}[\Psi_\kappa(x,\upsilon)]\frac{\partial^2}{\partial x^2}A^{-1}[\Psi_\kappa(x,\upsilon)]\Big]+\\ &\frac{1}{\upsilon^{2\alpha}}x^2\Big(\frac{\partial^2}{\partial x^2}A^{-1}[\Psi_\kappa(x,\upsilon)]\Big)^2 + \frac{1}{\upsilon^{2\alpha}}\Psi_\kappa(x,\upsilon).\end{aligned} \tag{4.5}$$

To determine the unknown coefficients, substitute $\kappa = 2,3,4,5,6,7$ into Eq (4.5) and Eq (4.4), and solve the expression

$$lim_{\upsilon\to\infty}(\upsilon^{\kappa\alpha+2}ARes_\kappa(x,\upsilon)) = 0.$$

Thus we have

$$\begin{aligned}
\hbar_2(x) &= 0,\\
\hbar_3(x) &= -x^2,\\
\hbar_4(x) &= 0,\\
\hbar_5(x) &= x^2,\\
\hbar_6(x) &= 0,\\
\hbar_7(x) &= -x^2.
\end{aligned}$$

In other words, for each $n \in \mathbb{N}$, we have

$$\hbar_{2n}(x) = 0, \qquad \hbar_{2n+1}(x) = (-1)^n x^2.$$

The 7-th approximate solution of Eq (4.3) is formulated as

$$\Psi_7(x, \upsilon) = \frac{x^2}{\upsilon^{\alpha+2}} - \frac{x^2}{\upsilon^{3\alpha+2}} + \frac{x^2}{\upsilon^{5\alpha+2}} - \frac{x^2}{\upsilon^{7\alpha+2}}.$$

By utilizing the inverse Aboodh transform on above equation, we get 7-th-order approximate solution in the original space which takes the form

$$\Phi_7(x, \tau) = x^2\Big(\frac{\tau^{\alpha}}{\Gamma[\alpha+1]} - \frac{\tau^{3\alpha}}{\Gamma[3\alpha+1]} + \frac{\tau^{5\alpha}}{\Gamma[5\alpha+1]} - \frac{\tau^{7\alpha}}{\Gamma[7\alpha+1]}\Big).$$

For $\alpha = 1$, the 7-th-approximate solution becomes as follows:

$$\Phi_7(x, \tau) = x^2\Big(\frac{\tau}{\Gamma[2]} - \frac{\tau^3}{\Gamma[4]} + \frac{\tau^5}{\Gamma[6]} - \frac{\tau^7}{\Gamma[8]}\Big).$$

This characterizes the first four terms of expansion of the exact solution of $x^2 \sin\tau$. The similar result has been obtained by Khalouta and Kadem [58].

Problems 2. Consider the following nonlinear time-fractional wave-like equation, which has variable coefficients [58]:

$$\begin{aligned}
D_\tau^{2\alpha}\Phi(x, \tau) = \Phi^2(x, \tau)\frac{\partial^2}{\partial x^2}\Big(\Phi_x(x, \tau)\Phi_{xx}(x, \tau)\Phi_{xxx}(x, \tau)\Big)&\\
+ x^2\frac{\partial^2}{\partial x^2}(\Phi_{xx}(x, \tau))^3 - 18\Phi^5(x, \tau) + \Phi(x, \tau),&
\end{aligned} \tag{4.6}$$

where, $0 < \alpha \leq 1, \tau \geq 0, x \in \mathbb{R}$, with the initial conditions:

$$\Phi(x, 0) = \mathrm{e}^x, D_\tau^{\alpha}\Phi(x, 0) = \mathrm{e}^x.$$

Applying the Aboodh transform on Eq (4.6), we get

$$\begin{aligned}
A[D_\tau^{2\alpha}\Phi(x, \tau)] = A\Big[\Phi^2(x, \tau)\frac{\partial^2}{\partial x^2}\Big(\Phi_x(x, \tau)\Phi_{xx}(x, \tau)\Phi_{xxx}(x, \tau)\Big)&\\
+ x^2\frac{\partial^2}{\partial x^2}\Big(\Phi_{xx}(x, \tau)\Big)^3 - 18\Phi^5(x, \tau) + \Phi(x, \tau)\Big].&
\end{aligned} \tag{4.7}$$

Using the approach outlined in Section 3, we obtain the following results from Eq (4.7):

$$\begin{aligned}\Psi(x,\upsilon) =& \frac{e^x}{\upsilon^2} + \frac{e^x}{\upsilon^{\alpha+2}} + \frac{1}{\upsilon^{2\alpha}}A\Big[\Big(A^{-1}[\Psi(x,\upsilon)]\Big)^2\frac{\partial^2}{\partial x^2}\Big(\frac{\partial}{\partial x}A^{-1}[\Psi(x,\upsilon)]\frac{\partial^2}{\partial x^2}\\ & A^{-1}[\Psi(x,\upsilon)]\frac{\partial^3}{\partial x^3}A^{-1}[\Psi(x,\upsilon)]\Big)\Big] + \frac{1}{\upsilon^{2\alpha}}A\Big[x^2\frac{\partial^2}{\partial x^2}\Big(\frac{\partial^2}{\partial x^2}\\ & A^{-1}[\Psi(x,\upsilon)]\Big)^3 - 18\Big(A^{-1}[\Psi(x,\upsilon)]\Big)^5\Big] + \frac{1}{\upsilon^{2\alpha}}\Psi(x,\upsilon).\end{aligned} \tag{4.8}$$

Assume that Eq (4.8) has a series solution in the following form:

$$\Psi(x,\upsilon) = \sum_{r=0}^{\infty}\frac{\hbar_r(x)}{\upsilon^{r\alpha+2}},\ \upsilon > 0.$$

The κth-truncated expansion is as

$$\Psi_\kappa(x,\upsilon) = \sum_{r=0}^{\kappa}\frac{\hbar_r(x)}{\upsilon^{r\alpha+2}},\ \upsilon > 0.$$

By utilizing Lemma 2.7 and Theorem 2.9, we get

$$lim_{\upsilon\to\infty}(\upsilon^2\Psi(x,\upsilon)) = \Phi(x,0) = \hbar_0(x) = e^x,\ \hbar_1(x) = D_\tau^\alpha\Phi(x,0) = e^x.$$

So, Eq (4.8) becomes as follows:

$$\Psi_\kappa(x,\upsilon) = \frac{e^x}{\upsilon^2} + \frac{e^x}{\upsilon^{\alpha+2}} + \sum_{r=2}^{\kappa}\frac{\hbar_r(x)}{\upsilon^{r\alpha+2}}. \tag{4.9}$$

The ARF of Eq (4.8) is formulated as

$$\begin{aligned}ARes(x,\upsilon) =& \Psi(x,\upsilon) - \frac{e^x}{\upsilon^2} - \frac{e^x}{\upsilon^{\alpha+2}} - \frac{1}{\upsilon^{2\alpha}}A\Big[\Big(A^{-1}[\Psi(x,\upsilon)]\Big)^2\frac{\partial^2}{\partial x^2}\Big(\frac{\partial}{\partial x}A^{-1}[\Psi(x,\upsilon)]\frac{\partial^2}{\partial x^2}\\ & A^{-1}[\Psi(x,\upsilon)]\frac{\partial^3}{\partial x^3}A^{-1}[\Psi(x,\upsilon)]\Big)\Big] - \frac{1}{\upsilon^{2\alpha}}A\Big[x^2\frac{\partial^2}{\partial x^2}\Big(\frac{\partial^2}{\partial x^2}\\ & A^{-1}[\Psi(x,\upsilon)]\Big)^3 - 18\Big(A^{-1}[\Psi(x,\upsilon)]\Big)^5\Big] - \frac{1}{\upsilon^{2\alpha}}\Psi(x,\upsilon).\end{aligned}$$

The κth-truncated ARF of Eq (4.8) is given by

$$\begin{aligned}ARes_\kappa(x,\upsilon) =& \Psi_\kappa(x,\upsilon) - \frac{e^x}{\upsilon^2} - \frac{e^x}{\upsilon^{\alpha+2}} - \frac{1}{\upsilon^{2\alpha}}A\Big[\Big(A^{-1}[\Psi_\kappa(x,\upsilon)]\Big)^2\frac{\partial^2}{\partial x^2}\Big(\frac{\partial}{\partial x}A^{-1}[\Psi_\kappa(x,\upsilon)]\frac{\partial^2}{\partial x^2}\\ & A^{-1}[\Psi_\kappa(x,\upsilon)]\frac{\partial^3}{\partial x^3}A^{-1}[\Psi_\kappa(x,\upsilon)]\Big)\Big] - \frac{1}{\upsilon^{2\alpha}}A\Big[x^2\frac{\partial^2}{\partial x^2}\Big(\frac{\partial^2}{\partial x^2}\\ & A^{-1}[\Psi_\kappa(x,\upsilon)]\Big)^3 - 18\Big(A^{-1}[\Psi_\kappa(x,\upsilon)]\Big)^5\Big] - \frac{1}{\upsilon^{2\alpha}}\Psi_\kappa(x,\upsilon).\end{aligned} \tag{4.10}$$

To find unknown coefficients, put $\kappa = 2, 3, 4, 5, 6, 7$ in Eq (4.10) and Eq (4.9), and make some simple calculations on the following equation

$$lim_{\upsilon\to\infty}(\upsilon^2 ARes_\kappa(x,\upsilon)) = 0.$$

We get the following results:

$$\begin{aligned}
\hbar_2(x) &= e^x,\\
\hbar_3(x) &= e^x,\\
\hbar_4(x) &= e^x,\\
\hbar_5(x) &= e^x,\\
\hbar_6(x) &= e^x,\\
\hbar_7(x) &= e^x.
\end{aligned}$$

In other words, for each $n \in \mathbb{N}$, we have $\hbar_n(x) = e^x$. The 7th approximate solution of Eq (4.8) is as follows:

$$\Psi_7(x, \upsilon) = e^x\Big(\frac{1}{\upsilon^2} + \frac{1}{\upsilon^{\alpha+2}} + \frac{1}{\upsilon^{2\alpha+2}} + \frac{1}{\upsilon^{3\alpha+2}} + \frac{1}{\upsilon^{4\alpha+2}} + \frac{1}{\upsilon^{5\alpha+2}} + \frac{1}{\upsilon^{6\alpha+2}} + \frac{1}{\upsilon^{7\alpha+2}}\Big). \tag{4.11}$$

The 7th approximate solution in the original space is achieved by utilizing the inverse Aboodh transform on the Eq (4.11), and we have

$$\begin{aligned}
\Phi_7(x, \tau) = e^x\Big(1 &+ \frac{\tau^{\alpha}}{\Gamma[\alpha+1]} + \frac{\tau^{2\alpha}}{\Gamma[2\alpha+1]} + \frac{\tau^{3\alpha}}{\Gamma[3\alpha+1]} + \frac{\tau^{4\alpha}}{\Gamma[4\alpha+1]} + \frac{\tau^{5\alpha}}{\Gamma[5\alpha+1]}\\
&+ \frac{\tau^{6\alpha}}{\Gamma[6\alpha+1]} + \frac{\tau^{7\alpha}}{\Gamma[7\alpha+1]}\Big).
\end{aligned}$$

For $\alpha = 1$, we get

$$\Phi_7(x, \tau) = e^x\Big(1 + \frac{\tau}{\Gamma[2]} + \frac{\tau^2}{\Gamma[3]} + \frac{\tau^3}{\Gamma[4]} + \frac{\tau^4}{\Gamma[5]} + \frac{\tau^5}{\Gamma[6]} + \frac{\tau^6}{\Gamma[7]} + \frac{\tau^7}{\Gamma[8]}\Big). \tag{4.12}$$

Equation (4.12) characterizes the first eight terms of expansion of the exact solution, $\Phi(x, \tau) = e^{x+\tau}$. The similar result has been obtained by Khalouta and Kadem [58].

Problem 3. Consider the (2+1)-heat equation with variable coefficients [57]:

$$D_\tau^\alpha \Phi(x, y, \tau) = \frac{1}{2}y^2\Phi_{xx}(x, y, \tau) + \frac{1}{2}x^2\Phi_{yy}(x, y, \tau), \tag{4.13}$$

where, $0 < \alpha \leq 1,\ \tau \geq 0,\ (x, y, \tau) \in (\mathbb{R}^+)^3$, with the initial condition:

$$\Phi(x, y, 0) = y^2.$$

Applying the Aboodh transform on Eq (4.13), we get

$$A[D_\tau^\alpha \Phi(x, y, \tau)] = A[\frac{1}{2}y^2\Phi_{xx}(x, y, \tau) + \frac{1}{2}x^2\Phi_{yy}(x, y, \tau)].$$

Using the approach outlined in Section 3, we obtain the following results from the above equation:

$$\Psi(x, y, \upsilon) = \frac{y^2}{\upsilon^2} + \frac{y^2}{2\upsilon^\alpha}D_{xx}\Psi(x, y, \upsilon) + \frac{x^2}{2\upsilon^\alpha}D_{xx}\Psi(x, y, \upsilon). \tag{4.14}$$

Now introduce the series solution of Eq (4.14) as

$$\Psi(x, y.\upsilon) = \sum_{r=0}^{\infty} \frac{\hbar_r(x, y)}{\upsilon^{r\alpha+2}},\ \upsilon > 0.$$

Furthermore, the κth-truncated series is given by

$$\Psi_\kappa(x, y.\upsilon) = \sum_{r=0}^{\kappa} \frac{\hbar_r(x, y)}{\upsilon^{r\alpha+2}},\ \upsilon > 0. \tag{4.15}$$

By utilizing the Lemma 2.7 and Theorem 2.9, we get

$$\hbar_0(x) = lim_{\upsilon\to\infty}(\upsilon^2\Psi(x, y, \upsilon)) = \Phi(x, y, 0) = y^2.$$

So, Eq (4.15) becomes

$$\Psi_\kappa(x, y.\upsilon) = \frac{y^2}{\upsilon^2} + \sum_{r=1}^{\kappa} \frac{\hbar_r(x, y)}{\upsilon^{r\alpha+2}},\ \upsilon > 0. \tag{4.16}$$

The ARF of (4.14) is defined as

$$ARes(x, y, \upsilon) = \Psi(x, y, \upsilon) - \frac{y^2}{\upsilon^2} - \frac{y^2}{2\upsilon^\alpha} D_{xx}\Psi(x, y, \upsilon) - \frac{x^2}{2\upsilon^\alpha} D_{xx}\Psi(x, y, \upsilon).$$

The κth-truncated ARF of (4.14) is given as

$$ARes_\kappa(x, y, \upsilon) = \Psi_\kappa(x, y, \upsilon) - \frac{y^2}{\upsilon^2} - \frac{y^2}{2\upsilon^\alpha} D_{xx}\Psi_\kappa(x, y, \upsilon) - \frac{x^2}{2\upsilon^\alpha} D_{xx}\Psi_\kappa(x, y, \upsilon). \tag{4.17}$$

To determine the unknown coefficients $\hbar_\kappa(x, y)$, substitute $\kappa = 1, 2, 3, 4, 5, 6, 7$ into Eq (4.16) and Eq (4.17), and solve the expression

$$lim_{\upsilon\to\infty}(\upsilon^{\kappa\alpha+2} ARes_\kappa(x, y, \upsilon)) = 0.$$

Thus we have

$$\begin{aligned}
\hbar_1(x, y) =& x^2, \\
\hbar_2(x, y) =& y^2, \\
\hbar_3(x, y) =& x^2, \\
\hbar_4(x, y) =& y^2, \\
\hbar_5(x, y) =& x^2, \\
\hbar_6(x, y) =& y^2, \\
\hbar_7(x, y) =& x^2.
\end{aligned}$$

In other words, for each $n \in \mathbb{N}$, we have

$$\hbar_{2n}(x, y) = y^2, \qquad \hbar_{2n+1}(x, y) = x^2.$$

The 7th-order approximate solution of Eq (4.14) is as

$$\Psi_7(x,y,\upsilon) = \frac{y^2}{\upsilon^2} + \frac{x^2}{\upsilon^{\alpha+2}} + \frac{y^2}{\upsilon^{2\alpha+2}} + \frac{x^2}{\upsilon^{3\alpha+2}} + \frac{y^2}{\upsilon^{4\alpha+2}} + \frac{x^2}{\upsilon^{5\alpha+2}} + \frac{y^2}{\upsilon^{6\alpha+2}} + \frac{x^2}{\upsilon^{7\alpha+2}}. \tag{4.18}$$

By utilizing the inverse Aboodh transform on Eq (4.18), we get 7th-order approximate solution in the original space in the following form:

$$\begin{aligned}\Phi_7(x,y,\tau) = {} & y^2\Big(1 + \frac{\tau^{2\alpha}}{\Gamma[2\alpha+1]} + \frac{\tau^{4\alpha}}{\Gamma[4\alpha+1]} + \frac{\tau^{6\alpha}}{\Gamma[6\alpha+1]}\Big) \\ & + x^2\Big(\frac{\tau^{\alpha}}{\Gamma[\alpha+1]} + \frac{\tau^{3\alpha}}{\Gamma[3\alpha+1]} + \frac{\tau^{5\alpha}}{\Gamma[5\alpha+1]} + \frac{\tau^{7\alpha}}{\Gamma[\alpha+1]}\Big).\end{aligned}$$

When $\alpha = 1$, the 7th-approximate solution becomes

$$\Phi_7(x,y,\tau) = y^2\Big(1 + \frac{\tau^2}{\Gamma[3]} + \frac{\tau^4}{\Gamma[5]} + \frac{\tau^6}{\Gamma[7]}\Big) + x^2\Big(\frac{\tau^1}{\Gamma[2]} + \frac{\tau^3}{\Gamma[4]} + \frac{\tau^5}{\Gamma[6]} + \frac{\tau^7}{\Gamma[8]}\Big). \tag{4.19}$$

The Eq (4.19) corresponds to the first eight terms of the exact solution $y^2\cosh\tau + x^2\sinh\tau$. A similar result has been obtained by Khan et al. [57].

Problem 4. Consider the (3+1)-wave equation with variable coefficients [57, 59]:

$$\begin{aligned}D_\tau^{2\alpha}\Phi(x,y,z,\tau) = {} & \frac{1}{2}x^2\Phi_{xx}(x,y,z,\tau) + \frac{1}{2}y^2\Phi_{yy}(x,y,z,\tau) \\ & + \frac{1}{2}z^2\Phi_{zz}(x,y,z,\tau) + x^2 + y^2 + z^2,\end{aligned} \tag{4.20}$$

where, $0 < \alpha \leq 1,\ (x,y,z,\tau) \in (\mathbb{R}^+)^4$, with the initial condition:

$$\begin{aligned}\Phi(x,y,z,0) = {} & 0,\\ D_\tau^{\alpha}\Phi(x,y,z,0) = {} & x^2 + y^2 - z^2.\end{aligned}$$

By utilizing the Aboodh transform on Eq (4.20), we get

$$\begin{aligned}A[D_\tau^{2\alpha}\Phi(x,y,z,\tau)] = {} & A[\frac{1}{2}x^2\Phi_{xx}(x,y,z,\tau) + \frac{1}{2}y^2\Phi_{yy}(x,y,z,\tau) \\ & + \frac{1}{2}z^2\Phi_{zz}(x,y,z,\tau) + x^2 + y^2 + z^2],\end{aligned}$$

Using the approach outlined in Section 3, we obtain the following results from the above equation:

$$\begin{aligned}\Psi(x,y,z,\upsilon) = {} & \frac{1}{\upsilon^{\alpha+2}}(x^2+y^2-z^2) + \frac{x^2}{2\upsilon^{2\alpha}}D_{xx}\Psi(x,y,z,\upsilon) + \frac{y^2}{2\upsilon^{2\alpha}}D_{yy}\Psi(x,y,z,\upsilon) \\ & + \frac{z^2}{2\upsilon^{2\alpha}}D_{zz}\Psi(x,y,z,\upsilon) + \frac{1}{\upsilon^{2\alpha+2}}(x^2+y^2+z^2).\end{aligned} \tag{4.21}$$

Introduce a series solution of algebraic Eq (4.21) as follows:

$$\Psi(x,y,z,\upsilon) = \sum_{r=0}^{\infty}\frac{\hbar_r(x,y,z)}{\upsilon^{r\alpha+2}},\ \upsilon > 0.$$

The κth-truncated series is as

$$\Psi_{\kappa}(x, y, z, \upsilon) = \sum_{r=0}^{\kappa} \frac{\hbar_r(x, y, z)}{\upsilon^{r\alpha+2}}, \ \upsilon > 0. \tag{4.22}$$

By using Lemma 2.7 and Theorem 2.9, we have

$$\begin{aligned} lim_{\upsilon\to\upsilon}(\upsilon^2\Psi(x, y, z, \upsilon)) =& \hbar_0(x, y, z) = \Phi(x, y, z, 0) = 0, \\ \hbar_1(x, y, z) =& D_{\tau}^{\alpha}\Phi(x, y, z, 0) = x^2 + y^2 - z^2. \end{aligned}$$

So, Eq (4.22) becomes

$$\Psi_{\kappa}(x, y, z, \upsilon) = \frac{(x^2 + y^2 - z^2)}{\upsilon^{\alpha+2}} + \sum_{r=2}^{\kappa} \frac{\hbar_r(x, y, z)}{\upsilon^{r\alpha+2}}, \ \upsilon > 0. \tag{4.23}$$

The ARF of Eq (4.21) is defined as

$$\begin{aligned} ARes(x, y, z, \upsilon) =& \Psi(x, y, z, \upsilon) - \frac{1}{\upsilon^{\alpha+2}}(x^2 + y^2 - z^2) - \frac{x^2}{2\upsilon^{2\alpha}} D_{xx}\Psi(x, y, z, \upsilon) \\ &- \frac{y^2}{2\upsilon^{2\alpha}} D_{yy}\Psi(x, y, z, \upsilon) \frac{z^2}{2\upsilon^{2\alpha}} D_{zz}\Psi(x, y, z, \upsilon) - \frac{1}{\upsilon^{2\alpha+2}}(x^2 + y^2 + z^2). \end{aligned}$$

The κth-truncated ARF of Eq (4.21) is defined as

$$\begin{aligned} ARes_{\kappa}(x, y, z, \upsilon) =& \Psi_{\kappa}(x, y, z, \upsilon) - \frac{1}{\upsilon^{\alpha+2}}(x^2 + y^2 - z^2) - \frac{x^2}{2\upsilon^{2\alpha}} D_{xx}\Psi_{\kappa}(x, y, z, \upsilon) \\ &- \frac{y^2}{2\upsilon^{2\alpha}} D_{yy}\Psi_{\kappa}(x, y, z, \upsilon) \frac{z^2}{2\upsilon^{2\alpha}} D_{zz}\Psi_{\kappa}(x, y, z, \upsilon) - \frac{1}{\upsilon^{2\alpha+2}}(x^2 + y^2 + z^2). \end{aligned} \tag{4.24}$$

To find the unknown coefficients, use $\kappa = 2, 3, 4, 5, 6, 7$ in Eq (4.23) and Eq (4.24), and solve the following equation

$$lim_{\upsilon\to\infty}(\upsilon^{\kappa\alpha+2} ARes_{\kappa}(x, y, z, \upsilon)) = 0.$$

We have

$$\begin{aligned} \hbar_2(x, y, z) =& x^2 + y^2 + z^2, \\ \hbar_3(x, y, z) =& x^2 + y^2 - z^2, \\ \hbar_4(x, y, z) =& x^2 + y^2 + z^2, \\ \hbar_5(x, y, z) =& x^2 + y^2 - z^2, \\ \hbar_6(x, y, z) =& x^2 + y^2 + z^2, \\ \hbar_7(x, y, z) =& x^2 + y^2 - z^2. \end{aligned}$$

In other words, for each $n \in \mathbb{N}$, we have

$$\hbar_{2n}(x, y, z) = x^2 + y^2 + z^2, \qquad \hbar_{2n+1}(x, y, z) = x^2 + y^2 - z^2.$$

The 7th approximate solution of Eq (4.21) is as follows:

$$\begin{aligned}\Psi_7(x,y,z,\upsilon) =&\frac{x^2+y^2-z^2}{\upsilon^{\alpha+2}}+\frac{x^2+y^2+z^2}{\upsilon^{2\alpha+2}}+\frac{x^2+y^2-z^2}{\upsilon^{3\alpha+2}}+\frac{x^2+y^2+z^2}{\upsilon^{4\alpha+2}}\\&+\frac{x^2+y^2-z^2}{\upsilon^{5\alpha+2}}+\frac{x^2+y^2+z^2}{\upsilon^{6\alpha+2}}+\frac{x^2+y^2-z^2}{\upsilon^{7\alpha+2}}.\end{aligned}\tag{4.25}$$

As a result, the 7*th* approximate solution of Eq (4.21) in original space is obtained by using the inverse Aboodh transform on Eq (4.25) given as

$$\begin{aligned}\Phi_7(x,y,z,\tau) =&(x^2+y^2-z^2)\Big(\frac{\tau^{\alpha}}{\Gamma[\alpha+1]}+\frac{\tau^{3\alpha}}{\Gamma[3\alpha+1]}+\frac{\tau^{5\alpha}}{\Gamma[5\alpha+1]}+\frac{\tau^{7\alpha}}{\Gamma[7\alpha+1]}\Big)\\&+(x^2+y^2+z^2)\Big(\frac{\tau^{2\alpha}}{\Gamma[2\alpha+1]}+\frac{\tau^{4\alpha}}{\Gamma[4\alpha+1]}+\frac{\tau^{6\alpha}}{\Gamma[6\alpha+1]}\Big).\end{aligned}\tag{4.26}$$

When $\alpha = 1$, the Eq (4.26) becomes

$$\begin{aligned}\Phi_7(x,y,z,\tau) =&(x^2+y^2-z^2)\Big(\frac{\tau}{\Gamma[2]}+\frac{\tau^{3}}{\Gamma[4]}+\frac{\tau^{5}}{\Gamma[6]}+\frac{\tau^{7}}{\Gamma[8]}\Big)\\&+(x^2+y^2+z^2)\Big(\frac{\tau^{2}}{\Gamma[3]}+\frac{\tau^{4}}{\Gamma[5]}+\frac{\tau^{6}}{\Gamma[7]}\Big).\end{aligned}\tag{4.27}$$

The Eq (4.27) corresponds to the first seven terms of the exact solution $\Phi(x,y,z,\tau) = (x^2+y^2-z^2)\sinh\tau+(x^2+y^2+z^2)(\cosh\tau-1)$. A similar result has been obtained by [57, 59].

Problem 5. Consider the following equation, which is a two-dimensional nonlinear time-fractional wave-like equation involving variable coefficients [58]:

$$\begin{aligned}D_{\tau}^{2\alpha}\Phi(x,y,\tau) =&\frac{\partial^2}{\partial x\partial y}(\Phi_{xx}(x,y,\tau)\Phi_{yy}(x,y,\tau))\\&-\frac{\partial^2}{\partial x\partial y}(xy\Phi_{x}(x,y,\tau)\Phi_{y}(x,y,\tau))-\Phi(x,y,\tau),\end{aligned}\tag{4.28}$$

where, $0<\alpha\leq 1,\ (x,y,\tau)\in(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{+})$, with the initial conditions:

$$\Phi(x,y,0)=\mathrm{e}^{xy},\qquad D_{\tau}^{\alpha}\Phi(x,y,0)=\mathrm{e}^{xy}.$$

Applying the Aboodh transform on Eq (4.28), we get

$$\begin{aligned}A[D_{\tau}^{2\alpha}\Phi(x,y,\tau)] =&A\Big[\frac{\partial^2}{\partial x\partial y}\Big(\Phi_{xx}(x,y,\tau)\Phi_{yy}(x,y,\tau)\Big)\\&-\frac{\partial^2}{\partial x\partial y}\Big(xy\Phi_{x}(x,y,\tau)\Phi_{y}(x,y,\tau)\Big)-\Phi(x,y,\tau)\Big].\end{aligned}\tag{4.29}$$

Using the approach outlined in Section 3, we obtain the following results from Eq (4.29) as

$$\begin{aligned}\Psi(x,y,\upsilon) =&\frac{\mathrm{e}^{xy}}{\upsilon^2}+\frac{\mathrm{e}^{xy}}{\upsilon^{\alpha+2}}+\frac{1}{\upsilon^{2\alpha}}A\Big[\frac{\partial^2}{\partial x\partial y}\Big(D_{xx}A^{-1}[\Psi(x,y,\upsilon)]D_{yy}A^{-1}[\Psi(x,y,\upsilon)]\Big)\Big]\frac{1}{\upsilon^{2\alpha}}\\&-A\Big[\frac{\partial^2}{\partial x\partial y}\Big(xyD_{x}A^{-1}[\Psi(x,y,\upsilon)]D_{y}A^{-1}[\Psi(x,y,\upsilon)]\Big)\Big]-\frac{1}{\upsilon^{2\alpha}}\Psi(x,y,\upsilon).\end{aligned}\tag{4.30}$$

Assume that Eq (4.30) has a series solution in the following form:

$$\Psi(x,y,\upsilon)=\sum_{r=0}^{\infty}\frac{\hbar_r(x,y)}{\upsilon^{r\alpha+2}},\ \upsilon>0.$$

The κth-truncated expansion is as

$$\Psi_\kappa(x,y,\upsilon)=\sum_{r=0}^{\kappa}\frac{\hbar_r(x,y)}{\upsilon^{r\alpha+2}},\ \upsilon>0.$$

By using Lemma 2.7 and Theorem 2.9, we get

$$\begin{aligned}&lim_{\upsilon\to\infty}(\upsilon^2\Psi(x,y,\upsilon)) =\Phi(x,y,0)=\hbar_0(x)=\mathrm{e}^{xy},\\&\hbar_1(x)=D_\tau^\alpha\Phi(x,y,0) =\mathrm{e}^{xy}.\end{aligned}$$

So, Eq (4.31) becomes

$$\Psi_\kappa(x,y,\upsilon)=\frac{\mathrm{e}^{xy}}{\upsilon^2}+\frac{\mathrm{e}^{xy}}{\upsilon^{\alpha+2}}+\sum_{r=2}^{\kappa}\frac{\hbar_r(x,y)}{\upsilon^{r\alpha+2}},\ \upsilon>0. \tag{4.31}$$

The ARF of Eq (4.30) is defined as

$$\begin{aligned}ARes(x,y,\upsilon) =&\Psi(x,y,\upsilon)-\frac{\mathrm{e}^{xy}}{\upsilon^2}-\frac{\mathrm{e}^{xy}}{\upsilon^{\alpha+2}}-\frac{1}{\upsilon^{2\alpha}}A\Big[\frac{\partial^2}{\partial x\partial y}\Big(D_{xx}A^{-1}[\Psi(x,y,\upsilon)]\\&D_{yy}A^{-1}[\Psi(x,y,\upsilon)]\Big)\Big]\frac{1}{\upsilon^{2\alpha}}+A\Big[\frac{\partial^2}{\partial x\partial y}\Big(xyD_xA^{-1}[\Psi(x,y,\upsilon)]\\&D_yA^{-1}[\Psi(x,y,\upsilon)]\Big)\Big]+\frac{1}{\upsilon^{2\alpha}}\Psi(x,y,\upsilon).\end{aligned}$$

The κth-truncated ARF of Eq (4.30) is defined as

$$\begin{aligned}ARes_\kappa(x,y,\upsilon) =&\Psi_\kappa(x,y,\upsilon)-\frac{\mathrm{e}^{xy}}{\upsilon^2}-\frac{\mathrm{e}^{xy}}{\upsilon^{\alpha+2}}-\frac{1}{\upsilon^{2\alpha}}A\Big[\frac{\partial^2}{\partial x\partial y}\Big(D_{xx}A^{-1}[\Psi_\kappa(x,y,\upsilon)]\\&D_{yy}A^{-1}[\Psi_\kappa(x,y,\upsilon)]\Big)\Big]\frac{1}{\upsilon^{2\alpha}}+A\Big[\frac{\partial^2}{\partial x\partial y}\Big(xyD_xA^{-1}[\Psi_\kappa(x,y,\upsilon)]\\&D_yA^{-1}[\Psi_\kappa(x,y,\upsilon)]\Big)\Big]+\frac{1}{\upsilon^{2\alpha}}\Psi_\kappa(x,y,\upsilon).\end{aligned} \tag{4.32}$$

To determine the unknown coefficients $\hbar_\kappa(x,y)$, substitute $\kappa=1,2,3,4,5,6,7$ into Eq (4.31) and Eq (4.32), and solve the following expression

$$lim_{\upsilon\to\infty}(\upsilon^{\kappa\alpha+2}ARes_\kappa(x,y,\upsilon))=0.$$

We get

$$\begin{aligned}\hbar_2(x,y)=&-\mathrm{e}^{xy},\\\hbar_3(x,y)=&-\mathrm{e}^{xy},\end{aligned}$$

$$\hbar_4(x,y) = e^{xy},$$
$$\hbar_5(x,y) = e^{xy},$$
$$\hbar_6(x,y) = -e^{xy},$$
$$\hbar_7(x,y) = -e^{xy}.$$

In other words, for each $n \in \mathbb{N}$, we have

$$\hbar_n(x,y) = (-1)^n e^{xy}.$$

The 7th approximate solution of Eq (4.30) is as follows:

$$\Psi_7(x,y,\upsilon) = e^{xy}\Big(\frac{1}{\upsilon^2} + \frac{1}{\upsilon^{\alpha+2}} - \frac{1}{\upsilon^{2\alpha+2}} - \frac{1}{\upsilon^{3\alpha+2}} + \frac{1}{\upsilon^{4\alpha+2}} + \frac{1}{\upsilon^{5\alpha+2}} - \frac{1}{\upsilon^{6\alpha+2}} - \frac{1}{\upsilon^{7\alpha+2}}\Big). \tag{4.33}$$

The 7th approximate solution in the original space is achieved by utilizing the inverse Aboodh transform on the Eq (4.33) as

$$\begin{aligned}\Phi_7(x,y,\tau) =& e^{xy}\Big(1 + \frac{\tau^{\alpha}}{\Gamma[\alpha+1]} - \frac{\tau^{2\alpha}}{\Gamma[2\alpha+1]} - \frac{\tau^{3\alpha}}{\Gamma[3\alpha+1]} + \frac{\tau^{4\alpha}}{\Gamma[4\alpha+1]} \\ &+ \frac{\tau^{5\alpha}}{\Gamma[5\alpha+1]} - \frac{\tau^{6\alpha}}{6\Gamma[\alpha+1]} - \frac{\tau^{7\alpha}}{\Gamma[7\alpha+1]}\Big).\end{aligned}$$

For $\alpha = 1$, we get

$$\Phi_7(x,y,\tau) = e^{xy}\Big(1 + \frac{\tau}{\Gamma[2]} - \frac{\tau^2}{\Gamma[3]} - \frac{\tau^3}{\Gamma[4]} + \frac{\tau^4}{\Gamma[5]} + \frac{\tau^5}{\Gamma[6]} - \frac{\tau^6}{\Gamma[7]} - \frac{\tau^7}{\Gamma[8]}\Big). \tag{4.34}$$

Equation (4.34) characterizes the first eight terms of expansion of the exact solution $e^{xy}(\cos\tau + \sin\tau)$. The same exact solution has been obtained by Khalouta and Kadem [58].

5. Numerical simulation and discussion

In this section, we evaluate the graphic and numerical results of the approximate and exact solutions to the models discussed in Problems 1–5. Error functions can be used to determine the accuracy and capabilities of the numerical method. ARPSM provides an approximate analytical solution in terms of an infinite fractional power series, and it is necessary to give the errors of the approximate solution. We chose residual and absolute error functions to demonstrate the accuracy and capabilities of ARPSM.

Figures 1–5 demonstrate the 2D graph of the comparative study of the approximate solutions obtained by the proposed method and the exact solutions to Problems 1–5, respectively. It is observed from Figures 1–5 that the 5th-order approximate solutions at $\alpha = 0.6, 0.7, 0.8, 0.9$ and 1.0 converge to the exact solutions at $\alpha = 1.0$. Furthermore, the 5th-order approximate solutions at $\alpha = 1.0$ overlap with the exact solutions at $\alpha = 1.0$ and this confirms the validity and applicability of the proposed method. Tables 1–5 show absolute and relative errors at reasonable nominated grid points in the interval $\tau \in [0, 1]$ amongst the 5th-order approximate and exact solutions attained by means of ARPSM of Problems 1–5 at $\alpha = 1.0$. From Table 1, we can obtain that the range of magnitude of absolute and relative errors is from 1.408×10^{-9} to 1.615×10^{-3} and from 1.274×10^{-9} to 5.941×10^{-4}, respectively.

From Table 2, we can obtain that the range of magnitude of absolute and relative errors is from 3.830×10^{-9} to 4.390×10^{-3} and from 1.274×10^{-9} to 5.941×10^{-4}, respectively. From Table 3, we can obtain that the range of magnitude of absolute and relative errors is from 2.747×10^{-15} to 7.123×10^{-6} and from 2.752×10^{-14} to 6.123×10^{-6}, respectively. From Table 4, we can obtain that the range of magnitude of absolute and relative errors is from 2.058×10^{-11} to 2.764×10^{-4} and from 1.787×10^{-10} to 9.856×10^{-5}, respectively. From Table 5, we can obtain that the range of magnitude of absolute and relative errors is from 3.828×10^{-9} to 4.240×10^{-3} and from 1.286×10^{-9} to 1.129×10^{-3}, respectively. From the tables, it can be seen that the approximate solutions are in imminent agreement with the exact solutions, which validates the efficacy of the proposed method. The absolute and relative errors of the 5th-order approximate solutions obtained by ARPSM of Problems 1–5 at $\alpha = 1.0$ are also compared in Tables 1–5 to the absolute error of the 5th-order approximate solutions obtained by NTDM [57] and RPSM [58, 59]. The comparison has confirmed that the suggested technique and [57–59] provide identical solutions, which indicates the effectiveness and reliability of the ARPSM.

Finally, from the numerical and graphical results, the following are the key advantages of the ARPSM: The proposed method is a systematic, powerful, and suitable tool for analytical approximate and exact solutions of FODEs. The proposed method is highly efficient and accurate with fewer calculations than existing numerical methods, so the small size of the computation of this technique is the strength of the scheme. The proposed method has an advantage over the homotopy perturbation method and the Adomian decomposition method in that it can solve nonlinear problems without the need for He's polynomials and Adomian polynomials. The suggested technique is based on a new version of Taylor's series that generates a convergent series as a solution. Establishing the coefficients for a series, like the RPSM, necessitates computing the fractional derivatives each time. As ARPSM just requires the concept of an infinite limit, we simply need a few computations to get the coefficients. The error analysis has confirmed the higher degree of accuracy. Therefore, we concluded that the proposed method is a useful and efficient algorithm for solving certain classes of FODEs with fewer calculations and iteration steps.

Figure 1 compares the 5th-order approximate solutions for various values of α to the exact solution for Problem 1 at $\alpha = 1.0$ in the interval $\tau \in [0,\ 1]$ when $x = 1.0$.

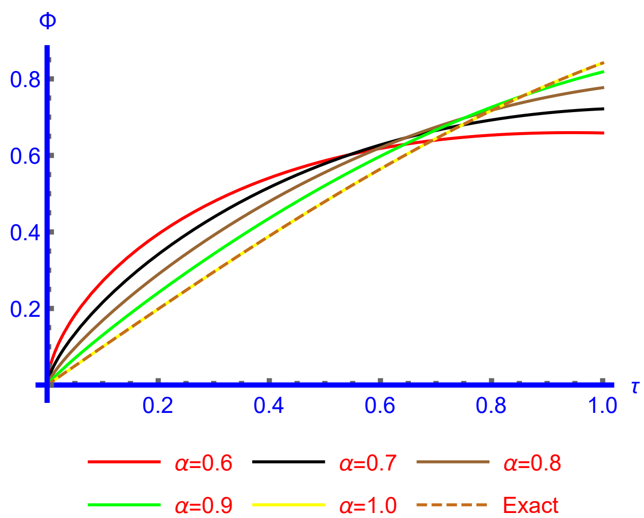

Figure 1. Comparison of approximate and exact solutions for different values of α.

Table 1 displays the absolute and relative errors at reasonable chosen grid points in the interval $\tau \in [0,\ 1]$ between the 5th-order approximate and exact solutions of Problem 1 at $\alpha = 1.0$ when $x = 1.0$ obtained by ARPSM and RPSM [58].

Table 1. The absolute and relative errors for Problem 1 at $\alpha = 1.0$ with $x = 1.0$.

τ	*Abs. Errors* [*ARPSM*]	*Abs. Errors* [58]	*Rel. Errors* [*ARPSM*]	*Rel. Errors* [58]
0.1	0.0000000014089809319273	0.0000000014089809319273	0.0000000012748986685070 1	0.0000000012748986685070 1
0.2	0.000000094935032181318	0.000000094935032181318	0.0000000749085447915238 6	0.0000000749085447915238 6
0.3	0.0000010575760032160980	0.0000010575760032160980	0.0000007834715729382329 2	0.0000007834715729382329 2
0.4	0.0000060309746037212621	0.0000060309746037212621	0.0000040426831740062081 2	0.0000040426831740062081 2
0.5	0.0000233540334615423011	0.0000233540334615423011	0.0000141649373223801672 1	0.0000141649373223801672 1
0.6	0.0000708003905087739601	0.0000708003905087739601	0.0000388560781512162223 0	0.0000388560781512162223 0
0.7	0.0001812908038099081401	0.0001812908038099081401	0.0000900263488845320700 1	0.0000900263488845320700 1
0.8	0.0004102618258010615001	0.0004102618258010615001	0.0001843425212040309600	0.0001843425212040309600
0.9	0.0008448611569500386002	0.0008448611569500386002	0.0003434949131092260502 1	0.0003434949131092260502 1
1.0	0.0016151623333331422012	0.0016151623333331422012	0.0005941850164695127132 2	0.0005941850164695127132 2

Figure 2 compares the 5th-order approximate solutions for various values of α to the exact solution for Problem 2 at $\alpha = 1.0$ in the interval $\tau \in [0,\ 1]$ when $x = 1.0$.

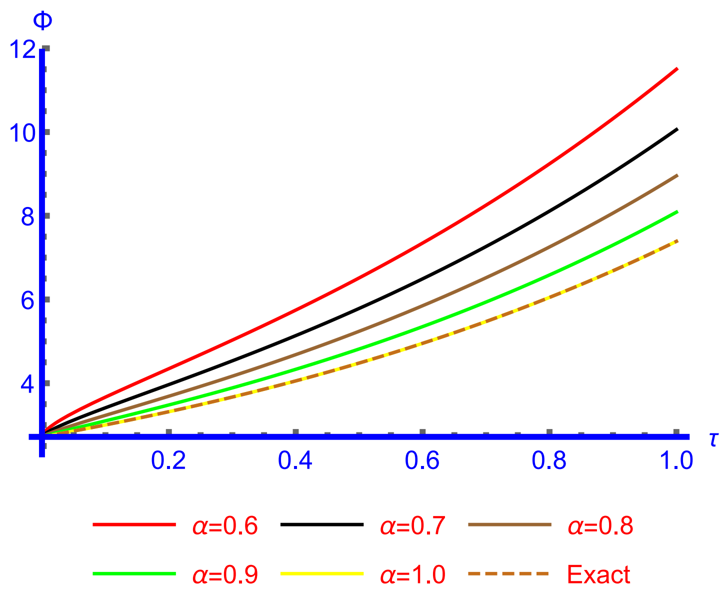

Figure 2. Comparison of approximate and exact solutions for different values of α.

Table 2 displays the absolute and relative errors at reasonable chosen grid points in the interval $\tau \in [0,\ 1]$ between the 5th-order approximate and exact solutions of Problem 2 at $\alpha = 1.0$ when $x = 1.0$ obtained by ARPSM and RPSM [58].

Table 2. The absolute and relative errors for Problem 2 at $\alpha = 1.0$ with $x = 1.0$.

τ	*Abs. Errors* [*ARPSM*]	*Abs. Errors* [58]	*Rel. Errors* [*ARPSM*]	*Rel. Errors* [58]
0.1	0.0000000038300078664121	0.0000000038300078664121	0.0000000012748988690647	0.0000000012748988690647
0.2	0.0000002487051271593543	0.0000002487051271593543	0.0000000749085447732857	0.0000000749085447732857
0.3	0.0000028747896321235092	0.0000028747896321235092	0.0000007834715730382094	0.0000007834715730382094
0.4	0.0000163938886732495353	0.0000163938886732495353	0.0000040426831740200271	0.0000040426831740200271
0.5	0.0000634828447791946410	0.0000634828447791946410	0.0000141649373222596122	0.0000141649373222596122
0.6	0.0001924554149681512213	0.0001924554149681512213	0.0000388560781512862201	0.0000388560781512862201
0.7	0.0004927994976648975211	0.0004927994976648975211	0.0000900263488848408502	0.0000900263488848408502
0.8	0.0011152072659852408015	0.0011152072659852408015	0.0001843425212039954304	0.0001843425212039954304
0.9	0.0022965707305075966735	0.0022965707305075966735	0.0003434949131091396002	0.0003434949131091396002
1.0	0.0043904637016884962317	0.0043904637016884962317	0.0005941846488086434001	0.0005941846488086434001

Figure 3 compares the 5th-order approximate solutions for various values of α to the exact solution for Problem 3 at $\alpha = 1.0$ in the interval $\tau \in [0,\ 1]$ when $x = 1.0$ and $y = 1.0$.

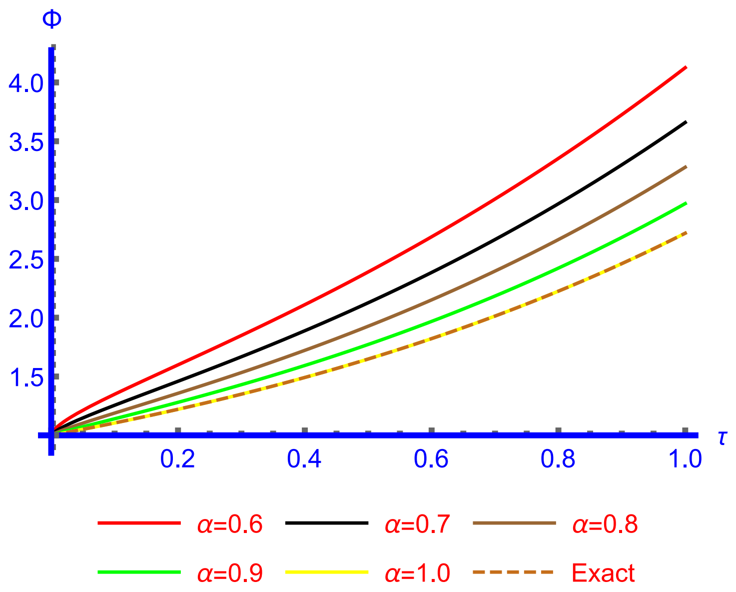

Figure 3. Comparison of approximate and exact solutions for different values of α.

Table 3 displays the absolute and relative errors at reasonable chosen grid points in the interval $\tau \in [0,\ 1]$ between the 5th-order approximate and exact solutions of Problem 3 at $\alpha = 1.0$ when $x = 1.0$ and $y = 1.0$ obtained by ARPSM and NTDM [57].

Table 3. The absolute and relative errors for Problem 3 at $\alpha = 1.0$ with $x = 1.0$ and $y = 1.0$.

τ	*Abs. Errors* [*ARPSM*]	*Abs. Errors* [57]	*Rel. Errors* [*ARPSM*]	*Rel. Errors* [57]
0.1	0.00000000000000002747801985	0.00000000000000002747801985	0.000000000000000275238700451 16	0.00000000000000027523870045116
0.2	0.0000000000001410399574908	0.0000000000001410399574908	0.0000000000007099231518341858	0.0000000000007099231518341858
0.3	0.00000000000541967026812 52	0.0000000000054196702681252	0.0000000001833942365347668 00	0.0000000001833942365347668 00
0.4	0.0000000007.2134892503683	0.0000000007.2134892503683	0.0000000018523753163765940 01	0.0000000018523753163765940 01
0.5	0.0000000053700759949926 18	0.0000000053700759949926 18	0.0000000112010636951611401 01	0.0000000112010636951611401 01
0.6	0.0000000276807496923936 40	0.0000000276807496923936 40	0.0000000490234989336831232 32	0.0000000490234989336831232 32
0.7	0.0000001107099133479394 00	0.0000001107099133479394 00	0.0000001718517133899986120 01	0.0000001718517133899986120 01
0.8	0.0000003.6772491962544500	0.0000003.6772491962544500	0.0000005126114133419281023 12	0.0000005126114133419281023 12
0.9	0.0000010598060549149581 00	0.0000010598060549149581 00	0.0000013529549947657942122 13	0.0000013529549947657942122 13
1.0	0.0000071233487501278671 10	0.0000071233487501278671 10	0.0000061233487501256230012 21	0.0000061233487501256230012 21

Figure 4 compares the 5th-order approximate solutions for various values of α to the exact solution for Problem 4 at $\alpha = 1.0$ in the interval $\tau \in [0, \ 1]$ when $x = 1.0$, $y = 1.0$, and $z = 1.0$.

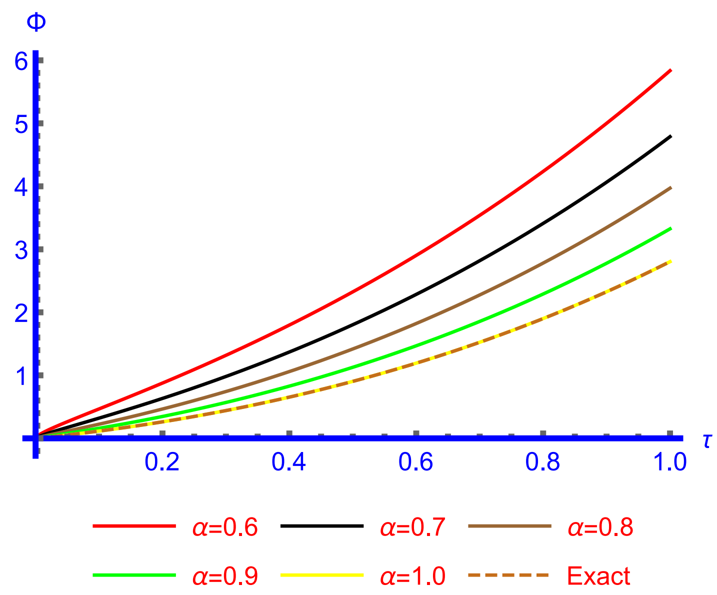

Figure 4. Comparison of approximate and exact solutions for different values of α.

Table 3 displays the absolute and relative errors at reasonable chosen grid points in the interval $\tau \in [0, \ 1]$ between the 5th-order approximate and exact solutions of Problem 4 at $\alpha = 1.0$ when $x = 1.0$, $y = 1.0$ and $z = 1.0$ obtained by ARPSM and RPSM [57].

Table 4. The absolute and relative errors for Problem 4 at $\alpha = 1.0$ with $x = 1.0$, $y = 1.0$, and $z = 1.0$.

τ	*Abs. Errors* [*ARPS M*]	*Abs. Errors* [57]	*Rel. Errors* [*ARPS M*]	*Rel. Errors* [57]
0.1	0.0000000000205878786241	0.0000000000205878786241	0.0000000000178746413747940	0.0000000000178746413747940
0.2	0.000000002731655179832 8	0.0000000027316551798328	0.000000001044465146695404 1	0.0000000010444651466954041
0.3	0.0000000483337240231307	0.0000000483337240231307	0.0000000109715759971395122	0.0000000109715759971395122
0.4	0.0000003746515125913063	0.0000003746515125913063	0.0000005728884087246288100	0.0000005728884087246288100
0.5	0.0000018469462227388430	0.0000018469462227388430	0.0000020431426733282040 01	0.0000020431426733282040 01
0.6	0.0000068368750438807520	0.0000068368750438807520	0.0000057305891765109310 11	0.0000057305891765109310 11
0.7	0.0000207645656957744512	0.0000207645656957744512	0.0000136242321015149560 12	0.0000136242321015149560 12
0.8	0.0000545544354908500401	0.0000545544354908500401	0.0000287066537851067260 41	0.0000287066537851067260 41
0.9	0.0001282945544978098220	0.0001282945544978098220	0.0000551620452717393702 12	0.0000551620452717393702 12
1.0	0.0002764323333335206021	0.0002764323333335206021	0.0000985694212986849423 22	0.0000985694212986849423 22

Figure 5 compares the 5th-order approximate solutions for various values of α to the exact solution for Problem 5 at $\alpha = 1.0$ in the interval $\tau \in [0,\ 1]$ when $x = 1.0$ and $y = 1.0$.

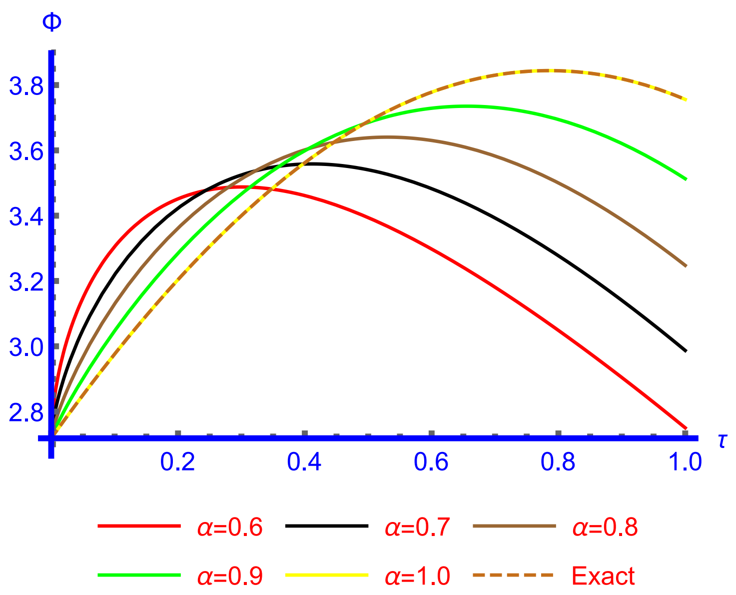

Figure 5. Comparison of approximate and exact solutions for different values of α.

Table 5 shows the absolute and relative errors at reasonable chosen grid points in the interval $\tau \in [0,\ 1]$ between the 5th-order approximate and exact solutions of Problem 5 at $\alpha = 1.0$ when $x = 1.0$ and $y = 1.0$ obtained by ARPSM and RPSM [58].

Table 5. The absolute and relative errors for Problem 5 at $\alpha = 1.0$ with $x = 1.0$ and $y = 1.0$.

τ	*Abs. Errors* [*ARPSM*]	*Abs. Errors* [58]	*Rel. Errors* [*ARPSM*]	*Rel. Errors* [58]
0.1	0.000000000382864451253791	0.000000000382864451253791	0.000000000128647356189609	0.000000000128647356189609
0.2	0.0000002.4835227829811025	0.0000002.4835227829811025	0.00000007750989571503319	0.00000007750989571503319
0.3	0.00000286564818985368042	0.00000286564818985368042	0.00000084279283008841991	0.00000084279283008841991
0.4	0.00001630159537713282722	0.00001630159537713282722	0.00000457620477589655102	0.00000457620477589655102
0.5	0.00006292687993214585012	0.00006292687993214585012	0.00001705922419745555301	0.00001705922419745555301
0.6	0.00019003969089403938034	0.00019003969089403938034	0.00005029697653130678003	0.00005029697653130678003
0.7	0.00048442177303043010043	0.00048442177303043010043	0.00012647355472679650002	0.00012647355472679650002
0.8	0.00109057399900969190051	0.00109057399900969190051	0.00028372131226548063006	0.00028372131226548063006
0.9	0.00223272085886838670072	0.00223272085886838670072	0.00058463274384336510012	0.00058463274384336510012
1.0	0.00424063624124793120131	0.00424063624124793120131	0.00112901508621710951200	0.00112901508621710951200

6. Conclusions

For the first time in research, we developed a new algorithm for solving time-fractional PDEs with variable coefficients in the sense of Caputo derivative using the Aboodh transform and RPSM. We proved some theorems on this method and solved some linear and nonlinear time-fractional PDEs with the help of the mentioned method. The efficiency of the ARPSM has been demonstrated by graphical and numerical results. We can observe from these graphs and tables that the approximate results obtained by ARPSM are in perfect agreement with their respective exact solutions. In addition, numerical results are also compared with other methods such as the NTDM and the RPSM. The comparison has confirmed that the suggested technique and NTDM and RPSM provide identical solutions.

In four important aspects, the ARPSM differs from other conventional numerical methods. This method has the advantage of not requiring any minor or major physical parametric assumptions in the problem. As a result, it applies to both weakly and strongly nonlinear problems, overcoming some of the inherent limits of traditional perturbation approaches. Second, while addressing nonlinear problems, the ARPSM does not require the He's polynomials or Adomian polynomials. To solve nonlinear PDEs, only a very small number of calculations are needed. As a consequence, it performs homotopy analysis and Adomian decomposition methods significantly better. Third, the ARPSM provides a simple and rapid way to find the coefficients of the recommended series as a solution to the problem. Unlike the traditional RPSM, establishing the coefficients for a series requires the computation of the fractional derivative every time. While the ARPSM only requires the concept of the limit at infinity in establishing the coefficients for the series. Finally, unlike conventional analytic approximation techniques, the ARPSM can create expansion solutions for linear and nonlinear fractional-order PDEs without the need for perturbation, linearization, or discretization.

Therefore, we concluded that our novel technique is simple to apply, accurate, adaptive, and efficient according to the results. Our goal in the future is to apply the ARPSM to other systems of FODEs that arise in other areas of science.

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Conflict of interest

The authors declare no conflict of interest.

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