Overlap Graph for Assembling and Scaffolding Algorithms: Paradigm Review and Implementation Proposals

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Abstract

Assembling Deoxyribonucleic Acid (DNA) *fragments* based on their *overlaps* remains the main *assembly* paradigm with long DNA fragments sequencing technologies, independently of the aim to resolve only one or several haplotypes. Since an overlap can be seen as a *succession relationship* between two oriented fragments, the *directed graph* structure has emerged as the more appropriate data structure for handling overlaps. However, this *graph* paradigm did not appear to take benefit of the *reverse* symmetry of the orientated fragments and their overlaps, which is a result of blind DNA double-strand sequencing. Thus, the *bi-directed graph* paradigm was introduced to be the one that reduces the graph size by handling the reverse symmetry, and since becomes the mainly used graph paradigm. Nevertheless, graph paradigms have never been contrasted before, and no implementations were described. Here we make a complete review on the existing overlap graph paradigms. Furthermore, we present different implementations that are theoretically compared in terms of memory, and their impact on the design and on the time of some basic graph algorithms. We also show that by adapting close logic implementations, a graph paradigm can be switched to another.

Keywords

Directed graph · *Overlaps*

Contents

1 Introduction

What is complementary strands?

Double-stranded Deoxyribonucleic Acid (DNA) molecules still cannot be entirely sequenced. In fact, every *sequencing* technology (*sequencer*) generates numerous overlapping genomic *fragments*.

1.1 Reads

A DNA fragment obtained from a sequencing stage is denoted by *read*.

▶ Definition 1.1: Raw reads

Let $\mathcal{R}_{aw}$ be the set of raw reads. $\forall r \in \mathcal{R}_{aw}$, let define the following attributes:

— $r_{rid} \in \mathbb{N}$, the read's integer identifier (index)

— $r_{seq} \in \Sigma^+$ the read's sequence, where $\Sigma = \{A, T, G, C\}$ is *nucleotide* alphabet

A read can be sequenced from one *strand* or from its complement.

▶ Axiom 1.1: Double-stranded DNA sequencing

i. The two DNA strands are both sequenced.

ii. A strand is read in reverse order to the order of its complementary strand.

iii. Given two reads, it is not possible to know *a priori* it they have been sequenced from the same strand.

The Definition 1.2 formalises Axiom 1.1 point iii.:

▶ Definition 1.2: Strand identifier

Let `strand_iid`: $\mathcal{R}_{aw} \to \{0; 1\}$ be the function which returns the DNA strand identifier for a read.

Then Axiom 1.1 point iii. can be mathematically formalised:

$$\exists\,(q,r) \in \mathcal{R}_{aw}{}^2, q \neq r \mid \texttt{strand\_iid}(q) \neq \texttt{strand\_iid}(r)$$

Axiom 1.1 suggests that combining two reads may result in a chimaera mix of the two strands because all the reads have not been sequenced from the same strand. Furthermore, only keeping the original sequence of the reads can result in the loss of strand parts during the *assembly* stage that aims to reconstruct the longest true parts of one strand. To solve these issues, it is sometimes necessary to consider the complementary sequence of some reads (i.e. the sequences on the complementary strand). For this, the sequence of the concerned reads are reversed (the end becomes the beginning, and vice versa) and each of its nucleotides is complemented ($A \rightleftarrows T, G \rightleftarrows C$). Arbitrarily, the original sequence of a read is denoted as the *forward*, while the *reverse* denotes the reverse-complement. In the following, determining what sequence is taken for a read is defined as *orienting* it. Finally, it is necessary to consider both the forward and the reverse orientation for all the reads. Figure 1 explains the meaning of forward and reverse orientations in the context of sequencing technology.

▶ Definition 1.3: Reversed reads

Let $\mathcal{R}_{ev} = \{u_r \mid u_f \in \mathcal{R}_{aw}\}$ denote the set of reversed reads. Thus $\mathcal{R} = \mathcal{R}_{aw} \cup \mathcal{R}_{ev}$. Let $u \in \mathcal{R}_{aw}$ be a read. Subscript is added to u to precise its orientation:

— u_f means that u is read in forward orientation (i.e. the original sequence of u, hence $u_f \in \mathcal{R}_{aw}$)

— u_r means that u is read in reverse orientation (i.e. the reverse-complemented sequence of u, hence $u_r \in \mathcal{R}_{ev}$)

Now the set of all oriented reads $\mathcal{R}$ was built, the reverse operation can be defined:

▶ Definition 1.4: The reverse operation for reads

For a given read $u \in \mathcal{R}$, $\overline{u}$ denotes its reverse, where:

— $\overline{u}_{rid} = u_{rid}$ (i.e. a read u and its reverse $\overline{u}$ have the same identifier)

— $\overline{u}_{seq} = \overline{u_{seq}} = \overline{b_1 b_2 \cdots b_n} = \overline{b_{n-1}} \cdots \overline{b_1}$ where $n \in \mathbb{N}$ is the length of the nucleotides sequence, and $\forall i \in [\![1;n]\!], b_i$ is the i^{th} nucleotide base and

$\overline{b_i}$ its complementary nucleotide base.

Note that $\overline{u_f} = u_r$ and $\overline{u_r} = u_f$.

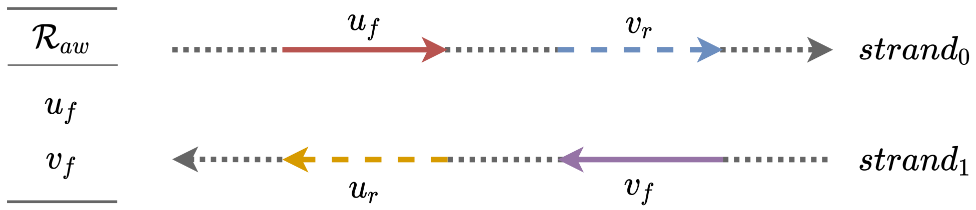


Figure 1 – Double-stranded DNA sequencing.
u_f, v_f are two reads from the raw reads set $\mathcal{R}_{aw}$ obtained from a double-stranded DNA sequencing stage. By default, f (forward) denotes the reading orientation of the raw reads' sequence that gives their sequence, while r (reverse) denotes the reading orientation of the raw reads' sequence that gives their reversed complemented sequence. In this illustration case, u_f was sequenced from strand $strand_0$ ($\texttt{strand\_iid}(u_f) = 0$) while v_f was sequenced from strand $strand_1$ ($\texttt{strand\_iid}(v_f) = 1$).

▶ **Axiom 1.2: Belonging to one strand**

The definition of $\mathcal{R}$ set and Axiom 1.1 point ii. give the following statement:

$$\forall r \in \mathcal{R},\ \texttt{strand\_iid}(r) = 1 - \texttt{strand\_iid}(\overline{r})$$

1.2 Overlaps

Reads must be assembled to reconstruct the entire *genome* since they are shorter than the length of the strand from which they have been sequenced (see Axiom 1.1). To assemble the reads means to find the original order of the (oriented) reads. Thus, the assembly stage is at least based on oriented reads *alignments* that define a *succession relationship* between them. One alignment type that answers this issue is *overlap*.

▶ **Definition 1.5: Overlap**

— An overlap between two nucleotides sequences (u, v) is a semi-global alignment or a close mapping between the end of sequence u_{seq} and the beginning of sequence v_{seq}.

— Let define $\mathcal{O}$ the set of overlaps.

All four scenarios for sequence overlap are depicted in Figure 2.

▶ Property 1.1: Overlap reverse symmetry

One overlap between two sequences always implies one overlap between the reversed sequences:

$$\forall (q, r) \in \mathcal{R}^2,\ (q, r) \in \mathcal{O} \iff (\overline{r}, \overline{q}) \in \mathcal{O}$$

▷ Proof

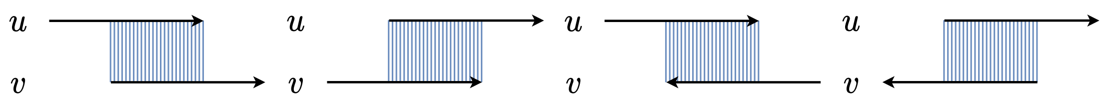
From Definition 1.3, it is sufficient to take one alignment between two sequences u and v, evaluated by an alignment score, that represents an overlap (hence succession relationship), and remark that:

i. the alignment score remains the same if the alignment is (reversely) read from v to u,

ii. the score remains the same if each nucleotide is transformed to its complement,

iii. combining points i. and ii. results in an alignment between $\overline{v}$ and $\overline{u}$ with the same alignment score as alignment u v

□



(a) $(u_f, v_f) \in \mathcal{O}$ **(b)** $(v_f, u_f) \in \mathcal{O}$ **(c)** $(u_f, v_r) \in \mathcal{O}$ **(d)** $(v_r, u_f) \in \mathcal{O}$

Figure 2 – Overlap cases between two sequences
u and v are two sequences, represented by an arrow that shows the forward orientation. Each subfigure illustrates one overlap case: **(a)** u_f overlaps v_f (hence v_r overlaps u_r) **(b)** v_f overlaps u_f (hence u_r overlaps v_r) **(c)** u_f overlaps v_r (hence v_f overlaps u_r) **(d)** v_r overlaps u_f (hence u_r overlaps v_f).

Analogously with Definition 1.4, Definition 1.6 extends the reverse operation to overlaps:

▶ **Definition 1.6: The reverse operation for overlaps**

For a given overlap $(q, r) \in \mathcal{O}$, $\overline{(q, r)}$ denotes its reverse i.e. $(\overline{r}, \overline{q})$.

2 Storing Overlaps in a Graph Structure

Overlaps between the reads can be represented in a *graph* structure. A graph is a mathematical object composed of *vertices* connected by *edges*. This graph structure would respect the following requirements:

1. Querying requirements
   — given an oriented read, getting every overlapping oriented reads
   — given two oriented reads, answering true if and only if they overlap
2. Dynamic requirements
   — adding a read/overlap
   — removing a read/overlap

Given all reads, the overlaps between them could be represented in a squared sparse matrix (of size $\mathcal{R}^2$). Sparse matrix compression as Compressed Sparse Row (CSR) or Compressed Sparse Column (CSC) are known for efficiently storing the matrix and enable fast overlap existence querying. Finally, they enable fast edges iteration, but suffer from requirements 2 and can not be immediately adapted for handling overlaps reverse symmetry.

Thus, adjacency list structure seems to be adapted to dynamically add vertices and edges.

Sections 2.1 to 2.3 below offer a complete review of graph paradigms, while Sections 3.1 to 3.3 propose graph implementations for each graph paradigm in the way of its philosophy. Figure 3 summarises the different graph paradigms and their implementations.

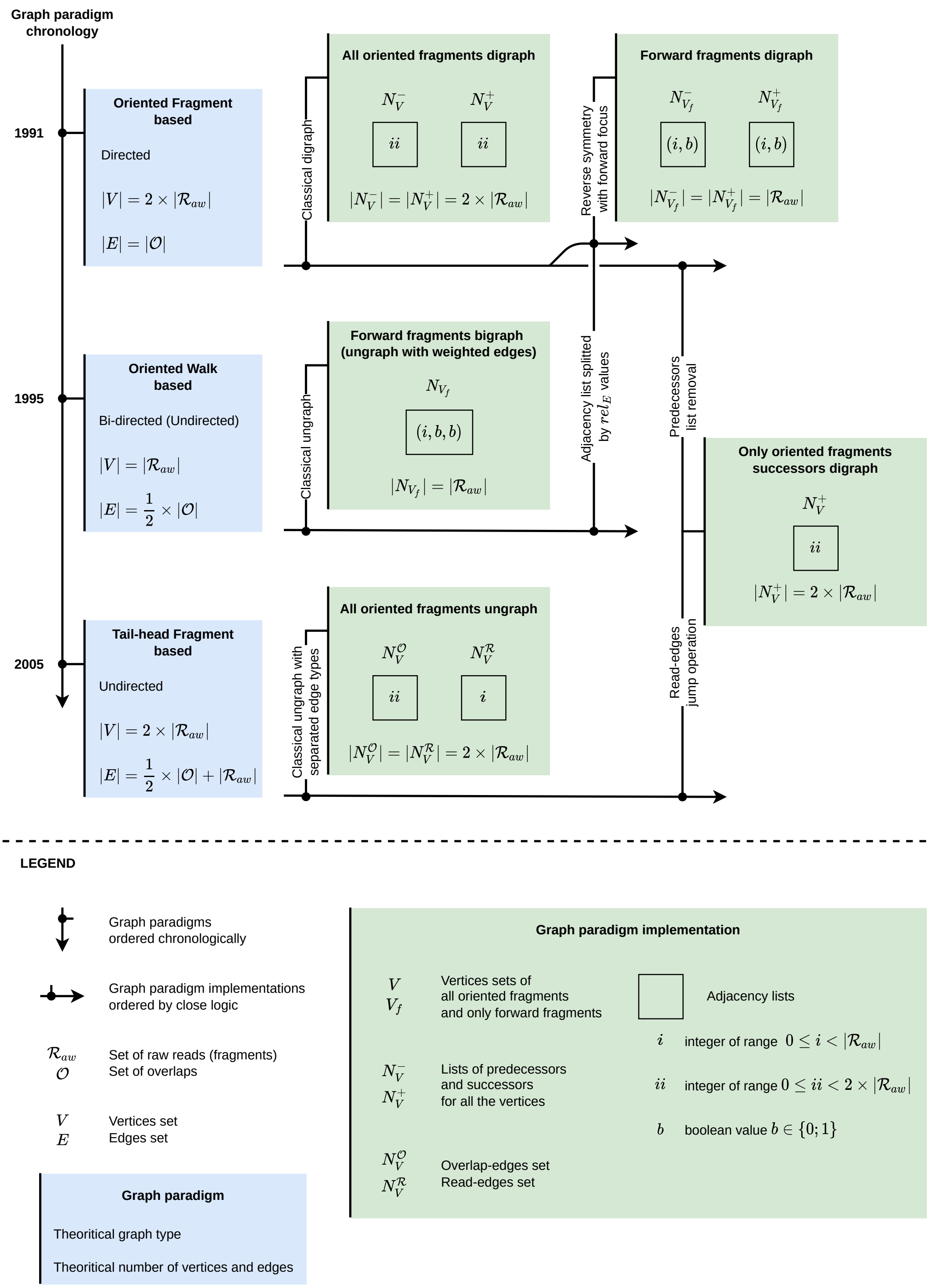

Graph paradigm chronology
1991
1995
2005
Oriented Fragment based
Directed
$|V| = 2 \times |\mathcal{R}_{aw}|$
$|E| = |\mathcal{O}|$
Oriented Walk based
Bi-directed (Undirected)
$|V| = |\mathcal{R}_{aw}|$
$|E| = \frac{1}{2} \times |\mathcal{O}|$
Tail-head Fragment based
Undirected
$|V| = 2 \times |\mathcal{R}_{aw}|$
$|E| = \frac{1}{2} \times |\mathcal{O}| + |\mathcal{R}_{aw}|$
Classical digraph
All oriented fragments digraph
N_V^- N_V^+
ii ii
$|N_V^-| = |N_V^+| = 2 \times |\mathcal{R}_{aw}|$
Reverse symmetry with forward focus
Forward fragments digraph
$N_{V_f}^-$ $N_{V_f}^+$
(i, b) (i, b)
$|N_{V_f}^-| = |N_{V_f}^+| = |\mathcal{R}_{aw}|$
Classical ungraph
Forward fragments bigraph (ungraph with weighted edges)
N_{V_f}
(i, b, b)
$|N_{V_f}| = |\mathcal{R}_{aw}|$
Adjacency list splitted by rel_E values
Predecessors list removal
Only oriented fragments successors digraph
N_V^+
ii
$|N_V^+| = 2 \times |\mathcal{R}_{aw}|$
Classical ungraph with separated edge types
All oriented fragments ungraph
$N_V^{\mathcal{O}}$ $N_V^{\mathcal{R}}$
ii i
$|N_V^{\mathcal{O}}| = |N_V^{\mathcal{R}}| = 2 \times |\mathcal{R}_{aw}|$
Read-edges jump operation
LEGEND
Graph paradigms ordered chronologically
Graph paradigm implementations ordered by close logic
$\mathcal{R}_{aw}$ Set of raw reads (fragments)
$\mathcal{O}$ Set of overlaps
V Vertices set
E Edges set
Graph paradigm
Theoritical graph type
Theoritical number of vertices and edges
Graph paradigm implementation
V V_f Vertices sets of all oriented fragments and only forward fragments
Adjacency lists
i integer of range $0 \leq i < |\mathcal{R}_{aw}|$
N_V^- N_V^+ Lists of predecessors and successors for all the vertices
ii integer of range $0 \leq ii < 2 \times |\mathcal{R}_{aw}|$
b boolean value $b \in \{0; 1\}$
$N_V^{\mathcal{O}}$ $N_V^{\mathcal{R}}$ Overlap-edges set Read-edges set

Figure 3 – Graph paradigms and their implementations respectively ordered chronologically and by logical adaptation.
Blue boxes: they correspond to graph paradigms. They are described by the type of graph they represent, the number of vertices and edges if they have to be drawn. **Green boxes:** they correspond to discussed graph implementations. After their name, the second line gives the adjacency list(s) definition set(s), following by the description of the type it (they) contain(s), finally the last line give the length of the adjacency list(s) (i.e. the number of vertices for which the *neighbours* are given). *Ungraph*, *digraph*, and *bigraph* respectively stand for *undirected graph*, *directed graph* and *bi-directed graph*.

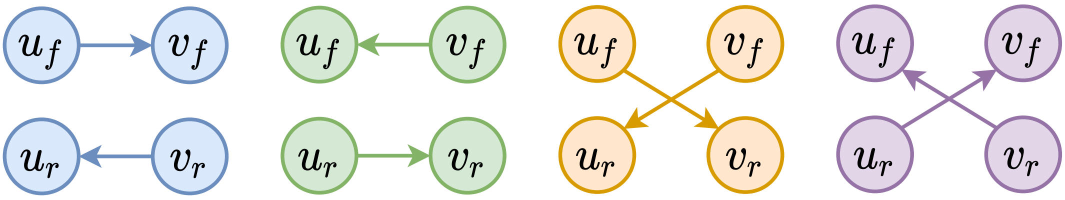
Fix Oriented FragmentS Based and Tail-head FragmentS based

2.1 Directed Graph: Oriented Fragments Based

The directed graph structure was the first idea to represent oriented reads and their overlaps, as overlaps are succession relationships. As far as we know, this structure was first mentioned in 1991 by Kececioglu [1]. Later, it was also described by Chin et al. [2], Kamath et al. [3], Andonov et al. [4], Shafin et al. [5] and Cheng et al. [6] in their respective assembly method.

> **▶ Definition 2.1: Oriented fragment based directed graph**
>
> Let $G = (V, E)$ a directed graph such that:
>
> — V is the set of vertices, where each $v \in V$ represents an oriented read. For each read, there are two vertices (one for the forward orientation, and one for the reverse orientation). Thus, each $v \in V$ has the following attributes:
>
> – $v_{rid} \in \mathbb{N}$ the identifier of the read it represents
>
> – $v_{or} \in \{0; 1\}$ the orientation of the read it represents (equals to 0 if forward, else equals to 1 if reverse)
>
> — E is the set of edges, where each $(u, v) \in E$ represents oriented read u overlapping oriented read v.

Figure 4 shows how each overlap case is represented in the directed graph.



(a) $(u_f, v_f) \in \mathcal{O}$ **(b)** $(v_f, u_f) \in \mathcal{O}$ **(c)** $(u_f, v_r) \in \mathcal{O}$ **(d)** $(v_r, u_f) \in \mathcal{O}$

Figure 4 – Overlap cases in the directed graph.
u and v are two sequences represented by two vertices each (u_f, u_r and v_f, v_r). Each subfigure illustrates one overlap impact on the directed graph: **(a)** $(u_f, v_f) \in E$ (hence $(v_r, u_r) \in E$) **(b)** $(v_f, u_f) \in E$ (hence $(u_r, v_r) \in E$) **(c)** $(u_f, v_r) \in E$ (hence $(v_f, u_r) \in E$) **(d)** $(v_r, u_f) \in E$ (hence $(u_r, v_f) \in E$).

▶ Property 2.1: Sizes of the oriented fragment based directed graph

Let $G = (V, E)$ be an oriented fragment based directed graph:

— $|V| = 2 \times |\mathcal{R}_{aw}| = |\mathcal{R}|$

— $|E| = |\mathcal{O}|$

Because there are two vertices for each read, such that one of them represents the reverse of the other one, it is possible to define a reverse operation for this graph. Definition 2.2 extends Definitions 1.4 and 1.6:

▶ Definition 2.2: The reverse operation for vertices and edges in the oriented fragment based directed graph

Vertices For a given vertex $v \in V$, $\overline{v}$ denotes its reverse, where:

— $\overline{v}_{rid} = v_{rid}$

— $\overline{v}_{or} = 1 - v_{or}$

Edges For a given edge $e = (u, v) \in E$, $\overline{e} = \left(\overline{v}, \overline{u}\right)$ denotes its reverse.

Definition 2.3 specifies what a valid *walk* (and a valid *path*) is in this graph:

▶ **Definition 2.3: Walking in oriented fragment based directed graph**

A valid walk results in a valid path $p = v_0 v_1 \cdots v_{n-1} \in V^n \mid n \in \mathbb{N}$ in the graph $G = (V, E)$ that respects the following criteria:

Contiguity Two consecutive vertices in the path must be connected by an edge in the graph:

$$\forall i \mid 0 \leq i < n - 1, (v_i, v_{i+1}) \in E$$

Exclusive orientation There is at most one orientation for each vertex in path p:

$$\forall v \in p, \overline{v} \notin p$$

Note that walking in this graph is quite similar that walking in a classic directed graph, except that it is not possible to walk through a vertex and its reverse. Figure 5 gives an example of a valid walk in the directed graph.

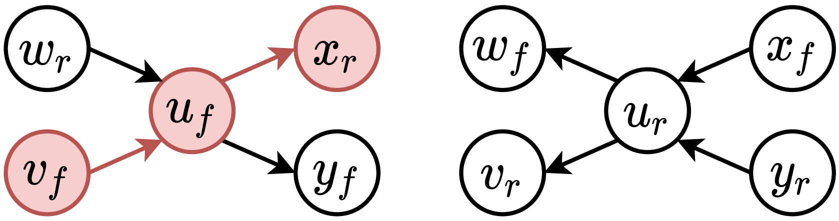


Figure 5 – Valid walk example in the oriented fragment based directed graph.
A valid walk that begins from vertex v_f is coloured in red. This walk results in a path $p = v_f u_f x_r$.

Finally, Property 2.2 suggests that this directed graph inherits a reverse symmetry property analogous to Property 1.1:

▶ **Property 2.2: Reverse symmetry for the vertices and the edges**

— $\forall v \in V, \overline{v} \in V$.

— $\forall (u, v) \in E, \overline{(u, v)} = (\overline{v}, \overline{u}) \in E$.

▷ Proof

By construction (see Definition 2.1), the first point is immediate. Concerning the second point, it is sufficient to note that if $(r, q) \in \mathcal{O}$ *(hence* $(u, v) \in E$*), then* $\left(\overline{q}, \overline{r}\right) \in \mathcal{O}$ *(hence* $\left(\overline{v}, \overline{u}\right) \in E$*), where* r *and* q *are two oriented reads in* $\mathcal{R}$*, respectively represented by vertices* u *and* v *in* V*.*

□

Figure 4 also illustrates Property 2.2.

2.2 Bi-directed Graph: Oriented Walk Based

In order to avoid creating two vertices for each read (and thus creating two edges for each overlap), the bi-directed graph structure to store overlaps was introduced by E. W. Myers in 1995 [7]. The key idea is to represent each read with only one vertex and keep the strict necessary overlap information between two reads on the edge connecting two vertices. Definition 2.4 provides a formal description. It was also described by Sommer et al. [8], Hernandez et al. [9] and Salmela et al. [10] in their respective assembly method.

▶ **Definition 2.4: Oriented walk based bi-directed graph**

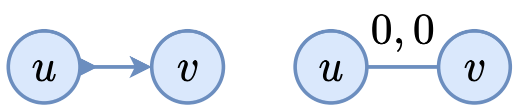
Let $G = (V, E, or_E, rel_E)$ a bi-directed graph such that:

— V is the set of vertices, where each $v \in V$ represents a non-oriented read. For each read, there are only one vertex. Thus, each vertex $v \in V$ has an identifier of the read it represents $v_{rid} \in \mathbb{N}$.

— E is the set of edges, where each $(u, v) \in E$ represents an overlap between oriented read u and oriented read v. The overlap information are given by or_{uv} and rel_{uv}:

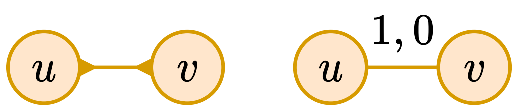
  - the orientation of u relatively to this of v is given by edge attribute or_E: if the orientations are the same, then $or_{uv} = 0$, else $or_{uv} = 1$
  - given that u identifier is lexicographically less that this of v, $rel_{uv} = 0$ if u in forward orientation overlaps v (in orientation given by or_{uv}), else, $rel_{uv} = 1$ if v (in orientation given by or_{uv}) overlaps u in forward orientation

While visually each edge has two extremities in the bi-directed graph, in memory the graph is undirected. Figure 6 shows how visually and how internally each overlap is stored in the bi-directed graph.

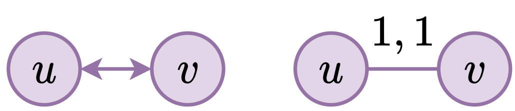


(a) $(u_f, v_f) \in \mathcal{O}$

(b) $(v_f, u_f) \in \mathcal{O}$



(c) $(u_f, v_r) \in \mathcal{O}$



(d) $(v_r, u_f) \in \mathcal{O}$

Figure 6 – Overlap cases in the bi-directed graph.
u and v are two unoriented sequences. Each subfigure illustrates one overlap impact on the bi-directed graph. For each, the first subfigure corresponds to the visualisation of a bi-directed edge, and the second shows the same information but with an undirected edge with or_{uv} and rel_{uv} attributes. **(a)** u and v have the same orientation in the overlap hence $or_{uv} = 0$, u in forward orientation overlaps v hence $rel_{uv} = 0$ **(b)** u and v have the same orientation in the overlap hence $or_{uv} = 0$, u in forward orientation is overlapped by v hence $rel_{uv} = 1$ **(c)** u and v do not have the same orientation in the overlap hence $or_{uv} = 1$, u in forward orientation overlaps v hence $rel_{uv} = 0$ **(d)** u and v do not have the same orientation in the overlap hence $or_{uv} = 1$, u in forward orientation is overlapped by v hence $rel_{uv} = 1$

> ▶ **Property 2.3: Sizes of the oriented walks based bi-directed graph**
>
> Let $G = (V, E)$ be an oriented walks based bi-directed graph:
>
> — $|V| = |\mathcal{R}_{aw}| = \frac{1}{2} \times |\mathcal{R}|$
>
> — $|E| = \frac{1}{2} \times |\mathcal{O}|$

As the vertices represent unoriented reads and the overlaps are succession relationships between two oriented reads, it is necessary to define a variable for each vertex that gives its orientation:

> ▶ **Definition 2.5: Vertex orientation variable**
>
> At each vertex $v \in V$ is associated a variable v_{or} that gives its orientation.

Given an overlap between two oriented reads, the variables of the two vertices that correspond to them (but unoriented) must respect their orientation. Definition 2.6 formalises this constraint that named co-validity constraint:

▶ Definition 2.6: Co-validity for the orientation variables of the edge' extremities

Let $(u, v) \in E$ be an edge. The orientation variables u_{or} and v_{or} are co-valid according the edge's attributes when:

— they are co-valid for or_{uv} attribute:

$$|u_{or} - v_{or}| = or_{uv}$$

— they are co-valid for rel_{uv} attribute:

$$u_{or} = \begin{cases} rel_{uv} & \text{if } u_{rid} < v_{rid} \\ |(1 - or_{uv}) - rel_{uv}| & \text{if } u_{rid} > v_{rid} \end{cases}$$

To know if the orientation variables are co-valid, the identifiers of the two consecutive vertices have to be compared. To avoid this verification, one may remark that the attribute rel_{uv} equals to $1 - rel_{vu}$, and can suggest a change in the attribute value, depending on the vertex u or v from which the edge (u, v) is considered. But this would contradict the bi-directed graph paradigm philosophy (which is a undirected graph paradigm) since having two different edge attributes rel_{uv} and rel_{vu} implies that the graph is directed.

As the graph is bi-directed, it is possible to define a reverse operation. Analogously to Definition 2.2, Definition 2.7 formalises the reverse operation in the bi-directed graph.

▶ Definition 2.7: The reverse operation for vertices and edges in the oriented walk based bi-directed graph

Vertices For a given vertex $v \in V$ and a fixed orientation variable v_{or}, $\overline{v}$ denotes its reverse, where:

— $\overline{v}_{rid} = v_{rid}$

— $\overline{v}_{or} = 1 - v_{or}$

Edges For a given edge $e = (u, v) \in E$ and fix co-valid orientation variables u_{or} and v_{or}, $\overline{e} = \left(\overline{v}, \overline{u}\right)$ denotes its reverse.

According to Definition 2.4, given $(u, v) \in E$ and one orientation for u or for v, it is necessary to check rel_{uv} value to determine the succession relationship between oriented u and oriented v. This particularity makes the bi-directed graph *oriented* *walk dependent.* Definition 2.8 specifies what a valid walk is in this graph.

▶ Definition 2.8: Walking in oriented walk based bi-directed graph

A valid walk results in a path $p = v_0 v_1 \cdots v_{n-1} \in V^n \mid n \in \mathbb{N}$ in the graph $G = (V, E, or_E, rel_E)$ that respects the following criteria:

Contiguity

— Two consecutive vertices in the path must be connected by an edge in the graph:

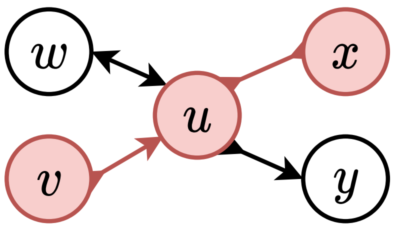
$$\forall i \mid 0 \leq i < n-1, (v_i, v_{i+1}) \in E$$

— Given the vertices' orientation $v_{i_{or}}$ and $v_{i+1_{or}}$, they must be co-valid.

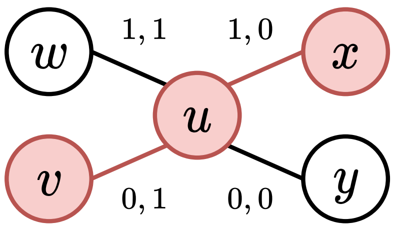
Exclusive orientation At most one orientation is chosen for each vertex:

$$\forall v_i, v_j \in p, v_i = v_j \implies i = j$$

Figure 7 gives an example of a valid walk in the bi-directed graph. Note that walking in this graph is not similar to walking in a classical undirected graph because the orientation variables must be co-valid.



(a) bi-directed graph view



(b) undirected graph view

Figure 7 – Valid walk example in the oriented walks based bi-directed graph. A valid walk is coloured in red. As the graph is undirected, the resulting path can be read in two possible ways: from vertex v or from vertex x. Let $p_v = vux$ the one that begins from v such that $v_{or} = 0, u_{or} = 0, x_{or} = 1$. Thus, $p_x = xuv$ is the one that begins from x such that $x_{or} = 0, u_{or} = 1, v_{or} = 1$. Note that $p_x = \overline{p_v}$. In **(a)** edges are drawn bidirected while in **(b)** they are drawn undirected.

Thanks to edges attributes or_E and rel_E, the bi-directed graph inherits a reverse symmetry property analogous to Property 1.1:

▶ **Property 2.4: Reverse symmetry for the edges**

Given an edge $(u, v) \in E$ and its attributes or_{uv} and rel_{uv}, an overlap and its reverse are encoded in the graph.

▷ Proof

Without any loss of generality, let suppose that u identifier is lexicographically less that this of v.

— *If $or_{uv} = 0$:*

  - *if $rel_{uv} = 0$, then: if u orientation is fixed forward, then overlap (u_f, v_f) is obtained; while if u orientation is fixed reverse, then (v_r, u_r) is obtained (because rel_{uv} determines if oriented v is after or before u* in forward orientation*)*
  - *else: if u orientation is fixed forward, then overlap (v_f, u_f) is obtained; while if u orientation is fixed reverse, then (u_r, v_r) is obtained.*

— *Else:*

  - *if $rel_{uv} = 0$, then: if u orientation is fixed forward, then overlap (u_f, v_r) is obtained; while if u orientation is fixed reverse, then (v_f, u_r) is obtained.*
  - *else: if u orientation is fixed forward, then overlap (v_r, u_f) is obtained; while if u orientation is fixed reverse, then (u_r, v_f) is obtained.*

□

2.3 Undirected Graph: Tail-Head Fragments Based

A new undirected graph structure was presented by E. W. Myers in 2005 [11]: one read is represented by its tail and its head. Both the tail and the head are vertices, and there is one edge from the tail to the head. Passing through first the tail and then the head corresponds to choose the read in forward orientation, while passing through first the head and then the tail corresponds to choose it in reverse orientation. This new type of edges are called read-edges, at the opposite of overlap-edges that correspond to overlaps. A walk in the graph must alternate between read-edges and overlap-edges. This representation was also described in the Mäkinen et al.'s book [12] and by Li [13] for its assembly method.

▶ Definition 2.9: Tail-head Fragments Based undirected graph

Let $G = (V, E)$ a undirected graph such that:

— V is the set of vertices, where each $v \in V$ represents one of the extremity of one read. For each read, there are two vertices v_t, v_h (respectively for the tail of the read, and for its head). Thus, each $v \in V$ has the following attributes:

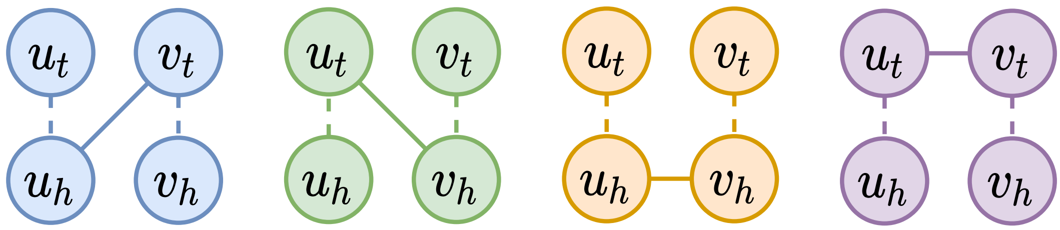
 - $v_{rid} \in \mathbb{N}$ the identifier of the read it represents
 - $v_{ext} \in \{0; 1\}$ the extremity of the read it represents (equals to 0 if it is the head, else equals to 1 if it is the tail[a])

— $E = E^{\mathcal{R}} \cup E^{\mathcal{O}}$ is the set of edges, such that:

 - each undirected read-edge $(v_t, v_h) \in E^{\mathcal{R}}$ encodes the two possible read orientations (forward if from v_t to v_h, else reverse)
 - each undirected overlap-edge $(u, v) \in E^{\mathcal{O}}$ represents which extremity of read u is overlapping which extremity of read v

[a]The values for the extremities are arbitrary: while selecting 0 for the tail and 1 for the head appears more natural, switching the values will be interesting later.

Figure 8 shows for all the overlap cases how they are represented in the undirected graph.



(a) $(u_f, v_f) \in \mathcal{O}$ **(b)** $(v_f, u_f) \in \mathcal{O}$ **(c)** $(u_f, v_r) \in \mathcal{O}$ **(d)** $(v_r, u_f) \in \mathcal{O}$

Figure 8 – Overlap cases in the undirected graph.
u_t, u_h and v_t, v_h are respectively the tail and the head of reads u and v. Dashed edges correspond to read-edges while plain edges correspond to overlap-edges. Each subfigure illustrates one overlap impact on the undirected graph: **(a)** $(u_h, v_t) \in E$ **(b)** $(v_h, u_t) \in E$ **(c)** $(u_h, v_h) \in E$ **(d)** $(v_t, u_t) \in E$.

▶ Property 2.5: Sizes of the tail-head fragment based undirected graph

Let $G = (V, E)$ be an oriented walks based bi-directed graph:

— $|V| = 2 \times |\mathcal{R}_{aw}| = |\mathcal{R}|$

— $|E| = \frac{1}{2} \times |\mathcal{O}| + |\mathcal{R}_{aw}|$

As for each read there are two vertices that represent their two extremities, it is possible to define a reverse operation. Analogously to Definitions 2.2 and 2.7, Definition 2.10 formalises the reverse operation in the undirected graph.

▶ Definition 2.10: The reverse operation for vertices and edges in the tail-head fragment based undirected graph

Vertices For a given vertex $v \in V$, $\overline{v}$ denotes its reverse, where:

— $\overline{v}_{rid} = v_{rid}$

— $\overline{v}_{or} = 1 - v_{or}$

Edges For a given edge $e = (u, v) \in E$, $\overline{e} = (v, u)$ denotes its reverse.

Because of the partition of edges set E in two subsets, walking in this graph does not correspond to a traditional undirected graph walk. Definition 2.11 specifies what a valid walk is in this graph.

▶ Definition 2.11: Walking in tail-head fragments based undirected graph

A valid walk results in a path $p = v_0 v_1 \cdots v_{2k-1} \in V^{2k} \mid k \in \mathbb{N}$ in the graph $G = \left(V, E^{\mathcal{R}} \cup E^{\mathcal{O}}\right)$ that respects the following criteria:

Contiguity Read-edges and overlap-edges alternate in the path:

— odd-numbered edges in the path are read-edges:

$$\forall i \mid 0 \leq i < k, (v_{2i}, v_{2i+1}) \in E^{\mathcal{R}}$$

— even-numbered edges in the path are overlap-edges:

$$\forall i \mid 1 \leq i < k, (v_{2i-1}, v_{2i}) \in E^{\mathcal{O}}$$

Exclusive orientation At most one orientation is chosen for each vertex:

$$\forall v_i, v_j \in p, v_i = v_j \implies i = j$$

Note that walking in this graph is not similar that walking in a classic undirected graph, because a valid walk must alternate between read-edges and overlap-edges. Figure 5 gives an example of a valid walk in the undirected graph.

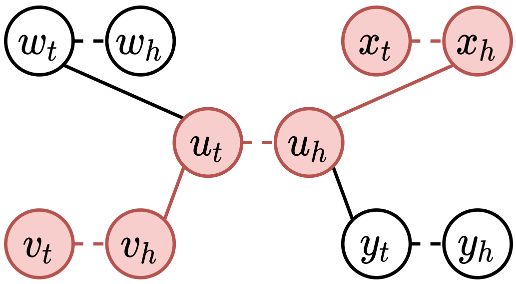


Figure 9 – Valid walk example in the tail-head fragment based undirected graph.
A valid walk is coloured in red. As the graph is undirected, the resulting path can be read in two possible ways: from vertex v_t or from vertex x_t. Let $p_v = v_t v_h u_t u_h x_h x_t$ the one that begins from v_t, thus $p_x = x_t x_h u_h u_t v_h v_t$ is the one that begins from x_t. Note that $p_x = \overline{p_v}$.

Because the set of vertices contains the two extremities of each read, the undirected graph inherits a reverse symmetry property analogous to Property 1.1:

▶ **Property 2.6: Reverse symmetry for the read-edges and the overlap-edges**

Given one overlap-edge $(u, v) \in E^{\mathcal{O}}$ and the two read-edges $(u_t, u_h) \in E^{\mathcal{R}}$ and $(v_t, v_h) \in E^{\mathcal{R}}$, one overlap and its reverse are encoded in the graph.

— $\forall (v_t, v_h) \in E, (v_h, v_t) \in E$, where v_t corresponds to the tail of read v and v_h corresponds to its head.

— $\forall (u, v) \in E, (v, u) \in E$.

▷ Proof

— *Read-edge (v_t, v_h) corresponds to reading read v in forward orientation, hence (v_h, v_t) correspond to reading it in reverse orientation.*

— *Let $(u, v) \in E^{\mathcal{O}}$ be an overlap-edge, hence read-edges $(\overline{u}, u)$ and $(v, \overline{v})$ are considered. Moreover, overlap-edge (v, u) is also in $E^{\mathcal{O}}$, hence read-edges $(u, \overline{u})$ and $(\overline{v}, v)$ are considered, and they correspond to the* *reverse of the previous read-edges.*

□

Figure 8 also illustrates Property 2.6.

3 Overlap Graph Implementations

For each graph paradigm in Section 2 at least one implementation is proposed in Sections 3.1 to 3.3. A map of all implementations and the graph paradigms they follow is illustrated in Figure 3.

In the following, let consider that all the overlaps in overlaps set $\mathcal{O}$ are kept, and reads in reads set $\mathcal{R}_{aw}$ participate in at least one overlap. Also let suppose that $min_{r \in \mathcal{R}_{aw}} r_{rid} = 0$ and $max_{r \in \mathcal{R}_{aw}} r_{rid} = |\mathcal{R}_{aw}| - 1$. The previous assertions permit to easily build an index on read identifiers rid. Also, mathematical terms as $G = (V, E)$ remain the same as in Section 2.

3.1 Directed Graph: Oriented Fragments Based

Section 3.1.1 describes the first implementation idea for the oriented fragment based directed graph, while Sections 3.1.2 and 3.1.3 takes the benefits of reverse symmetry to build a more clever implementation.

3.1.1 All Oriented Fragments Directed Graph (DGA)

For each read, there are two vertices in the directed graph. Remind that each read identifier r_{rid} corresponds to a unique integer identifier (it is an index, see Definition 1.1).

The All Oriented Fragments Directed Graph (DGA) implementation is the first and the simplest idea that follows a classical directed graph implementation. Thus, for each read $r \in \mathcal{R}_{aw}$ there are two vertices $v_f, v_r \in V$ such that the index of v_f is equal to $2 \times r_{rid}$ and the index of v_r is equal to $2 \times r_{rid} + 1$. Definition 3.1 formalises indices sets.

▶ **Definition 3.1: Graph indices set for DGA**

Let $Vind = [\![0; 2 \times |\mathcal{R}_{aw}|[\![$ be the set of indices for the vertices:

— $|Vind| = 2 \times |\mathcal{R}_{aw}|$

— $\forall v_{ind} \in Vind, v_{ind} = 2 \times v_{rid} + v_{or}$, where v_{rid} corresponds to the read's identifier and $v_{or} \in \{0; 1\}$ its orientation

Let $Eind = [\![0; |\mathcal{O}|[\![$ be the set of indices for the edges:

— $|Eind| = |\mathcal{O}|$

— $\forall e_{ind} \in Eind$, e_{ind} corresponds to the index of edge e, such that $\left|e_{ind} - \overline{e}_{ind}\right| = 1$

It is possible now to build the two adjacency lists: the first one for the *predecessors*, the second one for the *successors*. They both contain respectively the index of each predecessor, resp. of each successor vertices as the same way as a classic directed graph implementation. They also contain the index of the edge. Definition 3.2 details the way the implementation is built.

add edge index

▶ Definition 3.2: DGA implementation

DGA graph implementation is built by:

Predecessor list N_V^- $\forall v \in V, \forall u \in N_v^-$, u_{ind} is the index of a predecessor of vertex v

Successor list N_V^+ $\forall v \in V, \forall w \in N_v^+$, w_{ind} is the index of a successor of vertex v

Figure 10 illustrates the impact of overlap cases on the structure. For clarity's sake, only the indices of the predecessors (successors) are written.

Proposition 3.1 gives the amount of octet DGA implementation consumes.

▶ Proposition 3.1: DGA memory consumption

The memory size of the graph $Mem(DGA)$ (in octets) is equals to:

$$\begin{aligned} Mem(DGA) &= 2 \times (2 \times |\mathcal{R}_{aw}| + 1) \times P \\ &+ 2 \times |\mathcal{O}| \times \left(\left\lceil \frac{1 + \log_2 |\mathcal{R}_{aw}|}{8} \right\rceil + \left\lceil \frac{\log_2 |\mathcal{O}|}{8} \right\rceil \right) \end{aligned}$$

where P is the memory size of a memory address.

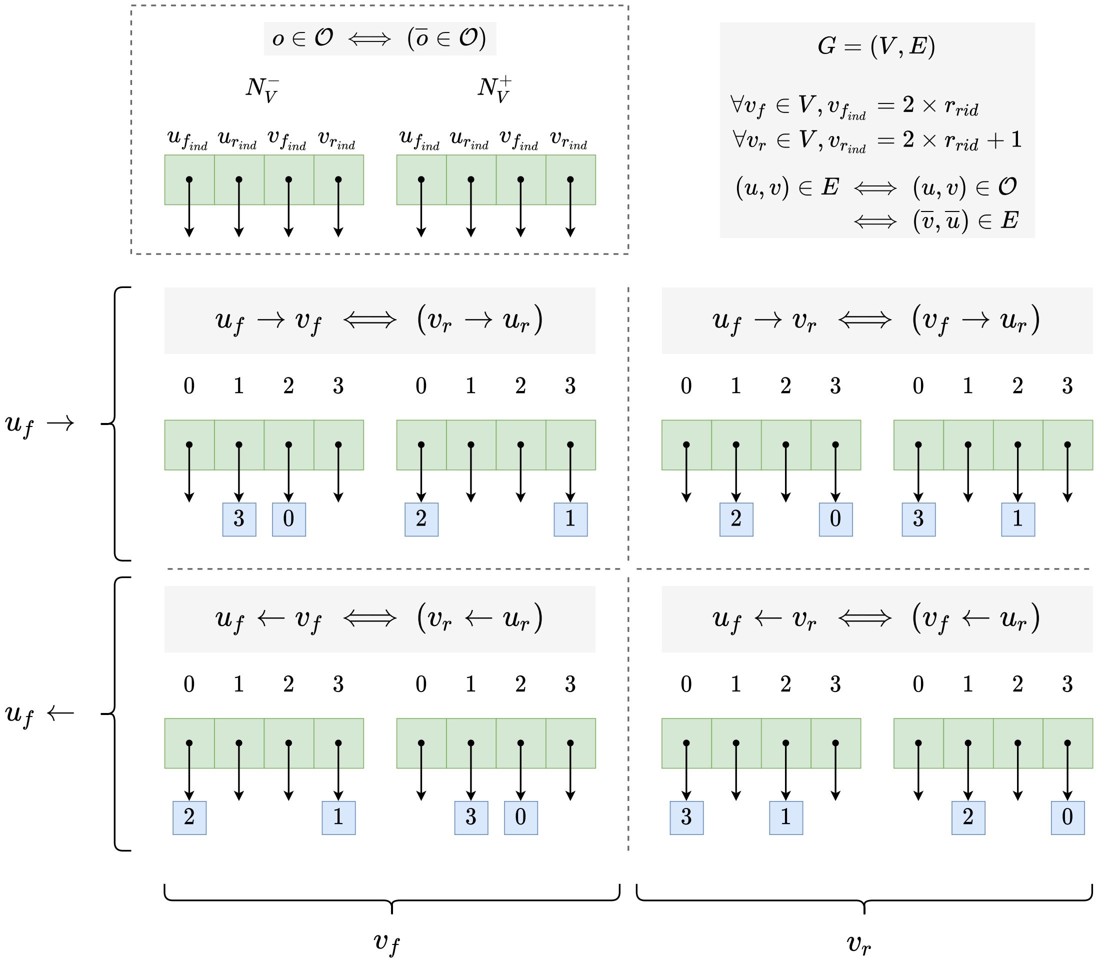


Figure 10 – All oriented fragments directed graph implementation.
At the top left, the dot lined box corresponds to the legend for each of the four overlap cases. The grey mathematical formula above provides the overlap and its reverse symmetric overlap, under parenthesis, for the illustrated case. Under, there are the two adjacency lists (the first one contains the predecessors, the second one contains the successors): they both contain the indices of the predecessors/successors. At the top right, the grey box gives some graph properties.

▷ Proof

proof here

□

3.1.2 Only Oriented Fragments' Successors Directed Graph (DGS)

DGA implementation suffer from redundancies in the data because it does not take benefits from the reverse symmetry. Only Oriented Fragments' Successors Directed Graph (DGS) implementation takes benefits from the symmetry hence while the structure follows the one of DGA, only the successor list is kept. Definition 3.3 details how it is built in memory.

add edge index

▶ **Definition 3.3: DGS implementation**

DGS graph implementation is built by the successor list N_V^+ such that $\forall v \in V, \forall w \in N_v^+$, w_{ind} is the index of a successor of vertex v. Also, for each successor w of vertex v

Figure 11 illustrates the impact of each overlap type on DGS implementation. For clarity's sake, only the indices of the successors are written, but not the edge indices.

Proposition 3.2 gives the amount of octet DGS implementation consumes.

▶ **Proposition 3.2: DGS memory consumption**

The memory size of the graph $Mem(DGS)$ (in octets) is equals to:

$$\begin{aligned} Mem(DGS) &= (2 \times |\mathcal{R}_{aw}| + 1) \times P \\ &+ |\mathcal{O}| \times \left(\left\lceil \frac{1 + \log_2 |\mathcal{R}_{aw}|}{8} \right\rceil + \left\lceil \frac{\log_2 |\mathcal{O}|}{8} \right\rceil \right) \end{aligned}$$

where P is the memory size of a memory address.

▷ Proof

For each vertex there is a pointer to its successor list, hence $2 \times |\mathcal{R}_{aw}|$ *pointers, and there is one pointer to the vertices' successor list. Then, for each successor, its index is provided (*$\lceil \log_2 2 \times |\mathcal{R}_{aw}| \rceil$ *bits), and the edge index is provided too (*$\lceil \log_2 |\mathcal{O}| \rceil$ *bits). These numbers are divided by eight and rounded up to the nearest integer to get the number of octets. Then the sum is multiplied by the number of overlaps (*$|\mathcal{O}|$*).*

□

verify if same with commented

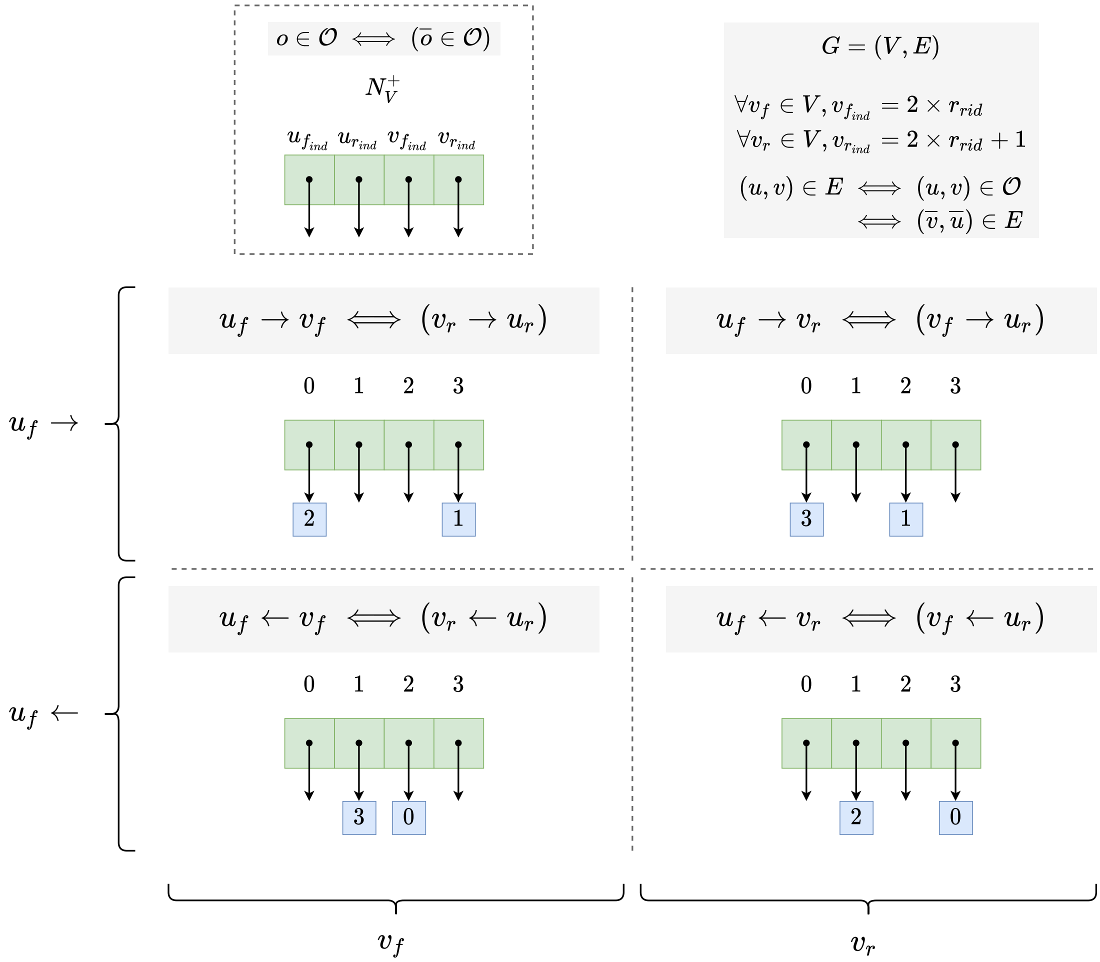


Figure 11 – Only oriented fragments successors directed graph implementation.
At the top left, the dot lined square corresponds to the legend for each of the four overlap cases. The grey mathematical formula above provides the overlap and its reverse symmetric overlap, under parenthesis, for the illustrated case. Under, there is the successor list: it contains the indices of the successors. At the top right, the grey square gives some graph properties.

Proposition 3.3 answer the issue of getting the predecessors.

▶ **Proposition 3.3: DGS reverse symmetry to retrieve predecessors**

For each vertex $v \in V$:

i. its predecessors are the reverse of its reverse' successors

ii. the edge index of $(u, v) \in E$ is the reverse of this of $(\overline{v}, \overline{u}) \in E$

▷ Proof

Let $v \in V$ be a vertex. Let $u \in V$ such that $(u, v) \in E$. Thus:

i. $(\overline{v}, \overline{u}) \in E$, hence $\overline{u}$ is the successor of $\overline{v}$ that is the reverse of v. Finally, $\overline{\overline{u}} = u$, that is the predecessor of v.

ii. If $\overline{e}_{ind}$ is the edge index of edge $(\overline{v}, \overline{u})$, then $e_{ind} = \overline{e}_{ind} + c$ (where $c = 1$ if $\overline{e}_{ind}$ is even, else $c = -1$) corresponds to the edge index of the reverse of $(\overline{v}, \overline{u})$ hence the edge index of (u, v).

□

3.1.3 Forward Fragments Directed Graph (DGF)

As for DGS implementation, the last implementation of the directed graph structure, Forward Fragments Directed Graph (DGF), takes benefits from the reverse symmetry. DGF focuses only on the neighbours of the vertices that represent the reads in forward orientation (hence the reads in raw reads set $\mathcal{R}_{aw}$). Therefore, the vertices indices set is not build as previously. Hence for each read $r \in \mathcal{R}_{aw}$, let $v_{ind} = r_{rid}$ be the index of the vertex that represents it. Because an overlap involves two oriented reads, the neighbours' orientation must be included to the adjacency lists. Hence for each neighbour $v \in V$ in the predecessors/successor lists of a forward vertex, let $v_{indor} = (v_{ind, v_{or}})$ be its oriented neighbour. Definition 3.4 describes the different sets, and Definition 3.5 details the DGF implementation. For clarity's sake, only the indices of the predecessors/successors are written, but not the edge indices.

▶ Definition 3.4: Graph indices set for DGA

Let $Vind = [\![0; |\mathcal{R}_{aw}|[\![$ be the set of indices for the vertices:

— $|Vind| = |\mathcal{R}_{aw}|$

— $\forall v_{ind} \in Vind, v_{ind} = v_{rid}$, where v_{rid} corresponds to the read's identifier

Let $Vindor = [\![0; |\mathcal{R}_{aw}|[\![\times \{0; 1\}$ be the set of indices enriched by a boolean value for the oriented vertices:

— $|Vindor| = 2 \times |\mathcal{R}_{aw}|$

— $\forall v_{indor} \in Vindor, v_{indor} = (v_{rid}, v_{or})$, where v_{rid} corresponds to the read's identifier and v_{or} to its orientation

Let $Eind = [\![0; |\mathcal{O}|[\![$ be the set of indices for the edges:

— $|Eind| = |\mathcal{O}|$

— $\forall e_{ind} \in Eind$, e_{ind} corresponds to the index of edge e, such that $\left|e_{ind} - \overline{e}_{ind}\right| = 1$

add edge index

▶ **Definition 3.5: DGF implementation**

DGF graph implementation is built by:

Predecessor list for the forward $N^-_{V_f}$ $\forall v \in V_f, \forall u \in N^-_v$, u_{ind} is the index of a predecessor of vertex v, and u_{or} gives its orientation

Successor list for the forward $N^+_{V_f}$ $\forall v \in V_f, \forall w \in N^+_v$, w_{ind} is the index of a successor of vertex v, and w_{or} gives its orientation

▶ **Proposition 3.4: DGF memory consumption**

The memory size of the graph $Mem(DGF)$ (in octets) is equals to:

$$\begin{aligned} Mem(DGF) &= 2 \times (|\mathcal{R}_{aw}| + 1) \times P \\ &\quad + |\mathcal{O}| \times \left(\left\lceil \frac{1 + \log_2 |\mathcal{R}_{aw}|}{8} \right\rceil + \left\lceil \frac{\log_2 |\mathcal{O}|}{8} \right\rceil \right) \end{aligned}$$

where P is the memory size of a memory address.

▷ Proof

For each (forward) vertex there is a pointer to its predecessor and successor lists, hence $2 \times |\mathcal{R}_{aw}|$ pointers, and there is one pointer to the vertices' predecessor and successor lists (hence $+2$). Then, for each predecessor/successor, its index and a boolean orientation value are provided ($\lceil \log_2 2 \times |\mathcal{R}_{aw}| \rceil$ bits), and the edge index is provided too ($\lceil \log_2 |\mathcal{O}| \rceil$ bits). These numbers are divided by eight and rounded up to the nearest integer to get the number of

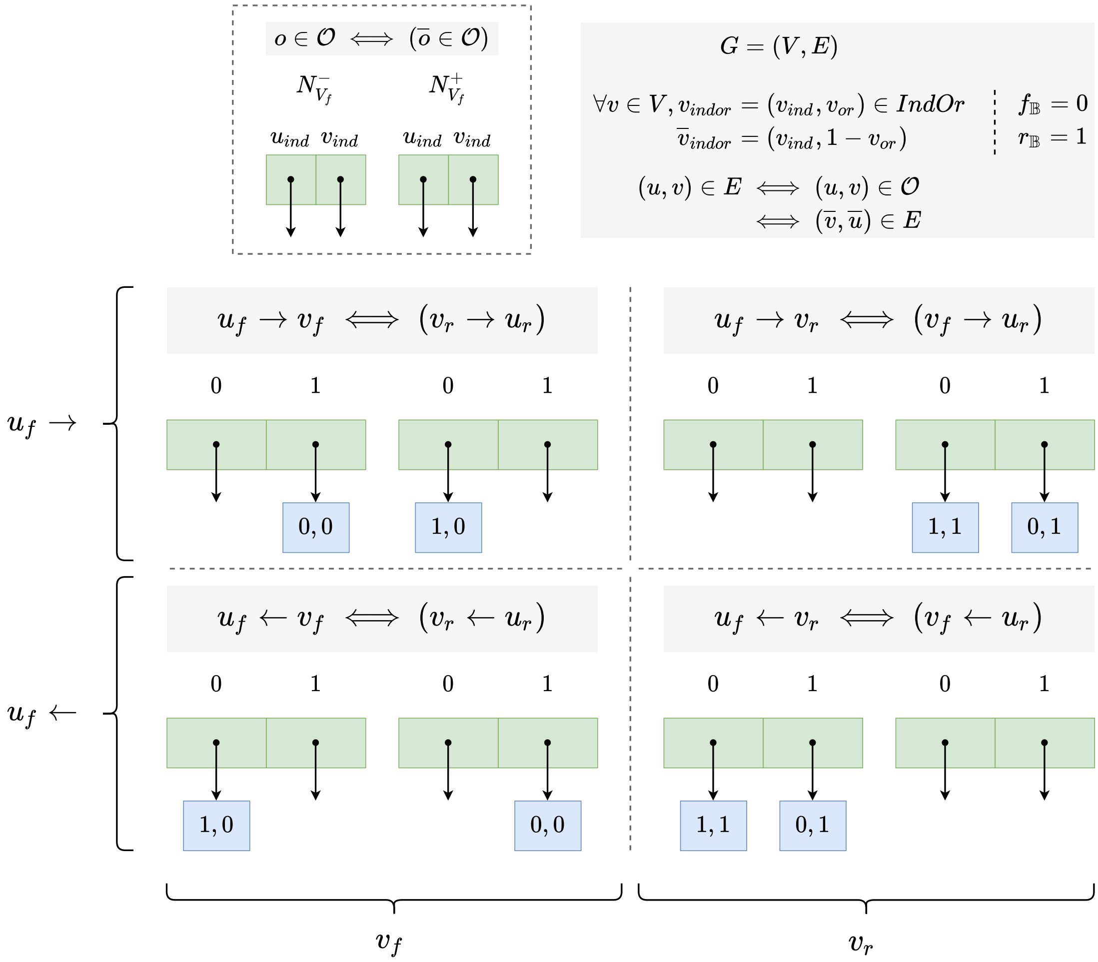


Figure 12 – Forward fragments directed graph implementation.
At the top left, the dot lined square corresponds to the legend for each of the four overlap cases. The grey mathematical formula above provides the overlap and its reverse symmetric overlap, under parenthesis, for the illustrated case. Under, there are the two adjacency lists for the forward (the first one contains the predecessors, the second one contains the successors): they both contain the couples index-orientation of the predecessors/successors. At the top right, the grey square gives some graph properties.

octets. Then the sum is multiplied by the number of overlaps $(|\mathcal{O}|)$. □

verify if same with commented

Note that in contrast to Proposition 3.2, the memory consumption of adjacency lists pointers is equals to $2 \times (|\mathcal{R}_{aw}| + 1)$ (versus $(2 \times |\mathcal{R}_{aw}| + 1)$). In fact, the extra P is due to an extra adjacency list.

Proposition 3.5 answer the issue of getting the predecessors/successors of the reverse.

▶ Proposition 3.5: DGF reverse symmetry to retrieve predecessors

For each vertex $v \in V_r$:

i. its predecessors are the reverse of its reverse' successors

ii. its successors are the reverse of its reverse' predecessors

iii. the edge index of $(u, v) \in E$ is the reverse of this of $(\overline{v}, \overline{u}) \in E$

▷ Proof

Let $v \in V_[r]$ be a vertex that represents a reverse read.

— *Let $u \in V$ such that $(u, v) \in E$. Thus:*

  i. *$(\overline{v}, \overline{u}) \in E$, hence $\overline{u}$ is the successor of $\overline{v}$ that is the reverse of v. Finally, $\overline{\overline{u}} = u$, that is the predecessor of v.*

  ii. *If $\overline{e}_{ind}$ is the edge index of edge $(\overline{v}, \overline{u})$, then $e_{ind} = \overline{e}_{ind} + c$ (where $c = 1$ if $\overline{e}_{ind}$ is even, else $c = -1$) corresponds to the edge index of the reverse of $(\overline{v}, \overline{u})$ hence the edge index of (u, v).*

— *Let $w \in V$ such that $(v, w) \in E$. Thus:*

  i. *$(\overline{w}, \overline{v}) \in E$, hence $\overline{w}$ is the predecessor of $\overline{v}$ that is the reverse of v. Finally, $\overline{\overline{w}} = w$, that is the successor of v.*

  ii. *If $\overline{e}_{ind}$ is the edge index of edge $(\overline{w}, \overline{v})$, then $e_{ind} = \overline{e}_{ind} + c$ (where $c = 1$ if $\overline{e}_{ind}$ is even, else $c = -1$) corresponds to the edge index of the reverse of $(\overline{w}, \overline{v})$ hence the edge index of (v, w).*

□

3.2 Bi-directed Graph: Oriented Walk Based

3.2.1 Forward Fragments Bi-directed Graph (BG)

In the bi-directed graph, there is one vertex for each raw read in $\mathcal{R}_{aw}$. Hence for each vertex v is associated an index $v_{ind} = v_{rid}$. An overlap and its reverse are represented by only one edge that carries strictly necessary overlap attributes, as described in Definition 2.4. Remind that the bi-directed graph is equivalent to a undirected graph: that's implemented by Unoriented Fragments Bi-directed Graph (BG) implementation. Definition 3.4 describes the different sets, and Definition 3.5 details the DGF implementation. For clarity's sake, only the indices of the predecessors/successors are written, but not the edge indices.

▶ Definition 3.6: Graph indices set for BG

Let $Vind = [\![0; |\mathcal{R}_{aw}|[\![$ be the set of indices for the vertices:

— $|Vind| = |\mathcal{R}_{aw}|$

— $\forall v_{ind} \in Vind, v_{ind} = v_{rid}$, where v_{rid} corresponds to the read's identifier

Let $Eind = \left[\!\left[0; \frac{1}{2} \times |\mathcal{O}|\right[\!\right[$ be the set of indices for the edges:

— $|Eind| = \frac{1}{2} \times |\mathcal{O}|$

— $\forall e_{ind} \in Eind$, e_{ind} corresponds to the index of edge e

put paragraph in definition

Concerning the edge indices e_{ind}, it is possible to use their euclidean quotient by two e'_{ind} in memory and retrieving them thanks to the orientation of the fewer vertex index (let say u). In fact, if the considered overlap uses u_f, then $e_{ind} = 2 \times e'_{ind}$, else $e_{ind} = 2 \times e'_{ind} + 1$

▶ Definition 3.7: BG implementation

BG graph implementation is built by:

Neighbours list for the forward N_V $\forall v \in V, \forall w \in N_v$, w_{ind} is the index of a neighbour of vertex v

Edges attributes for each neighbour w of a vertex v two boolean values are provided: or_{vw} and rel_{vw} (see Definition 2.4)

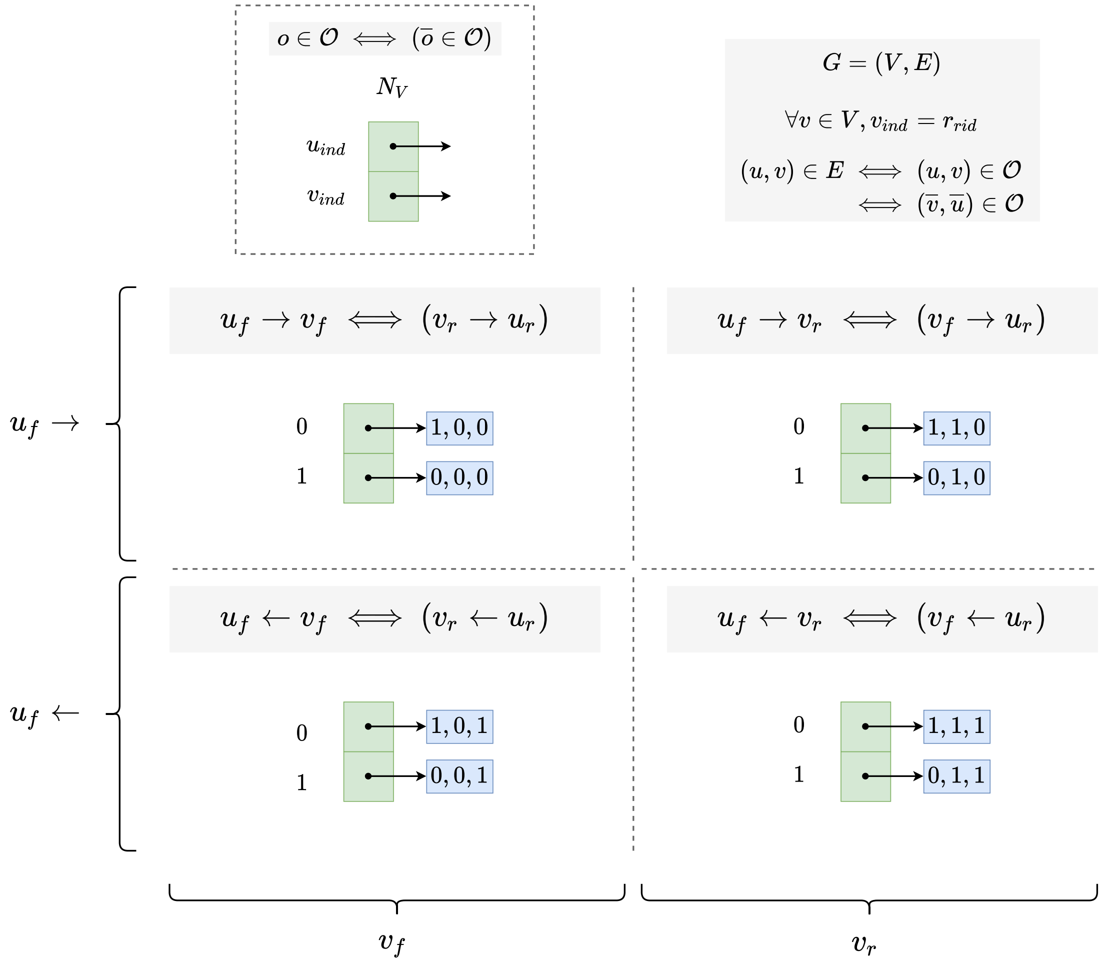


Figure 13 – caption title All four overlap cases structured in the bi-directed graph.
At the top, the dot lined square corresponds to the legend for the four overlap cases. The mathematical formula above the line is the overlap such that u is in forward orientation, and the reverse symmetric overlap is under parenthesis. Under the line, there is the one list that contains, in order, the index of the neighbour, the value of or_{uv} and the value of rel_{uv}.

▶ Proposition 3.6: BG memory consumption

The memory size of the graph $Mem(BG)$ (in octets) equals:

$$\begin{aligned} Mem(BG) = {} & (|\mathcal{R}_{aw}| + 2) \times P \\ & + |\mathcal{O}| \times \left(\left\lceil \frac{\log_2 |\mathcal{R}_{aw}|}{8} \right\rceil + \left\lceil \frac{\log_2 |\mathcal{O}| - 1}{8} \right\rceil \right) \\ & + \left\lceil \frac{|\mathcal{O}|}{8} \right\rceil \end{aligned}$$

where P is the memory size of a memory address.

▷ Proof

For each (unoriented) vertex there is a pointer to its neighbour list, hence $|\mathcal{R}_{aw}|$ *pointers, and there is one pointer to the vertices' neighbour list and one to the edges attribute list (hence* $+2$*). Then, for each neighbour, its index is provided (*$\lceil \log_2 |\mathcal{R}_{aw}| \rceil$ *bits), and the edge index is provided too (*$\left\lceil \log_2 \left(\frac{1}{2} \times |\mathcal{O}|\right) \right\rceil$ *bits). These numbers are divided by eight and rounded up to the nearest integer to get the number of octets. Then the sum is multiplied by the number of overlaps (*$|\mathcal{O}|$*). Finally, for each edge index (*$\frac{1}{2} \times |\mathcal{O}|$*), there are two boolean edge attributes.* $\frac{1}{2} \times |\mathcal{O}| \times 2$ *is divided by eight and rounded up to get the number of octets.*

□

verify if same as comments

The four combinations of edges' attributes *or* and *rel* let representing all the overlap cases and their reverse without any redundancy. To verify the reverse symmetry, it is sufficient (and easier) to choose an orientation of u and calculate the overlap case with the edges' attributes.

3.2.2 Forward Fragments Directed Graph (DGF)

description

ref figure from digraph implementation revsymg

memory

3.3 Undirected Graph: Tail-Head Fragments Based

3.3.1 All Oriented Fragments Undirected Graph (UGA)

fix for ungraph

For each read, there are two vertices in the directed graph. Remind that each read identifier r_{rid} correspond to a unique integer identifier (it is an index, see Definition 1.1). Thus, for each read $r \in \mathcal{R}_{aw}$ there are two vertices $v_f, v_r \in V$ such that the index of v_f is equal to $2 \times r_{rid}$ and the index of v_r is equal to $2 \times r_{rid} + 1$.

It is possible now to build the adjacency lists (one list for the predecessors, and another one for the successors).

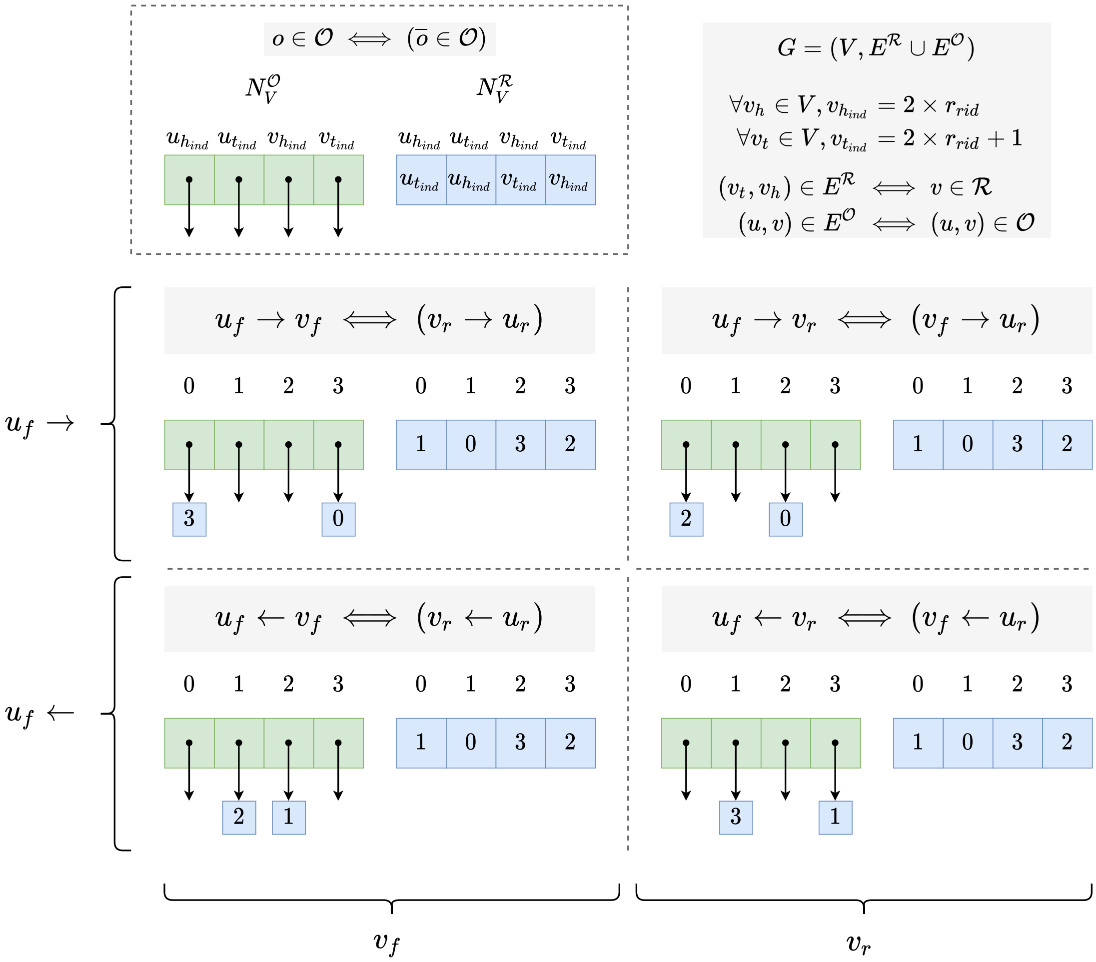


Figure 14 – caption title All four overlap cases structured in the directed graph.

At the top left, the dot lined square corresponds to the legend for the four overlap cases. The grey mathematical formula above is the overlap such that u is in forward orientation, and the reverse symmetric overlap is under parenthesis. Under, there are the two adjacency lists (the first one contains the neighbours for overlap-edges, the second one contains the neighbours for read-edge): they both contain the indices of the neighbours. Top right grey square gives some graph properties.

memory

3.3.2 Read-edges Jump Directed Graph (DGS)

description

Cref figure only oriented frag succs digraph

memory

remarque que l'on peut arriver visuellement au digraph

4 Memory and Time Costs

4.1 Algorithms

4.1.1 Subfunctions

▶ Algorithm 1: Reverse operation on index

Require: Vertex/edge index ind.
Ensure: Returns index of the reverse vertex/edge.

```
function REV(ind)
    if ind | 2 then
        return ind + 1
    return ind − 1
```

For clarity's sake, $\text{REV}(ind) = \overline{ind}$.

▶ Algorithm 2: Find place of edge index in the neighbour list. Cost: worst $3 \times |list|$, best 3, average $3 \times \left\lceil \frac{|list|}{2} \right\rceil$

Require: List of tuple $list$, edge index e_{ind}, index t_{ind} of the place of edge index in the tuple contained in $list$. The edge index must be in a tuple.
Ensure: Returns the index of the tuple containing e_{ind} in $list$.

```
function GET_INDEX(list, e_ind, t_ind)
    list_ind ← 0
```

while $list[list_{ind}][t_{ind}] \neq e_{ind}$ and $list_{ind} < |list| - 1$ **do**
    $list_{ind} \leftarrow list_{ind} + 1$
return $list_{ind}$

4.1.2 Iterating Over the Predecessors

▶ Algorithm 3: Iterate over the predecessors for DGS

Require: Graph $G = (V, E)$, vertex $v \in V$, empty list $preds$.
Ensure: The returned list contains the predecessors of v in graph G and the corresponding edge index.

function DGS_PREDS($G, v, preds$)
    for $(u, e_{ind}) \in N^+_{\overline{v}}$ **do**
        $preds$.APPEND($\overline{u}, \overline{e_{ind}}$)
    return $preds$

▶ Algorithm 4: Iterate over the predecessors for DGF

Require: Graph $G = (V, E)$, vertex $v \in V$, orientation of v $v_{or} \in \{0; 1\}$, empty list $preds$.
Ensure: The returned list contains the predecessors of v in graph G and the corresponding edge index.

function DGF_PREDS($G, v, v_{or}, preds$)
    if $v_{or} = 0$ **then** ▷ v *forward*
        for $(u, e_{ind}) \in N^-_v$ **do**
            $preds$.APPEND(u, e_{ind})
    else
        for $(u, u_{or}, e_{ind}) \in N^+_{\overline{v}}$ **do**
            $preds$.APPEND($u, 1 - u_{or}, \overline{e_{ind}}$)
    return $preds$

▶ Algorithm 5: Iterate over the predecessors for BGU

Require: Graph $G = (V, E)$, vertex $v \in V$, orientation of v $v_{or} \in \{0; 1\}$, empty list $preds$.
Ensure: The returned list contains the predecessors of v in graph G and the corresponding edge index.

function BGU_PREDS($G, v, v_{or}, preds$)
  $orientations \leftarrow (0, 1)$ ▷ *forward/reverse orientations*
  for $(u, e'_{ind}) \in N_v$ **do**
    if $v < u$ **then**
      if $v_{or} = 0$ **then**
        if $rel_{uv} = 1$ **then** ▷ $u_f \to v_f$ or $u_r \to v_f$
          $preds$.APPEND($u, orientations[or_{uv}], 2 \times e'_{ind}$)
      else if $rel_{uv} = 0$ **then** ▷ $u_f \to v_r$ or $u_r \to v_r$
        $preds$.APPEND($u, orientations[1 - or_{uv}], 2 \times e'_{ind} + 1$)
    else if $v_{or} = 0$ **then**
      if $or_{uv} = 0$ **then**
        if $rel_{uv} = 0$ **then** ▷ $u_f \to v_f$
          $preds$.APPEND($u, 0, 2 \times e'_{ind}$)
      else if $rel_{uv} = 1$ **then** ▷ $u_r \to v_f$
        $preds$.APPEND($u, 1, 2 \times e'_{ind} + 1$)
    else if $or_{uv} = 0$ **then**
      if $rel_{uv} = 1$ **then** ▷ $u_r \to v_r$
        $preds$.APPEND($u, 1, 2 \times e'_{ind} + 1$)
    else if $rel_{uv} = 0$ **then** ▷ $u_f \to v_r$
      $preds$.APPEND($u, 0, 2 \times e'_{ind}$)
  return $preds$

4.1.3 Iterating Over the Successors

▶ Algorithm 6: Iterate over the successors for DGS

Require: Graph $G = (V, E)$, vertex $v \in V$, empty list $succs$.
Ensure: The returned list contains the successors of v in graph G.

function DGS_SUCCS($G, v, succs$)
  for $(w, e_{ind}) \in N_v^+$ **do**
    $succs$.APPEND(w, e_{ind})
  return $succs$

▶ Algorithm 7: Iterate over the successors for DGF

Require: Graph $G = (V, E)$, vertex $v \in V$, orientation of v $v_{or} \in \{0; 1\}$, empty list $succs$.
Ensure: The returned list contains the successors of v in graph G.

function $\text{DGF\_SUCCS}(G, v, v_{or}, succs)$
  if $v_{or} = 0$ **then** ▷ *v forward*
    for $(w, e_{ind}) \in N_v^+$ **do**
      $succs.\text{APPEND}(w, e_{ind})$
  else
    for $(w, w_{or}, e_{ind}) \in N_{\overline{v}}^-$ **do**
      $succs.\text{APPEND}(w, 1 - w_{or}, \overline{e_{ind}})$
  return $succs$

▶ Algorithm 8: Iterate over the successors for BGU

Require: Graph $G = (V, E)$, vertex $v \in V$, orientation of v $v_{or} \in \{0; 1\}$, empty list $succs$.
Ensure: The returned list contains the successors of v in graph G.

function $\text{BGU\_SUCCS}(G, v, v_{or}, succs)$
  $orientations \leftarrow (0, 1)$ ▷ *forward/reverse orientations*
  for $(w, e'_{ind}) \in N_v$ **do**
    if $v < w$ **then**
      if $v_{or} = 0$ **then**
        if $rel_{vw} = 0$ **then** ▷ $v_f \to w_f$ *or* $v_f \to w_r$
          $succs.\text{APPEND}(w, orientations[or_{vw}], 2 \times e'_{ind})$
      else if $rel_{uv} = 1$ **then** ▷ $v_r \to w_f$ *or* $v_r \to w_r$
        $succs.\text{APPEND}(u, orientations[1 - or_{uv}], 2 \times e'_{ind} + 1)$
    else if $v_{or} = 0$ **then**
      if $or_{uv} = 0$ **then**
        if $rel_{uv} = 1$ **then** ▷ $v_f \to w_f$
          $succs.\text{APPEND}(w, 0, 2 \times e'_{ind})$
      else if $rel_{uv} = 0$ **then** ▷ $v_f \to w_r$
        $succs.\text{APPEND}(w, 1, 2 \times e'_{ind} + 1)$
    else if $or_{uv} = 0$ **then**
      if $rel_{uv} = 0$ **then** ▷ $v_r \to w_r$
        $succs.\text{APPEND}(w, 1, 2 \times e'_{ind} + 1)$
    else if $rel_{uv} = 1$ **then** ▷ $v_r \to w_f$
      $succs.\text{APPEND}(w, 0, 2 \times e'_{ind})$
  return $succs$

4.1.4 Adding a Vertex

▶ Algorithm 9: Add a new vertex in the graph for DGS

Require: Graph $G = (V, E)$.
Ensure: Returns the new vertex' index (in forward orientation if there is the choice).

function $\text{DGS\_ADD\_VERTEX}(G)$
  $N_V^+.\text{APPEND}(\text{EMPTY\_LIST}(\ ))$
  $N_V^+.\text{APPEND}(\text{EMPTY\_LIST}(\ ))$
  return $\left|N_V^+\right| - 2$

▶ Algorithm 10: Add a new vertex in the graph for DGF

Require: Graph $G = (V, E)$.
Ensure: Returns the new vertex' index (in forward orientation if there is the choice).

function $\text{DGF\_ADD\_VERTEX}(G)$
  $N_V^-.\text{APPEND}(\text{EMPTY\_LIST}(\ ))$
  $N_V^+.\text{APPEND}(\text{EMPTY\_LIST}(\ ))$
  return $\left|N_V^+\right| - 1$

▶ Algorithm 11: Add a new vertex in the graph for BGU

Require: Graph $G = (V, E)$.
Ensure: Returns the new vertex' index (in forward orientation if there is the choice).

function $\text{BGU\_ADD\_VERTEX}(G)$
  $N_V.\text{APPEND}(\text{EMPTY\_LIST}(\ ))$
  return $|N_V| - 1$

4.1.5 Adding an Edge

▶ Algorithm 12: Add a new edge and its reverse in the graph for DGS

Require: Graph $G = (V, E)$, $(u, v) \in V^2$. Note that the vertices are already in the graph.
Ensure: Returns the new edge's index (in forward orientation if there is the choice).

function $\mathrm{DGS\_ADD\_EDGE}(G, u, v)$
  ▷ *ind_edges is the number of edges. It is always even.* ◁
  $N_u^+.\mathrm{APPEND}(v, ind\_edges)$
  $N_{\overline{v}}^+.\mathrm{APPEND}(\overline{u}, ind\_edges + 1)$
  $ind\_edges \leftarrow ind\_edges + 2$
  $card\_edges \leftarrow card\_edges + 2$
  return $ind\_edges - 2$

▶ Algorithm 13: Add a new edge and its reverse in the graph for DGF

Require: Graph $G = (V, E)$, $(u, v) \in V^2$, with their orientation $(u_{or}, v_{or}) \in \{0; 1\}^2$. Note that the vertices are already in the graph.
Ensure: Returns the new edge's index (in forward orientation if there is the choice).

function $\mathrm{DGF\_ADD\_EDGE}(G, u, u_{or}, v, v_{or})$
  ▷ *ind_edges is the number of edges. It is always even.* ◁
  if $u_{or} = 0$ **then** ▷ $u_f \rightarrow v_f$ *or* $u_f \rightarrow v_r$
    $N_u^+.\mathrm{APPEND}(v, v_{or}, ind\_edges)$
  else ▷ $u_f \leftarrow v_f$ *or* $u_f \leftarrow v_r$
    $N_u^-.\mathrm{APPEND}(v, 1 - v_{or}, ind\_edges + 1)$
  if $v_{or} = 0$ **then** ▷ $u_f \rightarrow v_f$ *or* $u_r \rightarrow v_f$
    $N_v^-.\mathrm{APPEND}(u, u_{or}, ind\_edges)$
  else ▷ $u_f \leftarrow v_f$ *or* $u_r \leftarrow v_f$
    $N_v^+.\mathrm{APPEND}(u, 1 - u_{or}, ind\_edges + 1)$
  $ind\_edges \leftarrow ind\_edges + 2$
  $card\_edges \leftarrow card\_edges + 2$
  return $ind\_edges - 2$

▶ Algorithm 14: Add a new edge and its reverse in the graph for BGU

Require: Graph $G = (V, E)$, $(u, v) \in V^2$, with their orientation $(u_{or}, v_{or}) \in \{0; 1\}^2$. Note that the vertices are already in the graph.
Ensure: Returns the new edge's index (in forward orientation if there is the choice).

function $\text{BGU\_ADD\_EDGE}(G, u, u_{or}, v, v_{or})$
  ▷ *ind_edges is the number of edges. E_{attr} is a list of edges attributes.* ◁
  $or \leftarrow |u_{or} - v_{or}|$
  if $u < v$ **then**
    $rel \leftarrow u_{or}$
  else
    $rel \leftarrow 1 - v_{or}$
  $N_u.\text{APPEND}(v, ind\_edges)$
  $N_v.\text{APPEND}(u, ind\_edges)$
  $E_{attr}.\text{APPEND}(or, rel)$
  $ind\_edges \leftarrow ind\_edges + 1$
  $card\_edges \leftarrow card\_edges + 1$
  return $2 \times (ind\_edges - 1)$

4.1.6 Deleting a Vertex

▶ Algorithm 15: Delete a vertex for DGS

Require: Graph $G = (V, E)$, vertex $v \in V$ (in forward orientation).
Ensure: Delete the vertex and its reverse such that $V' = V \setminus' \{v, \overline{v}\}$, and $\forall v \in V', 0 \leq v_{ind} \leq |V'| - 2$.

function $\text{DGS\_DELETE\_VERTEX}(G, v)$
  ▷ *Delete it from its predecessors* ◁
  $v\_rev \leftarrow v + 1$
  for $(u, e_{ind}) \in N^+_{v\_rev}$ **do**
    $u\_rev \leftarrow \overline{u}$
    $adj\_ind \leftarrow \text{GET\_INDEX}(N^+_{u\_rev}, \overline{e_{ind}}, 1)$
    if $adj\_ind = \left|N^+_{u\_rev}\right| - 1$ **then**
      ▷ *Just delete the last element* ◁
      $N^+_{u\_rev}.\text{POP}(\ )$
    else
      ▷ *Replace the edge by the last in the neighbour list* ◁

```
            N+_u_rev[adj_ind] ← N+_u_rev.POP( )
        card_edges ← card_edges − 2
    DELETE(N+_v_rev)
    ▷ Delete it from its successors ◁
    for (w, e_ind) ∈ N+_v do
        w_rev ← w̄
        adj_ind ← GET_INDEX(N+_w_rev, ē_ind, 1)
        if adj_ind = |N+_w_rev| − 1 then
            ▷ Just delete the last element ◁
            N+_w_rev.POP( )
        else
            ▷ Replace the edge by the last in the neighbour list ◁
            N+_w_rev[adj_ind] ← N+_w_rev.POP( )
        card_edges ← card_edges − 2
    DELETE(N+_v)
    ▷ Delete the whole vertex ◁
    if v = |N+_V| − 2 then
        ▷ It is the last, just pop it and its reverse ◁
        N+_V.POP( )
        N+_V.POP( )
    else
        ▷ Replace it and its reverse by the last and its reverse ◁
        N+_v_rev ← N+_V.POP( )
        N+_v ← N+_V.POP( )
        ▷ Update the vertex index ◁
        for (w, e_ind) ∈ N+_v do
            w_rev ← w̄
            e_ind_rev ← ē_ind
            adj_ind ← GET_INDEX(N+_w_rev, e_ind_rev, 1)
            N+_w_rev[adj_ind] ← (v_rev, e_ind_rev)
        for (w, e_ind) ∈ N+_v_rev do
            w_rev ← w̄
            e_ind_rev ← ē_ind
            adj_ind ← GET_INDEX(N+_w_rev, e_ind_rev, 1)
            N+_w_rev[adj_ind] ← (v, e_ind_rev)
```

▶ Algorithm 16: Delete a vertex for DGF

Require: Graph $G = (V, E)$, vertex $v \in V$ (in forward orientation).
Ensure: Delete the vertex and its reverse such that $V' = V \setminus' \{v, \overline{v}\}$, and $\forall v \in V', 0 \le v_{ind} \le |V'| - 2$.

function $\text{DGF\_DELETE\_VERTEX}(G, v)$
  ▷ *Delete it from its predecessors* ◁
  for $(u, u_{or}, e_{ind}) \in N_v^-$ **do**
    if $u_{or} = 0$ **then**
      $adj\_ind \leftarrow \text{GET\_INDEX}(N_u^+, e_{ind}, 1)$
      if $adj\_ind = |N_u^+| - 1$ **then**
        ▷ *Just delete the last element* ◁
        $N_u^+.\text{POP}(\,)$
      else
        ▷ *Replace the edge by the last in the neighbour list* ◁
        $N_u^+[adj\_ind] \leftarrow N_u^+.\text{POP}(\,)$
    else
      $adj\_ind \leftarrow \text{GET\_INDEX}(N_u^-, \overline{e_{ind}}, 1)$
      if $adj\_ind = |N_u^-| - 1$ **then**
        ▷ *Just delete the last element* ◁
        $N_u^-.\text{POP}(\,)$
      else
        ▷ *Replace the edge by the last in the neighbour list* ◁
        $N_u^-[adj\_ind] \leftarrow N_u^-.\text{POP}(\,)$
    $card\_edges \leftarrow card\_edges - 2$
  $\text{DELETE}(N_v^-)$
  ▷ *Delete it from its successors* ◁
  for $(w, w_{or}, e_{ind}) \in N_v^+$ **do**
    if $w_{or} = 0$ **then**
      $adj\_ind \leftarrow \text{GET\_INDEX}(N_w^-, e_{ind}, 1)$
      if $adj\_ind = |N_w^-| - 1$ **then**
        ▷ *Just delete the last element* ◁
        $N_w^-.\text{POP}(\,)$
      else
        ▷ *Replace the edge by the last in the neighbour list* ◁
        $N_w^-[adj\_ind] \leftarrow N_w^-.\text{POP}(\,)$
    else
      $adj\_ind \leftarrow \text{GET\_INDEX}(N_w^+, \overline{e_{ind}}, 1)$
      if $adj\_ind = |N_w^+| - 1$ **then**
        ▷ *Just delete the last element* ◁

N_w^+.POP( )
else
▷ *Replace the edge by the last in the neighbour list* ◁
$N_w^+[adj\_ind] \leftarrow N_w^+$.POP( )
$card\_edges \leftarrow card\_edges - 2$
DELETE(N_v^+)
▷ *Delete the whole vertex* ◁
if $v = \left|N_V^+\right| - 1$ **then**
▷ *It is the last, just pop it and its reverse* ◁
N_V^-.POP( )
N_V^+.POP( )
else
▷ *Replace it and its reverse by the last and its reverse* ◁
$N_v^- \leftarrow N_V^-$.POP( )
$N_v^+ \leftarrow N_V^+$.POP( )
▷ *Update the vertex index* ◁
for $(u, u_{or}, e_{ind}) \in N_v^-$ **do**
if $u_{or} = 0$ **then**
$adj\_ind \leftarrow$ GET_INDEX($N_u^+, e_{ind}, 1$)
$N_u^+[adj\_ind] \leftarrow (v, 0, e_{ind})$
else
$u\_rev \leftarrow \overline{u}$
$e_{ind}\_rev \leftarrow \overline{e_{ind}}$
$adj\_ind \leftarrow$ GET_INDEX($N_{u\_rev}^-, e_{ind}\_rev, 1$)
$N_{u\_rev}^-[adj\_ind] \leftarrow (v, 1, e_{ind}\_rev)$
for $(w, w_{or}, e_{ind}) \in N_v^+$ **do**
if $w_{or} = 0$ **then**
$adj\_ind \leftarrow$ GET_INDEX($N_w^-, e_{ind}, 1$)
$N_w^-[adj\_ind] \leftarrow (v, 0, e_{ind})$
else
$w\_rev \leftarrow \overline{w}$
$e_{ind}\_rev \leftarrow \overline{e_{ind}}$
$adj\_ind \leftarrow$ GET_INDEX($N_{w\_rev}^+, e_{ind}\_rev, 1$)
$N_{w\_rev}^+[adj\_ind] \leftarrow (v, 1, e_{ind}\_rev)$

▶ Algorithm 17: Delete a vertex for BGU

Require: Graph $G = (V, E)$, vertex $v \in V$ (in forward orientation).

Ensure: Delete the vertex and its reverse such that $V' = V \setminus' \{v, \overline{v}\}$, and $\forall v \in V', 0 \leq v_{ind} \leq |V'| - 2$.

function $\text{BGU\_DELETE\_VERTEX}(G, v)$
  for $(w, e_{ind}) \in N_v$ **do**
    $adj\_ind \leftarrow \text{GET\_INDEX}(N_w, e_{ind}, 1)$
    if $adj\_ind = |N_w| - 1$ **then**
      ▷ *Just delete the last element* ◁
      $N_w.\text{POP}(\,)$
    else
      ▷ *Replace the edge by the last in the neighbour list* ◁
      $N_w[adj\_ind] \leftarrow N_w.\text{POP}(\,)$
  $\text{DELETE}(N_v)$
  ▷ *Delete the whole vertex* ◁
  if $v = |N_V| - 1$ **then**
    ▷ *It is the last, just pop it* ◁
    $N_V.\text{POP}(\,)$
  else
    ▷ *Replace it by the last* ◁
    $N_v \leftarrow N_V.\text{POP}(\,)$
    ▷ *Update the vertex index* ◁
    for $(w, e_{ind}) \in N_v$ **do**
      $adj\_ind \leftarrow \text{GET\_INDEX}(N_w, e_{ind}, 1)$
      $N_w[adj\_ind] \leftarrow (v, e_{ind})$
      if $w > v$ **then**
        ▷ *The last (the greatest) identifier becomes a fewer and breaks the identifier order* ◁
        $E_{attr}[e_{ind}] \leftarrow (E_{attr}[e_{ind}][0], 1 - E_{attr}[e_{ind}][1])$

4.1.7 Deleting an Edge

▶ Algorithm 18: Delete an edge for DGS

Require: Graph $G = (V, E)$, edge $(u, v) \in E$ and the edge's index e_{ind}.
Ensure: $|N_u^+ \setminus edge| = |N_u^+| - 1$ and $|N_v^- \setminus edge| = |N_v^-| - 1$.

function $\text{DGS\_DELETE\_EDGE}(G, u, v, e_{ind})$
  ▷ *Remove v from u's successors* ◁
  $adj\_ind \leftarrow \text{GET\_INDEX}(N_u^+, e_{ind}, 1)$
  if $adj\_ind = |N_u^+| - 1$ **then**

▷ *Just delete the last element* ◁
N_u^+.POP( )
else
▷ *Replace the edge by the last in the neighbour list* ◁
$N_u^+[adj\_ind] \leftarrow N_u^+$.POP( )
▷ *Remove u from v's predecessors.* ◁
$v\_rev \leftarrow \overline{v}$
$adj\_ind \leftarrow$ GET_INDEX$(N_{v_rev}^+, \overline{e_{ind}}, 1)$
if $adj\_ind = \left|N_{v\_rev}^+\right| - 1$ **then**
▷ *Just delete the last element* ◁
$N_{v\_rev}^+$.POP( )
else
▷ *Replace the edge by the last in the neighbour list* ◁
$N_{v\_rev}^+[adj\_ind] \leftarrow N_{v\_rev}^+$.POP( )
$card\_edges \leftarrow card\_edges - 2$

▶ Algorithm 19: Delete an edge for DGF

Require: Graph $G = (V, E)$, edge $(u, v) \in E$, with their orientation $(u_{or}, v_{or}) \in \{0; 1\}^2$, and the edge's index e_{ind}.
Ensure: $|N_u^+ \setminus edge| = |N_u^+| - 1$ and $|N_v^- \setminus edge| = |N_v^-| - 1$.
function DGF_DELETE_EDGE$(G, u, u_{or}, v, v_{or}, e_{ind})$
▷ *Remove v from u's successors* ◁
if $u\_or = 0$ **then**
$list\_succs \leftarrow N_u^+$
$adj\_ind \leftarrow$ GET_INDEX$(list_succs, e_{ind}, 2)$
else
$list\_succs \leftarrow N_u^-$
$adj\_ind \leftarrow$ GET_INDEX$(list_succs, \overline{e_{ind}}, 2)$
if $adj\_ind = |list\_succs| - 1$ **then**
▷ *Just delete the last element* ◁
$list\_succs$.POP( )
else
▷ *Replace the edge by the last in the neighbour list* ◁
$list\_succs[adj\_ind] \leftarrow list\_succs$.POP( )
▷ *Remove u from v's predecessors.* ◁
if $v\_or = 0$ **then**
$list\_preds \leftarrow N_v^-$
$adj\_ind \leftarrow$ GET_INDEX$(list_preds, e_{ind}, 2)$

```
    else
        list_preds ← N_v^+
        adj_ind ← GET_INDEX(list_preds, \overline{e_ind}, 2)
    if adj_ind = |list_preds| − 1 then
        ▷ Just delete the last element ◁
        list_preds.POP( )
    else
        ▷ Replace the edge by the last in the neighbour list ◁
        list_preds[adj_ind] ← list_preds.POP( )
    card_edges ← card_edges − 2
```

▶ Algorithm 20: Delete an edge for BGU

Require: Graph $G = (V, E)$, edge $(u, v) \in E$, with their orientation $(u_{or}, v_{or}) \in \{0; 1\}^2$, and the edge's index e_{ind}.

Ensure: $|N_u^+ \setminus edge| = |N_u^+| - 1$ and $|N_v^- \setminus edge| = |N_v^-| - 1$.

```
function BGU_DELETE_EDGE(G, u, u_or, v, v_or, e_ind)
    e'_ind ← e_ind ÷ 2                                   ▷ Euclidian division
    ▷ Remove v from u's neighbours. ◁
    adj_ind ← GET_INDEX(N_u, e'_ind, 1)
    if adj_ind = |N_u| − 1 then
        ▷ Just delete the last element ◁
        N_u.POP( )
    else
        ▷ Replace the edge by the last in the neighbour list ◁
        N_u[adj_ind] ← N_u.POP( )
    ▷ Remove u from v's neighbours. ◁
    adj_ind ← GET_INDEX(N_v, e'_ind, 1)
    if adj_ind = |N_v| − 1 then
        ▷ Just delete the last element ◁
        N_v.POP( )
    else
        ▷ Replace the edge by the last in the neighbour list ◁
        N_v[adj_ind] ← N_v.POP( )
    card_edges ← card_edges − 1
```

4.2 Time Complexities

4.2.1 Complexities Calculus Details

Table 1 details the calculus of algorithmic costs.

Table 1 – Calculus detail of basic operations costs. $c(x)$ is the cost of element x.

Operation	Cost	Description
x	0	Memory access
$x \pm y$	$1 + c(x) + c(y)$	Basic operation
$x \leftarrow y$	$c(y)$	Affectation
$\|x\|$	$1 + c(x)$	Absolute value or element's length
if $x \lesseqqgtr y$	$1 + c(x) + c(y)$	Conditional
EMPTY_LIST()	1	Empty list constructor
$list$.APPEND(x)	$1 + c(x)$	Add x to the end of the list $list$
$list$.POP()	1	Delete the last element of the list $list$
DELETE($list$)	$\|list\|$	Delete all the list
for $i \in [\![a; b]\!]$ **do** x	$(b - a + 1) \times c(x)$	For loop

best, worst and average cases are respectively rather lower and upper bounds, and equiprobable independent event's costs

4.2.2 Costs for Subfunctions

Table 2 gives the algorithmic costs of subfunctions in Algorithms 1 and 2.

Table 2 – Algorithmic costs for subfunctions.

	Best	Worst	Average
REV		2	
GET_INDEX	3	$3 \times \|list\|$	$3 \times \left\lceil \frac{\|list\|}{2} \right\rceil$

4.2.3 Iterating Over the Neighbours

Table 3 gives the algorithmic costs of Algorithms 3 to 8.

Table 3 – Algorithmic costs of iterating over the neighbours for DGS, DGF and BGU.

(a) Iterate over the predecessors

	Best	Worst	Average
DGS		$5 \times o_v^- + 2$	
DGF	$o_v^- + 1$	$4 \times o_v^- + 3$	$2.5 \times o_v^- + 2$
BGU	$3 \times o_v$	$6 \times o_v$	$4.75 \times o_v$

(b) Iterate over the successors

	Best	Worst	Average
DGS		o_v^+	
DGF	$o_v^+ + 1$	$4 \times o_v^+ + 3$	$2.5 \times o_v^+ + 2$
BGU	$3 \times o_v$	$6 \times o_v$	$4.75 \times o_v$

4.2.4 Costs for Dynamics

Table 4 gives the algorithmic costs of Algorithms 9 to 14. Table 5 gives the algorithmic costs of Algorithms 18 to 20. Table 6 gives the algorithmic costs of Algorithms 15 to 17.

Table 4 – Algorithmic costs of adding a vertex or an edge for DGS, DGF and BGU.

	Add a vertex			Add an edge		
	Best	Worst	Average	Best	Worst	Average
DGS		5			8	
DGF		5		6	10	8
BGU		3			9	

Table 5 – Algorithmic costs of deleting an edge for DGS, DGF and BGU.

(a) Best and Worst cases

	Best	**Worst**
DGS	17	$3 \times (o_u^+ + o_v^-) + 13$
DGF	17	$3 \times (o_u^- + o_v^+) + 17$
BGU	14	$3 \times (o_u + o_v) + 10$

(b) Average case

DGS	$3 \times \left(\left\lceil \frac{o_u^+}{2} \right\rceil + \left\lceil \frac{o_v^-}{2} \right\rceil\right) + 13 - \frac{1}{o_u^+} - \frac{1}{o_v^-}$
DGF	$\frac{3}{2} \times \left(\left\lceil \frac{o_u^-}{2} \right\rceil + \left\lceil \frac{o_u^+}{2} \right\rceil + \left\lceil \frac{o_v^-}{2} \right\rceil + \left\lceil \frac{o_v^+}{2} \right\rceil\right) + 13 - 2 \times \left(\frac{1}{o_u} + \frac{1}{o_v}\right)$
BGU	$3 \times \left(\left\lceil \frac{o_u}{2} \right\rceil + \left\lceil \frac{o_v}{2} \right\rceil\right) + 11 - \frac{1}{o_u} - \frac{1}{o_v}$

Table 6 – Algorithmic costs of deleting a vertex for DGS, DGF and BGU.

(a) Best cases

DGS	**DGF**	**BGU**
5	4	3

(b) Worst cases

DGS	$9 \times o_v + 3 \times \left(\sum_{u \in N_v^-} o_u^+ + \sum_{w \in N_v^+} o_w^- + \sum_{x \in N_z^-} o_x^+ + \sum_{y \in N_z^+} o_y^- \right) + 5 \times (o_z + 1)$
DGF	$9 \times o_v + 4 \times o_z + 3 \times \left(\sum_{u \in N_v^-} o_u^- + \sum_{w \in N_v^+} o_w^+ + \sum_{x \in N_z^-} o_x^- + \sum_{y \in N_z^+} o_y^+ \right) + 4$
BGU	$5 \times o_v + 3 \times o_z + 3 \times \left(\sum_{w \in N_v} o_w + \sum_{y \in N_z} o_y \right) + 4$

(c) Average cases (DGS, DGF and BGU)

$$10 \times o_v + \frac{10 \times \mathcal{R}_{aw} - 5}{2 \times \mathcal{R}_{aw}} \times o_z + 3 \times \left(\sum_{u \in N_v^-} \left\lceil \frac{o_u^+}{2} \right\rceil + \sum_{w \in N_v^+} \left\lceil \frac{o_w^-}{2} \right\rceil \right)$$
$$+ \frac{6 \times \mathcal{R}_{aw} - 3}{2 \times \mathcal{R}_{aw}} \times \left(\sum_{x \in N_z^-} \left\lceil \frac{o_x^+}{2} \right\rceil + \sum_{y \in N_z^+} \left\lceil \frac{o_y^-}{2} \right\rceil \right) - \left(\sum_{u \in N_v^-} \frac{1}{o_u^+} + \sum_{w \in N_v^+} \frac{1}{o_w^-} \right) + 5$$

$$\frac{15}{2} \times o_v + \frac{6 \times \mathcal{R}_{aw} - 3}{2 \times \mathcal{R}_{aw}} \times o_z + \frac{3}{2} \times \sum_{u \in N_v^- \cup N_v^+} \left(\left\lceil \frac{o_u^-}{2} \right\rceil + \left\lceil \frac{o_u^+}{2} \right\rceil \right)$$
$$+ \frac{6 \times \mathcal{R}_{aw} - 3}{4 \times \mathcal{R}_{aw}} \times \sum_{x \in N_z^- \cup N_z^+} \left(\left\lceil \frac{o_x^-}{2} \right\rceil + \left\lceil \frac{o_x^+}{2} \right\rceil \right)$$
$$- \frac{1}{2} \times \left(\sum_{u \in N_v^-} \frac{1}{o_u^+} + \sum_{w \in N_v^+} \frac{1}{o_w^-} \right) + \frac{6 \times \mathcal{R}_{aw} - 1}{\mathcal{R}_{aw}}$$

$$5 \times o_v + \frac{3 \times \mathcal{R}_{aw} - 3}{\mathcal{R}_{aw}} \times o_z + 3 \times \sum_{w \in N_v} \left\lceil \frac{o_w}{2} \right\rceil + \frac{3 \times \mathcal{R}_{aw} + 3}{2 \times \mathcal{R}_{aw}} \times \sum_{y \in N_z} \left\lceil \frac{o_y}{2} \right\rceil$$
$$- \sum_{w \in N_v} \frac{1}{o_w} + \frac{4 \times \mathcal{R}_{aw} - 1}{\mathcal{R}_{aw}}$$

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Acronyms

B | C | D

B

BG Unoriented Fragments Bi-directed Graph 31, 32

C

CSC Compressed Sparse Column 9

CSR Compressed Sparse Row 9

D

DGA All Oriented Fragments Directed Graph 22, 23, 25, 27

DGF Forward Fragments Directed Graph 27, 28, 30, 31

DGS Only Oriented Fragments' Successors Directed Graph 25–27

DNA Deoxyribonucleic Acid 1, 5, 7

Symbols

Attributes | **Functions** | **Graph** | **Operations** | **Sets**

Attributes

f Forward orientation 6–8, 12, 13, 15, 18, 19, 22, 28, 34, 37, 38, 40, *see* forward

ind Index 22, 23, 25, 27, 28, 30, 31, 35–38, 41–47

$indor$ Index with boolean orientation 27, 28

or Orientation 11, 12, 15–17, 20, 22, 23, 27, 28, 36–38, 40, 41, 43, 44, 46, 47, *see* forward & reverse

r Reverse orientation 6–8, 12, 13, 15, 18, 19, 22, 30, 34, 37, 38, 40, *see* reverse

rid Read identifier 5, 6, 11, 12, 14, 16, 19, 20, 22, 27, 28, 31, 33, 34

seq Nucleotide sequence 5–7

Functions

`strand_iid` Read identifier function 5–7

Graph

G Graph object 11–13, 15, 17, 20, 22, 36–41, 43–47, *see* undirected graph, directed graph, bi-directed graph, vertex & edge

V Vertices set 11–17, 19, 20, 22, 23, 25–28, 30, 31, 34, 36–47, *see* vertex

E Edges set 11–23, 27, 28, 30, 31, 36–41, 43–47, *see* edge

N Neighbours set 31, 37–39, 41, 45, 47, 51, *see* edge

N^- Predecesors set 23, 28, 36, 38–40, 43–47, 51, *see* edge, N & N^+

N^+ Successors set 23, 25, 28, 36–47, 51, *see* edge, N & N^-

Operations

$\overline{\cdot}$ Reverse operation 6–9, 12–14, 16, 17, 20, 21, 23, 27, 28, 30, 35, 36, 38, 40–47, *see* reverse & `rev`

Sets

$\mathbb{N}$ The set of natural (positive) integer 5, 6, 11, 13, 14, 17, 19, 20

$\mathcal{O}$ Set of overlaps 8, 9, 12, 14, 15, 19–23, 25, 28, 29, 31–33

$\mathcal{R}_{aw}$ Set of raw reads 5–7, 12, 15, 20, 22, 23, 25, 27, 28, 30–34, 51

$\mathcal{R}$ Set of reads 6–9, 12, 14, 15, 19–21

$\mathcal{R}_{ev}$ Set of reversed raw reads 6

Σ Alphabet set 5

Glossary

A | B | C | D | E | F | G | N | O | P | R | S | U | V | W

A

alignment Nucleotide comparison between at least two sequences 7, 8

assembly Method used to reconstruct genomes from fragments 1, 6, 7, 11, 14

assembly graph One of the output of genome assembly method *see* assembler

B

bi-directed graph Graph that contains 3 edge types 1, 11, 14–17, 20, 31, 32

C

contig Merge, consensus of reads obtain by assembly method(s) *see* assembler

D

directed graph Graph where the edges are directed 1, 11–13, 22–24, 26, 27, 29, 33, 34

E

edge Component of a graph that connects two vertices 9, 11–23, 25, 27, 28, 30, 31, 33, 34, *see* graph & vertex

F

forward Original sequence orientation 6–8, 11, 14, 15, 18, 19, 21, 27–29, 31, 32, 34, *see* reverse

fragment Generic nucleotide fragment 1, 5, 11–13, 20–22, 24, 26, 29

G

genome Entire set of DNA instructions found in a cell 7

graph Object composed of vertices connected by edges 1, 9, 11–14, 16–18, 20–29, 31, 34, *see* vertex & edge

N

neighbour Vertex connected to a given one 11, 27, 31–34

nucleotide Basic building block of nucleic acids (RNA and DNA) 5–8, *see* DNA

O

overlap Suffix-prefix alignment type between two sequences 1, 7–9, 11, 12, 14, 15, 18–27, 29, 31–34, *see* alignment

P

path A sequence of vertices such that two consecutive vertices are connected by an edge in the graph 12, 13, 17, 20, 21, *see* walk, graph, vertex & edge

predecessor In-vertex of one edge of one given vertex 23, 24, 26–31, 34, *see* successor

R

read DNA fragment from one DNA's strand, output by a sequencer 5–7, 9, 11, 12, 14, 15, 18–22, 27, 28, 30, 31, 33, 34, *see* sequencer

reverse Reverse-complement sequence orientation 1, 6, 7, 9, 11–13, 16, 18–22, 25–27, 30, 31, *see* forward

S

sequencer Sequencing technology machine 5

sequencing Method to generate nucleotide fragments 5–7

strand The DNA molecule is made up of two strands, each of which has a complementary sequence to the other 5–7

string graph Graph structure that stores overlaps between reads *see* overlap

succession relationship When moving from one to the other is feasible, two things have a succession relationship 1, 7, 8, 11, 15, 16

successor Out-vertex of one edge of one given vertex 23–31, 34, *see* predecessor

U

undirected graph Graph where the edges are undirected 11, 16–21, 31

V

vertex Component of a graph 9, 11–23, 25–28, 30, 31, 33, 34, *see* graph & edge

W

walk A sequence of vertices such that two consecutive vertices are connected by an edge in the graph 12, 13, 15–18, 20, 21, *see* path, graph, vertex & edge