

Article

On a Class of Multistage Stochastic Hierarchical Problems

Domenico Scopelliti 

Department of Economics and Management, University of Brescia, Contrada S. Chiara 50, 25122 Brescia, Italy; domenico.scopelliti@unibs.it

Abstract: In this paper, following the multistage stochastic approach proposed by Rockafellar and Wets, we analyze a class of multistage stochastic hierarchical problems: the Multistage Stochastic Optimization Problem with Quasi-Variational Inequality Constraints. Such a problem is defined in a suitable functional setting relative to a finite set of possible scenarios and certain information fields. The key of this multistage stochastic hierarchical problem turns out to be the nonanticipativity: some constraints have to be included in the formulation to take into account the partial information progressively revealed. In this way, we are able to study real-world problems in which the hierarchical decision processes are characterized by sequential decisions in response to an increasing level of information. As an application of this class of multistage stochastic hierarchical problems, we focus on the study of a suitable Single-Leader-Multi-Follower game.

Keywords: hierarchical problem; Optimization Problem with Quasi-Variational Inequality Constraints; multistage stochastic variational inequality; Single-Leader-Multi-Follower game

MSC: 49N30; 90C15; 91A20; 91A65; 91B70



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1. Introduction

Hierarchical problems are mathematical problems divided into several levels of hierarchy: they are of great interest since enabling to the description of real-world hierarchical decision processes. They are formulated as optimization problems with feasible sets that are implicitly defined as the solution sets of other problems. If the levels of hierarchy are two, the so-called upper and lower-level, such problems are known as *bilevel problems*; see, e.g., [1]. The origins of this research field are closely related to the economic problem of Stackelberg [2] who introduced the Leader-Follower game. Subsequently, bilevel problems have been extensively used to study several real-world applications: energy markets, transportations, resources optimal allocations, Eco-Industrial parks, etc; see, e.g., [1,3]. In particular, thanks to the hierarchical structure of the decision-making processes involved, recently great attention has captured the study of the so-called *Single-Leader-Multi-Follower game* (SLMF): it is a hierarchical problem with an upper-level in which a player acts as the leader of the game with a certain objective and a lower-level with players, acting as followers, that interact in a non cooperative way to reach a Generalized Nash Equilibrium, parametrized by the strategy of the leader; see, e.g., [4] for a state of the art on SLMF. In the literature, several approaches and concepts of solution are studied for such problems (For a comprehensive analysis and comparison of the different approaches and concepts of solution, the interested reader can refer to [1,3] and the references therein). If we focus on the classical *optimistic case*, the quoted SLMF game results to be a special case of a more general problem: the *Optimization Problem with Quasi-Variational Inequality Constraints* (OPQVIC); see, e.g., [3,5–7] and the references therein. The OPQVIC is a bilevel problem with an upper-level interrelated with a lower-level problem: the upper-level is an optimization problem having as a constraint the lower-level problem, that is, a Quasi-Variational Inequality problem (QVI). In this way, the QVI in the lower-level problem enables us to study a wide class of mathematical problems: equilibrium problems, optimization problems,

complementary problems, fixed point problems, projection problems, system of inequalities, etc; see, e.g., [8–11]. Indeed, the variational inequalities theory provides powerful and flexible tools to deal with the above described class of problems, in both analysis and computations; at the same time, we recall that a QVI can be seen also in terms of a suitable Ky Fan quasi-variational inequality; see, e.g., [12,13].

Although some practical problems involve only deterministic data, in most real-world applications problem data contain some uncertainty and randomness. This fact has motivated the examination of hierarchical problems complicated by uncertainty throughout different techniques and approaches; see, e.g., [1,14] and the references therein. Moreover, in many applications, the hierarchical decision processes could be characterized by sequential decisions in response to an increasing level of information. For this reason, in this paper, we focus on the study of a *Multistage Stochastic Optimization Problem with Quasi-Variational Inequality Constraints* (MSOPQVIC): it is a OPQVIC of multistage stochastic nature. In doing this, we follow the multistage stochastic approach proposed by Rockafellar and Wets in [15,16]. The key of this multistage formulation turns out to be the nonanticipativity: some constraints have to be included in the formulation of the MSOPQVIC to take into account the partial information progressively revealed. Indeed, stage by stage, these constraints impose the measurability with respect to the information field at that stage. In addition, nonanticipativity constraints can be dualized by suitable multipliers, providing important tools from a computational point of view. We point out that, by using the Rockafellar and Wets multistage stochastic approach, the study of a MSOPQVIC problem is a novelty in the literature.

The paper is organized as follows. Section 2 is devoted to the introduction of the MSOPQVIC in a suitable multistage-functional setting. Subsequently, in Section 3, the main properties of the MSOPQVIC are studied. In particular, without requiring any differentiability and (generalized-)monotonicity assumptions, the existence of solutions to the MSOPQVIC is proved. Moreover, as an application, the analysis of a suitable SLMF game of multistage stochastic nature is presented by using the results obtained in Section 3. Finally, a section with the conclusions is given.

2. Multistage Stochastic Hierarchical Problem

In this section, we aim to formally introduce the study of a class of multistage stochastic hierarchical problems: the MSOPQVIC. In doing this, we use the multistage stochastic approach proposed by Rockafellar and Wets in [15,16]. Indeed, it provides important tools to study real-world problems complicated by

- time, uncertainty, and risk;
- an increasing level of information progressively revealed.

To this goal, we firstly introduce the multistage framework and the functional space in which we operate; for further details, the interested reader can refer to [17,18] and the references therein.

2.1. Multistage Structure

Let $\mathcal{T} = \{1, \dots, t, \dots, T\}$ and $\mathcal{T}_0 = \{0\} \cup \mathcal{T}$ be the finite sets of stages, respectively, without and with the initial stage. At each stage $t \in \mathcal{T}$, $\Xi_t := \{\xi_t^1, \dots, \xi_t^{k_t}\} = \{\xi_t^{j_i}\}_{j_i=1, \dots, k_t}$ denotes the finite set of all uncertain situations that could occur, while ξ_0 represents the unique initial situation. Let us consider the following set:

$$\Omega \subseteq \{\xi_0\} \times \Xi_1 \times \dots \times \Xi_t \times \dots \times \Xi_T \text{ such that } \omega := (\xi_0, \xi_1^1, \dots, \xi_t^1, \dots, \xi_T^1) \in \Omega;$$

it is a discrete sample space of all *scenarios*, that is, the set of all the possible occurrences on the entire history. Let $\mathbb{P} = (\pi(\omega))_{\omega \in \Omega}$ be a probability measure on Ω such that each ω has an assigned strictly positive probability $\pi(\omega)$. Let

$$z := (z_0, z_1, \dots, z_t, \dots, z_T) \in \mathbb{R}^{m_0} \times \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_t} \times \dots \times \mathbb{R}^{m_T} = \mathbb{R}^m$$

where $m = m_0 + m_1 + \dots + m_t + \dots + m_T$. Then, the following $T + 1$ -stage pattern is considered

$$(\xi_0, z_0), (\xi_1^1, z_1), \dots, (\xi_t^t, z_t), \dots, (\xi_T^T, z_T)$$

where $\xi_t^{j_t} \in \Xi_t$ stands for the information revealed at the t -th stage when the decision z_t has to be made. To study such a multistage stochastic structure, authors in [16] introduced the following linear functional space

$$\mathcal{L}_m(\Omega, \mathbb{P}) := \mathcal{L}_m = \{\text{the collection of all functions } z : \Omega \rightarrow \mathbb{R}^m\}$$

equipped with the following *expectational inner product* and the associated norm:

$$\langle z, \hat{z} \rangle := \mathbb{E}[\langle z, \hat{z} \rangle_m] = \sum_{\omega \in \Omega} \pi(\omega) \langle z(\omega), \hat{z}(\omega) \rangle_m, \quad \|z\| := (\mathbb{E}[\langle z, z \rangle_m])^{\frac{1}{2}}, \quad (1)$$

where $\langle \cdot, \cdot \rangle_m$ identifies the classical inner product in the m -dimensional Euclidean space. The structure (1) makes \mathcal{L}_m a finite-dimensional Hilbert space. In particular:

$$\mathcal{L}_m = \mathcal{L}_{m_0} \times \mathcal{L}_{m_1} \times \dots \times \mathcal{L}_{m_t} \times \dots \times \mathcal{L}_{m_T};$$

for all $\omega \in \Omega$, it can be shown that $z(\omega) = (z_0(\omega), \dots, z_t(\omega), \dots, z_T(\omega))$. Stage by stage, we are interested to capture the recursive nature of the decision processes in this uncertain environment. By introducing suitable information fields, we can opportunely study stochastic decision processes in response to an increasing level of information.

Definition 1. A family of information-partitions of Ω is $\mathcal{P} := \{F_t\}_{t=0, \dots, T}$ where, for all $t \in \mathcal{T}_0$, $F_t := \{F_t^1, \dots, F_t^{k_t}\} = \{F_t^{j_t}\}_{j_t=1, \dots, k_t}$ is a partition of Ω such that

- (i) $F_0 = \{\Omega\}$;
- (ii) for all $t = 1, \dots, T$, $F_{t+1} \subset F_t$, that is: if $F_{t+1}^{j_{t+1}} \in F_{t+1} \Rightarrow F_{t+1}^{j_{t+1}} \subset F_t^{j_t}$ for some $F_t^{j_t} \in F_t$;
- (iii) $F_T = \Omega$.

For all $t \in \mathcal{T}_0$, the set $F_t^{j_t}$ is called elementary event and the partition F_t is called event.

Remark 1. Condition (i) means that at stage $t = 0$ no uncertainty is resolved; condition (ii) means that information is progressively revealed, that is, one has only partial information; (iii) indicates that all information is revealed at stage T . In order to link the multistage stochastic and the information partition, we can use a particular oriented graph, called event-tree; see, e.g., [18]. In this way, for each pair (ω, t) identified in \mathcal{P} corresponds a node $\xi_t^{j_t}$ and at each node $\xi_t^{j_t}$ of the oriented graph, we associate the elementary event $F_t^{j_t}$, that is, $F_t^{j_t} \cong \xi_t^{j_t}$.

If two scenarios $\tilde{\omega}, \tilde{\tilde{\omega}} \in \Omega$ are in the same elementary event $F_t^{j_t} \in F_t$, then they are indistinguishable at t on the basis of available information being that they share the same path up to t :

$$\tilde{\omega} \cong (\xi_0, \xi_1^{j_1}, \dots, \xi_t^{j_t}, \xi_{t+1}^s, \dots, \xi_T^s) \quad \tilde{\tilde{\omega}} \cong (\xi_0, \xi_1^{j_1}, \dots, \xi_t^{j_t}, \xi_{t+1}^c, \dots, \xi_T^c);$$

this fact is reflected on $z \in \mathcal{L}_m$ as follows.

Definition 2. Let $\mathcal{P} := \{F_t\}_{t \in \mathcal{T}_0}$ be an information-partitions of Ω . For any $F_{\bar{t}} \in \mathcal{P}$, we say that $z \in \mathcal{L}_m$ is $F_{\bar{t}}$ -measurable with respect to \mathcal{P} if, for all $j_{\bar{t}} = 1, \dots, k_{\bar{t}}$, it holds that:

$$\forall \tilde{\omega}, \tilde{\tilde{\omega}} \in F_{\bar{t}}^{j_{\bar{t}}} \quad z_t(\tilde{\omega}) = z_t(\tilde{\tilde{\omega}}) \quad \forall t = 0, \dots, \bar{t}.$$

We say that $z \in \mathcal{L}_m$ is measurable if it is F_t -measurable for all $F_t \in \mathcal{P}$ and $t \in \mathcal{T}_0$.

Let us denote by $\omega_{[t]} := (\xi_0, \xi_1^{j_1}, \xi_2^{j_2}, \dots, \xi_t^{j_t})$ the partial history of the scenario ω before stage $t + 1$; then, on the basis of Definitions 1 and 2, we introduce the following *nonanticipativity constraints subspace*:

$$\begin{aligned} \mathcal{N}_m &:= \{z \in \mathcal{L}_m : z_t \text{ is } F_t - \text{measurable } \forall t \in \mathcal{T}_0\} \\ &= \left\{ z \in \mathcal{L}_m : z(\omega) := (z_0, z_1(\omega_{[1]}), \dots, z_t(\omega_{[t]}), \dots, z_T(\omega)) \quad \forall \omega \in \Omega \right\}, \end{aligned}$$

where, for each $\omega \in \Omega$, $z_0(\omega) = z_0(\omega_{[0]}) = z_0$. Moreover, we denote by

$$\mathcal{M}_m := \{\rho \in \mathcal{L}_m : \langle \rho, z \rangle = 0 \quad \forall z \in \mathcal{N}_m\} := (\mathcal{N}_m)^\perp$$

the orthogonal subspace of \mathcal{N}_m , known as *nonanticipativity multipliers subspace*: according to the Riesz orthogonal decomposition, it follows that $\mathcal{L}_m = \mathcal{N}_m + \mathcal{M}_m$.

2.2. Hierarchical Problem

In this multistage stochastic framework, we introduce a suitable MSOPQVIC: it is a bilevel problem of multistage stochastic nature, with an upper-level interrelated with a lower-level problem. In particular, the lower-level problem is a suitable parametric *Multistage Stochastic Quasi-Variational Inequality* (MSQVI), which is parametrized by the upper-level variable: the upper-level is a multistage stochastic optimization problem having as a constraint the lower-level problem.

Let $\mathcal{C} := \{y \in \mathcal{L}_l : y(\omega) \in C(\omega) \quad \forall \omega \in \Omega\}$, with $C(\omega) \subseteq \mathbb{R}^l$ be nonempty, closed, and convex for each $\omega \in \Omega$. Let $\Phi : \mathcal{L}_u \times \mathcal{L}_l \rightrightarrows \mathcal{L}_l$ and $\mathcal{K} : \mathcal{L}_u \times \mathcal{C} \rightrightarrows \mathcal{C}$ be two set-valued maps so that

- Φ can be equivalently rewritten as

$$\Phi : \Omega \times \mathbb{R}^u \times \mathbb{R}^l \rightrightarrows \mathbb{R}^l \text{ such that } (\omega, x(\omega), y(\omega)) \rightarrow \Phi(\omega, x(\omega), y(\omega)) \subseteq \mathbb{R}^l;$$

- \mathcal{K} can be equivalently rewritten as

$$\mathcal{K} : \Omega \times \mathbb{R}^u \times C(\cdot) \rightrightarrows C(\cdot) \text{ such that } (\omega, x(\omega), y(\omega)) \rightarrow \mathcal{K}(\omega, x(\omega), y(\omega)) \subseteq C(\omega).$$

Then, for any, for any $x \in \mathcal{L}_u$, we introduce the following parametric MSQVI:

Find $\bar{y} \in \mathcal{K}(x, \bar{y}) \cap \mathcal{N}_l$ such that $\exists \varphi \in \Phi(x, \bar{y})$ and

$$\langle \varphi, y - \bar{y} \rangle \geq 0 \quad \forall y \in \mathcal{K}(x, \bar{y}) \cap \mathcal{N}_l; \tag{2}$$

it represents an extension of the Rockafellar and Wets multistage stochastic variational inequality problem when Φ and \mathcal{K} are suitable set-valued maps depending on a parameter $x \in \mathcal{L}_u$. From now on, we refer to the parametric problem (2) by using the abbreviation

$$MSQVI(\Phi(x, \cdot), \mathcal{K}(x, \cdot)) \equiv MSQVI(\Phi(x), \mathcal{K}(x));$$

in particular, although both Φ and \mathcal{K} depend on the couple $(x, y) \in \mathcal{L}_u \times \mathcal{L}_l$, in our abbreviation we want to emphasize the parametric dependence on $x \in \mathcal{L}_u$.

Let $\mathcal{X} := \{x \in \mathcal{L}_u : x(\omega) \in X(\omega) \quad \forall \omega \in \Omega\}$, with $X(\omega) \subseteq \mathbb{R}^u$ be nonempty and compact for each $\omega \in \Omega$. In the upper-level problem, we consider the expected value function $\mathcal{G} : \mathcal{L}_u \times \mathcal{L}_l \rightarrow \mathbb{R}$ defined as

$$\mathcal{G}(x, y) := \mathbb{E}[g(\omega, x(\omega), y(\omega))] = \sum_{\omega \in \Omega} \pi(\omega) g(\omega, x(\omega), y(\omega)), \tag{3}$$

where $g : \Omega \times \mathbb{R}^u \times \mathbb{R}^l \rightarrow \mathbb{R}$. Then, we introduce the following MSOPQVIC:

$$\begin{aligned} & \min_{x,y} \mathcal{G}(x, y) \\ & \text{s.t.} \begin{cases} x \in \mathcal{X} \cap \mathcal{N}_u \\ y \in \text{SOL}(\text{MSQVI}(\Phi(x), \mathcal{K}(x))), \end{cases} \end{aligned} \tag{4}$$

where $\text{SOL}(\text{MSQVI}(\Phi(x), \mathcal{K}(x)))$ represents the solution set of the parametric MSQVI (2); in particular, it follows that $\text{SOL}(\text{MSQVI}(\Phi(x), \mathcal{K}(x))) \subseteq \mathcal{N}_l$, that is, it is the set of all nonanticipative solutions to the MSQVI (2). Thanks to the general nature of the considered MSQVI, in the lower-level problem we could capture a wide class of mathematical problems of multistage stochastic nature: equilibrium problems, optimization problems, fixed point problems, projection problems, system of inequalities, etc. We point out that, by using the Rockafellar and Wets multistage stochastic approach, the definition of MSOPQVIC given in (4) is a novelty in the literature.

Remark 2. In the formulation of problem (4), we can opportunely incorporate constraints of expectational nature. In particular, we can replace \mathcal{X} with the following smaller set

$$\tilde{\mathcal{X}} := \left\{ x \in \mathcal{X} : \mathbb{E}[h_q(\omega, x(\omega))] \begin{cases} \leq 0 & q = 1, \dots, r \\ = 0 & q = r + 1, \dots, R \end{cases} \right\} \tag{5}$$

where $h_q(\omega, \cdot)$ is convex for $q = 1, \dots, r$ and affine for $q = r + 1, \dots, R$ so that $\tilde{\mathcal{X}}$ is still closed and convex; see, e.g., [16]. In the same way, we can replicate for \mathcal{C} the same structure of (5) so that we can introduce the set $\tilde{\mathcal{C}}$ and we can replace \mathcal{K} with the set-valued map $\tilde{\mathcal{K}} : \mathcal{L}_u \times \mathcal{C} \rightrightarrows \tilde{\mathcal{C}}$.

From an applicative point of view, it could be useful to perform a stochastic decomposition of the MSOPQVIC (4). By opportunely relaxing the nonanticipativity constraints \mathcal{N}_u and \mathcal{N}_l , respectively, on the upper and lower-level variables, we could get a decomposition of the starting multistage stochastic problems involved in the formulation (4) in a problem for each scenario throughout suitable nonanticipativity multipliers $\rho_u \in \mathcal{M}_u$ and $\rho_l \in \mathcal{M}_l$; see, e.g., [15] for the upper-level problem and [16] for the lower-level problem. In particular, relatively to the parametric multistage stochastic quasi-variational inequality problem (2), we can consider the *extensive formulation* of the parametric MSQVI($\Phi(x), \mathcal{K}(x)$):

Find $\bar{y} \in \mathcal{N}_l$ such that $\exists \varphi \in \Phi(x, \bar{y})$ and $\bar{\rho}_l \in \mathcal{M}_l$ so that

$$\forall \omega \in \Omega \quad \langle \varphi(\omega) + \bar{\rho}_l(\omega), y(\omega) - \bar{y}(\omega) \rangle_l \geq 0 \quad \forall y(\omega) \in \mathcal{K}(\omega, x(\omega), \bar{y}(\omega)), \tag{6}$$

where $\varphi(\omega) \in \Phi(\omega, x(\omega), \bar{y}(\omega))$. It provides an important tool to solve numerically the problem throughout suitable parallel procedures, such as the well-known Progressively Hedging Algorithm (PHA); see, e.g., [19,20] for the theoretical aspects and [18,21] for some applications.

3. Main Results

In this section, we aim to analyze the MSOPQVIC problem introduced in Section 2. To this goal, for the reader’s convenience, we preliminarily recall some classical definitions of set-valued analysis that will be used in the sequel; for further details, the interested reader can refer to [22] and the references therein.

Definition 3. A set-valued map $\Lambda : \mathcal{L}_m \rightrightarrows \mathcal{L}_c$ is said to be

- (i) lower semicontinuous at $z \in \mathcal{L}_m$ if for any sequence $\{z_n\}_{n \in \mathbb{N}} \subset \mathcal{L}_m$, with $z_n \rightarrow z$, and for any $s \in \Lambda(z)$, there exists a sequence $\{s_n\}_{n \in \mathbb{N}} \subset \mathcal{L}_c$, with $s_n \in \Lambda(z_n)$ for any $n \in \mathbb{N}$ and $s_n \rightarrow s$;

- (ii) upper semicontinuous at $z \in \mathcal{L}_m$ if for any open set V in \mathcal{L}_c containing $\Lambda(z)$, there exists an open neighborhood U of z in \mathcal{L}_m such that $\Lambda(\hat{z}) \subseteq V$ for all $\hat{z} \in U$;
- (iii) closed at $z \in \mathcal{L}_m$ if for any sequences $\{z_n\}_{n \in \mathbb{N}} \subset \mathcal{L}_m$, $\{s_n\}_{n \in \mathbb{N}} \subset \mathcal{L}_c$, with $z_n \rightarrow z$, $s_n \in \Lambda(z_n)$ and $s_n \rightarrow s$, then $s \in \Lambda(z)$;
- (iv) compact if $\Lambda(\mathcal{L}_m)$ is relatively compact in \mathcal{L}_c .

Focusing on the analysis of the MSOPQVIC (4), a key point is the study of the solutions set $SOL (MSQVI(\Phi(x), \mathcal{K}(x)))$ and its behavior varying the parameter $x \in \mathcal{X} \cap \mathcal{N}_u$, that is, varying the upper-level variable. Firstly, we denote by $\widetilde{SOL} (MSQVI(\Phi(x), \mathcal{K}(x))) \subseteq \mathcal{N}_l$ the solutions set of the extensive formulation of the parametric MSQVI (6) and we investigate on its relationship with $SOL (MSQVI(\Phi(x), \mathcal{K}(x)))$.

Proposition 1. For any $x \in \mathcal{L}_u$, the following inclusions hold:

- (i) $\widetilde{SOL} (MSQVI(\Phi(x), \mathcal{K}(x))) \subseteq SOL (MSQVI(\Phi(x), \mathcal{K}(x)))$;
- (ii) $SOL (MSQVI(\Phi(x), \mathcal{K}(x))) \subseteq \widetilde{SOL} (MSQVI(\Phi(x), \mathcal{K}(x)))$ if \mathcal{K} is with nonempty, closed, convex values and, for any $\bar{y} \in SOL (MSQVI(\Phi(x), \mathcal{K}(x)))$, there exists some $\hat{y} \in \mathcal{N}_l$ such that $\hat{y}(\omega) \in \text{relint } \mathcal{K}(\omega, x(\omega), \bar{y}(\omega))$ for all $\omega \in \Omega$ or, alternatively, $\mathcal{K}(\omega, x(\omega), \bar{y}(\omega))$ is polyhedral for all $\omega \in \Omega$.

Proof.

(i) For any $x \in \mathcal{L}_u$, let $\bar{y} \in \widetilde{SOL} (MSQVI(\Phi(x), \mathcal{K}(x)))$: for each $\omega \in \Omega$, we have

$$\begin{aligned} & \langle \varphi(\omega) + \bar{\rho}_l(\omega), y(\omega) - \bar{y}(\omega) \rangle_l \\ & = \langle \varphi(\omega), y(\omega) - \bar{y}(\omega) \rangle_l + \langle \bar{\rho}_l(\omega), y(\omega) - \bar{y}(\omega) \rangle_l \geq 0 \quad \forall y(\omega) \in \mathcal{K}(\omega, x(\omega), \bar{y}(\omega)). \end{aligned} \tag{7}$$

If we multiply each inequality (7) with the corresponding $\pi(\omega)$ and we sum up to ω , then, for all $y(\omega) \in \mathcal{K}(\omega, x(\omega), \bar{y}(\omega))$, it holds

$$\sum_{\omega \in \Omega} \pi(\omega) \langle \varphi(\omega), y(\omega) - \bar{y}(\omega) \rangle_l + \sum_{\omega \in \Omega} \pi(\omega) \langle \bar{\rho}_l(\omega), y(\omega) - \bar{y}(\omega) \rangle_l \geq 0. \tag{8}$$

Since $\bar{\rho}_l \in \mathcal{M}_l$, it follows that $\langle \langle \bar{\rho}_l, y - \bar{y} \rangle \rangle = 0$ for all $y \in \mathcal{K}(x, \bar{y}) \cap \mathcal{N}_l$; hence, from (8), it follows that

$$\langle \langle \varphi, y - \bar{y} \rangle \rangle \geq 0 \quad \forall y \in \mathcal{K}(x, \bar{y}) \cap \mathcal{N}_l,$$

that is, $\bar{y} \in SOL (MSQVI(\Phi(x), \mathcal{K}(x)))$.

(ii) For any $x \in \mathcal{L}_u$, let $\bar{y} \in SOL (MSQVI(\Phi(x), \mathcal{K}(x)))$: thanks to the assumptions on $\mathcal{K}(\omega, x(\omega), \bar{y}(\omega))$ and by using similar arguments to the ones of Theorem 3.2 in [16], the claim follows, that is, $\bar{y} \in \widetilde{SOL} (MSQVI(\Phi(x), \mathcal{K}(x)))$. \square

We emphasize that the relationship between the solution sets of the parametric multistage stochastic variational problems (2) and (6) provides us an important tool for numerically approaching the problem: under the assumptions of Proposition 1, we could approach the study of the extensive formulation (6) and, at the same time, we could be sure that it shares the same set of solutions with the starting problem (2).

At this point, a further step in the study of the MSOPQVIC (4) is to investigate the set-valued map:

$$x \rightarrow SOL (MSQVI(\Phi(x), \mathcal{K}(x))). \tag{9}$$

Proposition 2. Let $\mathcal{C} := \{y \in \mathcal{L}_l : y(\omega) \in C(\omega) \ \forall \omega \in \Omega\}$, with $\mathcal{C} \cap \mathcal{N}_l \neq \emptyset$ and $C(\omega) \subseteq \mathbb{R}^l$ nonempty, compact, and convex for each $\omega \in \Omega$. Then, the set-valued map (9) is closed with nonempty values if the following statements hold:

- (i) $\Phi : \mathcal{L}_u \times \mathcal{L}_l \rightrightarrows \mathcal{L}_l$ is upper semicontinuous with nonempty, compact, and convex values;
- (ii) $\mathcal{K} : \mathcal{L}_u \times \mathcal{C} \rightrightarrows \mathcal{C}$ is lower semicontinuous, closed with nonempty, compact, and convex values such that $\mathcal{K}(x, y) \cap \mathcal{N}_l \neq \emptyset$ for all $x \in \mathcal{L}_u$ and $y \in \mathcal{C}$.

Proof. Firstly, we introduce the set-valued map $\Gamma : \mathcal{L}_u \times \mathcal{C} \rightrightarrows \mathcal{C}$ defined by

$$(x, y) \rightarrow \Gamma(x, y) := \mathcal{K}(x, y) \cap \mathcal{N}_l;$$

in order to prove our thesis, we can equivalently focus on the study of the following set-valued map

$$x \rightarrow \text{SOL}(\text{MSQVI}(\Phi(x), \Gamma(x))). \tag{10}$$

▷ The set-valued map (10) is with nonempty values.

For any $x \in \mathcal{L}_u$ and $y \in \mathcal{C}$, $\Gamma(x, y) \neq \emptyset$ and \mathcal{N}_l is convex and closed as subspace of \mathcal{L}_l ; hence, as the intersection of closed/convex sets is closed/convex, it follows that Γ is with nonempty, closed, and convex values. Moreover, for any $x \in \mathcal{L}_u$ and $y \in \mathcal{C}$, since $\Gamma(x, y) \subseteq \mathcal{K}(x, y)$, $\Gamma(x, y)$ is compact, that is, Γ is with compact values. Since \mathcal{K} is lower semicontinuous and \mathcal{N}_l is closed, it follows that Γ is lower semicontinuous; see, e.g., [22]. Furthermore, Γ is closed as \mathcal{K} is closed. Then, thanks to Theorem A3 in the Appendix A, it follows that the parametric $\text{MSQVI}(\Phi(x), \Gamma(x))$ admits solutions, that is, the set-valued map (10) is with nonempty values.

▷ The set-valued map (10) is closed.

For any $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathcal{L}_u$, $\{y_n\}_{n \in \mathbb{N}} \subseteq \mathcal{C}$, with $x_n \rightarrow x$, $y_n \in \text{SOL}(\text{MSQVI}(\Phi(x_n), \Gamma(x_n)))$ and $y_n \rightarrow y$, we have to verify that $y \in \text{SOL}(\text{MSQVI}(\Phi(x), \Gamma(x)))$. Since $y_n \in \text{SOL}(\text{MSQVI}(\Phi(x_n), \Gamma(x_n)))$, for all $n \in \mathbb{N}$ there exists $\varphi_n \in \Phi(x_n, y_n)$ such that $\langle \langle \varphi_n, z_n - y_n \rangle \rangle \geq 0 \quad \forall z_n \in \Gamma(x_n, y_n)$. Moreover, as Γ is lower semicontinuous, for any $z \in \Gamma(x, y)$ there exists $\{z_n\}_{n \in \mathbb{N}}$ converging to z such that $z_n \in \Gamma(x_n, y_n)$ for all $n \in \mathbb{N}$. Furthermore, since Φ is upper semicontinuous with compact values, according to Lemma A1 in the Appendix A, there exists a subsequence $\{\varphi_v\}_{v \in \mathbb{N}} \subset \{\varphi_n\}_{n \in \mathbb{N}}$ converging to $\varphi \in \Phi(x, y)$; hence, one obtains

$$\langle \langle \varphi_v, z_v - y_v \rangle \rangle \geq 0 \quad \forall z_v \in \Gamma(x_v, y_v) \tag{11}$$

and, passing to the limit in (11), it follows that $y \in \text{SOL}(\text{MSQVI}(\Phi(x), \Gamma(x)))$, that is, the set-valued map (10) is closed. □

The results obtained in Proposition 2 for the set-valued map (9) gives us good properties on the constraints set of the upper-level problem of the MSOPQVIC (4): without requiring any (generalized-)monotonicity assumption on the problem, we prove that it admits solutions. In particular, the lack of (generalized-)monotonicity requirements could be of great interest in certain real-world applications: economic equilibrium problems are classical instances; see, e.g., [8].

Theorem 1. Let $\mathcal{X} := \{x \in \mathcal{L}_u : x(\omega) \in X(\omega) \quad \forall \omega \in \Omega\}$, with $\mathcal{X} \cap \mathcal{N}_u \neq \emptyset$ and $X(\omega) \subseteq \mathbb{R}^u$ nonempty and compact for each $\omega \in \Omega$. Let $\mathcal{C} := \{y \in \mathcal{L}_l : y(\omega) \in C(\omega) \quad \forall \omega \in \Omega\}$, with $\mathcal{C} \cap \mathcal{N}_l \neq \emptyset$ and $C(\omega) \subseteq \mathbb{R}^l$ nonempty, compact, and convex for each $\omega \in \Omega$. Then, the MSOPQVIC (4) admits solutions if the following statements hold:

- (i) $g(\omega, \cdot)$ is lower semicontinuous for all $\omega \in \Omega$;
- (ii) $\Phi : \mathcal{L}_u \times \mathcal{L}_l \rightrightarrows \mathcal{L}_l$ is upper semicontinuous with nonempty, compact, and convex values;
- (iii) $\mathcal{K} : (\mathcal{X} \cap \mathcal{N}_u) \times \mathcal{C} \rightrightarrows \mathcal{C}$ is lower semicontinuous, closed with nonempty, compact, and convex values such that $\mathcal{K}(x, y) \cap \mathcal{N}_l \neq \emptyset$ for all $x \in \mathcal{X} \cap \mathcal{N}_u$, $y \in \mathcal{C}$ and $\mathcal{K}(\mathcal{X} \cap \mathcal{N}_u, \mathcal{C})$ is bounded.

Proof. Firstly, $\mathcal{X} \cap \mathcal{N}_u$ is closed since it is the intersection of closed sets; hence, since $(\mathcal{X} \cap \mathcal{N}_u) \subseteq \mathcal{X}$, it follows that $\mathcal{X} \cap \mathcal{N}_u$ is compact. According to Proposition 2, the set-valued map (9) is closed with nonempty values. Notice that, for any $x \in \mathcal{X}$, since

$$\text{SOL}(\text{MSQVI}(\Phi(x), \mathcal{K}(x))) \subseteq \mathcal{K}(\mathcal{X} \cap \mathcal{N}_u, \mathcal{C}),$$

it follows that $SOL(MSQVI(\Phi(x), \mathcal{K}(x)))$ is a compact set. According to assumption (iii), the set-valued map (9) is compact. Moreover, from assumption (i), we have that \mathcal{G} is lower semicontinuous; see, e.g., [17]. Then, thanks to the Weierstrass theorem, the thesis follows. \square

4. Multistage Stochastic SLMF Game

In order to support the applicability of the class of multistage stochastic hierarchical problems introduced in Section 2 and the subsequent results proved about it, in this section we focus on the study of a suitable SLMF game of multistage stochastic nature: SLMF is a hierarchical game that recently has attracted great interest in the literature; see, e.g., [3,4,6]. In particular, the hierarchical structure of the game is such that it has

- an upper-level in which a player acts as the *leader* of the game with a certain objective;
- a lower-level with a finite set of players, acting as *followers*, that interact in a non cooperative way to reach an equilibrium, in terms of a *Generalized Nash Equilibrium Problem* (GNEP), parametrized by the strategy of the leader;

in this setting, the followers react to what the leader decides, while the leader chooses its own strategy taking into account the future reaction of the followers. All is studied in the multistage framework introduced in Section 2: we consider an uncertain environment that evolves in a finite number of stage so that, at each stage, a finite set of states is possible. The information on the evolution of this uncertain environment is progressively revealed at each stage: thanks to the information restriction that we consider, we are able to describe how the information influences the strategies of each player and how these strategies evolve over time.

Let $\mathcal{I} := \{1, \dots, i, \dots, I\}$ be the finite set of the players acting as followers; then, we consider a multistage stochastic hierarchical game with $I + 1$ players. The player acting as leader has control over the strategy $x \in \mathcal{L}_u$, while each player acting as follower has control over the strategy $y_i \in \mathcal{L}_i$. We denote by $y \in \mathcal{L}_l$ the vector comprehensive of all followers' strategies:

$$y := (y_i)_{i \in \mathcal{I}} = (y_i, y_{-i}) \quad \text{where} \quad y_{-i} := (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_I) \in \mathcal{L}_{l_{-i}};$$

the notation $-i$ represents all the followers in \mathcal{I} except i and $l := \sum_{i \in \mathcal{I}} l_i = l_{-i} + l_i = \sum_{\hat{i} \neq i} l_{\hat{i}} + l_i$.

Let $\mathcal{G} : \mathcal{L}_u \times \mathcal{L}_l \rightarrow \mathbb{R}$ be the expected payoff function of the leader depending on the variables (x, y) of all $I + 1$ players. We define \mathcal{G} as in (3) and we denote by $\mathcal{X} := \{x \in \mathcal{L}_u : x(\omega) \in X(\omega) \quad \forall \omega \in \Omega\}$ the constraints set of the leader strategy. For each $i \in \mathcal{I}$, let $\mathcal{U}_i : \mathcal{L}_u \times \mathcal{L}_l \rightarrow \mathbb{R}$ be the expected payoff function of the follower i defined as

$$\mathcal{U}_i(x, y) := \mathbb{E}[u_i(\omega, x(\omega), y(\omega))] = \sum_{\omega \in \Omega} \pi(\omega) u_i(\omega, x(\omega), y(\omega)), \quad \text{where}$$

$$\mathcal{U}_i(x, y) \equiv \mathcal{U}_i(x, y_i, y_{-i}) \quad \text{and} \quad u_i(\omega, x(\omega), y(\omega)) \equiv u_i(\omega, x(\omega), y_i(\omega), y_{-i}(\omega)),$$

with $u_i : \Omega \times \mathbb{R}^u \times \mathbb{R}^l \rightarrow \mathbb{R}$ and $\mathcal{U} := \prod_{i \in \mathcal{I}} \mathcal{U}_i$. Let $\mathcal{C}_i := \{y_i \in \mathcal{L}_i : y_i(\omega) \in C_i(\omega) \quad \forall \omega \in \Omega\}$ be the feasible region of the follower i and $\mathcal{C} := \prod_{i \in \mathcal{I}} \mathcal{C}_i = \mathcal{C}_{-i} \times \mathcal{C}_i = \prod_{\hat{i} \neq i} \mathcal{C}_{\hat{i}} \times \mathcal{C}_i \subseteq \mathcal{L}_l$; then, we introduce the constraints set-valued map

$$Y : \mathcal{L}_u \times \mathcal{C} \rightrightarrows \mathcal{C} \quad \text{such that} \quad Y(x, y) := \prod_{i \in \mathcal{I}} Y_i(x, y_{-i}), \quad \text{where} \quad Y_i(x, y_{-i}) \subseteq \mathcal{C}_i.$$

In particular, we define $Y_i : \mathcal{L}_u \times \mathcal{C}_{-i} \rightrightarrows \mathcal{C}_i$ such that

$$Y_i(x, y_{-i}) := \{y_i \in \mathcal{C}_i : y_i(\omega) \in Y_i(\omega, x(\omega), y_{-i}(\omega)) \quad \forall \omega \in \Omega\}$$

and we observe that, for all $i \in \mathcal{I}$, Y_i is independent of follower i 's strategy; hence, for mathematical convenience, we could also consider $\tilde{Y}_i : \mathcal{L}_u \times \mathcal{C} \rightrightarrows \mathcal{C}_i$ such that $\tilde{Y}_i(x, y) = Y_i(x, y_{-i})$.

The aim of each follower i , given the leader strategy x and other followers' strategies y_{-i} , is to find a strategy $\tilde{y}_i \in Y_i(x, y_{-i})$ that is nonanticipative and that minimizes the expected payoff function \mathcal{U}_i , that is

$$\min_{y_i \in Y_i(x, y_{-i}) \cap \mathcal{N}_i} \mathcal{U}_i(x, y_i, y_{-i}) := \mathcal{U}_i(x, \tilde{y}_i, y_{-i}). \tag{12}$$

Then, for any $x \in \mathcal{X} \cap \mathcal{N}_u$, a *parametric multistage stochastic GNEP*(x) is the following problem:

$$\text{Find } \tilde{y} \in Y(x, \tilde{y}) \cap \mathcal{N}_l \text{ such that } \forall i \in \mathcal{I} \quad \min_{y_i \in Y_i(x, \tilde{y}_{-i}) \cap \mathcal{N}_i} \mathcal{U}_i(x, y_i, \tilde{y}_{-i}) := \mathcal{U}_i(x, \tilde{y}_i, \tilde{y}_{-i}); \tag{13}$$

in this way, a vector \tilde{y} is an equilibrium solution for (13) if no follower can unilaterally decrease its expected payoff function by choosing a different strategy. For any $x \in \mathcal{X} \cap \mathcal{N}_u$, we denote by SOL ($GNEP(x)$) the solution set of the parametric multistage stochastic problem (13). In the literature, SOL ($GNEP(x)$) is also called *reaction map* because it contains the responses of the followers to an arbitrary action x of the leader; see, e.g., [5].

From now on, we assume that the followers will choose, among their equilibrium strategies, one that most benefits the leader: it is the so-called *optimistic case* to the considered hierarchical game that consists in considering the best equilibrium reaction of the followers with regards to the leader's expected payoff; see, e.g., [3,4]. Let the *Multistage Stochastic Single-Leader-Multi-Follower* game (MSSLMF) up to now described be denoted as follows

$$\mathcal{E} := MSSLMF(\mathcal{G}, \mathcal{X}, \text{SOL}(GNEP));$$

then, we can now formally introduce an equilibrium solution for it.

Definition 4. A vector $(\bar{x}, \bar{y}) \in \mathcal{L}_u \times \mathcal{L}_l$ is an *optimistic equilibrium* for \mathcal{E} if it solves the problem:

$$\begin{aligned} & \min_{x, y} \mathcal{G}(x, y) \\ & \text{s.t. } \begin{cases} x \in \mathcal{X} \cap \mathcal{N}_u \\ y \in \text{SOL}(GNEP(x)). \end{cases} \end{aligned} \tag{14}$$

Remark 3. We point out that MSSLMF scheme introduced in this section can capture a wide range of applications: energy market models, network models, water optimal allocation problems, demand-side management problems, medical supply competitions, etc. Indeed, at the upper level, one can always consider a leader with a certain objective: for instance, it could be a regulator. Instead, at the lower level, it is possible to fit a suitable GNEP describing the interactions among the followers in the considered application, parametrized by the leader strategy; see, e.g., [21,23,24] for problems in multistage stochastic settings.

In force of the results obtained in Section 3, we study the problem (14) as a suitable MSOPQVIC: in particular, *no differentiability assumptions* on the expected payoff functions of the followers will be required.

For any $x \in \mathcal{L}_u$, $i \in \mathcal{I}$, and $y_{-i} \in \mathcal{L}_{l_{-i}}$, if $u_i(\omega, x(\omega), \cdot, y_{-i}(\omega))$ is convex, then we can consider the subdifferential $\partial u_i(x, \cdot, y_{-i}) : \mathcal{L}_i \rightrightarrows \mathcal{L}_i$ so that, for any $y_i \in \mathcal{L}_i$, $\partial u_i(x, y_i, y_{-i})$ is such that

$$\begin{aligned} \partial u_i(x, y_i, y_{-i}) : \Omega & \rightrightarrows \mathbb{R}^l \\ \omega & \rightarrow \partial u_i(\omega, x(\omega), y_i(\omega), y_{-i}(\omega)), \end{aligned}$$

where, for each $\omega \in \Omega$, $\partial u_i(\omega, x(\omega), y_i(\omega), y_{-i}(\omega)) \equiv \partial u_i(\omega, x(\omega), y(\omega))$.

By using classical arguments, for any $x \in \mathcal{X} \cap \mathcal{N}_u$, $i \in \mathcal{I}$, and $y_{-i} \in \mathcal{C}_{-i} \cap \mathcal{N}_{l_{-i}}$, we can express the subdifferential $\partial \mathcal{U}_i(x, \cdot, y_{-i}) : \mathcal{L}_i \rightrightarrows \mathcal{L}_{l_i}$ in terms of the subdifferential $\partial u_i(x, \cdot, y_{-i}) : \mathcal{L}_i \rightrightarrows \mathcal{L}_{l_i}$.

Lemma 1. For any $x \in \mathcal{X} \cap \mathcal{N}_u$, $i \in \mathcal{I}$, and $y_{-i} \in \mathcal{C}_{-i} \cap \mathcal{N}_{l_{-i}}$, we assume that

- (i) $u_i(\omega, x(\omega), \cdot, y_{-i}(\omega))$ is convex for each $\omega \in \Omega$;
 - (ii) there exists some $\hat{y}_i \in \mathcal{N}_{l_i}$ such that $\hat{y}_i(\omega) \in \text{relint } Y_i(\omega, x(\omega), y_{-i}(\omega))$ for each $\omega \in \Omega$;
- then, for any $y_i \in \mathcal{C}_i \cap \mathcal{N}_{l_i}$, it holds:

$$\partial \mathcal{U}_i(x, y_i, y_{-i}) = \sum_{\omega \in \Omega} \pi(\omega) \partial u_i(\omega, x(\omega), y_i(\omega), y_{-i}(\omega))$$

Proof. Thanks to the considered assumptions, namely that $|\Omega| < +\infty$, the thesis follows by using similar arguments to the ones in [17]. \square

In this way, we can characterize the parametric problem (12) in terms of the following parametric multistage stochastic variational inequality problem:

Find $\bar{y}_i \in Y_i(x, y_{-i}) \cap \mathcal{N}_{l_i}$ such that $\exists \varphi_i \in \partial u_i(x, \bar{y}_i, y_{-i})$ and

$$\langle \langle \varphi_i, y_i - \bar{y}_i \rangle \rangle \geq 0 \quad \forall y_i \in Y(x, y_{-i}) \cap \mathcal{N}_{l_i}. \tag{15}$$

Proposition 3. For any $x \in \mathcal{X} \cap \mathcal{N}_u$, $i \in \mathcal{I}$, and $y_{-i} \in \mathcal{C}_{-i} \cap \mathcal{N}_{l_{-i}}$, let the assumptions of Lemma 1 be satisfied and $Y_i(x, y_{-i})$ be nonempty and convex. Then, $\bar{y}_i \in Y_i(x, y_{-i}) \cap \mathcal{N}_{l_i}$ solves (12) if and only if it is a solution to (15).

Proof. From Lemma 1 and thanks to the considered assumptions, the thesis follows by using similar arguments to the ones of Proposition 1.18 in [25]. \square

Let $\partial u := \prod_{i \in \mathcal{I}} \partial u_i$; then, we introduce the following MSOPQVIC:

$$\begin{aligned} & \min_{x,y} \mathcal{G}(x, y) \\ & \text{s.t. } \begin{cases} x \in \mathcal{X} \cap \mathcal{N}_u \\ y \in \text{SOL}(\text{MSQVI}(\partial u(x), Y(x))), \end{cases} \end{aligned} \tag{16}$$

where, for any $x \in \mathcal{X} \cap \mathcal{N}_u$, $\text{MSQVI}(\partial u(x), Y(x))$ denotes the following parametric problem:

Find $\bar{y} \in Y(x, \bar{y}) \cap \mathcal{N}_l$ such that $\exists \varphi := (\varphi_i)_{i \in \mathcal{I}} \in \partial u(x, \bar{y})$ and

$$\langle \langle \varphi, y - \bar{y} \rangle \rangle \geq 0 \quad \forall y \in Y(x, \bar{y}) \cap \mathcal{N}_l. \tag{17}$$

In particular, as made in the previous section, we could opportunely link the MSQVI (17) with its extensive formulation if we are interested to numerically solve the parametric multistage stochastic quasi-variational inequality problem. By using the results of Section 3, we prove the existence of an optimistic equilibrium for \mathcal{E} .

Theorem 2. Let $\mathcal{X} \cap \mathcal{N}_u \neq \emptyset$, with $X(\omega) \subseteq \mathbb{R}^u$ nonempty and compact for each $\omega \in \Omega$. For all $i \in \mathcal{I}$, let $\mathcal{C}_i \cap \mathcal{N}_{l_i} \neq \emptyset$, with $\mathcal{C}_i(\omega) \subseteq \mathbb{R}^{l_i}$ nonempty, compact, convex for each $\omega \in \Omega$, and the assumption (ii) of Lemma 1 be satisfied. Then, the MSSLMF (14) admits optimistic solutions if the following statements hold:

- (i) $g(\omega, \cdot)$ is lower semicontinuous for all $\omega \in \Omega$;
- (ii) for all $i \in \mathcal{I}$, $u_i(\omega, \cdot, \cdot)$ is convex for all $\omega \in \Omega$;
- (iii) for all $i \in \mathcal{I}$, $Y_i : (\mathcal{X} \cap \mathcal{N}_u) \times \mathcal{C}_{-i} \rightrightarrows \mathcal{C}_i$ is lower semicontinuous, closed with nonempty, compact, and convex values such that $Y_i(x, y_{-i}) \cap \mathcal{N}_{l_i} \neq \emptyset$ for all $x \in \mathcal{X} \cap \mathcal{N}_u$, $y_{-i} \in \mathcal{C}_{-i} \cap \mathcal{N}_{l_{-i}}$ and $Y_i(\mathcal{X} \cap \mathcal{N}_u, \mathcal{C}_{-i})$ is bounded.

Proof. For any $x \in \mathcal{X} \cap \mathcal{N}_u$, from Lemma 1 and Proposition 3, it follows that $\bar{y} \in Y(x, \bar{y}) \cap \mathcal{N}_l$ solves (13) if and only if it is a solution to (17). In this way, it is sufficient to verify that the MSOPQVIC (16) admits solutions. For any $i \in \mathcal{I}$, from Proposition A1 in the Appendix A, it follows that the subdifferential ∂u_i is upper semicontinuous with nonempty, compact, and convex values. According to Theorems A1 and A2 in the Appendix A, it follows that ∂u is upper semicontinuous with nonempty, compact, and convex values. Similarly, since we observed that $\tilde{Y}_i(x, y) = Y_i(x, y_{-i})$ for all $x \in \mathcal{X} \cap \mathcal{N}_u$ and $y \in \mathcal{C}$, it follows that the constraints set-valued map Y is lower semicontinuous, closed with nonempty, compact, convex values such that $Y(x, y) \cap \mathcal{N}_l \neq \emptyset$ for all $x \in \mathcal{X} \cap \mathcal{N}_u$, $y \in \mathcal{C} \cap \mathcal{N}_l$ and $Y(\mathcal{X} \cap \mathcal{N}_u, \mathcal{C})$ is bounded. Then, according to the considered assumptions and thanks to Theorem 1, the thesis follows. \square

5. Conclusions

In this paper, inspired by the approach proposed in [15,16], we introduce the analysis of an OPQVIC of multistage stochastic nature (4): this formulation is a novelty in the literature. The study is motivated by the fact that the MSOPQVIC (4) allows us to study several real-world problems in which hierarchical decision processes are characterized by sequential decisions in response to an increasing level of information. This is supported by the fact that the MSOPQVIC (4) has a general formulation. Indeed, at the lower-level, we could consider a wide class of mathematical problems of multistage stochastic nature such as equilibrium problems, optimization problems, fixed point problems, projection problems, system of inequalities, etc; moreover, the presence of the nonanticipativity constraints provides us important tools, from a computational point of view, to solve numerically the problem by performing a suitable stochastic decomposition of the starting formulation in a problem for each scenario in order to compute the solution by using the well-known Progressive Hedging Algorithm (see, e.g., [15] in the framework of stochastic programming and [19,20] for variational inequality problems). From an applicative point of view, another interesting aspect of the presence of the nonanticipativity constraints is that, both at the upper and lower-level of the MSOPQVIC (4), we could consider suitable risk measures without precluding the possibility of getting a suitable stochastic decomposition of the starting problem; see, e.g., [26].

Without requiring any monotonicity assumption on the problem, we prove that the MSOPQVIC (4) admits solutions. In view of this result, we focus on the study of a SLMF game of multistage stochastic nature. At the lower-level, the presence of a suitable GNEP of multistage stochastic nature describing the interactions among the followers and parametrized by the leader strategy could allow us to capture a wide range of applications: energy market models, network models, water optimal allocation problems, demand-side management problems, medical supply competitions, etc; see, e.g., [21,23,24,27]. For this reason, this paper is the starting point for future developments. In particular, it could be interesting to extend the analysis to multistage stochastic hierarchical problems in more general spaces and with a more general upper-level problem that could allow us to study Multi-Leader-Multi-Follower games and the related real-world applications, both from a theoretical and computational point of view.

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Appendix A

For the reader's convenience, we recall some results of set-valued and variational analysis used in the previous sections.

Lemma A1 ([25], Lemma 1.10). *Let Z be a topological space, S be a topological vector space and $\Lambda : Z \rightrightarrows S$ be a set-valued map such that $\Lambda(z)$ is nonempty and compact for all $z \in Z$. Then, Λ is upper semicontinuous at $z \in Z$ if and only if for any nets $\{z_\mu\} \subset Z$ with $z_\mu \rightarrow z$ and $\{s_\mu\} \subset S$ with $s_\mu \in \Lambda(z_\mu)$, there exists a subnet $\{s_\nu\} \subset \{s_\mu\}$ such that $s_\nu \rightarrow s$ for some $s \in \Lambda(z)$.*

Theorem A1 ([22], Tychonoff's Theorem). *The product of a family of topological spaces is compact in the product topology if and only if each factor is compact.*

Theorem A2 ([22]). *Let $I = \{1, \dots, n\}$ be a finite set, Z be a topological space, and S_i be a topological space for all $i \in I$. Let $\Lambda_i : Z \rightrightarrows S_i$ for all $i \in I$. The following statements hold:*

- (i) *if Λ_i is lower semicontinuous for all $i \in I$, then the Cartesian product $\Lambda = \prod_{i \in I} \Lambda_i$ is a lower semicontinuous mapping of Z into $S = \prod_{i \in I} S_i$;*
- (ii) *if Λ_i is upper semicontinuous for all $i \in I$, then the Cartesian product $\Lambda = \prod_{i \in I} \Lambda_i$ is an upper semicontinuous mapping of Z into $S = \prod_{i \in I} S_i$.*

At this point, we quote the classical Tan result for the existence of a QVI.

Theorem A3 ([11], Corollary to Theorem 3). *Let Z be a topological linear locally convex Hausdorff space and Z^* its dual. Let $C \subseteq Z$ be a nonempty, compact, and convex set. Let $\Phi : Z \rightrightarrows Z^*$ and $K : C \rightrightarrows C$ be two set-valued maps. Let us suppose that the following properties hold:*

- (i) *Φ is upper semicontinuous with nonempty, compact, and convex values;*
- (ii) *K is closed, lower semicontinuous with nonempty, compact, and convex values.*

Then, the QVI(Φ, K) admits solutions.

Finally, we recall some relevant properties of the subdifferential of a convex function.

Proposition A1 ([28], Propositions 2.1.2 and 2.1.5). *Let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be a convex function. Then, the subdifferential $\partial f : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is upper semicontinuous with nonempty, compact, and convex values.*

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