Properties for Voronoi Diagrams of Arbitrary Order on the Sphere

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Abstract

For a given set of points U on a sphere S, the order k spherical Voronoi diagram $SV_k(U)$ decomposes the surface of S into regions whose points have the same k nearest points of U. We study properties for $SV_k(U)$, using different tools: the geometry of the sphere, a labeling for the edges of $SV_k(U)$, and the inversion transformation. Hyeon-Suk Na, Chung-Nim Lee, and Otfried Cheong (Comput. Geom., 2002) applied inversions to construct $SV_1(U)$. We generalize their construction for spherical Voronoi diagrams from order 1 to any order k. We use that construction to prove formulas for the numbers of vertices, edges, and faces in $SV_k(U)$. Among the properties of $SV_k(U)$, we also show that $SV_k(U)$ has a small orientable cycle double cover.

1 Introduction

Let U be a set of n points on a sphere $S \subset \mathbb{R}^3$ such that no three of them lie in the same great circumference and no four of them are cocircular, i.e. U is in general position, and let $1 \leq k \leq n-1$ be an integer. The order k spherical Voronoi diagram $SV_k(U)$ decomposes the surface of S into regions whose points have the same k nearest points of U. Then, each of these regions is a face $f(P_k)$ of $SV_k(U)$ associated with a subset $P_k \subset U$ of size k: Each point in the interior of $f(P_k)$ has P_k as its k nearest neighbors from U.

Many researchers studied the nearest ($k = 1$) and the farthest ($k = n-1$) spherical Voronoi diagrams [2, 11, 10]. For these two diagrams it was seen that practically all algorithms in the plane can be adapted to the sphere. Spherical Voronoi diagrams of order different from $k = 1$ and $k = n-1$ have barely been studied. In this work we deepen in these diagrams and the properties and algorithms that we present are for Voronoi diagrams of arbitrary order k on the sphere. This abstract summarizes our main results on $SV_k(U)$; we refer the reader to the thesis of the second author [5] for more details and more properties. One of the most important tools that we use in our proofs is an edge labeling. This labeling is an extension to the sphere of the already defined edge labeling for Voronoi diagrams in the plane [4]. An edge that delimits a face of $SV_k(U)$ is a spherical segment of the perpendicular bisector (on the sphere) of two points i and j of U. This observation induces a natural labeling of the edges of $SV_k(U)$ with the following rule:

- **Edge rule:** An edge of $SV_k(U)$ which belongs to the perpendicular bisector of points $i, j \in U$ has labels i and j, where we put the label i on the side (half-sphere) of the edge that contains point i and we put label j on the other side. See Figure 1.

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Also, from this rule, we deduce two more rules of the labeling of $SV_k(U)$: one rule for the vertices and one rule for the faces. Vertices can be of type I, if they are centers of circles on the sphere passing through three points of U and enclosing $k-1$ points of U, or type II, if they are centers of circles on the sphere passing through three points of U and enclosing $k-2$ points of U. In the literature, vertices of type I (type II) are also called *new* (*old*) [7].

• **Vertex rule:** Let v be a vertex of $SV_k(U)$ and let $\{i, j, \ell\} \subset U$ be the set of labels of the edges incident to v. The cyclic order of the labels of the edges around v is i, i, j, j, ℓ, ℓ if v is of type I, and it is i, j, ℓ, i, j, ℓ if v is of type II.

• **Face rule:** In each face of $SV_k(U)$, the edges that have the same label i are consecutive, and these labels i are either all in the interior of the face, or are all in the exterior of the face.

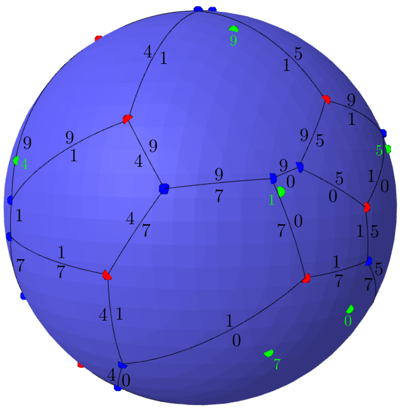


Figure 1 The edge labeling of $SV_2(U)$ for a set U of ten points $\{0, 1, \ldots, 9\}$ in general position (the visible ones are drawn in green color). Vertices of type I are drawn in blue, and vertices of type II in red.

Note that when walking along the boundary of a face, in its interior (exterior), a change in the labels of its edges appears whenever we reach a vertex of type II (type I), see Figure 1.

From this edge labeling, we observe that edges with same label i always form a cycle in $SV_k(i)$; see Figure 2. These edges with the same label i enclose a region $R_k(i)$ that consists of all the points of the sphere that have point $i \in U$ as one of their k nearest neighbors from U. We observe that $R_1(i)$ is contained in the kernel of this star-shaped set $R_k(i)$, and we identify the reflex (convex) vertices on the boundary $B_k(i)$ of $R_k(i)$ as vertices of type II (type I). See [4, 5] for details.

A *cycle double cover* [6] of a graph G is a collection of cycles $\mathcal{C}$ such that every edge of G belongs to precisely two cycles of $\mathcal{C}$. A double cover $\mathcal{C}$ is orientable if an orientation can be assigned to each element of $\mathcal{C}$ such that for every edge e of G, the two cycles that cover e are oriented in opposite directions.

Much research was done on finding small cycle double covers for several classes of graphs, see for instance [1, 12]. We show that every higher-order Voronoi diagram on the sphere admits an orientable double cover of its edges, using, precisely, the n cycles $B_k(i)$ in for $i = 1, \ldots, n$. We refer to [4] for related results on double covers of the edges of higher order Voronoi diagrams in the plane.

As one of our main results, we generalize to any order the construction of spherical

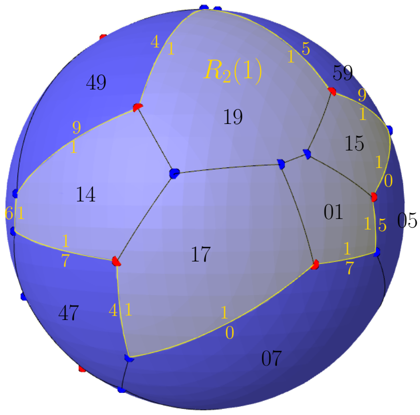


Figure 2 $SV_2(U)$ for the point set U of Figure 1; in each face, its two nearest neighbors are indicated. In yellow, the region $R_2(1)$ formed by all the faces of $SV_2(U)$ that have point 1 as one of their two nearest neighbors. The boundary $B_2(1)$ of $R_2(1)$ is formed by all the edges which have the label 1 and this label is always inside $R_2(1)$. The boundary vertices of $R_2(1)$ with an incident edge lying in the interior of $R_2(1)$ are of type II in $SV_2(U)$ and the remaining boundary vertices are of type I in $SV_2(U)$.

Voronoi diagrams defined by Hyeon-Suk Na, Chung-Nim Lee and Otfried Cheong [11], using precisely the regions $R_k(i)$ and the inversion transformation. Inversions for Voronoi diagrams were already applied in the classical work of Brown [2, 3]. In [11], $SV_1(U)$ is computed from two planar Voronoi diagrams after applying inversions to map U to the plane; two different inversion centers are used. In [11] it is also shown that $SV_1(U)$ is homeomorphic to the union of a nearest and a farthest Voronoi diagram, when glued together. We generalize this to $SV_k(U)$ being homeomorphic to the union of a planar Voronoi diagram of order k, and one planar Voronoi diagram of order $n-k$. Furthermore, these diagrams are linked via $R_k(i)$ in $SV_k(U \cup \{i\})$, with i the center of inversion, where the unbounded edges in the two planar Voronoi diagrams correspond to edges of $SV_k(U)$ intersected by $B_k(i)$. We further derive formulas for the numbers of vertices, edges and faces of $SV_k(U)$. The proof is based on the construction of $SV_k(U)$. Surprisingly, the obtained formulas seem to be new. We also obtain formulas for the number of vertices of type I and for the number of vertices of type II in $SV_k(U)$. The proof of Theorem 3.2 is omitted in this abstract, but also see [5].

2 Properties of $SV_k(U)$

▶ **Property 2.1.** Let u^* be the antipodal point of a point u on a sphere S. Then $SV_k(U) = SV_{n-k}(U^*)$, where $U^* = \{u^*|u \in U\}$.

The proof of this property is essentially the same as the one for the case $k = 1$ given in [2, 11].

Proof. The spherical distance for points $x, y \in S$ is $d(x,y) = \pi r - d(x, y^*)$ where r is the radius of the sphere. It follows that the k nearest neighbors of a point x must be the k farthest neighbors of x^*. Therefore, $x \in f(P_k)$ if and only $x \in f(U^* \setminus P_k^*)$ where $P_k^* = \{p^*|p \in P_k\}$, and the property follows. ◀

▶ **Property 2.2.** Let v be a vertex of type I of $SV_k(U)$. Then v^* is a vertex of type II of $SV_{n-k}(U)$. Similarly, if v is a vertex of type II of $SV_k(U)$ then v^* is a vertex of type I of $SV_{n-k}(U)$. See Figure 3.

Proof. If v is a vertex of type I of $SV_k(U)$, then it is the center of a disk D that passes through three points of U and contains $(k-1)$ points of U. From this, by the geometry of the sphere S, the remaining $(n-k-2)$ points are contained in the complementary disk $S \setminus D$ whose center is v^*. Therefore, v^* must be a vertex of type II of $SV_{n-k}(U)$. The symmetric argument works for v of type II. ◀

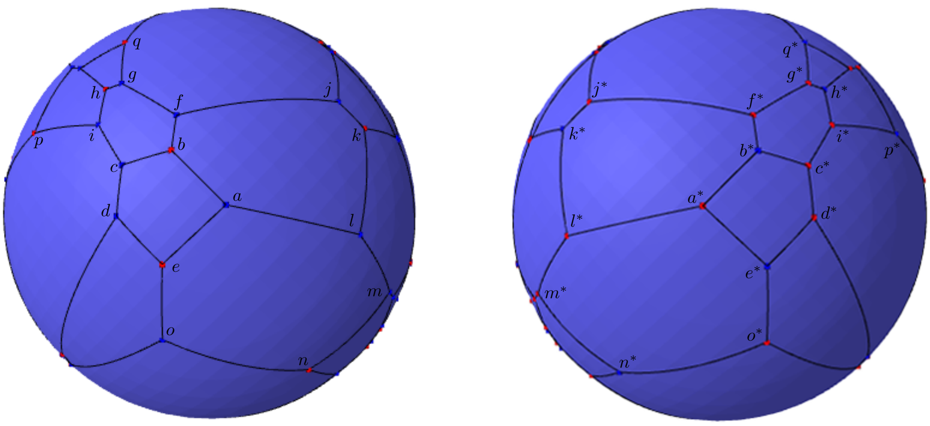


Figure 3 Two complementary Voronoi diagrams on an sphere $SV_k(U)$ and $SV_{n-k}(U)$, showing the homothetic relation between them and their corresponding antipodal points types. Type I vertices are blue and type II vertices are red.

▶ **Property 2.3.** Let $f(P_k)$ be a face of $SV_k(U)$ and let $f(U \setminus P_k)$ be its corresponding antipodal face in $SV_{n-k}(U)$. $f(P_k)$ and $f(U \setminus P_k)$ use the same labels but in opposite sides, i.e., if i is an interior label of an edge of $f(P_k)$ then it is an exterior label for the corresponding antipodal edge in $SV_{n-k}(U)$. See Figure 4.

Proof. It follows from Property 2.1 that $f(P_k)$ and $f(U \setminus P_k)$ are antipodal polygons. Then we just need to observe that antipodal polygons are defined by the complementary half-spheres defined by the same bisector, i.e, their edges are from the same bisectors but the antipodal polygons lie in opposite sides of those bisectors, see Figure 4. Therefore, by the edge rule, the statement is clear. ◀

▶ **Theorem 2.4.** *$SV_k(U)$ has an orientable double cover consisting of $|U| = n$ cycles.*

Proof. It is not difficult to see that for every $1 \leq i \leq n$, all the edges that have the label i in $SV_k(U)$ form one cycle (also see Property 6.1 in [5]). Since each label i, corresponding to a point $i \in U$, is inside the corresponding region $R_k(i)$, we can orient all the edges of a cycle with label i clockwise around point i; note that point i is also contained in $R_k(i)$. This shows that the cycle cover is orientable. Finally, as there is one cycle for each point of U, it follows that $SV_k(U)$ has an orientable double cover of n cycles. ◀

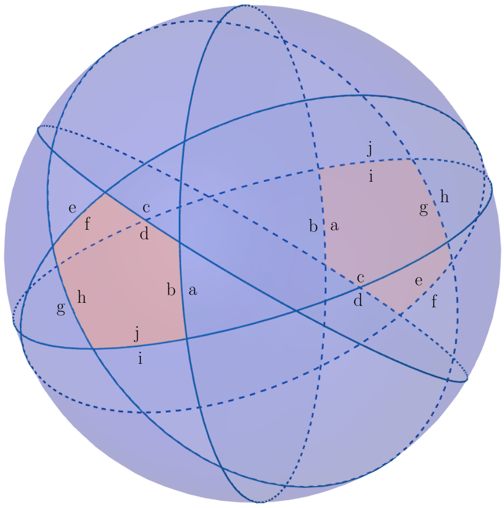


Figure 4 Two antipodal polygons, one has labels b, d, f, h, j in its interior, the other one has these labels in its exterior.

3 Relations between Planar and Spherical Voronoi Diagrams

In this section we generalize to Voronoi diagrams of arbitrary order k the construction given in [11] for the nearest and farthest Voronoi diagrams. We then prove some more properties using this construction.

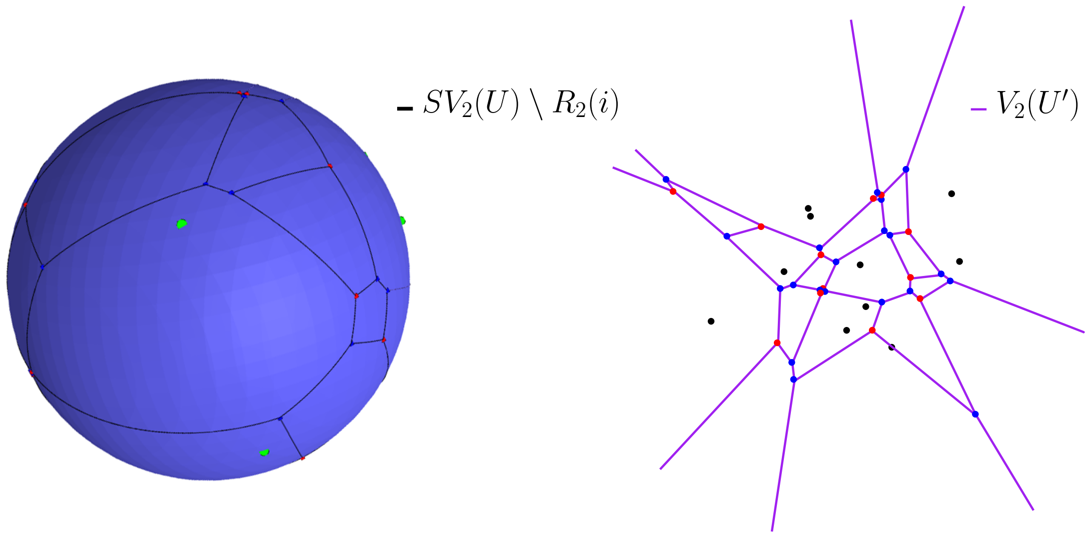
First, we need to define the inversion transformation, as it is the basis of the relation between Voronoi diagrams on the sphere and on the plane.

▶ **Definition 3.1.** The inversion transformation is determined by two parameters: The center of inversion O and the radius of inversion R. Two points P and P' in $\mathbb{R}^3$ are said to be inverses of each other if:

1. The points P and P' lie in the same half-line with origin in O.
2. The Euclidean distances $|\overline{OP}|$ and $|\overline{OP'}|$ in $\mathbb{R}^3$ satisfy $R^2 = |\overline{OP}||\overline{OP'}|$.

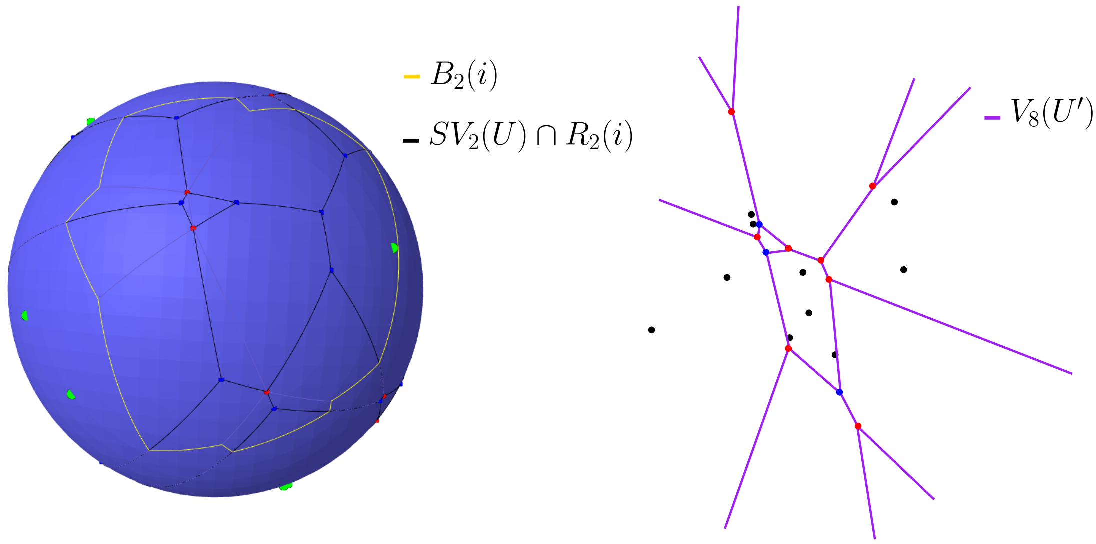
Now, we can proceed in a similar way to [3] to prove the construction for Voronoi diagrams on the sphere of arbitrary order, $SV_k(U)$. From now on, we denote by S' the plane inverse of the sphere S, by U' the set of points on the plane S' that are inverses of the points of $U \subset S$, and by $V_k(U')$ the Voronoi diagram of order k in the plane for the set of points U'.

▶ **Theorem 3.2.** *Let $i \notin U$ be a point on the sphere S such that $U \cup \{i\}$ is in general position. Let U' be the set of inverse points of U for a chosen inversion radius r and i the center of inversion. Then $SV_k(U)$ is homeomorphic to the union of $V_k(U')$ and $V_{n-k}(U')$, joined by the unbounded edges common to $V_k(U')$ and $V_{n-k}(U')$ (unbounded edges from the same bisector are glued together). Moreover, $R_k(i)$ in $SV_k(U \cup \{i\})$ partitions $SV_k(U)$ into two subgraphs that are homeomorphic to $V_k(U')$ and $V_{n-k}(U')$. The vertices of type I (type II) in $V_k(U')$ correspond to the vertices of type I (type II) in $SV_k(U)$ and the vertices of type I (type II) in $V_{n-k}(U')$ correspond to the vertices of type II (type I) in $SV_k(U)$. See Figures 5 and 6.*



(a) (b)

Figure 5 For a set U of ten points on the sphere (the visible ones are drawn in green color): The picture shows the homeomorphism between: (a) The induced graph by $SV_2(U)$ at the exterior of $R_2(i)$ in $SV_2(U \cup \{i\})$. (b) The planar Voronoi diagram of order 2 for the points of U' (black color).



(a) (b)

Figure 6 For a set U of ten points on the sphere (the visible ones are drawn in green color): The picture shows the homeomorphism between: (a) The induced graph by $SV_2(U)$ at the interior of $R_2(i)$ in $SV_2(U \cup \{i\})$. (b) The planar Voronoi diagram of order 8 for the points of U' (black color).

Theorem 3.2 tells us how to construct $SV_k(U)$: we just have to invert the points of U, compute planar Voronoi diagrams $V_k(U')$ and $V_{n-k}(U')$, and map them to the sphere as follows: each vertex $a'b'c'$ of either $V_k(U')$ or $V_{n-k}(U')$ corresponds to a vertex abc of $SV_k(U)$ (abc is center of the circle that passes through a, b and c on the sphere); vertices in $SV_k(U)$ are adjacent whenever the correspong vertices in $V_k(U')$ or in $V_{n-k}(U')$ are adjacent. Finally, the vertices of $SV_k(U)$ corresponding to vertices incident to an unbounded edge from the same bisector in $V_k(U')$ and $V_{n-k}(U')$ get connected.

Let us shortly also comment on the computational complexity of constructing higher order Voronoi diagrams on the sphere. The inversion is a linear time transformation and, once we have the planar Voronoi diagrams, mapping them to the sphere also only requires linear time. Therefore, the computational time for constructing the spherical Voronoi diagrams is bounded by the computational time for the planar ones. See [8] for a discussion on the several algorithms for higher order Voronoi diagrams.

Now, from these constructions, it is easy to see that properties proved for the plane [4] must be true for the sphere. We can prove easily some properties on the sphere using results from the plane, but also we can prove properties in the plane using the sphere. Next, we show that the number of vertices of type I (type II) in $SV_k(U)$ only depends on the number n of points of U, but not on their positions on the sphere.

▶ **Theorem 3.3.** *For a set U of n points on the sphere, the number of vertices of type I in $SV_k(U)$ is $2k(n-k-1)$ and the number of vertices of type II is $2(k-1)(n-k)$.*

Proof. By Theorem 3.2, we can define an inversion transformation such that there is a one-to-one correspondence between the vertices of $SV_k(U)$ and the vertices of $V_k(U')$ and $V_{n-k}(U')$. Vertices of type I of $SV_k(U')$ and vertices of type II of $V_{n-k}(U')$ correspond to the vertices of type I in $SV_k(U)$. Then, the number of vertices of type I in $SV_k(U)$ is the sum of type I vertices of $V_k(U')$ and type II vertices of $V_{n-k}(U')$ which correspond to the circles enclosing $k-1$ points of U' and circles enclosing $n-k-2$ points of U', respectively. We denote the number of such circles with c_{k-1} and c_{n-k-2}. By Theorem 5.3 of [9], we have

$$c_{k-1} + c_{n-k-2} = 2(k-1+1)(n-2-k+1) = 2k(n-k-1). \tag{1}$$

Then, the number of vertices of type I in $SV_k(U)$ is $2k(n-k-1)$. Similarly, we can compute the number of vertices of type II as the sum of vertices of type II in $V_k(U')$ and type I in $V_{n-k}(U')$, i.e., the number of the circles enclosing $k-2$ points of U', c_{k-2}, and enclosing $n-k-1$ points of U', c_{n-k-1}. Again, using Theorem 5.3 of [9], we have

$$c_{k-2} + c_{n-k-1} = 2(k-2+1)(n-2-k+2) = 2(k-1)(n-k). \tag{2}$$

Then, the number of vertices of type II in $SV_k(U)$ is $2(k-1)(n-k)$. ◀

▶ **Theorem 3.4.** *For a set U of n points on the sphere, the order k Voronoi diagram $SV_k(U)$ has $4kn-4k^2-2n$ vertices, $6kn-6k^2-3n$ edges and $2kn-2k^2-n+2$ faces.*

Proof. Vertices of spherical Voronoi diagrams are either of type I or type II, so the total number of vertices is the sum of vertices of the two types. Then, by Theorem 3.3, the number of vertices $|V|$ is

$$|V| = 2k(n-k-1) + 2(k-1)(n-k) = 4kn - 4k^2 - 2n. \tag{3}$$

Now, as each vertex has degree three in $SV_k(U)$, we can count the total number of edges. Since each edge is incident to two vertices, by double counting, the number of edges $|E|$ is

$$|E| = \frac{3}{2}\left(-4k^2 + 4kn - 2n\right) = 6kn - 6k^2 - 3n. \tag{4}$$

Finally, as $SV_k(U)$ is a planar graph, we can apply Euler's Formula to count the number of faces $|F|$, and we have

$$|F| = 2 - (-4k^2 + 4kn - 2n) + (-6k^2 + 6kn - 3n) = 2kn - 2k^2 - n + 2. \tag{5}$$

◀

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