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Author for correspondence:
Ramón Quintanilla
e-mail: ramon.quintanilla@upc.edu

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Decay of waves in strain gradient porous elasticity with Moore-Gibson-Thompson dissipation

J. R. Fernández[1], A. Magaña[2] and R. Quintanilla[2]

[1] Departamento de Matemática Aplicada I, Universidade de Vigo, Escola de Enxeñería de Telecomunicación, Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain
[2] Departamento de Matemáticas, Universitat Politècnica de Catalunya, ESEIAAT, Colom 11, 08222 Terrassa, Barcelona, Spain

We study a one-dimensional problem arising in strain gradient porous-elasticity. Three different Moore-Gibson-Thompson dissipation mechanisms are considered: viscosity and hyperviscosity on the displacements, and weak viscoporosity. Existence and uniqueness of solutions are proved. The energy decay is also shown, being polynomial for the two first situations, unless a particular choice of the constitutive parameters is made in the hyperviscosity case. Finally, for the weak viscoporosity, only the slow decay can be expected.

1. Introduction

Porous elastic materials are widely used in common life due to its low density and large surface, which give rise to a range of specific properties regarding the physical, mechanical, thermal, electrical and acoustic fields. The internal porous structure of the material highly determines its physical properties [17]. Applications of porous materials can be found in many areas, from biomedicine to the building industry. In the former, to repair injuries in bones, for example [35]. In the latter, to make light, hard and fire-resistant parts. As a matter of illustration we cite a sentence from the book of Liu and Chen [23]: "the use of porous metals in elevators can reduce energy consumption and absorb impacts, and their good specific stiffness makes them ideal to make cabin panels".

More applications can be found in the classical works of Biot [2,3] and also in the book of Straughan [32]. As these materials are so common and useful, it is necessary to study and understand how the porous structure of the material affects its elastic behavior.

Nunziato and Cowin [26] extended the classical theory of elasticity to the context of porous materials. They describe the behavior of elastic solids with voids supposing that the materials have a skeleton or matrix material that is elastic and the interstices are voids of material. A great amount of papers has been published since then analyzing this theory (without trying to be exhaustive, see, for instance, [5,7,8,13,14]). Even materials with a double porosity structure are being studying nowadays [9,19,20].

In order to obtain more detailed models for the configuration of the materials and their response to stimuli, some researchers proposed the inclusion of higher order gradients in the basic postulates of elasticity [15,24,34]. First, when referring to these new postulates the materials were called *non-simple*, but now the theories including the second gradient of the displacement (or the second gradient of the volume fraction field, or both of them) in the set of independent constitutive variables are called *strain gradient* theories.

On the other hand, a lot of attention has been paid recently to the Moore–Gibson–Thompson (MGT) equation, which arises in acoustics and accounts for the second sound effects and the associated thermal relaxation in viscous fluids [10,25,33]. Some authors track the MGT equation until the work of Stokes [31]. Regardless of its first appearance, the use of the MGT equation in the viscoelasticity theory seems to produce a model which is considered to be more faithful to reality than the usual Kelvin–Voigt one for the linear deformations of a viscoelastic solid [11,28, 29]. In fact, the linear Kelvin-Voigt viscoelasticity allows the instantaneous propagation of the mechanical waves (see [30], page 39), which contradicts the causality principle. However, the waves in the MGT equation propagate with finite velocity [27].

In this work we study a linear porous-elastic problem including three possible MGT dissipation mechanisms: two of them assumed to be on the displacement (leading to the viscoelasticity and hyperviscoelasticity cases), and the third one on the porosity (leading to the weak viscosity case). Existence and uniqueness are proved for the three cases by using the theory of linear semigroups. Generically, the energy decay is shown to be polynomial. Nevertheless, there is a particular choice of the constitutive parameters in the hyperviscoelasticity case that produces the exponential decay.

The structure of this paper is the following. In the next section we recall the evolution and constitutive equations we will use and we impose the boundary conditions for the variables. As we consider several problems, depending on the damping effect, the initial conditions will be established later. Then, in Section 3, the viscoelasticity case is considered, assuming that the MGT dissipation mechanism is included in the second-order term of the displacements. The existence and uniqueness of solutions and the polynomial energy decay are proved. A similar analysis is performed in Section 4 for the hyperviscoelasticity case. However, an exponential decay is obtained for a special choice of the constitutive parameters. Finally, in Section 5, we consider MGT dissipation mechanisms in the porosity. Since two cases can be analyzed as in the previous sections, only the weak viscoporosity case is studied. Again, the existence and uniqueness of solutions are proved. We also show that the decay of the solutions can only be slow. However, in this case we note that to clarify if the decay can be controlled by a polynomial is still an open question. Section 6 concludes the work.

2. Basic equations

In the context of the strain gradient porous-elasticity the evolution equations for the linear theory in the one-dimensional case are (see [18]):

$$\rho \ddot{u} = \vartheta_x - \mu_{xx}, \quad J\ddot{\varphi} = \chi_x - \sigma_{xx} + g, \quad x \in [0,\pi],\ t \geq 0.$$

Here, ρ is the mass density that must be a positive constant, J is the equilibrated inertia that must also be a positive constant, u is the displacement, φ is the fraction of volume, ϑ is the stress, μ is the hyperstress, χ is the equilibrated stress, σ is the equilibrated hyperstress and g is the equilibrated body force. Henceforth, the superscript dot means material time derivative.

The general form for the constitutive equations for a strain gradient viscoelastic solid are (see [18] for details)[1]:

$$
\begin{aligned}
\vartheta &= \int_{-\infty}^{t} \Big(a(t-s)\dot{u}_x(s) + b(t-s)\dot{\varphi}(s) + \beta(t-s)\dot{\varphi}_{xx}(s)\Big)\, ds,\\
\mu &= \int_{-\infty}^{t} \Big(k_1(t-s)\dot{u}_{xx}(s) + \gamma(t-s)\dot{\varphi}_x(s)\Big)\, ds,\\
\chi &= \int_{-\infty}^{t} \Big(\gamma(t-s)\dot{u}_{xx}(s) + \alpha(t-s)\dot{\varphi}_x(s)\Big)\, ds,\\
\sigma &= \int_{-\infty}^{t} \Big(\beta(t-s)\dot{u}_x(s) + d(t-s)\dot{\varphi}(s) + k_2(t-s)\dot{\varphi}_{xx}(s)\Big)\, ds,\\
g &= \int_{-\infty}^{t} \Big(-b(t-s)\dot{u}_x(s) - \xi(t-s)\dot{\varphi}(s) - d(t-s)\dot{\varphi}_{xx}(s)\Big)\, ds.
\end{aligned}
$$

In this paper, we are going to assume the boundary conditions

$$
\begin{aligned}
&u(0,t)=u(\pi,t)=u_{xx}(0,t)=u_{xx}(\pi,t)=0,\\
&\varphi_x(0,t)=\varphi_x(\pi,t)=\varphi_{xxx}(0,t)=\varphi_{xxx}(\pi,t)=0,
\end{aligned}
\tag{2.1}
$$

which are completely compatible with the boundary conditions proposed by Iesan [18].

The initial conditions will be set for each problem that we will consider.

3. First system: viscoelasticity

In this section we study the system obtained when

$$
\begin{aligned}
&a(s)=a+(a^*/\tau - a)e^{-s/\tau}, \quad b(s)=b, \quad \beta(s)=\beta, \quad k_1(s)=k_1,\\
&\gamma(s)=\gamma, \quad \alpha(s)=\alpha, \quad d(s)=d, \quad k_2(s)=k_2, \quad \xi(s)=\xi.
\end{aligned}
$$

Notice that, in this case, the damping is set in the gradient of the displacement.

If we denote by $\hat{\varphi}(x,t)=\varphi(x,t)+\tau\dot{\varphi}(x,t)$ and we assume that all the variables vanish at time $t=-\infty$ we get

$$
\begin{aligned}
&\rho(\ddot{u}+\tau\dddot{u}) = au_{xx} + a^*\dot{u}_{xx} + b\hat{\varphi}_x - \eta\hat{\varphi}_{xxx} - k_1(u_{xxxx}+\tau\dot{u}_{xxxx}),\\
&J\ddot{\hat{\varphi}} = \eta(u_{xxx}+\tau\dot{u}_{xxx}) - b(u_x+\tau\dot{u}_x) + \delta\hat{\varphi}_{xx} - \xi\hat{\varphi} - k_2\hat{\varphi}_{xxxx},
\end{aligned}
$$

where $\eta=\gamma-\beta$ and $\delta=\alpha-2d$. From now on, to simplify the notation, we will omit the hats over the variables. We assume that ρ, a, k_1, a^*, J, δ, ξ and k_2 are positive as well as we impose that $a\xi > b^2$, $k_1\delta > \eta^2$ and $a^* > \tau a$. These assumptions are usual in the studies of porous-elasticity. They guarantee that the energy of the system is positive definite (see (3.4)). At the same time, the last condition implies that the dissipation is positive (see (3.5)). We also suppose that $\eta \neq 0$, which implies a strong coupling. The initial conditions we will consider in this case are:

$$
u(x,0)=u^0(x),\ \dot{u}(x,0)=v^0(x),\ \ddot{u}(x,0)=c^0(x),\ \varphi(x,0)=\varphi^0(x),\ \dot{\varphi}(x,0)=\psi^0(x). \tag{3.1}
$$

To study our problem, it will be useful to change the variables. We will denote $u_1=u+\tau\dot{u}$, $v=\dot{u}$ and $u_3=\dot{u}+\tau\ddot{u}$. Therefore, our system of equations becomes

$$
\begin{aligned}
&\rho\ddot{u}_1 = au_{1xx} + \bar{a}\dot{v}_{xx} + b\varphi_x - \eta\varphi_{xxx} - k_1u_{1xxxx},\\
&J\ddot{\varphi} = \eta u_{1xxx} - bu_{1x} + \delta\varphi_{xx} - \xi\varphi - k_2\varphi_{xxxx},
\end{aligned}
\tag{3.2}
$$

[1] It is worth noting that these equations can be also obtained assuming the invariance of the entropy under time reversal, in the line proposed in [16] and [6]

where $\overline{a} = a^* - \tau a$. We study our problem in a suitable Hilbert space $\mathcal{H} = (H^2 \cap H_0^1) \times H_0^1 \times L^2 \times H_*^2 \times L_*^2$, where

$$L_*^2 = \{f \in L^2\,;\, \int_0^\pi f(x)\,dx = 0\}, \quad H_*^i = H^i \cap L_*^2 \text{ for } i = 1, 2,$$

and, to simplify, we write L^2 instead of $L^2(0,\pi)$, and analogously for the H's.

We define an inner product in $\mathcal{H}$: if we denote $U = (u_1, v, u_3, \varphi, \psi)$ and $U^* = (u_1^*, v^*, u_3^*, \varphi^*, \psi^*)$, therefore

$$\langle U, U^* \rangle = \frac{1}{2}\int_0^\pi \Big(\rho u_3 \overline{u_3^*} + J\psi\overline{\psi^*} + W\Big)\,dx, \tag{3.3}$$

where

$$W = k_1 u_{1xx}\overline{u_{1xx}^*} + a u_{1x}\overline{u_{1x}^*} + \tau\overline{a} v_x \overline{v_x^*} + \eta(u_{1xx}\overline{\varphi_x^*} + \overline{u_{1xx}^*}\varphi_x)$$
$$+ b(u_{1x}\overline{\varphi^*} + \overline{u_{1x}^*}\varphi_x) + \xi\varphi\overline{\varphi^*} + \delta\varphi_x\overline{\varphi_x^*} + k_2\varphi_{xx}\overline{\varphi_{xx}^*}.$$

As usual, a superposed bar on the variables denotes the conjugate of a complex number. It is clear that this inner product defines a norm that is equivalent to the usual one in the Hilbert space.

From the above inner product it is easy to write the equality of the energy for system (3.2):

$$E(t) + \int_o^\pi D(s)\,ds = E(0),$$

where

$$E(t) = \frac{1}{2}\int_0^\pi \Big(\rho|\dot{u}_1|^2 + J|\dot{\varphi}|^2 + k_1|u_{1xx}|^2 + a|u_{1x}|^2 + \tau\overline{a}|\dot{u}_x|^2 + 2\eta u_{1xx}\varphi_x + 2bu_{1x}\varphi$$
$$+ \xi|\varphi|^2 + \delta|\varphi_x|^2 + k_2|\varphi_{xx}|^2\Big)\,dx \tag{3.4}$$

and

$$D(s) = \overline{a}\int_0^\pi |\dot{u}_x|^2\,dx \tag{3.5}$$

We can write our problem in the following way:

$$\dot{u}_1 = u_3, \quad \dot{v} = \tau^{-1}(u_3 - v), \quad \dot{\varphi} = \psi,$$
$$\dot{u}_3 = \frac{1}{\rho}\Big[-k_1 D^4 u_1 + aD^2 u_1 + \overline{a}D^2 v + bD\varphi - \eta D^3\varphi\Big],$$
$$\dot{\psi} = \frac{1}{J}\Big[\eta D^3 u_1 - bDu_1 + \delta D^2\varphi - k_2 D^4\varphi - \xi\varphi\Big],$$

or more synthetic as

$$\frac{dU}{dt} = \mathcal{A}U, \quad U^0 = (u_1^0, v^0, u_3^0, \varphi^0, \psi^0), \tag{3.6}$$

where $u_1^0 = u^0 + \tau v^0$, $u_3^0 = v^0 + \tau c^0$ and

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & I & 0 & 0 \\ 0 & -\tau^{-1}I & \tau^{-1}I & 0 & 0 \\ \dfrac{aD^2 - k_1D^4}{\rho} & \dfrac{\overline{a}D^2}{\rho} & 0 & \dfrac{bD - \eta D^3}{\rho} & 0 \\ 0 & 0 & 0 & 0 & I \\ \dfrac{\eta D^3 - bD}{J} & 0 & 0 & \dfrac{\delta D^2 - k_2 D^4 - \xi}{J} & 0 \end{pmatrix}.$$

The domain of this operator is given by the elements in $\mathcal{H}$ such that

$$\overline{a}D^2 v - \eta D^3\varphi - k_1 D^4 u_1 \in L^2, \quad \eta D^3 u_1 - k_2 D^4\varphi \in L^2, \quad u_3 \in H^2 \cap H_0^1, \quad \psi \in H_*^2,$$

and

$$D^2u_1(0) = D^2u_1(\pi) = D^3\varphi(0) = D^3\varphi(\pi) = 0.$$

We first prove the existence of a semigroup of contractions generating the solutions to our problem. We note that the domain of the operator $\mathcal{A}$ is dense in the Hilbert space $\mathcal{H}$. On the other side, in view of the boundary conditions we can see that

$$\Re\langle \mathcal{A}U, U\rangle = -\frac{\overline{a}}{2}\int_0^\pi |v_x|^2\,dx \le 0, \tag{3.7}$$

for every U belonging to the domain of the operator $\mathcal{A}$, or $\mathcal{D}(\mathcal{A})$.

Lemma 3.1. *Zero belongs to the resolvent of the operator $\mathcal{A}$.*

Proof. For any $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$ we will find $U \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}\mathcal{U} = \mathcal{F}$. That is,

$$\begin{aligned}
&u_3 = f_1, \quad -v + u_3 = \tau f_2, \quad \psi = f_4,\\
&-k_1 D^4 u_1 + aD^2 u_1 + \overline{a}D^2 v + bD\varphi - \eta D^3\varphi = \rho f_3,\\
&\eta D^3 u_1 - bDu_1 + \delta D^2\varphi - k_2 D^4\varphi - \xi\varphi = J f_5.
\end{aligned}$$

We can obtain u_3, v and ψ in a straightforward way. So, it leads to the system:

$$\begin{aligned}
&-k_1 D^4 u_1 + aD^2 u_1 + bD\varphi - \eta D^3\varphi = \rho f_3 - \tau\overline{a}D^2 f_2 + \overline{a}D^2 f_1,\\
&\eta D^3 u_1 - bDu_1 + \delta D^2\varphi - k_2 D^4\varphi - \xi\varphi = J f_5.
\end{aligned} \tag{3.8}$$

Let us denote $F_1 = \rho f_3 - \tau\overline{a}D^2 f_2 + \overline{a}D^2 f_1$ and $F_2 = J f_5$. We can solve the above system by writing F_1 and F_2 in their expressions as Fourier series:

$$\begin{aligned}
F_1 &= \sum_{n=1}^{\infty} F_1^{(n)} \sin(nx) \text{ with } \sum_{n=1}^{\infty} \frac{\left(F_1^{(n)}\right)^2}{n^2} < \infty,\\
F_2 &= \sum_{n=1}^{\infty} F_2^{(n)} \cos(nx) \text{ with } \sum_{n=1}^{\infty} \left(F_2^{(n)}\right)^2 < \infty.
\end{aligned} \tag{3.9}$$

We look for solutions of the form

$$\begin{aligned}
u_1 &= \sum_{n=1}^{\infty} u_1^{(n)} \sin(nx),\\
\varphi &= \sum_{n=1}^{\infty} \varphi^{(n)} \cos(nx).
\end{aligned} \tag{3.10}$$

Replacing expressions (3.9) and (3.10) in system (3.8) and simplifying, we get for each n another system of equations:

$$\begin{aligned}
&(k_1 n^4 + an^2)u_1^{(n)} + (bn + \eta n^3)\varphi^{(n)} = -F_1^{(n)},\\
&(bn + \eta n^3)u_1^{(n)} + (k_2 n^4 + \delta n^2 + \xi)\varphi^{(n)} = -F_2^{(n)}.
\end{aligned} \tag{3.11}$$

We obtain solutions for u_1 and φ satisfying our required conditions. To be precise, we obtain

$$\begin{aligned}
u_1^{(n)} &= \frac{(bn + \eta n^3)F_2^{(n)} - (k_2 n^4 + \delta n^2 + \xi)F_1^{(n)}}{(k_1 n^4 + an^2)(k_2 n^4 + \delta n^2 + \xi) - (bn + \eta n^3)^2},\\
\varphi^{(n)} &= \frac{(bn + \eta n^3)F_1^{(n)} - (k_1 n^4 + an^2)F_2^{(n)}}{(k_1 n^4 + an^2)(k_2 n^4 + \delta n^2 + \xi) - (bn + \eta n^3)^2},
\end{aligned} \tag{3.12}$$

satisfying $\sum_{n=1}^{\infty} n^4 \left(u_1^{(n)}\right)^2 < \infty$ and $\sum_{n=1}^{\infty} n^4 \left(\varphi^{(n)}\right)^2 < \infty$.

Notice also that if $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5) = (0, 0, 0, 0, 0)$, therefore $u_3 = 0$, $v = 0$, $\psi = 0$ and $F_1 = F_2 = 0$, which implies $u_1 = 0$ and $\varphi = 0$. This proves the injectivity.

Moreover, we can see that

$$\|U\|_{\mathcal{H}} \le K\|\mathcal{F}\|_{\mathcal{H}}, \tag{3.13}$$

where K is a constant independent of U. □

The Lumer-Phillips corollary to the Hille-Yosida theorem says that to show that $\mathcal{A}$ generates a contraction semigroup it is enough to check that the domain of the operator is dense in the Hilbert space $\mathcal{H}$, that condition (3.7) holds and that zero belongs to the resolvent of the operator (see, for example, page 3 of reference [22]). As a consequence, the above results prove the following theorem.

Theorem 3.1. *The operator $\mathcal{A}$ generates a C^0-semigroup of contractions. Therefore, for each $U^0 \in \mathcal{D}(\mathcal{A})$ there exists a unique solution $U(t) \in C^1([0,\infty);\mathcal{H}) \cap C^0([0,\infty);\mathcal{D}(\mathcal{A}))$ to our problem.*

We analyze now the decay of the solutions. We first note that, whenever $b + \eta n^2 = 0$ for $n \in \mathbb{N}$, undamped solutions can be obtained: take for example, $u = 0$ and $\varphi(x,t) = \sin(\omega t)\cos(nx)$ with $J\omega^2 = \delta n^2 + k_2 n^4 + \xi$.

Therefore, from now on (in this section) we assume that $b + \eta n^2 \neq 0$ for every $n \in \mathbb{N}$.

We will see that the solutions cannot decay exponentially, To be precise, we prove the existence of solutions of the form

$$u = A_1 e^{\omega t} \sin nx, \quad \varphi = A_2 e^{\omega t} \cos nx, \quad n = 1, 2, 3 \ldots \tag{3.14}$$

such that $\Re(\omega) > -\epsilon$ for all positive ϵ sufficiently small.

We obtain nonzero solutions of this form whenever

$$\det \begin{pmatrix} k_1(1+\tau\omega)n^4 + (a + a^*\omega)n^2 + \rho(\omega^2 + \tau\omega^3) & \eta n^3 + bn \\ \eta(n^3 + \tau n^3 \omega) + b(n + \tau n \omega) & k_2 n^4 + \delta n^2 + J\omega^2 + \xi \end{pmatrix} = 0.$$

We denote by $p(x) = a_0 x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5$ the fifth degree polynomial obtained from this determinant replacing ω by x. Of course, its coefficients depend on the parameters of the system in the following way:

$$\begin{array}{l} a_0 = J\rho\tau, \quad a_1 = J\rho, \quad a_2 = (Jk_1\tau + k_2\rho\tau)n^4 + (Ja^* + \delta\rho\tau)n^2 + \rho\tau\xi, \\ a_3 = (Jk_1 + k_2\rho)n^4 + (Ja + \delta\rho)n^2 + \rho\xi, \\ a_4 = k_1 k_2 \tau n^8 + (\delta k_1 \tau - \tau\eta^2 + a^* k_2)n^6 + (a^*\delta - 2b\eta\tau + k_1\tau\xi)n^4 + (a^*\xi - b^2\tau)n^2, \\ a_5 = k_1 k_2 n^8 + (\delta k_1 - \eta^2 + a k_2)n^6 + (a\delta - 2b\eta + k_1\xi)n^4 + (a\xi - b^2)n^2. \end{array} \tag{3.15}$$

We will prove that there are roots of $p(x)$ as near to the imaginary axis of the complex plane as we desire, or equivalently, that there are roots of $p(x-\epsilon) = b_0 x^5 + b_1 x^4 + b_2 x^3 + b_3 x^2 + b_4 x + b_5$ with positive real part for any small $\epsilon > 0$. The coefficients of this polynomial depend on the constitutive parameters and ϵ (the full expression of each b_i can be seen in a little appendix at the end of the paper).

Thus, we will apply the Routh-Hurwitz theorem (see Dieudonné [12]), which states that, if $b_0 > 0$, then all the roots of polynomial $p(x-\epsilon)$ have negative real part if, and only if, all the leading diagonal minors of matrix

$$\begin{pmatrix} b_1 & b_0 & 0 & 0 & 0 \\ b_3 & b_2 & b_1 & b_0 & 0 \\ b_5 & b_4 & b_3 & b_2 & b_1 \\ 0 & 0 & b_5 & b_4 & b_3 \\ 0 & 0 & 0 & 0 & b_5 \end{pmatrix} \tag{3.16}$$

are positive.

The second leading minor of the above Routh-Hurwitz matrix is a fourth degree polynomial on n whose main coefficient is always negative. To be precise, if we denote by L_i the leading minors of this matrix,

$$\begin{array}{l} L_2 = -2\epsilon J\rho\tau^2(k_1 J + k_2\rho)n^4 + \left(J^2\rho(a^* - a\tau) - 2J\delta\epsilon\rho^2\tau^2 - 2J^2 a^*\epsilon\rho\tau\right)n^2 \\ \quad -40J^2\epsilon^3\rho^2\tau^2 + 24J^2\epsilon^2\rho^2\tau - 4J^2\epsilon\rho^2 - 2\xi J\epsilon\rho^2\tau^2. \end{array}$$

Therefore, it is clear that, for n large enough and ϵ small, L_2 will be negative.

We prove now that the solutions to our problem decay in a polynomial way. To this end, we use the result of Borichev and Tomilov [4], which guarantees that the solutions decay as $t^{-1/\alpha}$ $(\alpha > 0)$ whenever the imaginary axis is contained in the resolvent and the following asymptotic condition holds:

$$\overline{\lim_{|\lambda|\to\infty}} \lambda^{-\alpha}\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \tag{3.17}$$

where $\mathcal{L}(\mathcal{H})$ is the space of linear bounded maps from $\mathcal{H}$ to $\mathcal{H}$.

To show these conditions it is suitable to decompose $\mathcal{H} = \mathcal{K}^N \oplus \mathcal{K}$, where $\mathcal{K}^N$ is the finite dimensional subspace generated by

$$\Omega(i,j,k,l,m) = (\sin ix, \sin jx, \sin kx, \cos lx, \cos mx) \quad 1 \le i,j,k,l,m \le N.$$

It is worth noting that $\mathcal{K}^N$ is invariant under the semigroup and the solutions starting in this subspace always belong to it. In fact, a solution $U(t)$ can be written as $U_1(t) + U_2(t)$, where $U_1(t) \in \mathcal{K}^N$ and $U_2(t) \in \mathcal{K}$. We first study the solution $U_1(t)$ and we note that, since $\mathcal{K}^N$ is finite-dimensional, to prove the exponential decay of solutions it is sufficient to see that all the eigenvalues have negative real part.

Proposition 3.1. *All the eigenvalues of $\mathcal{A}$ restricted to $\mathcal{K}^N$ have negative real part.*

Proof. We set again $u = A_1 e^{\omega t}\sin nx$ and $\varphi = A_2 e^{\omega t}\cos nx$. Substituting these expressions in our system we obtain the polynomial $p(x)$ with coefficients a_i for $i = 0, 1, ..., 5$ as in (3.15). Let M_i for $i = 1, ..., 4$ be the leading minors associated with its Routh-Hurwitz matrix. Direct computation gives:

$$M_1 = J\rho, \quad M_2 = J^2\rho\bar{a}n^2, \quad M_3 = J^3\rho\bar{a}n^4(k_1n^2 + a), \quad M_4 = J^3n^6\rho\bar{a}^2\left(b + \eta n^2\right)^2,$$
$$M_5 = J^3n^8\rho\bar{a}^2\left(b + \eta n^2\right)^2\left(k_1k_2n^6 + (ak_2 + \delta k_1 - \eta^2)n^4 + (a\delta - 2b\eta + k_1\xi)\,n^2 + a\xi - b^2\right).$$

Since $\bar{a} = a^* - \tau a > 0$, $a\xi > b^2$, $\delta k_1 > \eta^2$ and $b + \eta n^2 \neq 0$ for every $n \in \mathbb{N}$ we see that $M_i > 0$ for every $i = 1, \dots, 5$. □

Lemma 3.2. *The imaginary axis is contained in the resolvent of the operator $\mathcal{A}$.*

Proof. We will assume that the thesis of the lemma does not hold and we will obtain a contradiction.

A standard argument (see [22], page 25) shows that, if we assume that there exists an element of the imaginary axis at the spectrum, there will exist a sequence of real numbers λ_n such that $\lambda_n \to \bar{\lambda} \in \mathbb{R}$, with $|\lambda_n| < |\bar{\lambda}|$, and a sequence of unit norm vectors $U_n = (u_{1n}, v_n, u_{3n}, \varphi_n, \psi_n) \in \mathcal{D}(\mathcal{A})$ such that $\|(i\lambda_n I - \mathcal{A})U_n\| \to 0$. This is equivalent to write the following convergences:

$$i\lambda_n u_{1n} - u_{3n} \to 0 \quad \text{in} \quad H^2, \tag{3.18}$$

$$i\tau\lambda_n v_n + v_n - u_{3n} \to 0 \quad \text{in} \quad H^1, \tag{3.19}$$

$$i\rho\lambda_n u_{3n} - [aD^2u_{1n} + \bar{a}D^2v_n + bD\varphi_n - \eta D^3\varphi_n - k_1D^4u_{1n}] \to 0 \quad \text{in} \quad L^2, \tag{3.20}$$

$$i\lambda_n\varphi_n - \psi_n \to 0 \quad \text{in} \quad H^2, \tag{3.21}$$

$$i\lambda_n J\psi_n - [\eta D^3u_{1n} - bDu_{1n} + \delta D^2\varphi_n - \xi\varphi_n - k_2D^4\varphi_n] \to 0 \quad \text{in} \quad L^2. \tag{3.22}$$

From the left hand side of inequality (3.7) we see that $v_n \to 0$ in H^1. Then, $u_{3n} \to 0$ and $\lambda_n u_{1n} \to 0$ in H^1. If we multiply convergence (3.20) by u_{1n} we obtain that $u_{1n} \to 0$ in H^2. Now, we multiply (3.20) by $D\varphi_n$ to find that

$$b\|D\varphi_n\|^2 + \eta\|D^2\varphi_n\|^2 - k_1 < Du_{1n}, D^4\varphi_n > \to 0,$$

but

$$D^4\varphi_n \sim \frac{1}{k_2}[i\lambda_n J\psi_n - \eta D^3u_{1n} + bDu_{1n} - \delta D^2\varphi_n - \xi\varphi_n]$$

and

$$< Du_{1n}, i\lambda_n J\psi_n > = < \lambda_n Du_{1n}, iJ\psi_n > \to 0,$$
$$< Du_{1n}, \eta D^3u_{1n} > = -\eta\|D^2u_{1n}\|^2 \to 0.$$

We write $\sim$ to indicate that the expression on the left hand side is equivalent to the one on the right hand side when n tends to infinity.

Therefore, we see that

$$b\|D\varphi_n\|^2 + \eta\|D^2\varphi_n\|^2 \to 0.$$

Notice that for $\varphi \in \langle \cos nx, \cos(n+1)x, \dots \rangle$ the Poincaré's constant is n^2. Therefore, if we assume that we work in a suitable subspace of $\mathcal{H}$, from the Poincaré's inequality we see that $D^2\varphi_n \to 0$.

Now, if we multiply convergence (3.22) by φ_n we also have $\psi_n \to 0$ and we obtain the contradiction. □

Lemma 3.3. *The asymptotic stability condition (3.17) holds with* $\alpha = 2$.

Proof. If we assume that the thesis of the lemma is not true, then there exist a sequence of real numbers λ_n with $|\lambda_n| \to \infty$ and a sequence of unit norm vectors U_n such that

$$\lambda_n^2\|(i\lambda_n I - \mathcal{A})U_n\| \to 0.$$

That is, we obtain the following convergences:

$$\begin{aligned}
&\lambda_n^2(i\lambda_n u_{1n} - u_{3n}) \to 0 \quad \text{in} \quad H^2,\\
&\lambda_n^2(i\tau\lambda_n v_n + v_n - u_{3n}) \to 0 \quad \text{in} \quad H^1,\\
&\lambda_n^2(i\rho\lambda_n u_{3n} - [aD^2u_{1n} + \overline{a}D^2v_n + bD\varphi_n - \eta D^3\varphi_n - k_1D^4u_{1n}]) \to 0 \quad \text{in} \quad L^2,\\
&\lambda_n^2(i\lambda_n\varphi_n - \psi_n) \to 0 \quad \text{in} \quad H^2,\\
&\lambda_n^2(iJ\lambda_n\psi_n - [\eta D^3u_{1n} - bDu_{1n} + \delta D^2\varphi_n - \xi\varphi_n - k_2D^4\varphi_n]) \to 0 \quad \text{in} \quad L^2.
\end{aligned}$$

Again, we find that $\lambda_n v_n \to 0$ in L^2. Therefore, $u_{3n} \to 0$ and $\lambda_n u_{1n} \to 0$ in L^2. From here, we can follow the same argument of the previous lemma to obtain a contradiction. □

Theorem 3.2. *The semigroup generated by the operator* $\mathcal{A}$ *is polynomially stable, that is, for every* $U(0) \in \mathcal{D}(\mathcal{A})$ *there exists a positive constant* C *independent of the initial data such that*

$$\|U(t)\|_{\mathcal{H}} \le Ct^{-1/2}\|U(0)\|_{\mathcal{D}(\mathcal{A})}.$$

Physically, it means that the solutions, and then the mechanical waves, decay as a polynomial: they slowly dampen as $t^{-1/2}$. This result highly differs from the one obtained by Liu *et al.* [21], where the mechanical dissipation was the usual one (not of MGT type) and there were not high order effects in the porosity.

4. Second system: Hyperviscoelasticity

We introduce now viscosity effect on the hyperstress. That is, we consider:

$$\begin{aligned}
&a(s) = a, \quad b(s) = b, \quad \beta(s) = \beta, \quad k_1(s) = k_1 + (k_1^*/\tau - k_1)e^{-s/\tau},\\
&\gamma(s) = \gamma, \quad \alpha(s) = \alpha, \quad d(s) = d, \quad k_2(s) = k_2, \quad \xi(s) = \xi.
\end{aligned}$$

Assuming again that the variables u and φ (and their derivatives until fourth order) vanish when the time goes to minus infinity, we obtain the system:

$$\begin{aligned}
&\rho(\ddot{u} + \tau\dddot{u}) = a(u_{xx} + \tau\dot{u}_{xx}) + b\varphi_x - \eta\varphi_{xxx} - k_1u_{xxxx} - k_1^*\dot{u}_{xxxx},\\
&J\ddot{\varphi} = \eta(u_{xxx} + \tau\dot{u}_{xxx}) - b(u_x + \tau\dot{u}_x) + \delta\varphi_{xx} - \xi\varphi - k_2\varphi_{xxxx},
\end{aligned}$$

where η and δ are defined as in the previous section. Moreover, we also assume that ρ, a, k_1, k_1^*, J, δ, ξ and k_2 are positive, and we also need to impose that $a\xi > b^2$, $k_1\delta > \eta^2$ and $k_1^* > \tau k_1$ to guarantee that the energy of the system and the dissipation are positive definite. We also assume that $\eta \neq 0$ to get a strong coupling.

In this section, we use the boundary conditions (2.1) and the initial conditions (3.1).

We study this problem in the Hilbert space $\mathcal{H}=(H^2\cap H_0^1)\times(H^2\cap H_0^1)\times L^2\times H_*^2\times L_*^2$. In this case, the elements of $\mathcal{H}$ are denoted by $U=(u,v,c,\varphi,\psi)$ and the inner product will be

$$\langle U,U^*\rangle=\frac{1}{2}\int_0^\pi\Big(\rho(v+\tau c)\overline{(v^*+\tau c^*)}+J\psi\overline{\psi^*}+W\Big)\,dx,$$

where

$$\begin{aligned}W=&\,k_1(u_{xx}+\tau v_{xx})\overline{(u^*_{xx}+\tau v^*_{xx})}+\tau\overline{k_1}v_{xx}\overline{v^*_{xx}}+a(u_x+\tau v_x)\overline{(u^*_x+\tau v^*_x)}+\delta\varphi_x\overline{\varphi^*_x}+\xi\varphi\overline{\varphi^*}\\&+k_2\varphi_{xx}\overline{\varphi^*_{xx}}+b((u_x+\tau v_x)\overline{\varphi^*}+\overline{(u^*_x+\tau v^*_x)}\varphi)+\eta((u_{xx}+\tau v_{xx})\overline{\varphi^*_x}+\overline{(u^*_{xx}+\tau v^*_{xx})}\varphi_x),\end{aligned}$$

and $\overline{k_1}=k_1^*-\tau k_1>0$. It is clear that it defines a norm that is equivalent to the usual norm in the Hilbert space. As in the previous section, we can write our problem in the form (3.6) where

$$\mathcal{A}=\begin{pmatrix}0 & I & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0\\ \dfrac{aD^2-k_1D^4}{\rho\tau} & \dfrac{\tau aD^2-k_1^*D^4}{\rho\tau} & -\dfrac{I}{\tau} & \dfrac{bD-\eta D^3}{\rho\tau} & 0\\ 0 & 0 & 0 & 0 & I\\ \dfrac{\eta D^3-bD}{J} & \dfrac{\tau(\eta D^3-bD)}{J} & 0 & \dfrac{\delta D^2-k_2D^4-\xi}{J} & 0\end{pmatrix}$$

and $U^0=(u^0,v^0,c^0,\varphi^0,\psi^0)$.

We note that the domain of this operator is given by the elements in $\mathcal{H}$ such that

$$\begin{aligned}&k_1D^4u+k_1^*D^4v+\eta D^3\varphi\in L^2,\quad c\in H_0^1\cap H^2,\\&\eta(D^3u+\tau D^3v)-k_2D^4\varphi\in L^2,\quad \psi\in H_*^2,\\&D^2(k_1u+k_1^*v)=D^3\varphi=0\quad\text{on}\quad\{0,\,\pi\}\end{aligned}$$

This domain is a dense subset of the Hilbert space $\mathcal{H}$ because it contains $[\mathcal{C}_0^\infty(0,\pi)]^5$, where $\mathcal{C}_0^\infty$ denotes the set of infinitely derivable functions with compact support.

Using integration by parts and the boundary conditions we see that

$$\Re\langle\mathcal{A}U,U\rangle=-\frac{\overline{k_1}}{2}\int_0^\pi|v_{xx}|^2\,dx\le 0, \tag{4.1}$$

for every $U\in\mathcal{D}(\mathcal{A})$.

Lemma 4.1. *The origin of the complex plane belongs to the resolvent of the operator.*

Proof. Let us consider $\mathcal{F}=(f_1,f_2,f_3,f_4,f_5)\in\mathcal{H}$. We want to find (u,v,c,φ,ψ) such that

$$\begin{aligned}&v=f_1,\quad c=f_2,\quad \psi=f_4,\\&-k_1D^4u-k_1^*D^4v+a(D^2u+\tau D^2v)+bD\varphi-\eta D^3\varphi-\tau c=\tau\rho f_3,\\&\eta D^3u+\tau\eta D^3v-bDu-\tau bDv+\delta D^2\varphi-\xi\varphi-k_2D^4\varphi=Jf_5.\end{aligned}$$

We can obtain v, c and ψ immediately and so, the above system becomes:

$$\begin{aligned}&-k_1D^4u+aD^2u+bD\varphi-\eta D^3\varphi=\tau\rho f_3+k_1^*D^4f_1-a\tau D^2f_1+\tau f_2,\\&\eta D^3u-bDu+\delta D^2\varphi-\xi\varphi-k_2D^4\varphi=Jf_5-\tau\eta D^3f_1-\tau bDf_1.\end{aligned}$$

This system can be solved using the same argument we have done in the proof of Lemma 3.1, and we obtain u and φ with the necessary regularity conditions. In fact, it can be shown that the inequality (3.13) also holds. □

Theorem 4.1. *The operator $\mathcal{A}$ generates a C^0-semigroup of contractions.*

Again, undamped solutions can be found whenever $b+n^2\eta=0$ for a natural number $n\in\mathbb{N}$. Therefore, we assume in this section that $b+\eta n^2\neq 0$ for every $n\in\mathbb{N}$.

We proceed as in the previous section and prove the existence of solutions of the form (3.14) such that $\Re(\omega) > -\epsilon$ for all positive ϵ sufficiently small whenever $Jk_1^* \neq k_2\rho\tau$. In this case, ω must satisfy the equation:

$$\det\begin{pmatrix} (k_1+k_1^*\omega)n^4 + an^2(1+\tau\omega) + \rho\omega^2(1+\tau\omega) & \eta n^3 + bn \\ \eta n^3(1+\tau\omega) + bn(1+\tau\omega) & k_2n^4 + \delta n^2 + J\omega^2 + \xi \end{pmatrix} = 0.$$

We abuse a little bit the notation and denote by $p(x)$ the polynomial obtained when ω is replaced by x in the above determinant. Afterwards, we replace x by $x-\epsilon$ and obtain polynomial $p(x-\epsilon) = b_0x^5 + b_1x^4 + b_2x^3 + b_3x^2 + b_4x + b_5$, to which we apply again the Routh-Hurwitz criterion. As before, coefficients b_i depend on the constitutive parameters and ϵ.

In this case, the fourth leading minor is a sixteenth degree polynomial on n. To be precise, we have

$$L_4 = 2J\epsilon k_2\rho(Jk_1^* - k_2\rho\tau)^2(2\epsilon k_1^*\tau - \overline{k_1})n^{16} + q_{14}(n),$$

where $q_{14}(n)$ is a fourteenth degree polynomial on n. Thus, if we take n large enough and ϵ small, L_4 will be negative.

In the remain of this section, we prove that the decay of the solutions to our problem can be controlled by a term of the form $t^{-1/2}$.

Proposition 4.1. *The eigenvalues of $\mathcal{A}$ restricted to $\mathcal{K}^N$ have negative real part.*

Proof. If we follow an argument similar to the previous section we find that

$$M_1 = J\rho, \quad M_2 = J^2\rho\overline{k_1}n^4, \quad M_3 = J^3\rho\overline{k_1}(a+k_1n^2)n^6, \quad M_4 = J^3\rho\overline{k_1}^2(b+\eta n^2)^2n^{10},$$
$$M_5 = J^3n^{12}\rho\overline{k_1}^2(b+\eta n^2)^2\left(k_1k_2n^6 + (ak_2+\delta k_1-\eta^2)n^4 + (a\delta - 2b\eta + k_1\xi)\,n^2 + a\xi - b^2\right).$$

Taking into account that $\overline{k_1} = k_1^* - k_1\tau > 0$, $a\xi > b^2$ and $\delta k_1 > \eta^2$, we conclude that these leading minors are all positive. □

Lemma 4.2. *The resolvent of operator $\mathcal{A}$ contains the imaginary axis.*

Proof. Again, we prove the result in a similar way to Lemma 3.2. If we assume that the thesis does not hold, there will exist a sequence of real numbers λ_n such that $\lambda_n \to \overline{\lambda} \in \mathbb{R}$, with $|\lambda_n| < |\overline{\lambda}|$, an a sequence of unit norm vectors $U_n = (u_n, v_n, c_n, \varphi_n, \psi_n) \in \mathcal{D}(\mathcal{A})$ such that

$$i\lambda_n u_n - v_n \to 0 \quad \text{in} \quad H^2, \tag{4.2}$$
$$i\lambda_n v_n - c_n \to 0 \quad \text{in} \quad H^2, \tag{4.3}$$
$$i\rho\tau\lambda_n c_n - [aD^2u_n + \tau aD^2v_n + bD\varphi_n - \eta D^3\varphi_n$$
$$-\rho c_n - k_1D^4u_n - k_1^*D^4v_n] \to 0 \quad \text{in} \quad L^2, \tag{4.4}$$
$$i\lambda_n\varphi_n - \psi_n \to 0 \quad \text{in} \quad H^2, \tag{4.5}$$
$$i\lambda_n J\psi_n - [\eta D^3u_n + \tau\eta D^3v_n - bDu_n - b\tau Dv_n + \delta D^2\varphi_n$$
$$-\xi\varphi_n - k_2D^4\varphi_n] \to 0 \quad \text{in} \quad L^2. \tag{4.6}$$

In view of the dissipation inequality (4.1) we obtain that $D^2v_n \to 0$. Then, $\lambda_n D^2u_n$ also tends to zero. If we multiply convergence (4.4) by v_n we obtain that $c_n \to 0$ in L^2. Now, we multiply convergence (4.4) by $D\varphi_n$ and, using a similar argument to the previous section, we also obtain that $D^2\varphi_n \to 0$ and, therefore, $\psi_n \to 0$. □

Lemma 4.3. *The asymptotic condition (3.17) holds with $\alpha = 2$.*

The proof is rather similar to the one used in Lemma 3.3, but considering the steps of the previous lemma. Hence, we have proved the following decay result.

Theorem 4.2. *The solutions to our problem satisfy, for every $U(0) \in \mathcal{D}(\mathcal{A})$, the estimate*

$$\|U(t)\|_{\mathcal{H}} \leq Ct^{-1/2}\|U(0)\|_{\mathcal{D}(\mathcal{A})},$$

where C is a constant which is independent of the initial data.

The comment after Theorem 3.1 applies, word by word, also here.

Remark. Generically, the solutions to our problem decay in an slow way and it can be controlled by $t^{-1/2}$. However if $Jk_1^* = k_2\rho\tau$, it is possible to prove the exponential decay. To do so, it is enough to show that the imaginary axis is contained in the resolvent of $\mathcal{A}$ and that the following asymptotic condition holds ([22]):

$$\overline{\lim_{|\lambda|\to\infty}}\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{4.7}$$

We have already proved the first condition. It we assume that (4.7) does not hold, therefore a sequence of real numbers λ_n, with $|\lambda_n| \to \infty$, and a sequence of unit norm vectors in $\mathcal{D}(\mathcal{A})$ exist such that (4.1)–(4.5) hold. The same arguments used in Lemma 4.2 show that D^2v_n, $\lambda_n D^2 u_n$ and c_n tend to zero. Multiplying (4.4) by $D\varphi_n$ we obtain

$$\langle i\rho\tau\lambda_n c_n, D\varphi_n\rangle - b\|D\varphi_n\|^2 - \eta\|D^2\varphi_n\|^2 + k_1\langle D^4u_n, D\varphi_n\rangle + k_1^*\langle D^4 v_n, D\varphi_n\rangle \to 0.$$

Notice that $\langle D^4u_n, D\varphi_n\rangle = -\langle Du_n, D^4\varphi_n\rangle$ and that, from (4.6),

$$D^4\varphi_n \sim \frac{1}{k_2}\left(i\lambda_n J\psi_n - \eta D^3u_n - \tau\eta D^3v_n - \delta D^3\varphi_n - \xi\varphi_n\right).$$

As $\langle Du_n, \lambda_n\psi_n\rangle = \langle\lambda_n Du_n, \psi_n\rangle \to 0$, we conclude that $\langle D^4u_n, D\varphi_n\rangle \to 0$. We concentrate now in

$$\langle i\rho\tau\lambda_n c_n, D\varphi_n\rangle + k_1^*\langle D^4v_n, D\varphi_n\rangle, \tag{4.8}$$

which is equivalent to

$$\langle Dv_n, -i\rho\tau\lambda_n\psi_n - k_1^* D^4\varphi_n\rangle.$$

Let us denote by $m = \frac{k_1^*}{k_2} = \frac{\rho\tau}{J}$. Therefore, the above expression can be written as

$$-m\langle Dv_n, iJ\lambda_n\psi_n + k_2D^4\varphi_n\rangle,$$

and from (4.6)

$$iJ\lambda_n\psi_n + k_2D^4\varphi_n \sim \eta D^3u_n + \tau\eta D^3v_n - \delta D^2\varphi_n - \xi_n.$$

Hence, expression (4.8) tends to zero. This implies that, in a sufficiently small Hilbert subspace, $D^2\varphi_n \to 0$.

If we multiply (4.6) by φ_n we also obtain that $\psi_n \to 0$, which leads to a contradiction about supposing that the vector is of unit norm. This proves the exponential decay when $Jk_1^* = \rho\tau k_2$.

It is worth mentioning that for this case the damping of the solutions and, therefore, of the mechanical waves, is controlled by a negative exponential. However, as it corresponds to a very specific situation, a singular case, the physical application of this result is quite irrelevant.

5. Porous viscosity

In this section, we sketch some results about the decay of the solution when we introduce several MGT-dissipation mechanisms on the microstructure.

The first case that we consider is obtained when

$$a(s)=a,\quad b(s)=b,\quad \beta(s)=\beta,\quad k_1(s)=k_1,\quad \gamma(s)=\gamma,$$
$$\alpha(s)=\alpha+(\alpha^*/\tau-\alpha)e^{-s/\tau},\quad d(s)=d+(d^*/\tau-d)e^{-s/\tau},\quad k_2(s)=k_2,\quad \xi(s)=\xi.$$

These assumptions give rise to the system:

$$\rho\ddot{u}=au_{xx}+b(\varphi_x+\tau\dot{\varphi}_x)-\eta(\varphi_{xxx}+\tau\dot{\varphi}_{xxx})-k_1u_{xxxx},$$
$$J(\ddot{\varphi}+\tau\dddot{\varphi})=\eta u_{xxx}-bu_x+\delta\varphi_{xx}+\delta^*\dot{\varphi}_{xx}-\xi(\varphi+\tau\dot{\varphi})-k_2(\varphi_{xxxx}+\tau\dot{\varphi}_{xxxx}).$$

Here, we have used the notations $\delta=\alpha-2d$, $\delta^*=\alpha^*-2d^*$ and $\eta=\gamma-\beta$. As before, we assume that ρ, a, k_1, J, δ, ξ, k_2 are positive constants and that $a\xi>b^2$, $k_1\delta>\eta^2$, $\delta^*>\tau\delta$ and $\eta\neq 0$.

We impose the boundary conditions (2.1), but as initial conditions we take the following:

$$\left.\begin{aligned} &u(x,0)=u^0(x),\quad \dot{u}(x,0)=v^0(x),\quad \varphi(x,0)=\varphi^0(x),\\ &\dot{\varphi}(x,0)=\psi^0(x),\quad \ddot{\varphi}(x,0)=\phi^0(x).\end{aligned}\right\}\quad \text{for a.e. } x\in(0,\pi). \tag{5.1}$$

We note that this problem is very similar to the one studied in Section 3. Therefore, considering the Hilbert space $\mathcal{H}=(H^2\cap H_0^1)\times L^2\times H_*^2\times H_*^1\times L^2$ and proceeding as before, one can show the polynomial decay of type $t^{-1/2}$ for the solutions.

The second case that we consider is obtained when

$$a(s)=a,\quad b(s)=b,\quad \beta(s)=\beta,\quad k_1(s)=k_1,\quad \gamma(s)=\gamma,$$
$$\alpha(s)=\alpha,\quad d(s)=d,\quad k_2(s)=k_2+(k_2^*/\tau-k_2)e^{-s/\tau},\quad \xi(s)=\xi.$$

Now, the system of equations is

$$\rho\ddot{u}=au_{xx}+b(\varphi_x+\tau\dot{\varphi}_x)-\eta(\varphi_{xxx}+\tau\dot{\varphi}_{xxx})-k_1u_{xxxx},$$
$$J(\ddot{\varphi}+\tau\dddot{\varphi})=\eta u_{xxx}-bu_x+\delta(\varphi_{xx}+\tau\dot{\varphi}_{xx})-\xi(\varphi+\tau\dot{\varphi})-k_2\varphi_{xxxx}-k_2^*\dot{\varphi}_{xxxx}.$$

We can study the problem determined by this system with the boundary conditions (2.1), the initial conditions (5.1) and assuming that ρ, a, k_1, J, δ, ξ, k_2 are positive constants and that $a\xi>b^2$, $k_1\delta>\eta^2$, $k_2^*>\tau k_2$ and $\eta\neq 0$. Hence, it is very similar to the one studied in Section 4 and, therefore, the lack of exponential decay as well as the polynomial decay of type $t^{-1/2}$ for the solutions can be shown in the Hilbert space $\mathcal{H}=(H^2\cap H_0^1)\times L^2\times H_*^2\times H_*^2\times L_*^2$ whenever $\rho k_2^*\neq J\tau k_1$. If $\rho k_2^*=J\tau k_1$, then the exponential decay can be obtained.

Finally, we assume that

$$a(s)=a,\quad b(s)=b,\quad \beta(s)=\beta,\quad k_1(s)=k_1,\quad \gamma(s)=\gamma,$$
$$\alpha(s)=\alpha,\quad d(s)=d,\quad k_2(s)=k_2,\quad \xi(s)=\xi+(\xi^*/\tau-\xi)e^{-s/\tau}.$$

The corresponding system of equations is

$$\rho\ddot{u}=au_{xx}+b(\varphi_x+\tau\dot{\varphi}_x)-\eta(\varphi_{xxx}+\tau\dot{\varphi}_{xxx})-k_1u_{xxxx},$$
$$J(\ddot{\varphi}+\tau\dddot{\varphi})=\eta u_{xxx}-bu_x+\delta(\varphi_{xx}+\tau\dot{\varphi}_{xx})-\xi\varphi-\xi^*\dot{\varphi}-k_2(\varphi_{xxxx}+\tau\dot{\varphi}_{xxxx}).$$

We consider the problem determined by this system with boundary conditions (2.1) and initial conditions (5.1). We assume here that ρ, a, k_1, J, δ, ξ, k_2 are positive constants and that $a\xi>b^2$, $k_1\delta>\eta^2$, $\xi^*>\tau\xi$ and $\eta\neq 0$.

If we introduce the new variables $\varphi_1=\varphi+\tau\dot{\varphi}$ and $\varphi_3=\psi+\tau\dot{\psi}$ (recall that $\psi=\dot{\varphi}$), we can write our system as

$$\dot{u}=v,\quad \dot{\psi}=-\tau^{-1}\psi+\tau^{-1}\varphi_3,\quad \dot{\varphi}_1=\varphi_3,$$
$$\dot{v}=\frac{1}{\rho}\Big[aD^2u+bD\varphi_1-\eta D^3\varphi_1-k_1D^4u\Big],$$
$$\dot{\varphi}_3=\frac{1}{J}\Big[\eta D^3u-bDu+\delta D^2\varphi_1-\xi\varphi_1-\overline{\xi}\psi-k_2D^4\varphi_1\Big],$$

where $\overline{\xi}=\xi^*-\tau\xi$. We study this problem in the Hilbert space $\mathcal{H}=(H^2\cap H_0^1)\times L^2\times H_*^2\times L_*^2\times L_*^2$, which elements are denoted by $U=(u,v,\varphi_1,\psi,\varphi_3)$. We define the inner product

$$\langle U,U^*\rangle=\frac{1}{2}\int_0^\pi\left(\rho v\overline{v^*}+J\varphi_3\overline{\varphi_3^*}+W\right)dx,$$

where

$$W=k_1u_{xx}\overline{u_{xx}^*}+au_x\overline{u_x^*}+\eta(u_{xx}\overline{\varphi_{1x}^*}+\overline{u_{xx}^*}\varphi_{1x})+b(u_x\overline{\varphi_1^*}+\overline{u_x^*}\varphi_1)$$
$$+k_2\varphi_{1xx}\overline{\varphi_{1xx}^*}+\delta\varphi_{1x}\overline{\varphi_{1x}^*}+\xi\varphi\overline{\varphi^*}+\tau\overline{\xi}\psi\overline{\psi^*},$$

and the matrix operator

$$\mathcal{A}=\begin{pmatrix}0 & I & 0 & 0 & 0\\ \dfrac{aD^2-k_1D^4}{\rho} & 0 & \dfrac{bD-\eta D^3}{\rho} & 0 & 0\\ 0 & 0 & 0 & 0 & I\\ 0 & 0 & 0 & -\tau^{-1}I & \tau^{-1}I\\ \dfrac{\eta D^3-bD}{J} & 0 & \dfrac{\delta D^2-k_2D^4-\xi}{J} & -\dfrac{\overline{\xi}}{J} & 0\end{pmatrix}.$$

Again, the domain of the operator is dense and

$$\Re\langle\mathcal{A}U,U\rangle=-\frac{\overline{\xi}}{2}\int_0^\pi|\psi|^2\,dx\leq 0,$$

for every $U\in\mathcal{D}(\mathcal{A})$.

Following the arguments proposed in the previous sections we can show that the imaginary axis is contained in the resolvent of the operator. Therefore, we can obtain again the existence of a semigroup of contractions, and the existence and uniqueness of solutions is guaranteed whenever the initial data belongs to the domain of the operator.

It seems difficult to obtain estimates for the decay of the solutions. However, we can guarantee that the point spectrum is on the left hand side of the imaginary axis whenever $b+\eta n^2\neq 0$ for every $n\in\mathbb{N}$. This fact suggests the decay of the solutions. However, a careful proof would need a further study.

If we follow an argument similar to the previous sections we find that

$$M_1=J\rho,\quad M_2=J\rho^2\overline{\xi},\quad M_3=J\rho^3\overline{\xi}(k_2n^4+\delta n^2+\xi),\quad M_4=J\rho^3\overline{\xi}^2n^2(b+\eta n^2)^2,$$
$$M_5=Jn^4\rho^3\overline{\xi}^2(\eta n^2+b)^2\left(k_1k_2n^6+(ak_2+\delta k_1-\eta^2)n^4+(k_1\xi-2b\eta+a\delta)n^2+a\xi-b^2\right).$$

Keeping in mind that $\overline{\xi}=\xi^*-\tau\xi>0$, $a\xi>b^2$, $\delta k_1>\eta^2$ and $b+\eta n^2\neq 0$ for every $n\in\mathbb{N}$ we see that $M_i>0$ for every $i=1,\ldots,5$. Therefore, all the eigenvalues have negative real part and the decay of the solutions is proved.

Now, we show that the decay is not controlled by any exponential. Taking solutions of the form (3.14) such that $\Re(\omega)>-\epsilon$ for all positive ϵ sufficiently small, we get that ω satisfies the equation:

$$\det\begin{pmatrix}an^2+k_1n^4+\rho\omega^2 & \eta n^3(1+\tau\omega)+bn(1+\tau\omega)\\ \eta n^3+bn & (k_2+k_2\tau\omega)n^4+(\delta+\delta\tau\omega)n^2+J\tau\omega^3+J\omega^2+\xi^*w+\xi\end{pmatrix}=0.$$

As in the previous sections, we analyze the roots of the polynomial $p(x-\epsilon)$ by using the Routh-Hurwitz criterion. In this case, the second leading minor is a fourth-degree polynomial on n whose main coefficient is always negative. To be precise, we have

$$L_2=-2J\rho\tau^2\epsilon(Jk_1+k_2\rho)n^4-J\rho(2Ja\epsilon\tau^2+2\delta\epsilon\rho\tau^2)n^2$$
$$-J\rho(40J\rho\epsilon^3\tau^2-24J\rho\epsilon^2\tau+2\rho\xi^*\epsilon\tau+4J\rho\epsilon+\rho\xi\tau-\rho\xi^*).$$

Thus, if we take n large enough and ϵ small, L_2 will be negative.

6. Conclusions

Strain gradient models and the Moore-Gibson-Thompson equation are currently being used in the thermomechanical context. In this work we have studied the linear strain gradient porous–elastic problem in the one-dimensional case when several dissipation mechanisms of MGT type are introduced in the system of equations, and we have shown the following properties:

- If viscoelasticity is considered, the solutions decay in a polynomial way with respect to the time variable.
- If hyperviscoelasticity is considered, the solutions decay again in a polynomial way, except for a particular combination of the constitutive coefficients, which leads to the exponential stability, that is, the decay of the solutions can be controlled by a negative exponential.
- The same results are sketched when the viscosity or hyperviscosity effects act in the porous structure.
- For the weak viscoporosity, we have seen that the decay can only be slow. However, to clarify if the decay can be controlled by a polynomial is still an open question.

Let us end this study by comparing the decay of the waves when MGT-dissipation mechanisms are taken into account with their decay when the classical Kelvin-Voigt dissipation (KV) is considered. In a recent study [1], the authors prove that under KV viscoelasticity, the waves dampen uniformly, that is, the elements of the spectrum are quite far away from the imaginary axis. Instead, if MGT-dissipation is considered, the elements of the spectrum approach the imaginary axis. For the hyperviscoelasticity, the waves behave similarly under both dissipation mechanisms: the elements of the spectrum get close to the imaginary axis. The same happens when the dissipation is considered in the porous structure. Finally, for what can be called *weak porous dissipation*, in the KV model there are some singular cases where the elements of the spectrum are (again) far away from the imaginary axis, while in the MGT model they are closer, and that means that in the former model the waves dampen quicker than in the latter.

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Appendix

For the sake of completeness, we write here the full expressions of coefficients b_i, for $i=0,1,...,5$, of the polynomial $p(x-\epsilon)$ of Section 3.

$$\begin{aligned}
b_0 = {} & J\rho\tau \\
b_1 = {} & J\rho(1-5\tau\epsilon) \\
b_2 = {} & n^4\tau\,(Jk_1+k_2\rho)+n^2\,(a^*J+\delta\rho\tau)+2J\rho\epsilon(5\tau\epsilon-2)+\xi\rho\tau \\
b_3 = {} & n^4(1-3\tau\epsilon)\,(Jk_1+k_2\rho)+n^2\,(-3a^*J\epsilon+aJ+\rho(\delta-3\delta\tau\epsilon)) \\
& +2J\rho\epsilon^2(3-5\tau\epsilon)+\xi\rho-3\xi\rho\tau\epsilon \\
b_4 = {} & n^8k_1k_2\tau+n^6\left(a^*k_2-\eta^2\tau+\delta k_1\tau\right) \\
& +n^4\,(a^*\delta-2b\eta\tau+Jk_1\epsilon(3\tau\epsilon-2)+k_1\xi\tau+k_2\rho\epsilon(3\tau\epsilon-2)) \\
& +n^2\left(3a^*J\epsilon^2+a^*\xi-2aJ\epsilon-b^2\tau+3\delta\rho\tau\epsilon^2-2\delta\rho\epsilon\right) \\
& +5J\rho\tau\epsilon^4-4J\rho\epsilon^3+3\xi\rho\tau\epsilon^2-2\xi\rho\epsilon \\
b_5 = {} & n^8k_1k_2(1-\tau\epsilon)+n^6\left(ak_2-a^*k_2\epsilon+\delta k_1(1-\tau\epsilon)+\eta^2(\tau\epsilon-1)\right) \\
& +n^4\left(-a^*\delta\epsilon+a\delta+2b\eta(\tau\epsilon-1)+Jk_1\epsilon^2(1-\tau\epsilon)+k_2\rho\epsilon^2(1-\tau\epsilon)+k_1\xi(1-\tau\epsilon)\right) \\
& +n^2\left(-a^*J\epsilon^3-a^*\xi\epsilon+aJ\epsilon^2+a\xi+b^2(\tau\epsilon-1)-\delta\rho\epsilon^2(\tau\epsilon-1)\right) \\
& -J\rho\epsilon^4(\tau\epsilon-1)-\xi\rho\epsilon^2(\tau\epsilon-1)
\end{aligned} \tag{6.1}$$

Obviously, when $\epsilon=0$ these coefficients agree with the a_i's.