

MEAN FIRST PASSAGE TIME FOR DISTANCE-BIREGULAR GRAPHS

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In this presentation, we obtain the explicit expression for the Group inverse of the Laplacian matrix associated with distance-biregular graphs, ([2]). A bipartite graph is called distance-biregular (DBR) if all the vertices of the same partite set admit the same intersection array. So, this kind of graphs are characterized by having two intersection arrays instead of one as in the case of distance-regular graphs. Examples of this kind of graphs are complete bipartite graphs, subdivision graphs of minimal cages and some block designs, see [19]. As an application, we provide the mean first passage time for DBR graphs as well as the Kemeny constant. The above expression will be given in terms of the so-called equilibrium measure for a vertex $\{x\}$, see [6]. Finally, we provide some examples as star graphs.

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1. INTRODUCTION

One problem with the theory of distance-regular graphs is that it does not apply directly to the graphs of generalised polygons. Godsil and Shawe-Taylor [19,] overcame this difficulty by introducing the class of distance-regularised graphs, a natural common generalisation. These graphs are shown to either be distance-regular or distance-biregular. This family includes the generalised polygons and other interesting graphs. Distance-biregular graphs, which were introduced by Delorme [16] in 1983, can be viewed as a bipartite variant of distance-regular graph: the graphs are bipartite and for each vertex there exists an intersection array depending on the stable component of the vertex. Thus such graphs are to distance-regular graphs as bipartite regular graphs are to regular graphs. They also are to non-symmetric association schemes as distance-regular graphs are to symmetric association schemes. Since their introduction, distance-biregular graphs have received quite some attention, see [1, 14, 15, 17, 18, 20, 24] or [9, Chapter 4] for an overview.

First, we obtain the group inverse of the combinatorial Laplacian matrix of distance-biregular graphs, so the mean first passage time can be calculated in terms of that matrix. The group inverse matrix can be seen in the framework of discrete potential theory as the Green's functions associated with the Laplacian operator and it can be used to deal with diffusion-type problems on graphs, such as chip-firing, load balancing, and discrete Markov chains. For some graph classes, the group inverse is known. Instances of it are the work of Urakawa [25], Bendito *et al.* [6, 7] or more recently the study of the Green function for forests by Chung and Zeng [13]. Other generalized inverses, such as the Moore-Penrose inverse, have been studied. For instance, the Moore-Penrose inverse of the incidence matrix of several graphs has been investigated by Azami and Bapat [3, 4, 5]. Nevertheless, the problem of computing group inverses still remains wide open for most graph classes. We show an explicit expression for the group inverse of the combinatorial Laplacian matrix of a distance-biregular graph in terms of its intersection numbers. This result, together with the group inverse of a distance-regular graph found by

Bendito, Carmona and Encinas [7], and independently, by Chung and Yau [12], completes the corresponding study for distance-regularised graphs.

In matrix theory, the Laplacian matrix is known to be a symmetric M -matrix (a symmetric positive semi-definite matrix with non-positive off-diagonal elements). Nonnegative matrices and M -matrices have become a staple in contemporary linear algebra, and they arise frequently in its applications. Such matrices are encountered not only in matrix analysis, but also in stochastic processes, graph theory, electrical networks, and demographic models [22]. A fundamental problem related with M -matrices is the so-called inverse M -matrix problem, that consists in characterizing all non-negative matrices whose inverses are M -matrices. For singular matrices, the inverse problem was originally posted by Neumann, Poole and Werner as follows: Characterize all singular and irreducible M -matrices for which its group inverse is also an M -matrix.

This question has only been answered for specific matrix classes. In the graph setting, this question has been solved for weighted trees by Kirkland and Neumann [23], and for distance-regular graphs by Bendito, Carmona and Encinas [8]. In a more general setting, it has been investigated for nonnegative matrices having few eigenvalues by Kirkland and Neumann [22], for periodic and nonperiodic Jacobi matrices by Chen, Kirkland and Neumann [11] and for general symmetric M -matrices whose underlying graphs are paths by Bendito, Carmona and Encinas [8] and Carmona, Encinas and Mitjana [10]. Recently, matrices whose group inverses are M -matrices were investigated by Kalauch, Lavanya and Sivakumar [21].

We answer the question for distance-biregular graphs, completing, together with the known results for distance-regular graphs [8], the characterization of when the group inverse of the combinatorial Laplacian matrix of a distance-regularised graph is an M -matrix.

2. GROUP INVERSE FOR DISTANCE-BIREGULAR GRAPHS

The triple $\Gamma = (V, E, c)$ denotes a finite network; that is, a finite connected graph without loops or multiple edges, with vertex set V , whose cardinality equals $n \geq 2$, and edge set E , in which each edge $\{x, y\}$ has been assigned a *conductance* $c(x, y) > 0$. The conductance can be considered as a symmetric function $c: V \times V \rightarrow [0, +\infty)$ such that $c(x, x) = 0$ for any $x \in V$ and moreover, $x \sim y$, that is vertex x is adjacent to vertex y , iff $c(x, y) > 0$. We define the *degree function* k as

$$k(x) = \sum_{y \in V} c(x, y)$$

for each $x \in V$. The usual distance from vertex x to vertex y is denoted by $d(x, y)$ and $D = \max\{d(x, y) : x, y \in V\}$ stands for the *diameter* of Γ . We denote as $\Gamma_i(x)$ the set of vertices at distance i from vertex x , $\Gamma_i(x) = \{y : d(x, y) = i\}$, $0 \leq i \leq D$ and define $k_i(x) = |\Gamma_i(x)|$. Then,

$$B_i(x) = \sum_{j=0}^i k_j(x)$$

is the cardinal of the i -ball centered at x . The *complement* of Γ is defined as the graph $\bar{\Gamma}$ on the same vertices such that two vertices are adjacent iff they are not adjacent in Γ ; that is $x \sim y$ in $\bar{\Gamma}$ iff $c(x, y) = 0$. More generally, for any $i = 1, \dots, D$, we denote by Γ_i the graph whose vertices are those of Γ and in which two vertices are adjacent iff they are at distance i in Γ . Therefore for any $x \in V$, $\Gamma_i(x)$ is the set of adjacent vertices to x in Γ_i . Clearly Γ_1 is the graph subjacent to the network Γ and $\Gamma_2 = \bar{\Gamma}$ when $D = 2$.

The set of real-valued functions on V is denoted by $C(V)$. When necessary, we identify the functions in $C(V)$ with vectors in $\mathbb{R}^{|V|}$ and the endomorphisms of $C(V)$ with $|V|$ -order square matrices.

The *combinatorial Laplacian* or simply the *Laplacian* of the graph Γ is the endomorphism of $C(V)$ that assigns to each $u \in C(V)$ the function

$$(1) \quad L(u)(x) = \sum_{y \in V} c(x, y) (u(x) - u(y)) = k(x)u(x) - \sum_{y \in V} c(x, y) u(y), \quad x \in V.$$

It is well-known that L is a positive semi-definite self-adjoint operator and has 0 as its lowest eigenvalue whose associated eigenfunctions are constant. So, L can be interpreted as an irreducible, symmetric, diagonally dominant and singular M -matrix, that in the sequel will be denoted as L . Therefore, the Poisson equation $L(u) = f$ on V has solution iff

$$\sum_{x \in V} f(x) = 0$$

and, when this happens, there exists a unique solution $u \in C(V)$ such that $\sum_{x \in V} u(x) = 0$, see [6].

The *Green operator* is the linear operator $G : C(V) \rightarrow C(V)$ that assigns to any $f \in C(V)$ the unique solution of the Poisson equation $L(u) = f - \frac{1}{n} \sum_{x \in V} f(x)$ such that $\sum_{x \in V} u(x) = 0$. It is easy to prove that G is a positive semi-definite self-adjoint operator and has 0 as its lowest eigenvalue whose associated eigenfunctions are constant. Moreover, if P denotes the projection on the subspace of constant functions then,

$$L \circ G = G \circ L = I - P.$$

In addition, we define the *Green function* as $G : V \times V \rightarrow \mathbb{R}$ given by $G(x, y) = G(\varepsilon_y)(x)$, where ε_y stands for the Dirac function at y . Therefore, interpreting G , or G , as a matrix it is nothing else but $L^\#$ the group inverse inverse of L , that coincides with its Moore-Penrose inverse. In consequence, $L^\#$ is a M -matrix iff $L^\#(x, y) \leq 0$ for any $x, y \in V$ with $x \neq y$ and then $L^\#$ can be identified with the combinatorial Laplacian matrix of a new connected network with the same vertex set, that we denote by $\Gamma^\#$.

From now on we will say that a network Γ has the *M-property* iff $L^\#$ is an M -matrix; that is, if L provides an answer our question.

In [6] it was proved that for any $y \in V$, there exists a unique $\nu^y \in C(V)$ such that $\nu^y(y) = 0$, $\nu^y(x) > 0$ for any $x \neq y$ and satisfying

$$(2) \quad L(\nu^y) = -n\varepsilon_y \quad \text{on } V.$$

We call ν^y the *equilibrium measure of $V \setminus \{y\}$* and then we define *capacity* as the function $\in C(V)$ given by $(y) = \sum_{x \in V} \nu^y(x)$.

Following the ideas in [6, 8, 25], we define, for any $y \in V$, the *equilibrium array for y* as the set $\{\nu^y(x) : x \in V\}$ of different values taken by the equilibrium measure of y , and we consider the *length of the equilibrium array* to be $\ell(y) = |\{\nu^y(x) : x \in V \setminus \{y\}\}|$. Since Γ is connected and $n \geq 2$, we obtain that $\ell(y) \geq 1$ for any $y \in V$. On the other hand, since $0 = \nu^y(y)$ we obtain that $\{\nu^y(x) : x \in V\} = \{q_i(y) : i = 0, \dots, \ell(y)\}$, where $0 = q_0(y) < q_1(y) < \dots < q_{\ell(y)}(y)$. In addition, given $y \in V$ for any $i = 0, \dots, \ell(y)$, we define $m_i(y) = |\{x \in V : \nu^y(x) = q_i(y)\}|$. Clearly, for any $y \in V$ we have that

$$m_0(y) = 1, \quad n = \sum_{i=0}^{\ell(y)} m_i(y), \quad \text{and} \quad (y) = \sum_{i=1}^{\ell(y)} m_i(y) q_i(y).$$

The group inverse of the Laplacian matrix and the equilibrium measures provide an equivalent information about the network structure, since the expression of $\#$ can be obtained from equilibrium measures and conversely. Specifically, see [6, Proposition 3.9], the group inverse $L^\#$ is given by

$$(3) \quad L^\#(x, y) = \frac{1}{n^2} ((y) - n \nu^y(x)).$$

In addition, the symmetry of the group inverse leads to the following relation for the equilibrium measures

$$(4) \quad \nu^y(x) - \nu^x(y) = \frac{1}{n} ((y) - (x)) = n (\#(y, y) - \#(x, x)), \quad x, y \in V.$$

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