On the Threefold Minimal Model Program in Positive and Mixed Characteristic

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Declaration of originality

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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Abstract

This dissertation explores the Minimal Model Program (MMP) in positive and mixed characteristic in dimension three with a particular focus on outputs of the program. In purely positive characteristic we combine the program with a detailed study of conic bundles to prove a birational boundedness result. We show that given a suitable set of log Calabi-Yau varieties, we can construct a bounded family containing fibres birational to any member of the chosen set.

For threefolds over a base of dimension at least one, we resolve the Abundance Conjecture for klt pairs in joint work with F. Bernasconi and I. Brivio. Showing in particular that every klt minimal model in mixed characteristic admits an Iitaka Fibration. This is then applied to prove an Invariance of Plurigenera result for suitable families of surfaces.

Finally we consider outstanding questions around Mori fibrations in mixed characteristic. We show that every klt threefold MMP terminates and that any two Mori fibre space outputs of an MMP from the same starting pair are connected by a series of Sarkisov links. As part of this we prove a mixed characteristic Finiteness of Minimal Models result. While the proof is focused in dimension three, the arguments work in any generality given that the requisite MMP results are known.

Acknowledgements

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Outside of mathematics; I could not have finished this thesis without the assistance of my friends and family. My parents have provided unwavering support since long before I began this degree, even if they're still not quite sure what I do. I have also relied heavily on my wife Rose, whose patience has been stretched thin but never quite snapped. My daughter Alice has been less helpful, but I am thankful for her nonetheless. And finally I am grateful for Shanosky's attempts to keep me sane and grounded.

Contents

Notation

- Rings will often be denoted by R. They will always be excellent and admitting a dualising complex. We will often use the ring interchangeably with its spectrum, e.g. Let R be a ring and $X \to R$ be an R scheme.
- If R is a local ring we denote the residue field of the closed point by k and the fraction field by K, unless otherwise stated.
- Schemes will always be excellent and Noetherian though we often state this explicitly. They will typically be denoted W, X, Y, Z. S and T are often used also, particularly for surfaces and the base of a pair respectively. In applications they will essentially always be quasi-projective over an excellent ring.
- If X is a scheme over a local ring we often write X_k for the closed fibre and X_K for the generic fibre.
- If X is an integral scheme we will write $K(X)$ for the fraction field
- A variety is a quasi-projective, integral scheme over a field.
- Given a scheme X we write:
  - $\mathrm{WDiv}_{\mathbb{K}}(X)$ for the group of Weil divisors tensored by $\mathbb{K} = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$ with the natural $\mathbb{K}$ module structure.
  - $\mathrm{Cl}_{\mathbb{K}}(X)$ for quotient of $\mathrm{WDiv}_{\mathbb{K}}(X)$ of by the submodule generated by principle divisors
  - $D \simeq_{\mathbb{K}} D'$ if $[D] = [D']$ inside $\mathrm{Cl}_{\mathbb{K}}(X)$
  - D is $\mathbb{K}$-Cartier if $[D] \in \mathrm{Cl}_{\mathbb{K}}(X)$ is contained in the subspace generated by $\{[L] \colon L \text{ is Cartier}\}$.
- Given a proper morphism of schemes $X \to T$:
  - A curve will always be an integral, one dimensional scheme proper over a closed point of T.
  - A $\mathbb{K}$ one cycle is formal sum of curves with coefficients in $\mathbb{K} = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. If no $\mathbb{K}$ is stated, we default to $\mathbb{R}$.
  - $N_1(X/T)$ for the space of one cycles modulo numerical equivalence
  - $N^1(X/T)$ for the space of $\mathbb{R}$-Cartier divisors modulo numerical equivalence
  - $\overline{NE}(X/T)$ is the closure of the cone of effective one-cycles. We sometimes call such one-cycles psuedo-effective, in analogy to divisors.

  - Two $\mathbb{K}$-Cartier divisors are numerically equivalent, written $D \equiv D'$ if they induce the same functional on $N_1(X/T)$.

- We say a proper morphism of schemes $f : X \to Y$ is a contraction has if $f_*\mathcal{O}_X = \mathcal{O}_Y$. When $\dim X > \dim Y$ such morphisms are also sometimes called fibrations.

- If $f : X \dashrightarrow Y$ is a birational map such that for any divisor E on Y, f^{-1} is an isomorphism near the generic point of E then f is a birational contraction. The notation is unfortunate, but we reassure ourselves that if X, Y are normal and f is a morphism then f is a contraction in the above sense also. If f^{-1} is also a birational contraction we say that f is small.

- If $f : X \dashrightarrow Y$ is a birational contraction with f^{-1} also a birational contraction we say that f is small.

- Given a fibration $X \to T$ and a property $\mathcal{P}$ we say it is a family of $\mathcal{P}$ varieties if the fibre over each closed point k is a k-variety with property $\mathcal{P}$.

- We largely consider X admitting a projective morphisms $X \to T$ of quasi-projective R-schemes. When $X \to T$ is part of the description of X in this fashion, we often say D is nef/ample/semiample etc, to mean that D is nef/ample/semiample over T. Since T need not contain any proper curves over R this should cause no confusion.

- We say an open immersion $U \hookrightarrow X$ is big if its image contains every point of codimension 1.

Chapter 1

Introduction

Algebraic Geometry is the study of geometric shapes described as solutions to polynomials, or perhaps more generally the study of geometric objects which are locally the spectra of rings.

While this set of definitions offers a rich and fruitful area of study and a firm mathematical foundation from which to approach, they are not particularly useful for describing the objects of interest. One would not expect to see a variety given by some list of equations, nor as a list of rings and gluing information, except perhaps in the very simplest of examples.

One of the key aims of modern birational geometry is to provide a language and structure to better understand and describe the geometric objects appearing as part of the wider study of algebraic geometry. This is a role taken on directly by flagship conjectures like the Minimal Model Program (MMP), but is also supported by the myriad of ideas developed in the study of birational geometry. Notions like klt, for instance, which were developed to better understand the singularities appearing in the MMP have quickly spread throughout the larger field of algebraic geometry.

We might summarise the key claim of the (klt) MMP as follows.

Conjecture 1. *Let (X, Δ) be a klt pair, projective over T. Then there is a $K_X + \Delta$ negative birational map of projective T-schemes $X \dashrightarrow X'$ inducing a klt pair $K_{X'} + \Delta'$ such that either*

1. *$K_{X'} + \Delta$ is nef; or*
2. *There is a $K_{X'} + \Delta'$ negative contraction $X' \to Z$ of relative Picard rank 1.*

In the first case X' is said to be a minimal model. In the latter we call $X' \to Z$ a Mori fibre space. We require some assumptions on T, in their most general we would ask for T integral, excellent, Noetherian and admitting a dualising complex.

Introduction

At its most reductive, the MMP can be understood to claim that projective varieties (more generally integral projective schemes) can be built from three types of geometric objects - each with their own distinct properties. These are Fano, Calabi-Yau and general type (or canonically polarised) varieties. For birational geometers it is natural to describe these in terms of their canonical bundle, these types of variety have K_X negative (anti-ample), K_X numerically trivial or K_X positive (ample) respectively.

This kind of numerical description is central to birational geometry. The numerical statement, K_X is positive, translates directly to an algebraic statement, K_X is ample. Indeed the following characterisation, due to Kleiman following Nakai-Moishezon, is the prototypical result for this kind of theorem.

Theorem. [Laz04, Corollary 1.4.11] *Let D be a $\mathbb{R}$-Cartier divisor on a scheme X, projective over T. Let $\overline{NE}(X/T)$ be the closure of the cone spanned by effective classes inside $N_1(X/T)$. Then D is ample over T if and only if there is $\epsilon > 0$ such that $D.C > \epsilon$ for any class $C \in \overline{NE}(X/T)$.*

Not every such condition is so easily interpreted, however. It is not clear even that K_X being numerically trivial ought to ensure that K_X is $\mathbb{Q}$-linearly trivial. Many results and theorems within the field therefore provide a means of turning numerical statements about divisors or pairs into algebraic ones. A more sophisticated example is the Basepoint Free Theorem.

Theorem. [KM98, Theorem 3.3] *Let (X, Δ) be a klt pair over a field of characteristic 0. Let M be a nef divisor and suppose that $M - (K_X + \Delta)$ is big and nef. Then M is semiample.*

Here nefness is an entirely numerical condition, and bigness has both numerical and cohomological characterisations, whereas semiampleness is an entirely algebraic phenomenon. We can also understand singularity conditions like klt and log canonical to be numerical conditions, though of a very different flavour to the intersection based conditions we impose upon divisors.

In characteristic zero, one of the most important tools for proving these kinds of results is Kawamata-Viehweg (KV) vanishing; a result which is known to fail in positive and mixed characteristic. Much of the difficulty of recreating the success of the MMP in characteristic zero comes from the need to find alternative methods of generating these kinds of translations from numerical conditions to algebraic. More philosophically, KV vanishing and similar results provide a clear impetus for the focus on the the canonical divisor, and more generally on klt pairs. It is less obvious in other settings exactly what the role of K_X should be.

In positive characteristic this role is taken on by applications of the Frobenius morphism. Ideas due to Keel [Kee99], provide a powerful semiampleness criteria along with a weaker, but often still useful, result on the existence of morphisms in the larger category of Algebraic Spaces.

In a different direction, a suite of Frobenius based singularities provide, amongst other important applications, a way to recover certain vanishing type theorems. These are often called F-singularities and they consist of local versions, F-pure and F-regular, as well as global versions, F-split and globally F-regular.

Together these ideas are sufficient to prove the bulk of MMP for threefolds in positive characteristic, at least for $p > 5$. This is done in [HX15], [Bir16a], [BW17]. It was the state of the art at the beginning of my PhD in 2018. **??** therefore focuses on some of the consequences of the MMP in this setting.

In mixed characteristic, the ideas of Keel are generalised immediately by [Wit20] through clever study of universal homeomorphisms in place of the Frobenius morphism. The full statement is as follows and captures both the mixed characteristic and the positive characteristic behaviour.

Theorem. [Wit20, Theorem 1.2] *Let $X \to T$ be a projective morphism of excellent, Noetherian schemes. Let D be a nef line bundle on X with $D|_{X_{\mathbb{Q}}}$ semiample. Let $\mathbb{E}[D]$ be the union of integral subschemes of X on which D is not big. Then D is semiample (resp. EWM) if and only if $D|_{\mathbb{E}[D]}$ is so.*

The F-singularities are less smoothly generalised, however. The BCM singularities of [MS21] are similar in some sense to F-regularity and provide important adjunction type results as in [MST$^+$19]. On the other hand the globally $+$-regular singularities of [BMP$^+$20] are inspired by, and analogous to, globally F-regular singularities. In particular they provide suitable vanishing type results for the proof of the existence of flips.

The rest of the thesis is devoted to exploring some of the remaining questions regarding the MMP for threefold klt pairs in mixed characteristic, with an emphasis on the structure and properties of the outputs of the MMP. In particular we prove Abundance Conjecture holds in this setting and show that Mori Fibrations are connected by Sarkisov links. A more thorough overview of the MMP is given in section 2.2. The results contained therein are largely known, but perhaps not in exactly the same generality as is presented there.

In addition, the theory of log pairs and the corresponding F-singularities are introduced in section 2.1. The log pairs are needed throughout the thesis. Only the notion of F-split is needed in **??**, however the globally F-regular condition provides important context. The local versions are considered in so far as they are equivalent to the global ones affine locally. No novel material appears in this section.

1.1 Boundedness

Once the MMP has been established for a particular class of objects, there is a natural follow up question of boundedness or birational boundedness. Loosely speaking, given that algebraic objects are constructed from certain building blocks, we might start to wonder how many of such building blocks there are, and how many different ways they might be put together.

More concretely we ask if certain sets of objects are bounded, that is if they fit into a flat family over some quasi-projective scheme. As well as being as an interesting area of study in its own rights, such result are often viewed as the first step towards construction of a moduli space.

Perhaps the most famous boundedness conjecture is the Borislav-Alexeev Boundedness Conjecture. This claims the following.

Conjecture 2. *Let $d \in \mathbb{N}$ and $\epsilon > 0$. Fix a field κ. Then the set of projective varieties X admitting an ϵ-log canonical pair (X, Δ) with $-(K_X + \Delta)$ big and nef form a bounded a family.*

In characteristic zero, it is proven in [Bir16b, Theorem 1.1].

There are similar results and conjectures for varieties of general type, see for instance [HMX18]. Log Calabi-Yau varieties, however, are somewhat more subtle. Even in dimension 2 there are issues: complex $K3$ surfaces are bounded but projective ones are not. We can understand the issue to be that the projective $K3$ surfaces consist of infinitely many lines inside the space of complex differential ones.

In somewhat greater generality we might expect to be able to replace the bigness condition of the BAB conjecture with some form of rational connectedness on the underlying variety X, at least in characteristic zero. In positive characteristic it is unclear such a result would hold, even for dimension 2 as rational connectedness fails to rule out the possibility that X is a $K3$ surface. To the best of my knowledge it is unknown if rationally connected $K3$ surfaces are bounded or not.

?? attempts to circumvent these issues by imposing the additional criteria that X be F-split. As well as preventing the aforementioned issue in dimension 2, it also provides a sufficient vanishing type result to make use of inductive style arguments. Roughly speaking, one runs an MMP to reduce to the case that X is a Mori Fibration, and applies the lower dimensional boundedness results on the general fibres and the base to infer the result on the total space. We prove the following.

Theorem A. ?? *Fix $0 < \delta, \epsilon < 1$. Let $S_{\delta,\epsilon}$ be the set of threefolds satisfying the following conditions*

- *X is a projective variety over an algebraically closed field of characteristic $p > 7, \frac{2}{\delta}$;*
- *X is terminal, rationally chain connected and F-split;*
- *(X, Δ) is ϵ-klt and log Calabi-Yau for some boundary Δ; and*
- *The coefficients of Δ are greater than δ.*

Then there is a set $S'_{\delta,\epsilon}$, bounded over $\mathrm{Spec}(\mathbb{Z})$ such that any $X \in S_{\delta,\epsilon}$ is either birational to a member of $S'_{\delta,\epsilon}$ or to some $X' \in S_{\delta,\epsilon}$, Fano with Picard number 1.

In practice we were only able to prove a birational boundedness result with this method. Furthermore it was necessary to bound the coefficients of the pair below to prevent pathologies appearing in the Mori fibration. This in turn necessitates working with terminal underling varieties. The condition that X is terminal allows us to reduce to the case that X is a terminal Mori fibre space. While we might normally achieve this by taking a terminalisation $\tilde{X} \to X$, we cannot do so while also ensuring that the coefficients of $\tilde{\Delta}$ are still bounded below. In fact while bounding the coefficients below is used to prove a canonical bundle formula for Mori fibre spaces of relative dimension 1 it is in many ways the relative dimension 2 case that forces the assumption X is terminal.

If $(X, \Delta) \to S$ is a klt Mori fibre space with coefficients bounded below by $\frac{2}{p}$ then we may freely take a terminalisation and run an MMP to obtain a tame conic bundle, which is what we require for our boundedness proof. If however the relative dimension is 2 then after taking a terminalisation and running an MMP we may end with a Mori fibration of relative dimension 1, where we cannot easily control the singularities of the base. This happens whenever X is singular along a curve C which maps inseparably onto the base and we expect this is the only way it might happen.

The result we prove is rather pleasantly independent of the base field, so long as the characteristic is sufficiently large. This mirrors well the understanding that F-split varieties should in some sense 'look like' they come from characteristic 0. These kinds of boundedness results are one possible path towards a more concrete description of this analogy.

1.2 Abundance

In keeping with the earlier theme of birational geometers seeking to turn numerical criteria into algebraic ones, the Abundance Conjecture claims the following.

Conjecture 3. *Let (X, Δ) be a klt pair. Then if $K_X + \Delta$ is nef, it is semiample.*

In many ways the conjecture provides the link between the modern formulation of the MMP and the original goal of classification. The fibration induced by the abundance conjecture is by definition $K_X + \Delta$ trivial, yielding a log Calabi-Yau fibration over a lower dimensional base. The conjecture remains open in most settings. Even in characteristic zero the result is fully known only in dimension three and below, though several key cases are known in greater generality. In particular the case that Δ is big is covered by the Basepoint Free Theorem.

The case of surfaces defined over a field was proven in increasing generality in [FT12, Tan14, Tan20] while the case of threefolds over a perfect field of characteristic $p > 5$ is still open, though the non-vanishing conjecture has been settled in [XZ19, Wit18a] and various cases have been verified ([DW19a, Zha20]).

In **??** we prove the case of a klt threefold over base of dimension at least 1. There is a further assumption that the base has no points of characteristic $p \leq 5$ but this is a limitation of the current MMP results, not the method of proof.

Theorem B. **??** *Let R be an excellent ring of finite Krull dimension, equipped with a dualising complex and whose residue fields of closed points have characteristic $p > 5$. Let $\pi\colon (X, B) \to T$ be a projective morphism of quasi-projective R-schemes such that $\pi(X)$ is positive dimensional. Suppose (X, B) is a three-dimensional klt pair with $\mathbb{R}$-boundary. If $K_X + B$ is π-nef, then it is π-semiample.*

They key idea is to first show that there is a fibration in the category of algebraic spaces with the correct numerical properties. This follows from abundance on the generic fibre of $X \to T$. Next we apply the MMP to reduce to the case that the fibration is equidimensional. The motivation being that such fibrations are generally well behaved with respect to semiampleness, at least in the category of schemes. Finally we restrict to a horizontal slice of X over the base and infer the result here.

The Keel-Witaszek theorem is a key ingredient in both the first and the last step, providing the relevant criteria to show that $K_X + \Delta$ is EWM and then semiample.

Although the key focus of this Chapter is mixed characteristic schemes, the proof as given applies more widely to schemes over a positive dimensional base containing points of positive characteristic. In particular it covers some cases of purely positive characteristic.

As a further application of **??**, we study the invariance of plurigenera for families of klt surface pairs in mixed characteristic. It is well-known that invariance of plurigenera might fail over DVR of positive or mixed characteristic as shown in [KU85,Suh08,Bri20]. However it was proven in [EH21] that an asymptotic version of invariance of plurigenera holds for log smooth surface pairs if the Kodaira dimension is not one. Using techniques of [HMX18], we use the MMP and the abundance **??** to show an asymptotic invariance of plurigenera for families of klt surface pairs (possibly even defined over imperfect fields), extending the work of Egbert and Hacon.

Theorem C. **??** *Let R be an excellent DVR such that the residue field k has characteristic $p > 5$. Let (X, B) be a three-dimensional klt R-pair. Suppose that the following conditions are satisfied:*

(1) $(X, X_k + B)$ is plt with X_k integral and normal;

(2) if V is a non-canonical centre of $(X, B + X_k)$ contained in $\mathbf{B}_-(K_X + B)$, then $\dim(V_k) = \dim(V) - 1$.

Suppose further that at least one of the following holds:

1. $\kappa(K_{X_k} + B_k) \neq 1$; or

2. B_k is big over $\mathrm{Proj}(K_{X_k} + B_k)$

Then there is $m_0 \in \mathbb{N}$ such that

$$h^0(X_K, m(K_{X_K} + B_K)) = h^0(X_k, m(K_{X_k} + B_k))$$

for all $m \in m_0\mathbb{N}$.

In this setting $p > 5$ cannot be avoided, even with more general MMP results. The proof relies on adjunction type results which are unknown in low characteristic. Indeed they are known to fail in characteristic 2 by [CT19], even over a closed field.

Our new result covers a broader class of singularities as well as allowing for points with imperfect residue fields. It is natural to discuss this problem in terms of the Kodaira dimension. In this sense we provide a full characterisation of the Kodaira dimensions for which the result holds.

The failure of invariance of plurigenera when X_k has Kodaira dimension 1 is closely related to super-singularity of elliptic fibres of the Iitaka fibration induced by Abundance. We might therefore reasonably expect there to be additional characterisations in terms of F-splitness to describe when such invariance results hold. For example we might hope that if $X \to R$ is a flat, terminal family over a DVR and the Iitaka fibration $X_k \to Z_k$ is relatively F-split then Invariance of Plurigenera holds on X. These ideas are not explored further in this thesis.

1.3 Mori Fibrations in Positive Characteristic

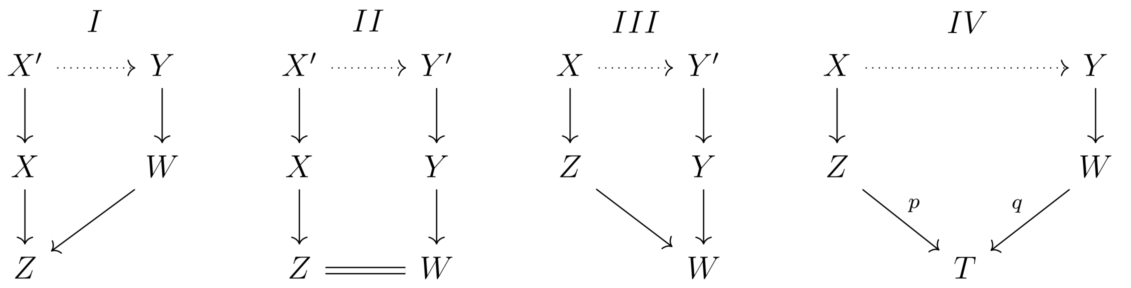
Where the Abundance Conjecture is needed to establish the existence of K_X trivial fibrations for minimal models, Mori fibrations come readily equipped with a K_X negative fibration. Conversely however they come with a more complex relationship between outputs of the MMP.

Terminal minimal models are connected by flops, due to the arguments of [Kaw08]. Mori fibrations however are expected to be linked by several different kinds of birational transformations. We prove this in dimension 3.

Theorem D (??). *Let R be an excellent ring of finite Krull dimension, equipped with a dualising complex and whose residue fields of closed points have characteristic $p > 5$. Fix an integral quasi-projective scheme T over R. Let $g_1 : Y_1 \to Z_1$ and $g_2 : Y_2 \to Z_2$ be two Sarkisov related, klt Mori fibre spaces of dimension 3, projective T. If the Y_i have positive dimension image in T, then they are connected by Sarkisov links.*

These links are characterised diagrammatically as follows.

Suppose that $f : X \to Z$, $g : Y \to W$ are two Mori Fibre Spaces over R. A Sarkisov link $s : X \dashrightarrow Y$ is one the following.

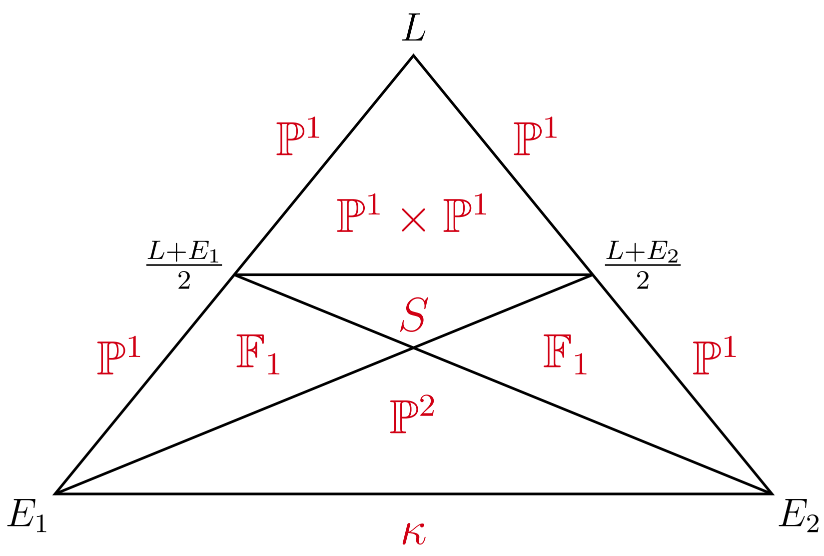


Such that the following holds:

- There is a klt pair (X, Δ) or (X', Δ') as appropriate such that the horizontal map is a sequence of flops for this pair
- Every vertical morphism is a contraction
- If the target of a vertical morphism is X or Y then it is an extremal divisorial contraction
- Either p, q are both Mori Fibre Spaces (this is type IV_m) or they are both small contractions (type IV_s)

A key insight, due to [HM09], is that the existence of these links can be seen in the structure of Shokurov polytopes. The classic example here is the following.

Let S be the blowup of $\mathbb{P}^2_\kappa$ at two points. Write E_1, E_2 for the exceptional curves and let L be the strict transform of the line between the blown up points. Then the effective cone is spanned by E_1, E_2, L and after fixing suitable $A \sim -K_X$ we can run a $D \sim K_X + A + D$ MMP for any D in the triangle, T, formed by L, E_1, E_2. We can then decompose T according to the output of the D MMP as follows, where $\mathbb{F}_1$ is the blowup of $\mathbb{P}^2$ at a single point.



The decomposition of T describes the geometry of the Mori fibre spaces.

1. Triangles inside T with a side along the boundary correspond to Mori fibre spaces
2. Shared sides of interior triangles correspond to blowups
3. All the morphisms (blowups and Mori fibrations) are induced by Abundance for pairs on the corresponding polygon

The Sarkisov links between the various Mori fibrations can be seen in the decomposition by composing the birational transformations coming from interior lines meeting at an exterior vertex.

The ample divisor $A \sim -K_X$ plays an important role here. Indeed the key result needed to recreate this kind of decomposition is the following Finiteness of Minimal Models result.

Theorem E (**??**). *Let R be an excellent ring of finite Krull dimension, equipped with a dualising complex and whose residue fields of closed points have characteristic $p > 5$ and take X a threefold over R. Let A be an ample $\mathbb{Q}$-Cartier divisor and C be a rational polytope inside $\mathcal{L}_A(V)$. Suppose there is a boundary $A+B \in \mathcal{L}_A(V)$ such that $(X, A+B)$ is a klt pair. Then the following hold:*

1. *There are finitely many birational contractions $\phi_i : X \dashrightarrow Y_i$ such that*
$$\mathcal{E}(C) = \bigcup \mathcal{W}_i = \mathcal{W}_{\phi_i}(C)$$
where each $\mathcal{W}_i$ is a rational polytope. Moreover if $\phi : X \to Y$ is a wlc model for any choice of $\Delta \in \mathcal{E}(C)$ then $\phi = \phi_i$ for some i, up to composition with an isomorphism.
2. *There are finitely many rational maps $\psi_j : X \dashrightarrow Z_j$ which partition $\mathcal{E}(C)$ into subsets $\mathcal{A}_{\psi_j}(C) = \mathcal{A}_i$.*
3. *For each W_i there is a j such that we can find a morphism $f_{i,j} : Y_i \to Z_j$ and $W_i \subseteq \overline{A_j}$.*
4. *$\mathcal{E}(C)$ is a rational polytope and A_j is a union of the interiors of finitely many rational polytopes.*

The keys ideas of the proof come from [BCHM10]. The main difficulty in mixed characteristic is the lack of appropriate Bertini type theorems. There are sufficient results to prove the result over local rings, with some modifications to the original proof. Some work is needed, however, to translate a local version of the result to a more general one.

Though it does seem it should be possible, we take a slightly different approach. First introducing a notion of an rlt pair, one which is replaceable by a klt pair locally over the base. This allows us to essentially extend the local Bertini result to a global one, at the cost of a slightly more complicated type of pair. With this accounting system in place, the finiteness result is essentially no harder to prove over an arbitrary base than over a local ring. The proofs given rely only on the MMP, and will generalise immediately to higher dimensions if the appropriate MMP results are known.

These rlt pairs are also useful for working with Sarkisov links. Once again there are not sufficiently strong Bertini type results to produce a klt pair corresponding to the flops of a Sarkisov link, instead an rlt pair is needed.

In this chapter we also give a short proof of termination for klt threefold pairs in mixed characteristic; showing that any MMP from a pair with $K_X + \Delta$ not psuedo-effective eventually terminates with a Mori Fibration.

Theorem F (**??**)**.** *Let R be an excellent ring of finite Krull dimension, equipped with a dualising complex and whose residue fields of closed points have characteristic $p > 5$. Let $f : (X, \Delta) \to T$ be a threefold dlt pair over R, then any $K_X + \Delta$ MMP terminates.*

1.4 The Augmented Base Locus

In addition to the earlier results related to the MMP and its applications we also study some more technical birational geometry results. The focus is largely on nef line bundle on mixed characteristic schemes which are semiample in characteristic 0.

The augmented base locus is well studied for schemes over a field. It is defined as follows.

Definition 1.4.1. *Let L be a line bundle on a projective Noetherian scheme X. Then base locus is given as*

$$\mathbf{B}(L) = \bigcap_{s \in H^0(X,L)} Z(s)_{red}$$

where $Z(s)$ is the zero set of s equipped with the obvious scheme structure. The stable base locus is then

$$\mathbf{SB}(L) = \bigcap_{m \geq 0} \mathbf{B}(mL).$$

Fix an ample line bundle A. The augmented base locus is given as

$$\mathbf{B}_+(L) = \bigcap_{m \geq 0} \mathbf{SB}(mL - A)$$

and is independent of the choice of A.

An important characterisation of the augmented base locus, first noted for smooth varieties of characteristic 0 by Nakayame [Nak00], is that for a nef line bundle L the augmented base locus $\mathbf{B}_+(L)$ agrees with the exceptional locus $\mathbb{E}(L)$.

Since then the result has been shown to hold for projective schemes over a field, first in positive characteristic by Cascini-M^c^Kernan-Mustaţă [CMM14], and then for $\mathbb{R}$-divisors over any field by Birkar [Bir17]. Similar results are given for non-nef divisors in [ELM+09] and for Kähler manifolds in [CT15].

We make use of methods developed in [Wit20] together with ideas from the positive characteristic proof to show that $\mathbf{B}_+(L) = \mathbb{E}(L)$ for a nef line bundle on a projective scheme over an excellent Noetherian base, so long it holds true on the characteristic zero part of the scheme. In particular the result holds in the following cases.

Theorem 1.4.2 (**??**). *Let X be a projective scheme over an excellent Noetherian base S with L a nef line bundle on X. Suppose that one of the following holds:*

1. *$S_{\mathbb{Q}}$ has dimension 0;*
2. *$L|_{X_{\mathbb{Q}}}$ is semiample;*

Then $\mathbf{B}_+(L) = \mathbb{E}(L)$.

We also extend the semiampleness result of [Wit20] to show that there is an equality of stable base loci when the characteristic 0 part is semiample.

Theorem 1.4.3 (**??**). *Suppose that X is a projective scheme over an excellent Noetherian base with L a nef line bundle on X. Then $\mathbf{SB}(L) = \mathbf{SB}(L|_{\mathbb{E}(L)})$ so long as $L|_{X_{\mathbb{Q}}}$ is semiample.*

Chapter 2

Preliminaries

2.1 Singularity Theory

We begin by collecting relevant notions of singularities for the minimal model program in mixed and positive characteristic. These include classic notions coming from the characteristic 0 setting, as well as algebraic singularity conditions developed in the positive characteristic setting.

2.1.1 Singularities of pairs

Here $\mathbb{K}$ will be taken to mean either $\mathbb{R}$ or $\mathbb{Q}$. If no field is specified, it is taken to be $\mathbb{R}$, i.e. a log pair is always a log pair with $\mathbb{R}$ boundary.

Definition 2.1.1. *A sub-log pair (X, Δ) with $\mathbb{K}$ boundary is an excellent, Noetherian, integral, normal scheme X admitting a dualising complex together with an $\mathbb{K}$-divisor Δ such that $(K_X + \Delta)$ is $\mathbb{K}$-Cartier. If Δ is effective, we say (X, Δ) is a log pair.*

In practice we study these almost exclusively in the following context.

Definition 2.1.2. *A sub R-pair $(X, \Delta)/T$ with $\mathbb{K}$-boundary will be the following data:*

- *A sub log pair (X, Δ) with $\mathbb{K}$ boundary;*
- *An excellent, normal ring R of finite dimension which admits a dualising complex and whose residue fields have characteristic at least* 5*;*
- *A quasi-projective R-scheme T; and*
- *A projective contraction $f\colon X \to T$.*

The dimension of such a pair is the dimension of X. Equally the pair is said to $\mathbb{Q}$-factorial if X is.

If $\Delta \geq 0$ we call it an R-pair with $\mathbb{K}$ boundary.

Note that $f\colon X \to T$ is a contraction ensures that $X \to T$ is surjective and T is integral and normal. We include this assumption for notational simplicity. All results extend to the case f is not a contraction by taking a Stein factorisation, though for some results this may require assumptions on the dimension of T be replaced with corresponding assumptions on the dimension of $f(X)$.

In practice we will often have $T = R$. In this case we may omit T from the notation and say only that (X, Δ) is an R-pair. If further $R = \kappa$ is a field, we often say one that (X, Δ) is a pair over a field or just that (X, Δ) is a pair, depending on context. Finally if $\Delta = 0$ we just say that X/T is an R-pair.

We will often ask that $X \to T$ has positive dimensional image, or equally that T is positive dimensional. Partly, this is because many results for threefolds are not known in greater generality than this, for example much is unknown when X is a variety over an imperfect field. Also many of the arguments will rely on lifting results from the general fibre, which only works for positive dimensional bases.

Since $K_X + \Delta$ is $\mathbb{R}$-Cartier, we may pull it back along any morphism $\pi\colon Y \to X$. If π is birational then there is a unique choice of $\Delta_Y = \sum -a(Y, E, X, \Delta)E$ which agrees with Δ away from the exceptional locus of π such that $\pi^*(K_X + \Delta) \sim_{\mathbb{R}}= K_Y + \Delta_Y$. In a slight abuse of notation we write $f^*(K_X + \Delta) = (K_Y + \Delta_Y)$.

Suppose that there are normal, integral schemes Y_i with $f_i : Y_i \to X$ birational and there is a some normal, integral scheme Z with $g_i : Z \to Y_i$. If E_i are divisors on Y_i with a common strict transform E on Z then $a(Z, E, X, \Delta) = a(Z, E, Y_i, \Delta_{Y_i}) = a(Y_i, E_i, X, \Delta)$ since we have $g_i^* f_i^* r(K_X + \Delta) = g_i^* r(K_{Y_i} + \Delta_{Y_i})$.

We may view, then, the values of $a(Y, E, X, \Delta)$ as being independent of the model Y and write $a(E, X, \Delta)$ instead.

For every prime divisor E on a birational model, Y, of X we have an associated DVR $\mathcal{O}_{Y,E}$, the stalk at the generic point of E which gives a valuation, ν_E on the function field $K(X)$. If $f\colon Y \to X$ is a birational morphism and D a prime divisor on Y together with a choice of generator inside $K(X)$ then pulling back D and looking at its coefficient at E is equivalent to asking for the valuation under ν_E.

In general, the converse is false. Not every valuation can be applied to K_X in this fashion.

For example suppose X is a proper normal variety over a field which is not $\mathbb{Q}$-Gorenstein. Let U be the smooth locus and P a point at which X is not $\mathbb{Q}$-Gorenstein. We may blowup X at P to give $Y \to X$ with E lying over P. Then U is smooth and birational to Y, but we cannot take the valuation of K_U with respect to E since no multiple of K_U is Cartier on X. If we wish to think of the $a(X, \Delta, E)$ as coming from valuations we must, therefore, consider only those with non-empty center on X.

Definition 2.1.3. *Let A be an integral domain with Frac$(A) = K$ and R a DVR in K with maximal ideal m_R. Then the center of R in A is $m_R \cap A$. We extend the definition to normal, integral schemes in the natural fashion.*

If X is of finite type over a locally Noetherian scheme T then X is proper over T if and only if every T-valuation has non-empty centre on X, by the valuative criterion of properness [Sta, Tag 0208].

Equally for a prime divisor E on a birational model Y of X, we can think of it as having non-empty centre on X if there is a dominating model $Z \to X, Y$ such that the generic point of E is contained in the image of Z on Y. This is the same as asking for the valuation it induces to have non-empty centre on X. In fact we can realise the centre of the valuation as the closure of the strict transform of E.

For simplicity, we will always think of a divisor E with non-empty centre on X as lying on a model Y which dominates X. Since the valuation does not depend on the birational model, we can always choose a higher model to ensure this is a valid assumption.

Definition 2.1.4. *Let $\pi \colon Y \to X$ be a proper birational morphism of integral, normal schemes. A divisor E on Y is said to be exceptional if π is not an isomorphism at the generic point of E, or equally if the centre of E is not a divisor on X.*

Given a sub-pair (X, Δ) we define the discrepancy

$$\text{Disc}(X, \Delta) := \inf\{a(E, X, \Delta) \text{ such that } E \text{ is exceptional and has non-empty center on } X\}$$

and the total discrepancy

$$\text{TDisc}(X, \Delta) := \inf\{a(E, X, \Delta) \text{ such that } E \text{ has non-empty center on } X\}$$

We then use this define a suite of singularities.

Definition 2.1.5. *Let (X, Δ) be a (sub)-log pair then we say that (X, Δ) is*

- *(sub) terminal if Disc$(X, \Delta) > 0$*
- *(sub) canonical if Disc$(X, \Delta) \geq 0$*
- *(sub) plt if Disc$(X, \Delta) \geq -1$*
- *(sub) ϵ-klt if TDisc$(X, \Delta) > \epsilon - 1$*
- *(sub) ϵ-lc if TDisc$(X, \Delta) \geq \epsilon - 1$*

Remark 2.1.6. *Klt is short for Kawamata log terminal and lc is short for log canonical.*

For $\epsilon = 0$ we say klt, lc respectively. We also say X has singularities of type $\mathcal{P}$ to mean $(X, 0)$ has such. An equivalent formulation of lc is that Disc$(X, \Delta) \geq -1$ as this condition ensures that Δ has coefficients bounded above by 1.

Lemma 2.1.7. *Let (X, Δ) be a (sub)-log pair with Disc$(X, \Delta) \geq -1$, then X is lc.*

Proof. Suppose for contradiction X has $\text{Disc}(X, \Delta) \geq -1$ but not $\text{TDisc}(X, \Delta) \geq -1$. Let D be in the support of Δ with $\text{Coeff}_D(\Delta) > 1$. Since X is normal we may localise at Q a point of codimension 2 inside the smooth locus of X and D which meets no other component of Δ. This reduces us to the case that X is smooth of dimension 2 and the support of D is a smooth curve, C. Now by assumption we have that $D = (1+\epsilon)C$ for $\epsilon > 0$. If we blow up the closed point, we get an exceptional divisor E with $a(E, X, \Delta) = -\epsilon$. Blowing up the intersection of E and the strict transform of Δ gives E_2 with $a(E_2, X, \Delta) = -2\epsilon$. Continuing in this fashion we can find E_n with $a(E_n, X, \Delta) = -n\epsilon < -1$ for some suitably large n.

Since $\text{Disc}(X, \Delta) \geq -1$ no such E_n can exist, so the result holds by contradiction. □

Note that if (X, B) and (X, Δ) are log pairs with $\Delta \geq B$ then clearly $a(E, X, \Delta) \leq a(E, X, B)$. So (X, B) cannot have singularities which are worse, in the above sense, than (X, Δ). Moreover if (X, Δ) is sub ϵ-lc and (X, B) is sub ϵ-klt then so is any sub-log pair (X, D) with $D \leq \delta B + (1-\delta)\Delta$ for any $1 > \delta > 0$.

When we have resolution of singularities there is another, more practical version of these definitions.

Definition 2.1.8. *We say (X, Δ) is log regular if X is a regular scheme and $\Delta = \sum d_i D_i$ is a divisor with normal crossing support*

If (X, Δ) is a sub-log pair and $\pi \colon Y \to X$ is projective, birational morphism with exceptional locus E such that $(Y, \pi_*^{-1}\Delta + E)$ is log regular then $\pi \colon Y \to X$ is a log resolution of (X, Δ). In this case, we sometimes say $\pi \colon Y \to (X, \Delta)$ is a log resolution.

When R is a closed field we often say log smooth instead of log regular.

Remark 2.1.9. *In principle it is enough for a log resolution to be proper for the purposes of these valuative notions of singularity. In practice we will often want projective log resolutions for other reasons and we do not separate the notions.*

Lemma 2.1.10. *Suppose that (X, Δ) is log regular. Let E be a prime divisor with center $V \neq E$ on X, write P for the generic point of V. Let $\Delta = \sum d_i D_i$. Then*

1. $a(E, X, D) \geq \textit{codim}(P, X) - 1 - \sum_{i: P \in D_i} d_j$
2. $\textit{TDisc}(X, \Delta) = \min\{0, -d_i\}$
3. $\textit{Disc}(X, \Delta) = \min\{1, 1 - d_i, 1 - d_i - d_j \colon D_i \cap D_j \neq \emptyset\}$

Proof. Let $Y \to X$ be a birational morphism such that E is a divisor on Y, let Q be its generic point. Localise at P in X so we may suppose that P is closed and given by the vanishing of $x_1, ...x_n$ where $n = \text{codim}(P, X)$. Similarly we may suppose E is given as the vanishing of a local coordinate y_1 on Y. Since (X, Δ) is log regular we may, after reordering, suppose $D_1, ..D_k$ contain P and each is given as the vanishing of a local coordinate x_i. Further can write $f^*x_i = y_1^{a_i}u_i$ where u_i does not vanish at Q and $a_i \in \mathbb{Z}_{>0}$.

We then have
$$f^*dx_i = a_i y_i^{a_i-1}u_i dy_1 + y_1^{a_i}du_i$$
by the chain rule where $du_i = w_i$ are regular at Q.

Putting $c_i = d_i$ for $i \leq k$ and $c_i = 0$ otherwise gives
$$f^*\frac{dx_i}{x_i^{d_i}} = a_i y_1^{(1-c_i)a_i-1}u_i^{1-c_i}dy_1 + y_1^{(1-c_i)a_i}w_i.$$

However then we see that the only possible poles of
$$f^*\frac{dx_1 \wedge ... \wedge dx_n}{x_1^{c_1}...x_n c^n}$$
at Q come from

$$y_1^{A_i}dy_1 \wedge w_1 \wedge ... \wedge w_{i-1} \wedge w_{i+1} \wedge ... \wedge w_n$$
with
$$A_i = -1 + \sum_1^n (1-c_j)a_j \geq -1 + \sum_1^n a_j - \sum_1^k d_j a_j \geq n - 1 - \sum_1^k d_j,$$
giving (1).

For any E with center V we have $a(E, x, D) \geq \operatorname{codim}(V, X) - 1 - \sum_{V \subseteq D_i} d_i$ and since $d_i \leq 1$ for every i the smallest value occurs when $V = E$ has codimension 1 and we obtain $\operatorname{TDisc}(X, \Delta) = \min\{0, -d_i\}$. Similarly if E is required to be exceptional we must have the smallest values when V has dimension 2 so that $\operatorname{Disc}(X, D) \geq \min\{1, 1 - d_i, 1 - d_i - d_j \text{ such that } D_i \cap D_j \neq \emptyset\}$.

Suppose however we blow up $V \subseteq D_i$ of codimension 2 and label the exceptional divisor E. It is an easy calculation that $a(E, X, D) = 1 - d_i$ if $V \not\subseteq D_j$ for all j else $a(E, X, D) = 1 - d_i - d_j$ where $V \subseteq D_j$ giving 2. Similarly by blowing up V of codimension 2 not contained in any D_i we see that there is some E with $a(X, E, D) = 1$ so (3) holds.

□

Corollary 2.1.11. *Let (X, Δ) be a (sub)-log pair and $\pi\colon Y \to X$ a log resolution of (X, Δ). Let $-d_i$ be the coefficients of Δ_Y and $d = \min d_i$. Then (X, Δ) is*

- *(sub) terminal iff $d > -1$ and $d_i + d_j > -1$ if $D_i \cap D_j \neq \emptyset$.*
- *(sub) canonical iff $d \geq -1$ and $d_i + d_j \geq -1$ if $D_i \cap D_j \neq \emptyset$.*
- *(sub) plt iff $d \geq -1$ and $d_i + d_j > -2$ if $D_i \cap D_j \neq \emptyset$.*

- *(sub)* $\epsilon - klt$ *iff* $d > \epsilon - 1$.
- *(sub)* $\epsilon - lc$ *iff* $d \leq \epsilon - 1$.

In particular we see that klt and lc may be checked in terms of the total discrepancy coming from a single log resolution. Terminal and canonical may also be checked in terms of discrepancy of a single resolution if $\Delta = 0$. If X is $\mathbb{Q}$-factorial then (X, Δ) is plt only when $\lfloor \Delta \rfloor$ is disjoint by [BMP+20, Lemma 2.28]. In this setting we may say, with notation as above, that (X, Δ) is plt if and only if $\lfloor \Delta \rfloor$ is disjoint, $\lfloor \Delta_Y \rfloor = \pi_*^{-1} \lfloor \Delta \rfloor$ and $d \geq -1$.

These calculations also give rise to an additional notion of singularity.

Definition 2.1.12. *An R pair* (X, Δ) *is called dlt if it lc and there is a closed subscheme* $Z \subseteq Z$ *such that:*

- $X \setminus Z$ *is regular,*
- $\Delta|_{X \setminus Z}$ *is simple normal crossing*
- *If* E *is an exceptional divisor with centre in* Z *then* $a(E, X, \Delta) > -1$.

Roughly speaking this says a dlt pair is an lc pair with is klt away from the locus where it is log smooth.

Note that if (X, Δ) is plt then it is also dlt.

Remark 2.1.13. *We can also characterise dlt with reference to a log resolution as follows. A pair* (X, Δ) *is dlt if there is a log resolution* $\pi \colon Y \to X$ *of* (X, Δ) *with* $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$ *such that* $\mathrm{Coeff}_E(\Delta_Y) < 1$ *for every* E *exceptional. The converse implication holds if sufficiently strong resolution results are known.*

This definition is not independent of the resolution. Consider for example X *a smooth surface with* $\Delta = C_1 + C_2$ *with connected log smooth support. This is trivially dlt, however if we blow up a point* P *in* $C_1 \cap C_2$ *then the pullback of* $K_X + \Delta$ *has coefficient* 1 *at the exceptional divisor.*

Allowing sub-pairs, being klt, lc etc pulls back naturally along birational morphisms. The following lemma allows us to push forward along them as well.

Lemma 2.1.14 (Negativity Lemma)**.** [BMP+20, Lemma 2.14] *Let* $f \colon X \to Y$ *be a projective birational morphism of normal, excellent, integral schemes. Let* D *be an* $\mathbb{R}$ *Cartier divisor on* X *with* $-D$ *nef over* Y. *Then* D *is effective if and only if* $f_* D$ *is.*

Lemma 2.1.15. *Suppose* $(X, \Delta), (X', \Delta')$ *are log pairs equipped with projective birational morphisms* $f \colon X \to Y$ *and* $f' : X' \to Y$ *with* $f_* \Delta = f'_* \Delta'$.

Suppose further that $-(K_X + \Delta)$ *is* f *nef and* $(K_{X'} + \Delta')$ *is* f' *nef. Then* $a(E, X, \Delta) \leq a(E, X', \Delta')$ *for any* E *with non-trivial center on* Y.

If in fact $-(K_X+\Delta)$ *is* f*-ample and* f *is not an isomorphism above the generic point of* $\text{centre}_X(E)$*, then*

$$a(E,X,\Delta) < a(E,X',\Delta').$$

Proof. Let Z be a normal, integral scheme with projective, birational morphisms $g\colon Z \to X$ and $g' : Z \to X'$, write $h = f \circ g = f' \circ g'$. Let $D = g^*(K_X+\Delta) - g'^*(K_{X'}+\Delta')$ which is exceptional by construction. Further since nefness is preserved under pulllback, $-D$ is nef over Y and hence we may apply the negativity lemma to see that D is effective. Thus $g^*(K_X+\Delta) \leq g'^*(K_{X'}+\Delta')$. In particular if E is any divisor on Z, then $a(E,X,\Delta) > a(E,X,\Delta')$.

Suppose now E is a valuation with non-trivial center on Y. There is some $Z \to Y$ with E a divisor on Z. We may then resolve the indeterminacy of $Z \to X$ and $Z \to X'$ and assume wlog that Z lies over X, X' also and the first part of result follows.

In the latter case, we see that E is covered by curves C with $D.C < 0$. Hence we must have that E is in the support of D and $a(E,X,\Delta) > a(E,X,\Delta')$. □

This is exactly the result that shows these notions of singularity are preserved under a $(K_X+\Delta)$ MMP.

2.1.2 Frobenius singularities

This section will focus on Frobenius singularities in positive characteristic. These will only be needed for schemes over a field, though one can make sense of these definitions in a more general context. We will often work with varieties over a field κ, which here will mean just mean integral, quasi-projective κ-schemes.

2.1.2.1 Frobenius singularities of pairs

Definition 2.1.16. *Given a* κ *algebra* R *over positive characteristic we denote the Frobenius morphism by* $F\colon R \to R$ *sending* $x \to x^p$*. Any* R *module* M *then has an induced module structure, denoted* F_*M *where* R *acts as* $r.m = F(r)m = r^p m$*. Finally* R *is said to be* F*-finite if* F_*R *is a finite* R *module. This is a particularly important notion in the case that* $R = \kappa$*.*

These definitions naturally extend to schemes over κ*.*

Note that all perfect fields are F-finite. Moreover any finitely generated algebra over an F-finite field is itself F-finite. In particular varieties over an F-finite field are F-finite.

In this context we can view the Frobenius morphism as a map of R modules $F\colon R \to F_*R$. We will also write $F^e : R \to F^e_*R$ for the e^{th} iterated Frobenius.

We have the following well known result due to Kunz.

Theorem 2.1.17. [Sta, Tag 0EC0][Kun69] *Let R be a reduced Noetherian local ring of characteristic $p > 0$, then R is regular if and only if F_*R is a flat R module.*

It is natural then to try and understand the singularities of a scheme via flatness conditions on F_*R. In the first instance we have the following definitions.

Definition 2.1.18. *Let X be a normal variety over an F-finite field. We say X is:*

- *F-pure if the Frobenius morphism $\mathcal{O}_X \to F_*\mathcal{O}_X$ is pure, or equivalently locally split.*
- *(Globally) F-split if the Frobenius morphism $\mathcal{O}_X \to F_*\mathcal{O}_X$ is split.*

Here for a morphism $f\colon R \to S$ to be pure means the induced map $M \to M \times S$ is injective for every R module M. When S is a finite R module, f is pure if and only if it is split. That is there is a morphism $g\colon S \to R$ of R modules with $g \circ f = id$.

Remark 2.1.19. *This purity condition is closed related to both flatness and effective descent. Roughly speaking every flat morphism is an effective descent morphism, but in general an effective descent morphism need only be pure. In fact purity turns out to be a sufficient condition also* [Sta, Tag 08WE].

In particular regular varieties are F-pure.

While these are useful definitions in their own right, for the purposes of the MMP we would like ones which can be more naturally applied to pairs (X, Δ).

Take X a normal variety over an F-finite field. To mirror the notion of a boundary we introduce pairs $(\mathcal{L}, \phi)$ where $\mathcal{L}$ is a line bundle and $\phi : F^e_*\mathcal{L} \to \mathcal{O}_X$. By applying duality on the regular locus, which contains all the codimension 1 points, we observe that $\mathrm{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{L}, \mathcal{O}_X) = H^0(X, \mathcal{L}^{-1}((1-p^e)K_X))$. Therefore such a pair corresponds to a divisor $\Delta_\phi \geq 0$ with $(1-p^e)(K_X + \Delta_\phi) \sim \mathcal{L}$. Reversing this procedure is slightly more involved. If $(1-p^e)(K_X + \Delta) \sim \mathcal{L}$ (we sometimes write this $K_X + \Delta \sim_{\mathbb{Z}_{(p)}} \mathcal{L}^{-1}$) we may obtain $\phi_\Delta : F^e_*\mathcal{L} \to \mathcal{O}_X$, however we could also write say $(1-p^{2e})(K_X + \Delta) \sim \mathcal{L}'$ where $\mathcal{L}' \not\sim \mathcal{L}$. We introduce, therefore, the following notion of equivalence.

First, we say that two such pairs, $(\mathcal{L}, \phi)$ and $(\mathcal{L}', \phi')$ are equivalent if:

- There is an isomorphism $\psi : \mathcal{L} \to \mathcal{L}'$ such that following diagram commutes; or

$$\begin{array}{ccc} F^e_*\mathcal{L} & \xrightarrow{F^e_*\psi} & F^e_*\mathcal{L}' \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi'} & \\ & \mathcal{O}_X & \end{array}$$

- $\mathcal{L} = \mathcal{L}^{p^{e'}+1}$ and $\phi'\colon F_*^{e+e'}\mathcal{L}^{p^{e'}+1} \to \mathcal{O}_X$ is the precisely the map given by

$$F_*^{e+e'}(\mathcal{L} \otimes \mathcal{L}^{p^{e'}}) \xrightarrow{F_*^e\phi} F_*^e\mathcal{L} \xrightarrow{\phi} \mathcal{O}_X.$$

We then expand the notion of equivalence to allow any finite combination of the above equivalences, more precisely we take the transitive closure of our initial relation.

The need for first part of this is clear. The second comes from the following lemma

Lemma 2.1.20. *Suppose that $(\mathcal{L}, \phi)$ and $(\mathcal{L}', \phi')$ are pairs as above. Then we have the following map*

$$\psi = \phi' \circ_F \phi : F_*^{e+e'}(\mathcal{L} \otimes (\mathcal{L}')^{p^e}) \cong F_*^{e'}(F_*^e\mathcal{L} \otimes \mathcal{L}') \to F_*^{e'}\mathcal{L}' \to \mathcal{O}_X$$

and the associated divisor is $\Delta_\psi = \frac{p^e-1}{p^{e+e'}-1}\Delta_\phi + \frac{p^e(p^{e'}-1)}{p^{e+e'}-1}\Delta_{\phi'}$.

Proof. The statement is local, so we may suppose that $\mathcal{L} = \mathcal{L}' = \mathcal{O}_X$ and $X = \mathrm{Spec}R$. Fix $\Phi : F_*R \to R$ the generating map of $\mathrm{Hom}_R(F_*R, R)$ as an F_*R module. Hence we have $\phi = x.\Phi^e$ and $\phi' = x'.(\Phi)^{e'}$. Hence we clearly have

$$\psi(r) = \phi' \circ F_*^{e'}(\phi)(r) = \Phi^{e'} \circ (x'(F_*^{e'}(x.\Phi^e))(r) = \Phi^{e+e'}(x(x')^{p^e})r).$$

Hence we see that the divisor is

$$\begin{aligned}\Delta_\psi &= \frac{1}{p^{e+e'}-1}(\mathrm{div}(x) + p^e\mathrm{div}(x')) \\ &= \frac{p^e-1}{p^{e+e'}-1}\Delta_\phi + \frac{p^e(p^{e'}-1)}{p^{e+e'}-1}\Delta_{\phi'}.\end{aligned}$$

Since we must have $\Delta_{(x.\Phi^k)} = \frac{1}{p^k-1}\mathrm{div}(x)$ under the identification $\mathrm{Hom}_R(F_*^eR, R) \cong F_*^eR$.

□

We write ϕ^n for $\phi^{n-1} \circ_F \phi$. Note that by the above calculation, $\Delta_{\phi^n} = \Delta_\phi$, which is why we require the second part of the equivalence relation.

Remark 2.1.21. *We might ask if this construction still makes sense for $e = 0$. Obviously we cannot divide by $p^e - 1$ but if we run through the correspondence, we are simply identifying* $\mathrm{Hom}(\mathcal{L}, \mathcal{O}_X)$ *with* $\mathrm{Hom}(\mathcal{O}_X, \mathcal{L}^{-1})$. *So a morphism $\phi : \mathcal{L} \to \mathcal{O}_X$ induces a divisor D_ϕ with $\mathcal{O}_X(D_\phi) \simeq \mathcal{L}^{-1}$. Then we get the formula*

$$\Delta_{\phi' \circ_F \phi} = \frac{1}{p^{e'}-1}D_\phi + \Delta_{\phi'}$$

Similarly when $e' = 0$ we get $\Delta_{\phi' \circ_F \phi} = \Delta_{\phi' \circ \phi} = \Delta_\phi + \frac{p^e}{p^e-1}D_{\phi'}$ and if $e = e' = 0$ we recover the usual composition formula $D_{\phi' \circ \phi} = D_\phi + D_{\phi'}$.

Suppose $\phi : \mathcal{L}\mathcal{L}'^{-1} \to \mathcal{O}_X$ *then* ϕ *corresponds to a divisor* $D \sim \mathcal{L}'\mathcal{L}^{-1}$ *in the usual sense. The result is than that* $\psi = \phi \circ_F \phi' : F^{e'}_*(\mathcal{L}\mathcal{L}'^{-1} \otimes \mathcal{L}') = F^{e'}_*(\mathcal{L}) \to F^{e'}_*(\mathcal{L}') \to \mathcal{O}_X$ *has* $\Delta_\psi = \frac{1}{p^e-1}D + \Delta_{\phi'}$ *from above. Equally of course we may view* ϕ *as a morphism* $\mathcal{L} \to \mathcal{L}'$*.*

Note that Lemma 2.1.20 and Remark 2.1.21 can be applied in the opposite direction. Suppose that ϕ, ψ have $\Delta_\phi \geq \delta_\psi$. Let $E = \Delta_\phi - \Delta_\psi$. Then we get an induced map $F^e_* i_{(p^e-1)E} \colon F^e_*\mathcal{L}_\phi \to F^e_*\mathcal{L}_\psi$. Now $\psi \circ_F i_{(p^e-1)E}$ has induced boundary $\frac{1}{p^e-1}(p^e-1)E + \Delta_\psi = \Delta_\phi$. Hence in fact $\phi = \psi \circ_F i_{(p^e-1)E}$.

In particular then, every ϕ is of the form $F^e_*\mathcal{L} \to F^e_*\omega_X^{\otimes(1-p^e)} \to \mathcal{O}_X$ where $F^e_*\mathcal{L} \to F^e_*\omega_X^{\otimes(1-p^e)}$ is the pushforward of the inclusion $\mathcal{O}_X((1-p^e)(K_X+\Delta_\phi)) \to \mathcal{O}_X((1-p^e)(K_X))$ induced by $(1-p^e)\Delta_\phi = D_\phi$. This can also be seen directly from the construction of D_ϕ if one takes care.

We see then that ϕ is dual to the map $\mathcal{O}_X \to F^e_*\mathcal{O}_X \to F^e_*\mathcal{O}_X((p^e-1)\Delta_\phi)$. We can study the same kinds of pairs by working with such maps instead. This is the setup of [SS10] for example.

Lemma 2.1.22. *Two pairs* $(\mathcal{L}, \phi)$ *and* $(\mathcal{L}', \phi')$ *are equivalent if and only if* $\Delta_\phi = \Delta_{\phi'}$*. In particular then there is a bijection between equivalence classes of such pairs and* $\Delta \geq 0$ *with* $(K_X + \Delta)$ $\mathbb{Z}_{(p)}$*-Cartier.*

Proof. From above we have that if $(\mathcal{L}, \phi)$ and $(\mathcal{L}', \phi')$ are equivalent then $\Delta_\phi = \Delta_{\phi'}$ so we prove only the converse statement.

By taking higher powers of these maps we may assume wlog that $e = e'$. This does not change Δ_ϕ or $\Delta_{\phi'}$ by Lemma 2.1.20, moreover the equivalence classes of $(\mathcal{L}, \phi)$ and $(\mathcal{L}', \phi')$ are unchanged by definition.

However if $D = \Delta_\phi - \Delta_{\phi'}$ then $(p^e-1)D \sim 0$ defines an isomorphism

$$i : \mathcal{O}_X((p^e-1)(K_X + \Delta_\phi)) \to \mathcal{O}_X((p^e-1)(K_X + \Delta_{\phi'})).$$

Let $\psi = \phi' \circ i$ so we have $\Delta_\psi = D + \Delta_{\phi'} = \Delta_\phi$ but this says exactly that $\psi = \phi \circ u$ for some automorphism u of $\mathcal{L}$ and hence $(\mathcal{L}, \psi) \sim (\mathcal{L}', \phi')$.

□

To extend this framework to allow for sub pairs we can instead work with morphisms $F^e_*\mathcal{L} \to K(X)$ where we view $K(X)$ as a constant sheaf on X. Given such a morphism ϕ, we can always find $E \geq 0$ Cartier such that when we twist by E we obtain

$$\phi' := F^e_*(\mathcal{L}((1-p^e)E)) \to \mathcal{O}_X$$

and thus associate a divisor $\Delta_{\phi'}$ with $(1-p^e)(K_X + \Delta_{\phi'}) \sim \mathcal{L}((1-p^e)E$. We then take $\Delta_\phi = \Delta_{\phi'} - E$.

Lemma 2.1.23. *With the notation as above,* Δ_ϕ *does not depend on the choice of* E*.*

Proof. Suppose E_1, E_2 are two choices of E, suppose wlog that $E_1 \leq E_2$. Write $\phi_i := F^e_*(\mathcal{L}((1-p^e)E_i)) \to \mathcal{O}_X$ for their twists. Let i be the inclusion $\mathcal{L}((1-p^e)E_2) \to \mathcal{L}((1-p^e)E_1)$. Then by Lemma 2.1.20 since $\phi_2 = \phi_1 \circ F^e_* i$ we have that $\Delta_{\phi_2} = \Delta_{\phi_1} + (E_2 - E_1)$ so that $\Delta_{\phi_2} - E_2 = \Delta_{\phi_1} - E_1$. □

Definition 2.1.24. *A sub $\mathbb{Z}_{(p)}$-pair is a κ-pair (X, B) where κ is F-finite, $(K_X + B)$ is $\mathbb{Z}_{(p)}$-Cartier and the coefficients of B are less than 1. We write $\phi_B : F^{e_B}_* \mathcal{L}_{e,B} \to K(X)$ for the associated morphism dropping the dependence on B when it remains clear. If B is effective (X, B) is called a $\mathbb{Z}_{(p)}$ pair and we view ϕ_B as being a morphism to $\mathcal{O}_X$.*

Let (X, B) be a (sub) $\mathbb{Z}_{(p)}$ pair, then (X, B) is

- *(sub) F-pure if $\mathcal{O}_X \subseteq Im(\phi^e)$ for some e*
- *(sub) F-split if $1 \in Im(H^0(X, \phi^e))$ for some e*
- *(sub) F-regular if for every $D \geq 0$ there is some e with $\mathcal{O}_X \subseteq \phi^e(F^e_*(\mathcal{L}_e(-D))$*
- *globally (sub) F-regular if for every $D \geq 0$ there is some e with $1 \in \mathrm{Im}(H^0(X, \phi^e|_{F^e_*(\mathcal{L}_e(-D))})$*

Remark 2.1.25. *We can also extend the definitions to log pairs in the sense of Definition 2.1.1. Roughly we speaking we say (X, Δ) satisfies the definition if there is $B \geq \Delta$ such that (X, B) is a sub $\mathbb{Z}_{(p)}$ pair satisfying the definition in question. Alternatively one can work with reflexive sheaves in the place of line bundles. By Lemma 2.1.26 the two are equivalent.*

Being F-split is also sometimes called globally F-split, to distinguish it from the case of local splittings.

Some immediate consequences of Lemma 2.1.20 and Remark 2.1.21 are the following.

Lemma 2.1.26. *Let (X, Δ) and (X, B) be globally F-split pairs. Then for $0 \leq t << 1$ we have that $(X, t\Delta + (1-t)B)$ is F-split and for $0 \leq \lambda \leq 1$ we have that $(X, \lambda\Delta)$ is F-split also. Moreover if (X, Δ) is in fact globally F-regular then*

1. *$(X, \lambda\Delta)$ is globally F-regular for all $0 \leq \lambda \leq 1$.*
2. *$(X, t\Delta + (1-t)B)$ is globally F-regular for $0 \leq t << 1$.*
3. *For any $D \geq 0$, $(X, \Delta + tD)$ is globally F-regular for $0 \leq t << 1$.*
4. *$B = \Delta + D$ then $(X, \Delta + tD)$ is globally F-regular for $0 \leq t < 1$.*

Proof. Let (X, Δ) and (X, B) be globally F-split pairs. By composing the associated morphisms $\phi_\Delta \colon F^e_* \mathcal{L}_\Delta \to \mathcal{O}_X$ and $\phi_B \colon F^{e'}_* \mathcal{L}_B \to \mathcal{O}_X$ as in Lemma 2.1.20 we obtain a split morphism $\psi = \phi_\Delta \circ_F \phi_B$ with associated divisor $(1 - \frac{p^e-1}{p^{e+e'}-1})\Delta + \frac{p^e-1}{p^{e+e'}-1} B$. Taking $e' >> e$ yields the result.

To see that $(X, \lambda\Delta)$ is F-split for $\lambda \geq 0$, we can assume $\lambda \in \mathbb{Z}_{(p)}$. Then by Lemma 2.1.20 and the discussion following it, we see that we have a factorisation

$$\phi_\Delta \colon F^e_* \mathcal{L} \to F^e_* \mathcal{L}_{\lambda\Delta} \to \mathcal{O}_X$$

for $e >> 0$. This yields the result.

Now suppose that (X, Δ) is globally F-regular. Then part (1) follows exactly as above. We now prove (2). To this end, let H be a Cartier divisor whose support contains B.

From the definition of globally F-regular and previous parts we have small $\epsilon, \delta > 0$ with $(X, \epsilon\Delta + (1-\epsilon)B + \delta H)$ F-split. We claim that $(X, \epsilon\Delta + (1-\epsilon)B)$ is globally F-regular. Certainly it is F-split so let $\phi \colon \mathcal{L} \to \mathcal{O}_X$ be the associated morphism. This pair is globally F-regular on $U = X \setminus H$, since $H \geq B$. So if we fix $D \geq 0$ then we have a splitting of $\phi_U \colon F^e_* \mathcal{L}_U(-D) \to \mathcal{O}_U$, say $i_U \colon \mathcal{O}_U \to F^e_* \mathcal{L}_U(-D)$. We now seek to extend i_U to an appropriate splitting on X.

By [Har77, Chapter II, Lemma 5.14(b)] i_u extends to a section $i : \mathcal{O}_X \to (F^e_* \mathcal{L}) \otimes \mathcal{O}_X(mH)$ such that the following composition

$$\mathcal{O}_X \xrightarrow{i} (F^e_* \mathcal{L}(-D)) \otimes \mathcal{O}_X(mH) \xrightarrow{\phi \otimes \mathcal{O}_X(mH)} \mathcal{O}_X(mH)$$

is natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(mH)$ induced by H. Tensoring by $\mathcal{O}_X(-mH)$ yields

$$\mathcal{O}_X(-mH) \xrightarrow{i \otimes \mathcal{O}_X(-mH)} F^e_* \mathcal{L}(-D) \xrightarrow{\phi} \mathcal{O}_X.$$

Again this is the natural inclusion of $\mathcal{O}_X(-mH)$ into $\mathcal{O}_X$. Finally we tensor by $F^{e'}_* \mathcal{L}$ to yield

$$F^{e'}_* \mathcal{L}(-mH) \to F^{e'+e}_* \mathcal{L}(-D) \to F^{e'}_* \mathcal{L}.$$

By assumption, for $e' >> 0$ the morphism $F^{e'}_* \mathcal{L}(-mH) \to F^{e'}_* \mathcal{L} \to \mathcal{O}_X$ splits. Hence so too does $F^{e'+e}_* \mathcal{L}(-D) \to F^{e'}_* \mathcal{L} \to \mathcal{O}_X$. Thus $(X, \epsilon\Delta + (1-\epsilon)B)$ is globally F-regular as claimed.

Now for (3) fix a $D \geq 0$. Then

$$F^e_* \mathcal{L}(-D) \to F^e_* \mathcal{L}(-D) \to \mathcal{O}_X$$

splits for $e >> 0$. The associated divisor is precisely $(X, \Delta + \frac{1}{p^e-1}D)$ so $(X, \Delta + tD)$ is F-split for small t. By (1) with $B = \Delta + tD$, we may shrink t and assume the pair is globally F-regular.

The final part follows straight from (2) since $t\Delta + (1-t)B = \Delta + (1-t)D$.

□

Locally to a point of codimension 1 these definitions are particularly well-behaved.

Lemma 2.1.27. *Let R be a regular DVR with parameter t, then a sub $\mathbb{Z}_{(p)}$ pair $(R, \lambda t)$ is sub F-pure iff $\lambda \leq 1$ and sub F-regular iff $\lambda < 1$.*

Proof. After choosing an isomorphism $\mathcal{L} \simeq R$ we may suppose that λt defines a morphism $F^e_* R \to R$. By Remark 2.1.21 this factors $F^e_* R \to < t > \to R$ if and only if $\Delta_\phi \geq \frac{p^e}{p^{e-1}} t$. This happens for $e >> 0$ if and only if $\lambda > 1$. This gives the F-pure case.

Since every divisor D on R is of the form μt, for μ a unit, the F-regular result follows also. □

In particular we see that the coefficient of Δ_ϕ at E depends only on ϕ near E.

Corollary 2.1.28. *Suppose $\phi : F^e_* \mathcal{L} \to k(X)$ has associated divisor Δ then $\mathit{Coeff}_E(\Delta) = \inf\{t : (X, \Delta + tE)$ is F sub pure at the generic point of $E\}$.*

While these definitions do not pullback along birational morphisms as obviously as the usual MMP singularities, it is still possible.

Lemma 2.1.29. *Suppose that $f \colon X \to Y$ is a birational morphism with X normal and (Y, Δ) $\mathbb{Z}_{(p)}$ pair then there is Δ' on X making (X, Δ') a $\mathbb{Z}_{(p)}$ pair such that $(K_X + \Delta') = f^*(K_Y + \Delta)$. If (Y, Δ) is sub F-split so too is (X, Δ').*

Proof. Take the corresponding map $\phi \colon F^e_* \mathcal{L} \to K(Y)$, we may freely view $\mathcal{L}$ as a subsheaf of $K(Y)$ via some $i \colon \mathcal{L} \hookrightarrow K(Y)$ and so extend ϕ to a map $\bar{\phi} \colon F^e_* K(Y) \to K(Y)$. Taking the inverse image gives $f^{-1}(\bar{\phi}) \colon f^{-1} F^e_* K(Y) \to f^{-1} K(Y)$ and $f^{-1}(i) \colon f^{-1} F^e_* \mathcal{L} \to f^{-1} K(Y)$. Since f is birational we obtain an isomorphism $f^{-1} K(Y) \to K(X)$. We then have the following situation.

$$\begin{array}{ccccc} f^{-1} F^e_*(\mathcal{L}) \otimes_{f^{-1} F^e_* \mathcal{O}_Y} \mathcal{O}_X & \hookrightarrow & F^e_* K(X) & \longrightarrow & K(X) \\ \uparrow & & \uparrow \sim & & \uparrow \sim \\ f^{-1} F^e_*(\mathcal{L}) & \xrightarrow{f^{-1}(i)} & f^{-1} F^e_* K(Y) & \xrightarrow[f^{-1}(\bar{\phi})]{} & f^{-1} K(Y) \end{array}$$

Note however that $f^{-1} F^e_*(\mathcal{L}) \otimes_{f^{-1} F^e_* \mathcal{O}_Y} \mathcal{O}_X = F^e_* f^* \mathcal{L}$ and hence we obtain the desired map $\tilde{\phi} : F^e_* f^* \mathcal{L} \to K(X)$. This induces a divisor Δ' on X with $(p^e - 1)(K_X + \Delta') \sim f^* \mathcal{L} \sim (p^e - 1) f^*(K_Y + \Delta)$. The coefficient of Δ' at a codimension one point can be recovered from $\tilde{\phi}$ by working locally around that point. wherever f is an isomorphism, ϕ and $\tilde{\phi}$W agree and therefore the coefficients of Δ and Δ' agree on this locus also.

Hence in fact we have an actual equality of divisors $f^*(K_Y + \Delta) = (K_X + \Delta')$ as required. Moreover commutativity of the earlier diagram gives that whenever $1 \in \mathrm{Im}(H^0(Y, \phi))$ then it is also in the image of $H^0(X, \tilde{\phi})$, and hence (X, Δ) is sub F-split. □

Note that a pair (X, Δ) is sub F-pure if and only if there is an open cover $\{U_i\}$ with $(U_i, \Delta|_{U_i})$ sub F-split. Hence in fact this shows we may also lift sub F-pure pairs in the same fashion.

Similarly a pair (X, Δ) is (globally) sub F-regular if and only if for every $D \geq 0$ there is $\epsilon < 0$ with $(X, \Delta + \epsilon D)$ sub F-pure (F-split). Further if $f \colon Y \to X$ is birational with

$D \geq 0$ on Y there is always some $D' \geq 0$ on X with $f^*D' \geq D$. Therefore pulling back $(X, \Delta + \epsilon D')$ to $(Y, \Delta' + \epsilon f^*D')$ we see that $(Y, \Delta' + \epsilon D)$ is sub F-pure (F-split)and so (Y, Δ') is (globally) sub F-regular.

Theorem 2.1.30. *Let (X, Δ) be a sub F-pure pair. Then (X, Δ) is sub-lc. Moreover if (X, Δ) is sub F-regular then in fact it is sub-klt.*

Proof. Let $Y \to X$ be a proepr birational morphism of integral normal schemes and Δ_Y the induced boundary on Y. From above we see that (Y, Δ_Y) is sub F-pure. However by Corollary 2.1.28 we see that this ensures $\mathrm{Coeff}_D(\Delta_Y) \leq 1$ for every prime divisor D on X. Hence (Y, Δ_Y) is sub-lc and therefore so too is (X, Δ). An identical calculation completes the F-regular case. □

In general we cannot push forward the local forms of these singularities, however the global ones often can be pushed forward, even along morphisms which are not birational.

Lemma 2.1.31. *Suppose that (X, Δ) is sub F-split and $f\colon X \to Y$ has $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $K_X + \Delta \sim_{\mathbb{Z}_{(p)}} f^*\mathcal{L}$. If every component of Δ which dominates Y is effective then there is Δ_Y with (Y, Δ_Y) sub F-split and $K_Y + \Delta_Y \sim_{\mathbb{Z}_{(p)}} \mathcal{L}$.*

Proof. This is the inverse construction of Lemma 2.1.29. By assumption the pair (X, Δ) corresponds to a morphism $\phi\colon F^e_* f^*\mathcal{L} \to K(X)$. Since the dominant part of Δ is effective we may view this as a morphism $\phi : f^*\mathcal{L} \to f^*\mathcal{O}_X(D)$ where D is some divisor on Y with $(1 - p^e)\Delta \geq -f^*D$.

This then pushes forward to a non-zero morphism $\phi_Y : F^e_*\mathcal{L} \to \mathcal{O}_Y(D) \subseteq K(Y)$ which canonically induces a pair (Y, Δ_Y). Note further that we have natural isomorphisms

$$\begin{array}{ccc} H^0(X, F^e_* f^*\mathcal{L}) & \xrightarrow{H^0(\phi^e)} & H^0(X, f^*\mathcal{O}_Y(D)) \\ \downarrow \simeq & & \downarrow \simeq \\ H^0(Y, F^e_*\mathcal{L}) & \xrightarrow{H^0(\phi^e_Y)} & H^0(Y, \mathcal{O}_Y(D)) \end{array}$$

so that (X, Δ) is sub F-split if and only if (Y, Δ_Y) is so. □

If in fact (X, Δ) is globally F-regular then so too is (Y, Δ_Y). Indeed if D is a divisor on Y, then there is $\epsilon > 0$ with $(X, \Delta + \epsilon f^*D)$ globally F-split but then $(Y, \Delta + \epsilon D)$ is globally F-split also.

By Corollary 2.1.28 if $f\colon X \to Y$ is birational then the conditions are automatically satisfied and the induced Δ_Y is just the pushforward $f_*\Delta$. Therefore if X is sub F-split so is every X' birational to X. Further if X is F-split and X' is obtained by taking a terminalisation or running a $K_X + B$ MMP for any B then X' is F-split.

2.1.2.2 Global Frobenius Singularities

Pairs (X, Δ) which are globally F-split or globally F-regular can always be modified slightly to assume a particularly nice form.

Lemma 2.1.32. *Suppose that (X, Δ) is a globally F-split pair, then we have $\Delta' \geq \Delta$ such that (X, Δ') is globally F-split and $K_X + \Delta \sim_{\mathbb{Z}_{(p)}} 0$.*

If instead (X, Δ) is globally F-regular, then we have $\Delta' \geq \Delta$ such that (X, Δ') is globally F-regular and $-(K_X + \Delta)$ is ample.

Proof. Suppose (X, Δ) is a globally F-split $\mathbb{Z}_{(p)}$ pair. Let $\phi \colon F^e_* \mathcal{L} \to \mathcal{O}_X$ be the corresponding morphism with $1 \in \mathrm{Im}(H^0(\phi))$. Then by assumption we have a section $s : \mathcal{O}_X \to F^e_* \mathcal{L}$ which is a splitting of ϕ.

However we get an induced section $F^e_* \mathcal{O}_X \to F^e_* \mathcal{L}$ given locally by $r \to r \times s(1)$, hence in fact s factors $s : \mathcal{O}_X \to F^e_* \mathcal{O}_X \to F^e_* \mathcal{L}$. The composition $F^e_* \mathcal{O}_X \to F^e_* \mathcal{L} \to \mathcal{O}_X$ induces an F-split pair (X, Δ') with $K_X + \Delta' \sim_{\mathbb{Z}_{(p)}} 0$. Moreover we have $\Delta' \geq \Delta$ by Remark 2.1.21.

Now suppose that (X, Δ) is a globally F-regular $\mathbb{Z}_{(p)}$ pair. First, from above, we may take B with $(X, \Delta + B)$ F-split and $K_X + \Delta + B \sim_{\mathbb{Z}_{(p)}} 0$.

Now choose $H \geq B$ an ample divisor. Then we have that the composition $F^e_* \mathcal{L}(-H) \to F^e_* \mathcal{O}_X \xrightarrow{\phi} \mathcal{O}_X$ splits. As before the section $\mathcal{O}_X \to \mathcal{L}(-H)$ factors $\mathcal{O}_X \to F^e_* \mathcal{O}_X(-H) \to F^e_* \mathcal{L}(H)$. The split morphism $F^e_* \mathcal{O}_X(-H) \to \mathcal{O}_X$ induces a globally F-split pair $(X, \Delta + D)$ with $K_X + \Delta + D \sim_{\mathbb{Z}_{(p)}} \delta H$ where $\delta = \frac{1}{p^e - 1}$. Moreover the first part of the lemma applied to $(X, \Delta + D)$ yields the F-split pair $(X, \Delta + D + \delta H)$, again by Remark 2.1.21.

We now apply Lemma 2.1.26 to $(X, \Delta + B)$ and $(X, \Delta + D + \delta H)$ to see that $(X, \Delta + tD + (1 - t + \delta')B)$ is F-split. By the same lemma we can choose t small enough that $(X, \Delta + tD)$ is globally F-regular. Applying the lemma one more time to these two new pairs, we see that $(X, \Delta + tD + (1-t)B)$ is globally F-regular. By construction this pair has $K_X + \Delta + tD + (1 - t)B \sim_{\mathbb{Z}_{(p)}} -tH$ as required.

□

Lemma 2.1.33. *Let (X, Δ) be a globally F-split pair. Then $H^i(X, K_X + \Delta + A) = 0$ for A an ample $\mathbb{Q}$-Cartier divisor and $i > 0$. In particular $H^i(X, A) = 0$ for $i > 0$. Moreover if (X, Δ) is globally F-regular then we may suppose only that A is big and nef instead.*

Proof. Suppose first that (X, Δ) is F-split and A is ample. Then we have a split map $F^e_* \mathcal{L} \to \mathcal{O}_X$ where $\mathcal{L} = \mathcal{O}_X((1 - p^e)(K_X + \Delta))$. Tensoring by $\mathcal{O}_X(K_X + \Delta + A)$ yields $F^e_* \mathcal{O}_X(K_X + \Delta + p^e A) \to \mathcal{O}_X(K_X + \Delta + A)$. Taking cohomology then gives a surjection $H^i(X, K_X + \Delta + p^e A) \to H^i(X, K_X + \Delta + A)$ for $i \geq 0$ where the left hand side vanishes for $e >> 0$ and $i > 0$ by Serre vanishing. Hence in fact $H^i(X, K_X + \Delta + A) = 0$ as claimed. From above, we can assume that $K_X + \Delta \sim \_\mathbb{Z}_{(p)} 0$, so we have A' ample with $K_X + \Delta + A' = A$ and the second part follows.

Now suppose that (X, Δ) is globally F-regular. Choose $E \geq 0$ with $nA - E$ ample for $n >> 0$. Then we have $F^e_*\mathcal{L}(-E) \to \mathcal{O}_X$ split. Again we tensor by $K_X + \Delta + A$ to yield a split map $F^e_*\mathcal{O}_X(K_X + \Delta + p^eA - E) \to \mathcal{O}_X(K_X + \Delta + A)$. From the first part, the cohomology of $\mathcal{O}_X(K_X + \Delta + p^eA - E)$ vanishes for $e >> 0$ and the result follows.

□

If (X, Δ) is F-split (resp. globally F-regular) in the sense of Remark 2.1.25 one needs to be slightly more careful. In this case we have a $\mathbb{Z}_{(p)}$ pair (X, B) with $B \geq 0$ and $\mathcal{L} = \mathcal{O}_X((1-p^e)(K_X+B))$ which is F-split (resp. globally F-regular). Then the morphism $F^e_*\mathcal{L} \to \mathcal{O}_X$ must factor $F^e_*\mathcal{L} \to F^e_*\mathcal{O}_X((1-p^e)(K_X+\Delta) \to \mathcal{O}_X$, as in Lemma 2.1.26 and the result follows exactly as above.

2.2 The Minimal Model Program

2.2.1 Overview of the Minimal Model Program

In it's original incarnation the Minimal Model Program seeks to modify a smooth complex variety to a simpler (or minimal) birational model. The last few decades have seen a shift away from this paradigm, however.

The Minimal Model Program now consists of a suite of useful tools in its own right, focused on the birational modification of pairs R-pairs $(X, B)/T$ over a suitable base, and having mild singularities - typically $\mathbb{Q}$-factorial and klt, or more generally dlt or log canonical singularities might be permitted. We will focus mainly on the klt case here.

The acronym MMP is often used to refer to both the specific process of running a series of birational modifications to a pair as well as the overall research area. For the avoidance of confusion MMP will be refer to the process and Minimal Model Program to the area of study.

The key structural result of the Minimal Model Program is the Cone Theorem. In its most general form we might expect the following.

Conjecture 2.2.1 (Cone Theorem)**.** *Take an excellent ring R admitting a dualising complex. Let $(X, \Delta)/T$ be a dlt $\mathbb{Q}$-factorial R-pair of dimension n. Then there is a countable collection of curves $\{C_i\}$ on X such that:*

1. $$\overline{NE}(X/T) = \overline{NE}(X/T)_{K_Y+\Delta\geq 0} + \sum_i \mathbb{R}[C_i]$$

2. *The rays C_i do not accumulate in $(K_Y + \Delta)_{<0}$.*

3. *For each i there is d_{C_i} with*

$$0 < -(K_X + \Delta).C_i \leq 2nd_{c_i}$$

and d_{C_i} divides $L \cdot_k C_i$ for every Cartier divisor L on X.

If the field is algebraically closed we can take $d_{C_i} = 1$ for every i, but not in general even if the field is perfect. See for example [Tan18a, Example 7.3].

An MMP is then run by contracting extremal $K_X + \Delta$ negative curves. The existence of such contractions is a key application of the Basepoint Free Theorem.

Conjecture 2.2.2 (Basepoint Free Theorem)**.** *Let $(X, \Delta)/T$ be a klt R-pair. Suppose that D is a $\mathbb{Q}$-Cartier divisor, nef over T, such that $D - (K_X + \Delta)$ is big and nef over T. Then D is semiample.*

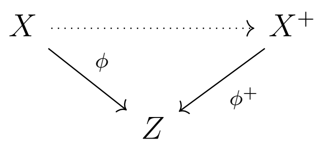
When we contract an extremal ray via $\phi\colon X \to X'$ we have three mutually exclusive possibilities.

1. Mori Fibration: $\dim X' < \dim X$ and ϕ is a $K_X + \Delta$ negative fibration of relative Picard rank 1
2. Divisorial Contraction: ϕ contracts exactly one prime divisor on X
3. Flipping (or Small) Contraction: ϕ contracts a locus of codimension at least 2

The first case is considered an output of the MMP and the process terminates here. If the second occurs then the process may continue unobstructed. The final case, however, always yields a very singular X'. In particular since the dimension of $N^1(X/T)$ falls but no Weil Divisor is contracted, X' cannot be $\mathbb{Q}$-factorial.

The solution to this is to construct a flip. This is a pair (X^+, Δ^+) admitting a small $K_{X^+} + \Delta^+$ positive contraction $\phi^+ : X^+ \to X'$ of relative Picard rank 1 such that the Δ^+ is the strict transform of Δ under the induced map $X \dashrightarrow X^+$.

Conjecture 2.2.3 (Existence of flips)**.** *Let $(X, \Delta)/T$ be a klt R-pair and suppose $\phi\colon X \to Z$ is a $(K_X + \Delta)$ negative flipping contraction. Then there exists a flip.*



Divisorial contractions always reduce the Picard rank, so there can only be finitely many. Flips, however, do not have such a clearly associated invariant and it is not immediately clear that there can be no infinite sequence of flips. Nonetheless this is expected to be true.

Conjecture 2.2.4 (Termination of flips)**.** *Let $(X, \Delta)/T$ be a $\mathbb{Q}$-factorial klt R-pair. Then there is no infinite sequence of $(K_X + \Delta)$ flips $X \dashrightarrow X_1 \dashrightarrow \ldots$ over T.*

Together these conjectures form the key results of the Minimal Model Program and are sufficient to run a terminating MMP from any klt pair. The output (Y, B) of any such MMP can be one of two things.

1. Minimal Model: $K_Y + B$ is nef
2. Mori Fibre Space: Y admits a $K_Y + B$ negative Mori Fibration

A closely related conjecture is the following

Conjecture 2.2.5 (Special termination)**.** *Let $(X, \Delta)/T$ be a $\mathbb{Q}$-factorial dlt R-pair. Then there is no infinite sequence of $(K_X + \Delta)$ flips $X \dashrightarrow X_1 \dashrightarrow \ldots$ over T whose flipping or flipped locus meet $\lfloor \Delta \rfloor$.*

By [Fuj07, 4.2.1], this holds in dimension n if termination of flips holds in dimensions $\leq n-1$.

For threefolds over a positive dimensional base, the current state of the art is the following:

Theorem 2.2.6. [BMP$^+$20] *Let $(X, \Delta)/T$ be a $\mathbb{Q}$-factorial three-dimensional dlt pair over a ring R. Suppose that the closed points of R have residue field of characteristic $p = 0$ or $p > 5$. Suppose further that* $\dim T > 0$. *Then the Cone and Basepoint Free Theorems hold.*

Moreover there exists a $(K_X + \Delta)$-MMP over T that terminates. If $K_X + \Delta$ is pseudo-effective then every MMP terminates.

In particular there is a sequence of birational maps of three-dimensional integral, normal and $\mathbb{Q}$-factorial schemes:

$$X =: X_0 \overset{\varphi_0}{\dashrightarrow} X_1 \overset{\varphi_1}{\dashrightarrow} \cdots \overset{\varphi_{\ell-1}}{\dashrightarrow} X_\ell$$

such that if Δ_i denotes the strict transform of Δ on X_i, then the following properties hold:

1. *For any $i \in \{0, \ldots, \ell\}$, (X_i, Δ_i) is dlt, $\mathbb{Q}$-factorial and projective over Z.*
2. *For any $i \in \{0, \ldots, \ell-1\}$, $\varphi_i \colon X_i \dashrightarrow X_{i+1}$ is either a $(K_{X_i} + \Delta_i)$-divisorial contraction over Z or a $(K_{X_i} + \Delta_i)$-flip over Z.*
3. *If $K_X + \Delta$ is pseudo-effective over Z, then $K_{X_\ell} + \Delta_\ell$ is nef over Z.*
4. *If $K_X + \Delta$ is not pseudo-effective over Z, then there exists a $(K_{X_\ell} + \Delta_\ell)$-Mori fibre space $X_\ell \to Y$ over Z.*

Over a positive characteristic field slightly less is known, even if the field is algebraically closed.

Theorem 2.2.7. [BW17, Theorem 1.7][Bir16a][Wal18, Theorem 1.6][HNT17, Proposition 6.7] *Let (X, Δ) be a $\mathbb{Q}$-factorial three-dimensional dlt pair, projective over a closed field κ. Suppose that κ has characteristic $p > 5$, then the Cone and Basepoint Free Theorems hold.*

Moreover there exists a $(K_X + \Delta)$-MMP over κ that terminates. Moreover if $K_X + \Delta$ is effective then every MMP terminates.

Note that by [Wit18b, Theorem 2], if (X, Δ) is klt and $K_X + \Delta$ is pseudo-effective then in fact $K_X + \Delta$ is effective. These results extend more generally to the case that κ is a perfect field by base change.

Terminating MMP's can also be run for certain fourfolds birational to their base or semistable over a curve [HW20]. Some of the conjectures of the MMP and the existence of log terminal models are also known for threefold pairs over an imperfect field [DW19b] or over perfect characteristic five fields [HW19]. Finally many results are also known in the log canonical setting due to [Wal18], [HNT17].

2.2.2 Birational Modifications

A particularly useful application of the Minimal Model Program is to find modifications with suitably mild singularities. We will explore some of these modifications and their consequences in this section. In particular we always assume the existence of log resolutions as well as the conjectures of subsection 2.2.1.

We can largely avoid termination arguments, i.e. termination of klt flips and special termination. This is done where possible, largely for the sake of generality. For the results of this section to hold, it suffices to know only that an MMP with scaling terminates for klt R-pairs $(X, B)/T$ with $K_X + B$ pseudo-effective and B big.

In fact slightly less is likely fine - that such pairs have a log terminal model (see Definition 2.2.28). If R is not of finite type over a field then some care is needed. In some places we would like to take a log terminal model for pairs which are only rlt (see **??**). It is not immediate that such models exists, even if they do for each witness, though in practice one would not expect this to be an issue. Some modifications to Theorem 2.2.12 would also be needed with such assumptions.

In any case, the required results are all known in the settings of Theorem 2.2.7 and Theorem 2.2.6, which is where we will apply them. They also hold on any excellent surface pair by [Tan18b], which is needed for some inductive arguments. Finally, they are also satisfied if R is a field of characteristic 0 by [BCHM10]. This will never be needed but provides a natural motivation for assumptions.

A vital ingredient in these results is the negativity lemma, Lemma 2.1.14.

The approach for all the modifications is the same - take a log resolution, choose a suitable pair on the resolution, run an MMP for this new pair. We typically then conclude it is a crepant modification, in some sense, using the negativity lemma. We begin with the case of terminalisation.

Lemma 2.2.8. *Let $(X, \Delta)/T$ be a klt R-pair Then there is a terminal pair (Y, Δ_Y) admitting a birational morphism, called a terminalisation, $\pi \colon Y \to X$ with $\pi^*(K_X+\Delta) = K_Y + \Delta_Y$.*

Proof. Let $f \colon Y' \to X$ be a log resolution extracting every divisor with discrepancy at most 0. Write $f^*(K_X + \Delta) + E = K_{Y'} + \Delta_{Y'}$ where E is exceptional and $E, \Delta_{Y'} \geq 0$ share no support. Blowing up further, if needed, we can assume that $\Delta_{Y'}$ has disjoint support, so that $(Y', \Delta_{Y'})$ is terminal.

Then we can run a $K_{Y'} + \Delta_{Y'}$ MMP over X to get $\phi \colon Y \dashrightarrow Y'$ where $\pi \colon Y \to X$ has (Y, Δ_Y) terminal and $K_Y + B_Y$ π nef. By negativity, $G = \pi^*(K_X + \Delta) - K_Y + \Delta_Y$ has $G \geq 0$, since G is exceptional and $-G$ is π-nef. On the other hand $G = \phi_* E$, so $G \geq 0$. Thus $G = 0$ and we have $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$ as required.

□

Perhaps the most useful form of modification is a dlt modification. The main difficulty versus a terminalisation arises from the need to run an MMP for a pair which is not klt. The following proof comes from [Fuj09, Theorem 10.4], but is largely due to Hacon.

Theorem 2.2.9. *Let (X, Δ) an R pair with coefficients bounded above by 1. Write Δ' for the divisor with $\mathrm{Coeff}_E(\Delta') = \mathrm{Min}(\mathrm{Coeff}_E(\Delta), 1)$. Then there is a birational morphism $f \colon Y \to X$, called a dlt modification, such that the following holds:*

- *Y is $\mathbb{Q}$-factorial,*
- *$a(E, X, \Delta) \leq -1$ for every f exceptional divisor E,*
- *If $\Delta_Y = f_*^{-1}\Delta' + \sum_{E \text{ exceptional}} E$ then (Y, Δ_Y) is dlt, and*
- *$K_Y + \Delta_Y + F = f^*(K_X + \Delta)$ where $F = \sum_{E \colon a(E,X,\Delta) < -1} -(a(E, X, \Delta) + 1)E$.*

Here $\mathrm{Nklt}(Y, \Delta_Y) = f^{-1}(\mathrm{Nklt}(X, \Delta))$, Supp$(F) = f^{-1}(\mathrm{Nlc}(X, \Delta))$ and $f_* F = \Delta - \Delta'$. Moreover if (X, Δ) is plt then this is a small morphism.

Proof. Take a log resolution $\pi \colon Y \to X$ of (X, Δ) admitting an ample exceptional divisor $-C$, which exists by [KW21, Theorem 1]. Note that by the negativity lemma, as $-C$ is nef we have that $C \geq 0$, justifying the choice of sign.

Roughly speaking we would like to say that $\pi^*(K_X + \Delta) = K_Y + \pi_*^{-1}\Delta' + E$ and run an MMP for the dlt pair $(Y, S + \pi_*^{-1}\Delta')$ where $S = \mathrm{Supp}(E)$. Indeed, if such an MMP exists, then we can replace Y with the output so that $N = \pi^*(K_X + \Delta) - (K_Y + E + \pi_*^{-1}\Delta') =$

$\pi_*^{-1}(\Delta - \Delta') + (E - F)$ has $\pi_* N = \Delta - \Delta' \geq 0$ and $-N$ nef. That is, $N \geq 0$ by the negativity lemma and the result follows immediately taking $F = N$.

When this MMP is not known to exist, the same result is achieved by making small perturbations by suitable ample divisors. In general we do not have sufficiently strong Bertini theorems to create klt pairs from such perturbations. However they are always rlt by **??**. This is sufficient to run a terminating MMP, see further **??**.

To this end, let

$$D = \sum_{\substack{E \text{exceptional} \\ a(E,X,\Delta) > -1}} E$$

and

$$G = \sum_{\substack{E \\ a(E,X,\Delta) \leq -1}} -a(E,X,\Delta)E$$

Let S be the support of G, so that $\pi_*(G - S) = \Delta - \Delta' \geq 0$. Let A be sufficiently ample on X, so that $H = -C + \pi^* A$ is ample. Note that for small $s > 0$ we still have that $sS - C + \pi^* A = H_s$ is ample.

Then $(Y, (1 - rs)S + (1 - t)D + rH_s + \pi_*^{-1}\Delta_{<1})$ is rlt for small $r, s, t > 0$ by **??**. We may choose t sufficiently small that $a(E, X, \Delta) > t - 1$ for each E in the support of D. Write $\pi^*(K_X + \Delta) = K_Y + B$, and then choose N as follows.

$$\begin{aligned} -N &= K_Y + S + (1-t)D + \pi_*^{-1}\Delta_{<1} + rH - \pi^*(K_X + \Delta + A) \\ &= S + (1-t)D + \pi_*^{-1}\Delta_{<1} - rC - B \end{aligned}$$

From the choice of t, we have that for each E in the support of D that $\mathrm{Coeff}_E(N) = (t - 1) + a(E, X, \Delta) < 0$.

Let $f \colon Y' \to X$ be the output of an MMP for $(Y, (1 - rs)S + (1 - t)F + rH_s + \pi_*^{-1}\Delta_{<1})$. By construction Y' is $\mathbb{Q}$-factorial and is also a minimal model for the pair $(Y, S + (1 - t)F + \pi_*^{-1}\Delta_{<1} + rH)$. In particular, letting S', F', H', H_s', D' be the strict transforms of the corresponding divisors on Y, we have that $(Y', S' + (1 - t)D' + f_*^{-1}\Delta_{<1})$ is dlt and $M = K_Y + S' + (1 - t)D' + f_*^{-1}\Delta_{<1} + rH'$ is nef over X.

Note then that $N' = f^*(K_X + \Delta + A) - M$, so that $-N'$ is nef over X. On the other hand $f_* N \geq 0$ and hence by negativity $N' \geq 0$.

Every component of D' has negative coefficient inside N' by construction. Thus in fact $D' = 0$, since $N' \geq 0$, and in particular every exceptional divisor on Y over X has discrepancy at most -1. Hence we have contracted every E exceptional with $a(E, X, \Delta) > -1$ and therefore $S' = \mathrm{Exc}(\pi)$. Moreover the pair $(Y, \Delta_Y = S' + f_*^{-1}\Delta_{<1})$ is dlt by construction.

Consider then

$$F = f^*(K_X + \Delta) - (K_Y + \Delta_Y) = B' - S' - f_*^{-1}\Delta_{<1} = N' - rC' \geq 0.$$

If E is exceptional over X then we have $\mathrm{Coeff}_E F = \mathrm{Coeff}_E(B' - S) = -(a(E, X, \Delta) + 1)$. Suppose then E is not exceptional. Then we get $\mathrm{Coeff}_E F = \mathrm{Coeff}_E(B' - f_*^{-1}\Delta_{\leq 1}) = \mathrm{Coeff}_{f_*E}(\Delta - \Delta')$. If $\mathrm{Coeff}_{f_*E}\Delta \geq 1$ then this yields $-(a(E, X, \Delta) + 1)$, otherwise we get $\mathrm{Coeff}_E F = 0$.

If X is plt, then $\lfloor \Delta_Y \rfloor = F = 0$ and there are no exceptional divisors. □

Remark 2.2.10. *Note that in the construction above we can choose r sufficiently small that $\mathrm{coeff}_E(F) > 0$ ensures that $\mathrm{coeff}_E(N') > 0$ also. So we may assume N' and F have the same support. In particular if C is a curve on Y contracted over X if it meets F it must be contained in it. Otherwise $N'.C > 0$, contradicting nefness of $-N'$.*

The main consequence of this is that if F dominates $x \in X$ then it contains the fibre over x also.

The case that (X, Δ) is klt is particularly important and is called a (small) $\mathbb{Q}$-factorialisation. One would like to be able to say that the dlt modification is small if (X, Δ) is dlt. This requires quite strong resolution of singularity assumptions, however. If (X, Δ) is dlt and admits a log resolution which is an isomorphism overt the snc locus, then it admits a small $\mathbb{Q}$-factorialisation.

A useful application of DLT modifications is the study of the non-klt and non-lc loci. In particular we have following generalisation of the Cone Theorem as well as a connectedness result for suitable pairs.

Theorem 2.2.11 (Nlc Cone Theorem)**.** *Let $(X, B)/T$ be an R-pair. Then write $\overline{NE}(X/T)_{nlc}$ for the cone spanned by curves contained in the non log canonical locus of X. Then we have the following decomposition*

1. $$\overline{NE}(X/T) = \overline{NE}(X/T)_{K_Y+\Delta\geq 0} + \overline{NE}(X/T)_{nlc} + \sum_i \mathbb{R}_{>0}[C_i]$$
2. *The rays C_i do not accumulate in $(K_Y + \Delta)_{<0}$.*
3. *For each i there is d_{C_i} with*
$$0 < -(K_X + \Delta).C_i \leq 2nd_{c_i}$$
and d_{C_i} divides $L \cdot_k C_i$ for every Cartier divisor L on X.
4. *For each C_i we have $\mathbb{R}_{>0}[C_i] \cap \overline{NE}(X/T)_{nlc} = 0$.*

Proof. If (X, B) is dlt then it is the limit of klt pairs $(X, \frac{n}{n+1}B)$ and the Cone Theorem follows immediately from the klt case.

Suppose next that $\Delta = B + F$ where (X, B) is dlt and F has support contained in $\lfloor B \rfloor$. Note that if C is an irreducible curve with $F.C < 0$ then $C \subseteq F$. Therefore any effective curve C can be written $C = C_0 + C_F$ where $F.C_0 \geq 0$ and $C_F \subseteq F$. Thus by compactness of the unit ball in a finite dimensional vector space, any $[\gamma] \in \overline{NE}(X/T)$ can be written $[\gamma] = [\gamma_0] + [\gamma_F]$ with $F.\gamma_0 \geq 0$ and $[\gamma_F] \in \overline{NE}(F/T)$ in the same fashion.

Take any $K_X + \Delta$ negative extremal ray L. Take a non-zero $[\gamma] \in L$, then as L is extremal we have $[\gamma_F], [\gamma_0] \in L$. If $[\gamma_F] \neq 0$ then $L \subseteq \overline{NE}(F/T)$. Otherwise if $[\gamma_F] = 0$ then L is $K_X + B$ negative. Hence we can conclude the result from the Cone Theorem for dlt pairs.

Suppose finally that X is not dlt. Let $\pi \colon Y \to X$ be a dlt modification of (X, B) with (Y, B_Y) dlt and $K_Y + B_Y + F = \pi^*(K_X + B)$. Take any $K_X + B$ negative extremal ray, L, such that $L \cap \overline{NE(X)}_{nlc} = \{0\}$. Take any class γ with $[\gamma] \in L \setminus \{0\}$ and choose $[\gamma'] \in \overline{NE}(Y/T)$ with $f_*[\gamma'] = [\gamma]$. Then by the projection formula we have that $(K_Y + B_Y + F).\gamma' = (K_X + B).f_*\gamma' = (K_X + B).\gamma < 0$.

From above, we can write $\gamma' = C_0 + C_F + \sum \lambda_i C_i$ where $\lambda_i > 0$, $(K_Y + B_Y + F).C_0 \geq 0$ $C_F \in \overline{NE}(F/T)$ and the C_i each generate $(K_Y + B_Y + F)$ negative extremal rays with $-(K_Y + B_Y + F).C_i \leq 2nd_{c_i}$. From our choice of R we must have $f_*C_0 = f_*C_F = 0$ and hence it follows that $[f_*C_k] \in R \setminus \{0\}$ for some k. Thus $(K_X + B).f_*C_k = (K_Y + B_Y + F).C_k \geq -2nd_{c_i}$.

Since each R is the pushforward of a $(K_Y + B_Y)$ negative extremal ray, there are only countably many generating curves C_i and they cannot accumulate in $(K_X + \Delta)_{<0}$ else they would accumulate on Y also. □

Theorem 2.2.12 (Weak Connectedness Lemma)**.** *Let $(X, \Delta)/T$ be an R-pair with $f_*\mathcal{O}_X = \mathcal{O}_T$. Then if $-(K_X + \Delta)$ is big and nef. Suppose that* $\mathrm{Nklt}(X, \Delta)$ *is vertical over T then for any $t \in T$, $f^{-1}t \cap$* $\mathrm{Nklt}(X, \Delta)$ *is connected. Otherwise* $\mathrm{Nklt}(X, \Delta)$ *dominates T and it is connected.*

In particular $\mathrm{Nklt}(X, \Delta)$ *is always connected in a neighbourhood of any $t \in T$.*

Proof. If (X, Δ) is klt over T then the result is trivial so assume otherwise.

Writing $-K_X + \Delta = A + E$ for suitably small E such that $\mathrm{Nklt}(X, \Delta) = \mathrm{Nklt}(X, \Delta + E)$, we may replace Δ with $\Delta + E$ and assume that $-(K_X + \Delta)$ is ample.

We prove this by induction. Suppose first that (X, Δ) has dimension 1, then R is a field. If $-(K_X + \Delta)$ is big and nef then so is $-K_X$. Then we have $\deg K_X = -2$ by [Tan18b, Corollary 2.8] giving that $\deg \Delta < 2$. The non-klt locus of (X, Δ) is precisely the support of $\lfloor \Delta \rfloor$ and hence can contain at most one point.

Now suppose that the result holds when the total dimension of X is less than n, take X of dimension n.

Let $f \colon (Y, \Delta_Y) \to (X, \Delta)$ be a dlt modification. Then $-L := K_Y + \Delta_Y + F = f^*(K_X + \Delta)$ with (Y, Δ_Y) dlt and L nef and big. We may further write $L = A + E$ with A ample and E effective and exceptional over X. In particular E has support contained inside

$S_Y = \lfloor \Delta_Y \rfloor$. Note that S_Y maps surjectively onto $\text{Nklt}(X, \Delta)$ so it is sufficient to show that S_Y is connected.

Take a general $G_Y \sim \epsilon A + (1-\epsilon)L - \delta S_Y$, then for small δ we may assume G_Y is ample. It may not quite true that we can choose G_Y such that $(X, \Delta_Y + G_Y)$ is dlt. However $\Delta_Y + G_Y = \Delta_Y - \delta S_Y + \epsilon A + (1-\epsilon)L$ so the pair is rlt and we may still run a terminating $K_Y + \Delta_Y + G_Y$ MMP. Moreover the pair $(Y, \Delta_Y + G_Y)$ generalised dlt, which is preserved by this MMP. In particular (Y, Δ_Y) remains dlt during this MMP. By the same logic, if in fact (Y, Δ_Y) is plt then it remains so throughout the MMP.

Write $K_Y + \Delta_Y + G_Y \sim -P_Y = -(\epsilon E + F + \delta S_Y)$ and note $\text{Supp}(P_Y) = S_Y$. In particular $K_Y + \Delta_Y + G_Y$ is not pseudo-effective. Let $Y \dashrightarrow Y'$ be a $(Y, \Delta_Y + G_Y)$ LMMP. If $\dim T < \dim X$ then this terminates in a Mori fibre space $Y' \to Z$. Otherwise we have that Y' such that $-P_Y \simeq K_Y + \Delta_Y + G_Y$ is nef over T. These two possibilities correspond to the verticality conditions. If $\text{Nklt}(X, \Delta)$ dominates T, so does P_Y and we must end with a Mori Fibration. Otherwise $\text{Nklt}(X, \Delta)$ is vertical over T, then $-P_Y$ is psuedo-effective and we end with $-P_{Y'}$ nef.

We claim that on the induced pair $(Y', \Delta_{Y'})$, $\text{Nklt}(Y', \Delta_{Y'}) = \text{Supp}(\lfloor \Delta_{Y'} \rfloor) = \text{Supp}(P_{Y'})$ has the same number of connected components as $\text{Nklt}(X, \Delta)$. Indeed P_Y has the same number of components, so suppose for contradiction there is an MMP step which reduced the number of connected components. Replacing Y with the first point of failure, we can assume there is a step $\pi : Y \dashrightarrow \hat{Y}$ such that $P_{\hat{Y}}$ has one fewer connected components.

Since $\text{Supp}(P_Y) = \lfloor \Delta_Y \rfloor$, we can subtract components of P_Y from Δ and assume that $\lfloor \Delta_Y \rfloor$ contains only two components S_1, S_2 which are disjoint on Y but whose strict transforms meet on $\hat{Y}$. However (Y, Δ_Y) is then $\mathbb{Q}$-factorial plt, and thus so too must $(\hat{Y}, \Delta_{\hat{Y}})$ be. In particular $\lfloor \Delta_{\hat{Y}} \rfloor$ consists of disconnected divisors, a contradiction.

The only possibility then is that $\pi : Y \to \hat{Y}$ is divisorial and contracts a connected component of P_Y. Let $P_Y = \sum P_Y^i$ be the decomposition into connected components. Then we can assume P_Y^1 is the contracted component, in which case it is a prime divisor. Thus $P_Y^1.C < 0$ for any contracted curve, since $\hat{Y}$ is $\mathbb{Q}$-factorial. On the other hand $P_Y^j.C = 0$ for any such C, since P_Y^j does not meet P_Y^1. Thus $P_Y.C < 0$. This is a $-P_Y$ MMP however, so this cannot be the case. Hence, as claimed, the number of connected components of $P_{Y'}$ is the same as P_Y.

Suppose first that $-P_{Y'} \simeq K_{Y'} + \Delta_{Y'} + G_{Y'}$ is nef over T. Then $P_{Y'} \geq 0$ has $-P_Y$ nef over T. Thus for any $t \in T$, if $P_{Y'}$ meets the fibre over t it must contain the entire fibre. Otherwise there would be some curve C mapped to t and meeting P_Y but not contained in it, contradicting nefness of $-P_Y$.

Otherwise we assume that $Y' \to Z$ is a Mori Fibration. Suppose then that $\dim Z = 0$. Then Y' is a variety over a field with $\rho(Y') = 1$. In particular if D, D' are effective and H ample, then $H^{n-2}.D.D' > 0$, so certainly $D \cap D' \neq \emptyset$. Thus $P_{Y'}$ cannot have disconnected support.

Otherwise have that $\dim Z > 0$. Let F be the generic fibre. We must have $P_{Y'}|_F > 0$

since $Y' \to Z$ is a $P_{Y'} \sim -(K_{Y'} + \Delta_{Y'} + G_{Y'})$ positive contraction. However $P_{Y'}$ has the same support as $\lfloor \Delta_{Y'} \rfloor$ so at least one connected component must dominate Z. Suppose then, for contradiction, there is a second connected component. We claim it must also dominate Z. Indeed let S_1, S_2 be the two connected components and assume that S_1 dominates Z. Then $S_1.C > 0$ for any contracted curve C. If we choose C contained entirely in S_2 we see that it meets S_1, so no such curve exists and S_2 is not vertical.

Consider then $(F, \Delta_F = \Delta_{Y'}|_F)$. Since $F \to Y'$ is flat, the pullback of $\Delta_{Y'}$ is just the inverse image, and in particular $\lfloor \Delta_T \rfloor$ contains the pullback of both connected components. Suppose L is the extremal ray whose contraction induces the Mori fibration. Then we have $-(K_{Y'} + \Delta_{Y'} + G_{Y'}).L > 0$, but since L is spanned by a nef curve, as contracting it defines a fibration, and $G_{Y'}$ is effective, we must have $G_{Y'}.L \geq 0$. Hence in fact $-(K_{Y'} + \Delta_{Y'}).L > 0$ also, and so $-K_F + \Delta_F$ is ample. Then, however, the non-klt locus of (L, Δ_L) must be connected by induction, a contradiction.

□

In practice we have essentially run a $K_Y + B_Y - \delta S_Y + M$ MMP for $M = -f^*(K_X + B)$ big and nef which preserves dltness of $K_Y + B_Y$. Working with generalised pairs instead, one can push this result quite far for pairs with rational coefficients. Thinking of $(Y, B_Y + M)$ as a generalised dlt pair and instead running a $K_Y + B_Y + M \simeq_T -F$ MMP we obtain the same result for the nlc locus. This proof works even if M is only nef, however termination in this case requires special termination for dlt pairs. This then generalises Remark 2.2.10. Many of these ideas are explored for positive characteristic pairs in [FW20].

2.2.3 Adjunction

Dlt modifications are also closely related to the study of adjunction. We work under the same assumptions as subsection 2.2.2, however the main focus is on the setting of Theorem 2.2.6. In particular we have the following easy application.

Theorem 2.2.13. *Let $(X, S + B)$ be a log-pair where S is a prime divisor not contained in the support of $B \geq 0$. If $(X, S + B)$ is plt near S if and only if (S^N, B_{S^N}) is klt, where $S^N \to S$ is the normalisation of S and B_{S^N} is the different* [Kol13, Definition 4.2]. *Similarly $(X, S + B)$ is lc near S if and only if (S^N, B_{S^N}) is lc.*

Proof. The question is local on X so we may assume it is affine with $X = \mathrm{Spec}(R)$, and hence that X is an R-pair. Now, one direction is [Kol13, Lemma 4.8], so suppose that (S^N, B_{S^N}) is klt. Let $\pi \colon Y \to X$ be a dlt modification, so that $\pi^*(K_X + S + B) = K_Y + S_Y + B_Y + F$. Suppose that E is a a divisor exceptional over X with $a(E, X, S + B) \geq 1$. Let T be the normalisation of S_Y. Now we have that the induced pair $(T, B_T) \to (S^N, B_{S^N})$ is crepant. Since (T, B_T) is sub-klt it cannot be that S_Y meets E by [Kol13, Claim 4.7.3]. On the other hand, the non-klt locus of $(Y, S_Y + B_Y)$ is connected in a neighbourhood of the fibre over any point by Theorem 2.2.12. Hence $\pi(E)$ does not meet S.

The lc case is identical, using Remark 2.2.10 in place of Theorem 2.2.12 to see that if $\pi(F)$ meets S then F meets S_Y. □

Remark 2.2.14. *If it is known that one can run an lc MMP, then a similar argument can be made for the lc case that does not use log resolutions. Assume for contradiction that $(X, S+B)$ is not lc. Then there is $Y \to X$ extracting E lying over $s \in S$ with $a(E, X, S+B) > 1$. Let F be the reduced exceptional divisor and run a $K_Y + \pi_*^{-1}(S+B) + F$ MMP. This does not contract E because the discriminant at E cannot increase, so we can replace Y with the output and assume that $K_Y + \pi_*^{-1}(B+S) + F + G = \pi^*(K_X + B + S)$ for $G \geq 0$ exceptional with $-G$ nef over X. By assumption Supp$(E) \subseteq$ Supp(G). Then as G is nef it contains the fibre over s, and hence meets $\pi_*^{-1}S$, contradicting* [Kol13, Claim 4.7.3].

The plt/klt case is slightly more involved, but can also be proven with a modification of the arguments of Theorem 2.2.12 *so long as we can run suitable klt MMPs. We may assume as above there is $\pi\colon Y \to X$ extracting E with $a(E, X, B+S) = 1$ lying over $s \in S$ such that $K_Y + \pi_*^{-1}(B+S) + F$ is nef over Y. Then we can run a $-(F + \pi_*^{-1}S)$ MMP by perturbing $K_Y + \pi_*^{-1}(B+S)$ as in Theorem 2.2.12. Thus we may assume that $-(F + \pi_*^{-1}S)$ is nef. This yields a contradiction, however, as then $F + \pi_*^{-1}S$ contains the fibre over s, and some component of F meets S.*

In practice we often wish to know more than this, that if $(X, S+B)$ is plt then in fact (S, B_S) is klt. From above it is enough to know that S is normal. While normality of plt centres is in general an open problem, it is known that the result holds up to universal homeomorphism for prime Q-Cartier centres.

Lemma 2.2.15. [HW20, Lemma 2.1]

Let $(X, D+B)$ be a plt log-pair with D prime and $\mathbb{Q}$-Cartier. Then the normalisation $D^N \to D$ is a universal homeomorphism.

More is understood in the case X has dimension 3.

Theorem 2.2.16. [BMP+20, Corollary 7.17] *Let $(X, S+\Delta)$ be a plt log-pair. Suppose that Δ has standard coefficients all less than* 1. *Take any $x \in S$ with* char $k(x) > 5$ *S is normal at x. In particular the same holds if $K_X + S$ is $\mathbb{Q}$-Cartier.*

When S is the special fibre of X over a DVR yet more can be said. In this case normality is closely related to Cohen-Macaulay-ness and rationality of klt singularities over the residue field. The important characterisation to keep in mind is the following.

Theorem 2.2.17. [Kov17, Theorem 1.16] *Let X be a scheme admitting a dlt pair (X, Δ), then X has rational singularities if and only if X is Cohen-Macaulay.*

The first result, due to [HW20], lets us extend the previous theorem. Roughly speaking it says that if $X \to R$ is a fibration such that (X, X_k) is plt and the normalisation of X_k is Cohen-Macaulay then X_k is normal. In particular this holds if klt singularities are Cohen-Macaulay over k, in dimension $\dim X_k$.

The key observation we will use is the following.

Lemma 2.2.18. *Let R be a complete, excellent DVR and suppose $\mathcal{X} \to R$ is an integral, normal R scheme. Let X be the special fibre, and $X^N \to X$ be the normalisation map. If X^N admits a formal lift over R then $X^N \to X$ is an isomorphism.*

Proof. The morphism $X^N \to X$ is necessarily finite. Thus by [Sta, Tag 09ZT] there is an algebraic lift $\bar{\mathcal{X}}$ of X^N, endowed with a corresponding finite morphism $\bar{\mathcal{X}} \to \mathcal{X}$. On the other hand $\bar{X} \to \mathcal{X}$ is an isomorphism over the generic point of X inside $\mathcal{X}$, and hence a birational morphism. Since X is normal, $\bar{\mathcal{X}} \to \mathcal{X}$ must be an isomorphism. In particular, so too is $X^N \to X$. □

The normality of a special fibre, therefore, is equivalent to liftability of the normalisation. We then have the following liftability characterisation.

Lemma 2.2.19. [Zda18, Lemma A.23]

Let $U \to X$ be an open immersion of k-schemes. Let $Z = X \setminus U$ and suppose that Z has codimension at least 3 in X. Then if X is S_3 at every point of Z, the morphism of deformation functors $Def_X \to Def_U$ is smooth, and in particular $Def_X(A) \to Def_U(A)$ is surjective for any local, Artinian ring A.

Lemma 2.2.20. *Let R be a complete DVR with residue field of characteristic $p > 5$ and suppose $\mathcal{X} \to R$ is an integral, normal R scheme. Let X be the special fibre, and $X^N \to X$ be the normalisation map. If $(\mathcal{X}, X)$ is a plt R-pair, and X^N is Cohen-Macaulay, or even just S_3, then X is normal.*

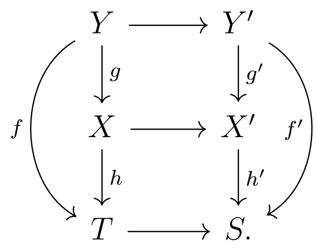
Proof. Then by Lemma 2.2.18 it suffices to check that X^N admits a formal lift. By Lemma 2.2.19, since X^N is S_3, we need only check this away from a closed subset of codimension at least 3. By localising at codimension 2 points of X and applying Theorem 2.2.13, however, we see that X is normal in codimension 2. Therefore $X^N \to X$ is an isomorphism away from a closed subset of codimension 3 and the result follows, since X lifts.

□

Note that X^N is always klt under these assumptions. This result does not use the results of the MMP, however if R is not complete then the existence of log resolutions is needed to ensure that the plt condition is preserved by base change to the completion. Alternatively if plt inversion of adjunction is known, then base change to the completion preserves pltness - since the fibre is not changed.

While this is a very useful characterisation, it cannot be applied to the case that $(X, X_k + B)$ is a plt pair over a DVR with X not $\mathbb{Q}$-Gorenstein unless B has standard coefficients. However we also have a very similar set of results coming from vanishing of certain cohomology classes. For this we need the following liftability result.

Proposition 2.2.21. *Let S be a local Artinian ring and $T \hookrightarrow S$ be a closed immersion defined by a square-zero ideal I. Let $f\colon Y \to T$, and $h\colon X \to T$ be flat morphisms and let $g\colon Y \to X$ be a morphism of T-schemes. Suppose that $g_*\mathcal{O}_Y = \mathcal{O}_X$, $R^1 g_*\mathcal{O}_Y = 0$ and Y has a flat lifting $f'\colon Y' \to S$. Then there exists a flat lifting X' over S and a morphism $g'\colon Y' \to X'$ making the following commutative diagram:*



Moreover, $g'_*\mathcal{O}_{Y'} = \mathcal{O}_{X'}$ and $R^1 g'_*\mathcal{O}_{Y'} = 0$.

Proof. This is essentially the construction of [CvS09, Theorem 3.1].

As Y' has the same underlying topological space of Y, we may see the sheaf $\mathcal{O}_{Y'}$ as a sheaf on the topological space Y. Now we define X' to coincide with X as a topological space and the natural map g' coinciding with g. The schematic structure on X' is given by the sheaf $g_*\mathcal{O}_{Y'}$. This construction fits naturally in a commutative diagram as above and we are only left to check that X' is a flat lifting of X over S.

Since this can be checked locally, we may assume that X, X' are affine. The defining short exact sequence of the extension is

$$\mathcal{E}\colon 0 \to I \to S \to T \to 0$$

Since $\mathcal{O}_{Y'}$ is flat over S, this induces a corresponding short exact sequence of $\mathcal{O}_{Y'}$ modules on Y'.

$$\mathbf{L}f'^*\mathcal{E}\colon 0 \to f'^*I \to \mathcal{O}_{Y'} \to \mathcal{O}_Y \to 0$$

We now push this forward by g' onto X'. Since the pushforward is a topological in nature we have $\mathbf{R}g'_*\mathcal{O}_Y = \mathbf{R}g_*\mathcal{O}_Y$. Similarly since I has the natural structure of an R module, induced by $I^2 = 0$, we have an identification $f^*I = f'^*I$ as group sheaves. Thus we obtain the following.

$$0 \to h^*I \to g'_*\mathcal{O}_{Y'} \to \mathcal{O}_X \to \mathbf{R}^1 g_*\mathcal{O}_Y \otimes h^*I \to \mathbf{R}^1 g'_*\mathcal{O}_{Y'} \to \mathbf{R}^1 g_*\mathcal{O}_Y \to$$

By assumption $\mathbf{R}^1 g_*\mathcal{O}_Y = 0$ and so we have

$$\mathbf{R}g'_*\mathbf{L}f'^*\mathcal{E}\colon 0 \to h'^*I \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

viewed here as a sequence of $\mathcal{O}_{X'}$ modules.

Moreover we have $\mathbf{R}g'_*\mathbf{L}f'^*\mathcal{E} = \mathbf{L}h'^*\mathcal{E}$, and thus we see that there is a canonical identification $\mathcal{O}_{X'} \otimes R = \mathcal{O}_X$. That is $X' \times_S T = X$. We also see that $\mathrm{Tor}^i(\mathcal{O}_{X'}, R) = 0$, since

$\mathbf{L}h'^{*}\mathcal{E}$ is nothing but $\mathcal{O}_{X'} \otimes^{L} \mathcal{E}$. Since $\mathcal{O}_X = \mathcal{O}_{X'}/I\mathcal{O}_{X'}$ is flat over R, by assumption, we must have by [Sta, Tag 0AS8] that $\mathcal{O}_{X'}$ is a flat S module, as required.

□

Theorem 2.2.22. *Let R be a DVR and let X be a normal projective R-scheme such that X_k is normal. Let $f\colon X \to Z$ be a contraction over R and suppose that*

$$f_k\colon X_k \xrightarrow{g_1} Y_1 \xrightarrow{h_1} Z_k$$

is the Stein factorisation of f_k. If $R^1g_{1,*}\mathcal{O}_{X_k} = 0$, then Z_k is normal and h_1 is an isomorphism. In particular $f_{k,*}\mathcal{O}_{X_k} = g_{1,*}\mathcal{O}_{X_k} = \mathcal{O}_{Z_k}$.

Proof. Since we are only interested in the special fibre, we can replace R with its completion at its maximal ideal $\mathfrak{m}$ without any loss of generality. Write $R_i = R/\mathfrak{m}^i$ where m is the maximal ideal of R, then let $X_i = X \times R_i$, $Z_i = Z \times R_i$ and $f_i = f \times R_i\colon X_i \to Z_i$. Then f_1 factors as $f_1\colon X_1 \xrightarrow{g_1} Y_1 \xrightarrow{h_1} Z_1$ where $R^ig_{1,*}\mathcal{O}_{X_1} = 0$, so by Proposition 2.2.21 we can lift $g_1\colon X_1 \to Y_1$ to $g_i\colon X_i \to Y_i$ over R_i such that the following diagram commutes.

$$\begin{array}{ccccc}
X_1 & \longrightarrow & X_2 & \longrightarrow & \dots \\
\downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \\
Y_1 & \longrightarrow & Y_2 & \longrightarrow & \dots \\
\downarrow{\scriptstyle h_1} & & \vdots{\scriptstyle h_2} & & \\
Z_1 & \longrightarrow & Z_2 & \longrightarrow & \dots
\end{array}$$

Here the h_i are defined as follows. The underlying topological map is just h_1 and the map $\mathcal{O}_{Z_i} \to h_{i,*}\mathcal{O}_{Y_i}$ comes from the map $\mathcal{O}_{Z_i} \to f_{i,*}\mathcal{O}_{X_i}$ and the identification $f_{i,*}\mathcal{O}_{X_i} = h_{i,*}g_{i,*}\mathcal{O}_{X_i} \simeq h_{i,*}\mathcal{O}_{Y_i}$. Each h_i is finite, and thus by [Sta, Tag 09ZT] we have that the compatible system $\{Y_i \to Z_i\}$ lifts to a finite morphism $Y \to Z$ over R. By [Sta, Tag 0A42] there is a factorisation $f\colon X \xrightarrow{g} Y \xrightarrow{h} Z$, where $g_*\mathcal{O}_X = \mathcal{O}_Y$, because $g_{i,*}\mathcal{O}_{X_i} = \mathcal{O}_{Y_i}$ for all i. Similarly h is a finite morphism.

Therefore $f\colon X \xrightarrow{g} Y \xrightarrow{h} Z$ is the Stein factorisation for f, but since f is a contraction of normal schemes we conclude that h has to be an isomorphism. In particular, h_1 is an isomorphism and $Z_k = Y_k$, thus concluding. □

Remark 2.2.23. *The key observation in previous proof is that we can think of Y_i as the lift of Y_1 over Z_i rather than simply over R_i. This construction can be thought of as a generalisation of Proposition 2.2.21.*

Although Kawamata-Viehweg Vanishing fails in positive characteristic, we often have sufficiently strong vanishing type results in low dimensions.

Lemma 2.2.24. *Let R be an excellent DVR. Let $(X, X_K + \Delta)$ be a plt R-pair. Suppose that*

1. *X_k is normal;*
2. $\dim Z_k \geq 1$*;*
3. *there is a contraction $f\colon X \to Z$ over R such that $-(K_{X_k} + \Delta_k)$ is f_k-big and f_k-nef;*
4. *X has dimension at most 3, or k is perfect of characteristic $p \geq 7$ and X has dimension at most 4.*

Then Z_k is normal and $f_{k,*}\mathcal{O}_{X_k} = \mathcal{O}_{Z_k}$. Further, if f is birational and $B := f_*\Delta$, then $(Z, Z_k + B)$ is plt and (Z_k, B_k) is klt.

Proof. Since X_k is normal, the pair (X_k, Δ_k) is klt by adjunction. Then we can replace R with its completion to prove the first claim, as this leaves the special fibre unchanged. Let

$$f_k\colon X_k \xrightarrow{\bar{f}_k} \bar{Z}_k \xrightarrow{h_k} Z_k$$

be the Stein factorisation. We can assume $\dim Z_k > 0$ else there is nothing to prove. Since $-(K_{X_k} + \Delta_k)$ is $\bar{f}_k$-big and $\bar{f}_k$-nef, we conclude $R^i \bar{f}_{k,*}\mathcal{O}_{X_k} = 0$ for $i > 0$ by [Tan18b, Theorem 3.3] if $\dim X = 3$ and [BK20, Theorem 25] otherwise. By Theorem 2.2.22 h_k is an isomorphism, $f_{k,*}\mathcal{O}_{X_k} = \mathcal{O}_{Z_k}$ and Z_k is normal.

Suppose now f is birational. As $(X, \Delta + X_k)$ is plt, so is $(Z, B + Z_k)$ as the plt centre X_k is not contracted. Hence (Z_k, B_k) is klt by adjunction. □

Remark 2.2.25. *If* $\dim(Z_k) = 0$ *and* $\dim X \leq 3$*, then* $H^1(X_k, \mathcal{O}_{X_k}) = 0$ *if k is perfect (*[NT20, Proposition 2.20]*) or $p \geq 7$ (*[BT22, Theorem 5.7]*). Under these assumptions, the proof of Lemma 2.2.24 still holds, though it is typically less interesting in this setting.*

We are now able to prove the normality of the special fibre in a plt family which is not $\mathbb{Q}$-Gorenstein, assuming that klt pairs have rational singularities over the residue field and the base is complete.

Theorem 2.2.26. *Let R be a complete, excellent DVR with residue field, k, of characteristic $p > 5$. Suppose that every klt pair of dimension* $\dim X_k$ *has rational singularities. If $(X, \Delta + X_k)$ is a plt R-pair then X_k is normal and (X_k, Δ_k) is a klt pair.*

Proof. Let $f\colon (Y, \Delta_Y) \to (X, \Delta)$ be a small $\mathbb{Q}$-factorialisation. Then $(Y, \Delta_Y + Y_k)$ is a $\mathbb{Q}$-factorial plt pair and hence Y_k is normal by Lemma 2.2.20, since Y_k^N has klt, and hence Cohen-Macaulay, singularities by assumption. In particular $Y_k = Y^N$. Then since X_k^N is also klt, it has rational singularities and so by Theorem 2.2.22, X is normal also and hence (X_k, Δ_k) is a klt pair.

□

In particular the result holds when X has dimension 3, even if R is not complete, without any further assumptions besides those on the characteristic.

Corollary 2.2.27. *Let R be an excellent DVR with residue field, k, of characteristic $p > 5$. Suppose $(X, \Delta + X_k)$ is a plt R-pair and that X has dimension 3. Then X_k is normal and (X_k, Δ_k) is klt.*

Proof. Let $f: (Y, \Delta_Y) \to (X, \Delta)$ be a small $\mathbb{Q}$-factorialisation. Then $(Y, \Delta_Y + Y_k)$ is a $\mathbb{Q}$-factorial plt pair and hence Y_k is normal by Theorem 2.2.16. By construction f is $(K_Y + \Delta_Y)$-trivial so Lemma 2.2.24 ensures the result. □

The result also holds in dimension 4 when the residue field is perfect of char $p > 5$, under the assumption that resolutions exist by [HW20] together with [HW17, Theorem 1.1], [ABL20, Corollary 1.3].

2.2.4 Rational Polytopes of Boundaries

In this section we recall relevant information about rational polytopes and their application to different kinds of birational models.

A non-exhaustive list of important kinds of birational models is as follows.

Definition 2.2.28. *Let $\phi: X \dashrightarrow Y$ be a birational contraction. Take a divisor D and write $D' = \phi_* D$.*

We say it is D-non-positive (resp. D-negative) if there is a common resolution $p: W \to X$, $q: W \to Y$ where

$$p^* D = q^* D' + E$$

and $E \geq 0$ is q exceptional (resp. $E \geq 0$ is q exceptional and contains the strict transform of every ϕ exceptional divisor in its support).

If $(X, \Delta)/T$ is a psuedoeffective lc R-pair then ϕ is a weak log canonical (wlc) model if ϕ is a $K_X + \Delta$ non-positive birational contraction over T with $K_Y + \Delta_Y$ nef, where $\Delta_Y = \phi_* \Delta$. As ϕ is non-positive (Y, Δ_Y) is always lc and if (X, Δ) is klt then so is (Y, Δ_Y).

If in fact ϕ is $K_X + \Delta$ negative, Y is $\mathbb{Q}$-factorial, and (Y, Δ_Y) is dlt then ϕ is a log terminal model. Again if (X, Δ) is dlt then the dlt condition on (Y, Δ_Y) is automatic as ϕ is negative. If $K_Y + \Delta_Y$ is semiample then ϕ is said to be a good log terminal model.

If instead $\phi: X \dashrightarrow Y$ is a rational map then it is an ample model for D if there is H ample on Y such that $p^* D \sim_{\mathbb{R}} q^* H + E$ where $E \geq 0$ is such that $E \leq B$ for any $p^* D \sim_{\mathbb{R}} B \geq 0$.

Wlc models are not in general unique, but they are crepant. In particular we have the following.

Lemma 2.2.29. *Suppose that (X, B) is a pseudo-effective log canonical pair, projective over T. Let $\phi_i \colon (X, B) \dashrightarrow (Y_i, B_i)$ be wlc models for (X, B) over T. Then $K_{Y_1} + B_1$ is semi-ample over T if and only if $K_{Y_2} + B_2$ is so.*

Proof. Let $f \colon Z \to X$ be a projective birational contraction of normal schemes together with proper birational contractions $g_i \colon Z \to Y_i$. We can write

$$K_Z + \Delta_Z = g_i^*(K_{Y_i} + B_i) + E_i,$$

where E_i are effective and g_i-exceptional divisors. Consider

$$g_1^*(K_{Y_1} + B_1) - g_2^*(K_{Y_2} + B_2) = E_2 - E_1.$$

In particular, $E_2 - E_1$ is g_2-nef and therefore by the negativity lemma, Lemma 2.1.14, we conclude that $E_2 - E_1 \leq 0$. By symmetry, we conclude that $E_2 = E_1$. Therefore $g_1^*(K_{Y_1} + B_1) = g_2^*(K_{Y_2} + B_2)$. In particular, $K_{Y_1} + B_1$ is semi-ample over T iff $K_{Y_2} + B_2$ is so. □

Ample models, on the other hand, are always unique. If $X \dashrightarrow Y$ and $X \dashrightarrow Z$ are two ample models, then on some common resolution W of both maps we have $f \colon W \to Y$, $g \colon W \to Z$ and $h \colon W \to X$. Now there are ample divisors A_Y, A_Z with $f^*A_Y + E_Y \sim_{\mathbb{R}} h^*D \sim_{\mathbb{R}} g^*A_Z + E_Z$. But by definition $E_Z = E_Y$ and hence $f^*A_Y \sim_{\mathbb{R}} g^*A_Z$, so there is an isomorphism $i \colon Z \to Y$ with $i \circ f = g$ as required.

If (X, Δ) is a pair then we say $\phi \colon X \dashrightarrow Y$ is an ample model of (X, Δ) if it is an ample model for $K_X + \Delta$. We can often replace pairs with linearly equivalent versions.

Lemma 2.2.30. [BCHM10, Lemma 3.6.8] *Let $\phi \colon X \to Y$ be a rational map. Suppose (X, Δ) and (X, Δ') are two pairs and D, D' two $\mathbb{R}$-Cartier divisors on X. Take $t > 0$ a positive real number.*

- *If $D \equiv tD'$ and ϕ_*D, ϕ_*D' are both $\mathbb{R}$-Cartier then ϕ is D negative (resp D non-negative) if and only if it is D' negative (resp. non-negative)*
- *If both pairs are lc and $K_X + \Delta \sim_{\mathbb{R}} t(K_X + \Delta')$ then ϕ is a wlc model for (X, Δ) if and only if it is a wlc model for (X, Δ').*
- *If both pairs are dlt and $K_X + \Delta \equiv t(K_X + \Delta')$ then ϕ is a log terminal model for (X, Δ) if and only if it is a log terminal model for (X, Δ').*
- *If $D \sim_{\mathbb{R}} tD$ then ϕ is an ample model for D if and only if it is an ample model for D'.*

An import tool for studying different outputs of the MMP and associated models on a scheme are rational polytopes of divisors. We recall the definition of the various polytopes we will need.

Definition 2.2.31. *Let X be a normal, $\mathbb{Q}$-factorial, integral scheme and let $f\colon X \to T$ be a projective morphism such that the image of X in T is positive dimensional. Fix a $\mathbb{Q}$-divisor $A \geq 0$. Let V be a finite dimensional, rational affine subspace of $WDiv_{\mathbb{R}}(X)$ containing no components of A.*

We have the following subsets of $WDiv_{\mathbb{R}}(X)$.

$$V_A = \{A + B \colon B \in V\};$$
$$\mathcal{L}_A(V) = \{\Delta = A + B \in V_A \colon (X, \Delta) \text{ is an lc pair}\};$$
$$\mathcal{N}_A(V) = \{\Delta \in \mathcal{L}_A(V) \colon K_X + \Delta \text{ is nef over } T\}.$$

Given a birational contraction $\phi\colon X \dashrightarrow Y$ we also define

$$\mathcal{W}_\phi(C) = \{\Delta \in \mathcal{E}(C) : \phi \textit{ is a weak log canonical (wlc) model of } (X, \Delta)\}$$

and given a rational map $\psi\colon X \dashrightarrow Z$

$$\mathcal{A}_\phi(C) = \{\Delta \in \mathcal{E}(C) : \phi \textit{ is the ample model of } (X, \Delta)\}$$

Remark 2.2.32. *The polytope $\mathcal{L}_A(V)$ does not depend on the morphism $X \to T$, however all the other polytopes introduced above do. We typically consider the projective morphism $X \to T$ as part of the data of X and omit any reliance on it from the notation.*

Recall that as long as there is a projective log resolution of (X, A) together with (the support of) V the set $\mathcal{L}_A(V)$ is a rational polytope by the work of Shokurov [Sho92], in particular this is true when $\dim X \leq 3$. Further if $(X, A + B)$ is klt and $(X, A + B')$ is lc then $(X, A + tB + (1-t)B')$ is klt for any $0 \leq t < 1$, so the set of klt pairs is open in $\mathcal{L}_A(V)$. In fact if $\mathcal{L}_A(V)$ contains a klt pair, the entire interior consists of klt boundaries and the same is true for any sub-polytope.

The cone theorem, even the slightly weaker form proved in mixed characteristic in [BMP+20], implies that $\mathcal{N}_A(V)$ is a rational polytope. We record the result in dimension 3.

Lemma 2.2.33. [BMP+20, Proposition 9.31] *Suppose that R is an excellent threefold whose closed points have residue fields of characteristic $p = 0$ or $p > 5$. Fix a $\mathbb{Q}$-divisor $A \geq 0$ such that $(X, A)/T$ is a $\mathbb{Q}$-factorial klt three-dimensional R-pair. Then $\mathcal{N}_A(V)$ is a rational polytope.*

The further study of these objects will largely be deferred till **??**, where we will introduce a slightly more flexible notion of a pair in order to better work with such polytopes.

We include now, however, one important application. We can prove abundance for pairs with $\mathbb{R}$-boundaries given the appropriate results for $\mathbb{Q}$-boundaries.

Proposition 2.2.34. *Suppose that R is an excellent threefold whose closed points have residue fields of characteristic $p = 0$ or $p > 5$. Let $X \to T$ be a threefold R pair where $\dim T \geq 1$. Suppose that for every $\mathbb{Q}$-divisor such that (X, B) is klt and $K_X + B$ nef, then $K_X + B$ semiample. Then $K_X + \Delta$ is semiample for every $\mathbb{R}$-divisor Δ such that (X, Δ) is klt and $K_X + \Delta$ is nef.*

Proof. Let $\Delta = \sum_1^n t_i B_i$ and V be the $\mathbb{R}$-linear span of B_i in $\mathrm{WDiv}_{\mathbb{R}}(X)$. By Lemma 2.2.33 we have that $\mathcal{N}_0(V)$ is a rational polytope. Hence there are rational boundaries $D_i \in \mathcal{N}_0(V)$ such that $\Delta = \sum \lambda_i D_i$ where $\sum \lambda_i = 1$. Since (X, Δ) is klt, by choosing D_i sufficiently close to Δ we may suppose that each (X, D_i) is a klt pair with $\mathbb{Q}$-boundary and $K_X + D_i$ f-nef. By assumption $K_X + D_i$ is f-semiample and thus so is $K_X + \Delta = \sum \lambda_i (K_X + D_i)$. $\square$

Chapter 3

Boundedness of Globally F-split varieties

This chapter focuses on boundedness results for globally F-split varieties admitting a Log Fano pair. This work also appears in [Sti20]. In this chapter we generally work over a field. By variety we will always mean an integral quasi-projective scheme over a field.

In this direction, we prove the following.

Theorem 3.0.1. *Fix* $0 < \delta, \epsilon < 1$*. Let* $S_{\delta,\epsilon}$ *be the set of threefolds satisfying the following conditions*

- *X is a projective variety over an algebraically closed field of characteristic* $p > 7, \frac{2}{\delta}$*;*
- *X is terminal, rationally chain connected and F-split;*
- (X, Δ) *is* ϵ*-klt and log Calabi-Yau for some boundary* Δ*; and*
- *The coefficients of* Δ *are greater than* δ*.*

Then there is a set $S'_{\delta,\epsilon}$*, bounded over* $\mathrm{Spec}(\mathbb{Z})$ *such that any* $X \in S_{\delta,\epsilon}$ *is either birational to a member of* $S'_{\delta,\epsilon}$ *or to some* $X' \in S_{\delta,\epsilon}$*, Fano with Picard number* 1*.*

In addition to the main result we prove along the way, essentially in **??** and **??**, the following result. This in turn drew heavily on the arguments of Jiang in [Jia14].

Theorem 3.0.2. *Fix* $0 < \delta, \epsilon < 1$ *and let* $T_{\delta,\epsilon}$ *be the set of threefold pairs* (X, Δ) *satisfying the following conditions*

- *X is projective over a closed field of characteristic* $p > 7, \frac{2}{\delta}$*;*
- *X is terminal, rationally chain connected and F-split;*
- (X, Δ) *is* ϵ*-klt and LCY;*

- *The coefficients of Δ are greater than δ; and*
- *X admits a Mori fibre space structure $X \to Z$ where Z is not a point.*

Then the set $\{Vol(-K_X)\}$ is bounded above.

Remark 3.0.3. *Together with the observation that taking a terminalisation and running a K_X-MMP can only increase the anti-canonical volume, we reduced weak BAB for varieties in $S_{\Delta,\epsilon}$ to the case of prime Fano varieties of ϵ-LCY type. Over a fixed field, however, this is essentially superseded by the result of* [Das18]*, which gives weak BAB for varieties X with $K_X + \Delta \equiv 0$ for some boundary Δ taking coefficients in a DCC set and making (X, Δ) klt.*

3.1 Preliminaries

We will be interested in LCY varieties in which general points can be connected by rational curves in the following senses.

Definition 3.1.1. *Let X be a variety over a field κ. Then X is said to be:*

- *Uniruled if there is a proper family of connected curves $f: U \to Y$ where the generic fibres have only rational components together with a dominant morphism $U \to X$ which does not factor through Y.*
- *Rationally chain connected (RCC) if there is $f: U \to Y$ as above such that $u^2: U \times_Y U \to X \times_k X$ is dominant.*
- *Rationally connected if there is $f: U \to Y$ as above witnessing rational chain connectedness such that the general fibres are irreducible.*
- *Separably rationally connected if f as above is separable.*

If $X \to X'$ is a dominant morphism from X uniruled/RCC/rationally connected then we may compose $U \to X \to X'$ to see that X' is uniruled/RCC/rationally connected.

Theorem 3.1.2. [PZ21, Theorem 1.2] *Let X be a normal, Cohen Macaulay variety with $W\mathcal{O}$-rational singularities over a perfect field of positive characteristic. Then X cannot simultaneously satisfy all the following conditions.*

1. *X is uniruled.*
2. *X is F-split.*
3. *X has trivial canonical bundle.*

If in fact X is smooth then we may replace $K_X \sim 0$ with $K_X \equiv 0$.

Corollary 3.1.3. *Let X be a uniruled, F-split surface over a closed field of positive characteristic. If $K_X \equiv 0$ then X has worse than canonical singularities.*

Proof. Suppose for contradiction that X has canonical singularities. Then we can replace X with its minimal resolution and suppose that X is smooth. In particular it is Cohen-Macaulay and has $W\mathcal{O}$-rational singularities and we may apply **??** to obtain the result.

$\square$

Lemma 3.1.4. [Jia18, Lemma 2.5] *Suppose X is projective and normal, D is an $\mathbb{R}$-Cartier divisor and S is a basepoint free normal and prime divisor. Then for any $q > 0$,*

$$Vol(X, D + qS) \leq Vol(X, D) + q \dim(X) \, Vol(S, D|_S + qS|_S).$$

Lemma 3.1.5. [Kol13, Proposition 4.37] *Suppose that (S, B) is a klt surface and $(K_S + B + D) \sim 0$ for D effective, integral and disconnected, then D has exactly two connected components.*

Theorem 3.1.6. [Tan17, Theorem 1] *Let (X, Δ) be a log canonical (resp. klt) pair where Δ is an effective $\mathbb{Q}$-divisor. Suppose D is a semiample divisor on X then there is an effective divisor $D' \sim D$ with $(X, \Delta + D')$ log canonical (resp. klt).*

Corollary 3.1.7. *Suppose that (X, Δ) is a sub klt pair together with D a divisor on X and $\pi\colon (X', \Delta') \to X$ a log resolution of (X, Δ). Further assume that there is some D' on X' with $\pi_* D' = D$, $-(K_{X'} + \Delta' + D')$ π-nef, (X, Δ') sub klt and D' semiample. Then there is $E \sim D$ on X effective with $(X, \Delta + E)$ sub klt. If in fact (X, Δ) is ϵ-klt then we may choose E such that $(X, \Delta + E)$ is also.*

Proof. We may write $\Delta' = \Delta_p - \Delta_n$ as the difference of two effective divisors. Since (X', Δ') is log smooth we must have that (X', Δ_p) is klt. Thus by the proceeding theorem we have that there is some $E' \sim D'$ with $(X', \Delta_p + E')$ klt. Then we must also have that $(X', \Delta' + E')$ is sub klt Write $E = \pi_* E'$, then $R = \pi^*(K_X + \Delta + E) - (K_{X'} + \Delta' + E') \equiv_f -(K_{X'} + \Delta' + D')$ is π-nef and exceptional. Hence by the negativity lemma we have that $-R$ is effective, and $\pi^*(K_X + \Delta + E) \leq (K_{X'} + \Delta' + E')$ giving that $(X, \Delta + E)$ is klt.

If (X', Δ) is ϵ-klt then so is (X', Δ_p). Let $\delta = \min(1 - \epsilon - c_i)$ where c_i are the coefficients of Δ_p and take $m \in \mathbb{N}$ such that $\frac{1}{m} < \delta$. Applying the previous theorem to mD' instead of D', yields $E'' \sim mD$ with $(X', \Delta' + E'')$ klt. Taking $E' = \frac{1}{m} E$ then continuing as above gives the required divisor. $\square$

Theorem 3.1.8. [PW17, Corollary 1.6] *Let $f\colon X \to Z$ be a projective fibration of relative dimension 2 from a terminal variety with $f_* \mathcal{O}_X = \mathcal{O}_Z$ over a perfect field of positive characteristic $p \geq 11$, such that $-K_X$ is ample over Z. Then a general fibre of f is smooth.*

Theorem 3.1.9 (Bertini for residually separated morphisms)**.** [CGM86, Theorem 1] *Let $f\colon X \to \mathbb{P}^n$ a residually separated morphism of finite type from a smooth scheme. Then the pullback of a general hyperplane H on $\mathbb{P}^n$ is smooth.*

3.1.1 Boundedness

Definition 3.1.10. *We say that a set $\mathfrak{X}$ of varieties is birationally bounded over a base S if there is a flat, projective family $Z \to T$, where T is a reduced quasi-projective scheme over S, such that every $X \in \mathfrak{X}$ is birational to some geometric fibre of $Z \to T$. If the base is clear from context, say if every $X \in \mathfrak{X}$ has the same base, we omit dependence on S.*

If for each $X \in \mathfrak{X}$ the map to a geometric fibre is an isomorphism we say that $\mathfrak{X}$ is bounded over S.

If $S = \mathrm{Spec}R$ we often just say (birationally) bounded over R. In practice we characterise boundedness over $\mathbb{Z}$ via the following result, coming from existence of the Hilbert and Chow schemes.

Lemma 3.1.11. [Tan19, Proposition 5.3] *Fix integers d and r. Then there is a flat projective family $Z \to T$ where T is a reduced quasi-projective scheme over $\mathbb{Z}$ satisfying the following property. If*

1. *κ is a field;*
2. *X is a geometrically integral projective scheme of dimension r over κ; and*
3. *there is a closed immersion $j \colon X \to \mathbb{P}^m_\kappa$ for some $m \in \mathbb{Z}$ such that $j^*(\mathcal{O}(1))^r \leq d$.*

Then X is realised as a geometric fibre of $Z \to T$

Corollary 3.1.12. *Suppose $\mathfrak{X}$ is a set of varieties over closed fields and there are positive real numbers d, V such that for every $X \in \mathfrak{X}$,*

- *X has dimension at most d; and*
- *There is M on X with $\phi_{|M|}$ birational and Vol$(M) \leq V$.*

Then $\mathfrak{X}$ is birationally bounded over $\mathbb{Z}$. If in fact each M is very ample then $\mathfrak{X}$ is bounded.

Conversely, if S is Noetherian then we may always choose H relatively very ample on $Z \to T$ with trivial higher direct images. The restriction of H to any geometric fibre is therefore very ample, and of bounded degree.

Theorem 3.1.13. [Ale94, Theorem 6.9] *Fix $\epsilon > 0$ and an algebraically closed field of arbitrary characteristic. Let S be the set of all projective surfaces X which admit a Δ such that:*

- *(X, Δ) is ϵ-klt;*

- *$-(K_X + \Delta)$ is nef; and*
- *Any of the following holds $K_X \not\equiv 0$, $\Delta \neq 0$, X has worse than Du Val singularities.*

Then S is bounded.

Alexeev shows boundedness over a fixed field, however it is not immediately clear if such varieties are collectively bounded over $\mathbb{Z}$. We briefly show that his methods can be extended, via the arguments of [Wit15] to give a boundedness result in mixed characteristic.

Theorem 3.1.14. *Fix ϵ a positive real number. Let S be the set of projective surfaces X such that following conditions hold:*

- *X has dimension d over some closed field κ;*
- *(X, B) is ϵ-klt for some boundary B;*
- *$-(K_X + B)$ is nef; and*
- *X is rationally chain connected and F-split (if κ has characteristic p).*

Then S is bounded.

Proof. We consider first $\hat{S} := \{X \in S \colon K_X \not\equiv 0\}$. Take any such $X \in \hat{S}$, then by Alexeev [Ale94, Chapter 6] we have the following:

- The minimal resolution $\tilde{X} \to X$ has $\rho(X) < A$, for some constant A, depending only on ϵ and admits a birational morphism to $\mathbb{P}^2$ or $\mathbb{F}_n$ for $n < \frac{2}{\epsilon}$. In particular there is a set T_ϵ bounded over $\mathbb{Z}$ such that every $\tilde{X}$ is a blowup of some $Y \in T_\epsilon$ along a finite length subscheme of dimension 0. That is the set of minimal desingularisations is bounded over $\mathbb{Z}$.
- We may run a K_X-MMP to obtain X' a Mori fibre space.
- There is an N, independent of the field of definition, such that $NK_{X'}$ is Cartier for any Mori fibre space X' obtained as above.
- $\mathrm{Vol}(-K_{X'})$ is bounded independently of the base field.
- If X' is such a Mori fibre space $X' \to \mathbb{P}^1$ and F a general fibre then $-K_X + (\frac{2}{\epsilon} - 1)F$ is ample.

It is sufficient then to show $S' = \{X' \text{ an } \epsilon - \text{LCY type, Mori fibre space }\}$ is bounded in mixed characteristic, then $\hat{S}$ is bounded by sandwiching as in Alexeev's original proof and the full result follows. In turn by **??** it is enough to find V such that every $X' \in S'$ has a very ample divisor, H, satisfying $H^2 \leq V$. We do this first for positive characteristic varieties.

Fix, then, $m > \frac{2}{\epsilon} - 1$ and suppose $X' \to \mathbb{P}^1$ is a Mori fibre space in positive characteristic. Then $A = -K_{X'} + mF$ is ample and NA is Cartier. Further we have $A' = 7NK_{X'} + 27N^2A = (7N - 27N^2)K_{X'} - 27N^2mF$ is very ample by [Wit15, Theorem 4.1]. Since F is base point free, we may add further multiples of F and consider the very ample Cartier divisor $\hat{A} = (27N^2 - 7N)(-K_{X'} + 2F)$. Then

$$\hat{A}^2 = \mathrm{Vol}(X', \hat{A}) \leq (27N^2 - 7N^2)(\mathrm{Vol}(X', -K_{X'}) + 2\mathrm{Vol}(F, -K_F))$$

which is bounded above, since $\mathrm{Vol}(X', -K_{X'})$ is bounded and $\mathrm{Vol}(F, -K_F) = 2$.

Similarly if X' has $\rho(X') = 1$ and $-K_{X'}$ ample then $-nK_{X'}$ is a very ample Cartier divisor with vanishing higher cohomology for some n fixed independently of X'. Then $(-nK_{X'})^2 = n^2\mathrm{Vol}(X, -K_{X'})$ is bounded and the result follows similarly.

Suppose then that $X \in S$ with $K_X \equiv 0$, then it must have worse than canonical singularities by **??**. Let $\pi\colon Y \to X$ be a minimal resolution, with $K_Y + B = \pi^*K_X \equiv 0$ and $B > 0$, then Y is still ϵ-klt, so $Y \in \hat{S}$. Consequently X has $\mathbb{Q}$-Cartier Index dividing N also. Moreover, there is H on Y very ample with H^2 bounded above. Let $H' = \pi_*H$, so that NH' is ample and Cartier on X. Applying [Wit15, Theorem 4.1] again we see that $A \equiv 27N^2H$ is very ample, since $K_X \equiv 0$, with A^2 bounded above.

The arguments in characteristic 0 are essentially the same, making use of Kollár's effective base-point freeness result [Kol93, Theorem 1.1, Lemma 1.2] instead of Witaszek's result, and the existence of very free rational curves on smooth rationally connected surfaces instead of **??**. □

Remark 3.1.15. *In particular we have an affirmative answer to Question 1 in dimension 2.*

3.2 Conic Bundles

In this section the ground field will always be algebraically closed of characteristic $p > 0$. In some results we put additional restrictions on the characteristic. We start with some useful results on finite morphisms and klt singularities.

Definition 3.2.1. *Take a finite, separable and dominant morphism of normal varieties $f\colon X \to Y$.*

If D is a divisor on Y then f is said to be tamely ramified over D if for every prime divisor D' lying over D the ramification index is not divisible by p and the induced residue field extension is separable.

Moreover f is said to be divisorially tamely ramified if for any proper birational morphism of normal varieties $Y' \to Y$ we have the following. If $X' \to X$ is the normalisation of the base change $X \times_Y Y'$, and $f'\colon X' \to Y'$ the induced map, then f' is tamely ramified over every prime divisor in Y'.

If instead f is generically finite, we say it is divisorially tamely ramified if the finite part of its Stein factorisation is so. Equally if either of X or Y is not normal, $f\colon X \to Y$ is said to be divisorially tamely ramified if the induced morphism on their normalisations is.

If f is generically finite of degree $d < p$ then it is always divisorially tamely ramified. If D' lies over a D then both the ramification index, $r_{D'}$ and the inertial degree, $e_{D'}$ are bounded by d, in fact $d = \sum_{f(D')=D} r_{D'}e_{D'}$ by multiplicativity of the norm. This remains the case on any higher birational model.

Lemma 3.2.2. *Let $f\colon Y \to X$ be a dominant, separable, finite morphism of normal varieties over char p. Suppose that K_X is $\mathbb{Q}$-Cartier then $K_Y = f^*K_X + \Delta$ where $\Delta \geq 0$. Further if f is divisorially tamely ramified, then for $Q \in Y$ a codimension 1 point lying over $P \in X$ we have $Coeff_Q(\Delta) = r_Q - 1$ where r_Q is the degree of $f|_Q\colon Q \to P$.*

Proof. By localising at the codimension 1 points of X we reduce to the case of Riemann-Hurwitz-Hasse to see that Δ exists as required and $\mathrm{Coeff}_Q(\Delta) = \delta_Q$ where $\delta_Q \geq r_Q - 1$ with equality when $p \nmid r_q$. In particular when f is divisorially tamely ramified, we ensure $\delta_Q = r_Q - 1$. □

Lemma 3.2.3. [Kol97, Proposition 3.16] *Let $f\colon X' \to X$ be a dominant, divisorially tamely ramified, finite morphism of normal varieties of degree d over char p. Fix Δ on X with $K_X + \Delta$ $\mathbb{Q}$-Cartier. Write $K_{X'} + \Delta' = f^*(K_X + \Delta)$ then the following hold:*

1. $1 + TDisc(X, \Delta) \leq 1 + TDisc(X', \Delta') \leq d(1 + TDisc(X, \Delta))$.
2. (X, Δ) *is sub klt (resp. sub LC) iff* (Y, Δ') *is sub klt (resp. sub LC).*

Proof. By restricting to the smooth locus of X, which contains all the codimension 1 points of X, we may suppose that K_X is Cartier and apply the previous lemma. Hence we get $\Delta' = f^*(K_X + \Delta) - K_{X'}$ where for $Q \in X'$ lying over $P \in X$ we have $\mathrm{Coeff}_Q(\Delta') = r_Q(\mathrm{Coeff}_P(\Delta)) - (r_Q - 1)$.

Suppose that we have proper birational morphisms $\pi\colon Y \to X$ and we write Y' for the normalisation of $Y \times_X X'$ so that we have the following diagram.

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

Let E' be a divisor on Y' exceptional over X' and E the corresponding divisor on Y.

At E' we can write

$$K_{Y'} = \pi'^*(K_{X'} + \Delta') + a(E', X', \Delta')E' = g^*\pi^*(K_X + \Delta) + a(E', X', \Delta')E'$$

essentially by definition. Conversely however we have $K_{Y'} = g^*K_Y + \delta_{E'}E'$ which may be rewritten as

$$K_{Y'} = g^*(\pi^*(K_X + \Delta) + a(E, X, \Delta)E) + \delta_{E'}E'.$$

In particular equating the two descriptions, as $\delta_{E'} = r_{E'} - 1$ by **??**, we have that

$$r_{E'}a(E, X, \Delta) + (r_{E'} - 1) = a(E', X', \Delta')$$

and thus $a(E, X, \Delta) + 1 = \frac{1}{r_{E'}}(a(E', X', \Delta') + 1)$ with $1 \leq r_{E'} \leq d$.

Since, by a theorem of Zariski [Kol96, Theorem VI.1.3], every valuation with center on X' is realised by some birational $Y' \to X'$ occurring as a pullback of a birational morphism $Y \to X$, this is sufficient to show that $1 + \mathrm{TDisc}(X, \Delta) \leq 1 + \mathrm{TDisc}(X', \Delta') \leq d(1 + \mathrm{TDisc}(X, \Delta))$. The second part then follows. □

Definition 3.2.4. *A conic bundle is a threefold sub pair (X, Δ) equipped with a morphism $f\colon X \to Z$ where Z is a normal surface, $f_*\mathcal{O}_X = \mathcal{O}_Z$, the generic fibre is a smooth rational curve and $(K_X + \Delta) = f^*D$ for some $\mathbb{Q}$-Cartier divisor on X. We will call it regular if X and Z are smooth and f is flat; and terminal if X is terminal and f has relative Picard rank 1. Further we call it (sub) ϵ-klt or log canonical if (X, Δ) is.*

If each horizontal component of Δ is effective and divisorially tamely ramified over Z then the conic bundle is said to be tame.

For P a codimension 1 point of Z we define

$$d_P = \max\{t\colon (X, \Delta + tf^*(P)) \text{ is lc over the generic point of } P\}.$$

The discriminant divisor of $f\colon X \to Z$ is $D_Z = \sum_{P \in X}(1 - d_P)$. The moduli part M_Z is then given by $D - D_Z - K_Z$.

In positive characteristic the discriminant divisor is not always well defined for a general fibration, it may be that $d_P \neq 1$ for infinitely many P. This can be caused by either a failure of generic smoothness or inseparability of the horizontal components of Δ over the base.

Suppose, however, that $(X, \Delta) \to Z$ is a tame conic bundle. We may take a log resolution $X' \to X$ as this does not change d_P and is still a tame conic bundle by the **??**. Thus we may suppose that Δ is an SNC divisor and hence near P, $\Delta + f^*P$ is also SNC for all but finitely many P, by generic smoothness of the fibres and as the horizontal components are divisorially tamely ramified over Z. Hence in fact B_Z is well defined in this case.

Lemma 3.2.5. *Let $f\colon (X, \Delta) \to Z$ be a tame conic bundle, and $X' \to X$ either a birational morphism from a normal variety or the base change by a divisorially tamely ramified morphism from a normal variety $g\colon Z' \to Z$. Then there is Δ' with (X', Δ') a tame conic bundle over Z or Z' as appropriate. Moreover in this case $X' \to X$ is also divisorially tamely ramified.*

Proof. If $\pi\colon X' \to X$ is a birational morphism with $K_{X'} + \Delta' = \pi^*(K_X + \Delta)$ then the only horizontal components of Δ' are the strict transforms of horizontal components of Δ. Take such a component D' then, normalising if necessary, it factors $D' \to D \to Z$ with $D \to Z$ divisorially tamely ramified but then it must itself be divisorially tamely ramified.

Suppose then $g\colon Z' \to Z$ is generically finite. From above, and by Stein factorisation we may freely suppose that g is finite. Then the base change morphism $g'\colon X' \to X$ is a finite morphism of normal varieties and we may induce Δ' with $g'^*(K_X + \Delta) = K_{X'} + \Delta'$. Again the horizontal components of Δ' are precisely the base changes of the horizontal components of Δ.

It suffices to show then that if D is a horizontal divisor on X such that $D \to Z$ is divisorially tamely ramified then $D' \to Z'$, the base change, is also divisorially tamely ramified. Certainly $D' \to Z'$ is still separable. Suppose C is any curve on Z and C' a curve on Z' lying over it. In turn take any $C_{D'}$ lying over C' on D'. Then $C_{D'}$ is the base change of some C_D. Since $C_D \to C$ is separable, so too is $C_{D'} \to C'$. Equally as the ramification indices of C', C_D are not divisible by p, neither can the ramification index of $C_{D'}$ over C_D be. This same argument holds after base change by any higher birational model of Z, and by [?kollar1999rational, Theorem VI.1.3] every valuation with centre on Z' is can be realised on the pullback of some such model. Thus $D' \to Z'$ is divisorially tamely ramified and hence $(X', \Delta') \to Z'$ is tame.

It is enough to show that $X' \to X$ is divisorially tamely ramified after base changing by a higher birational model of Z. In particular, after taking a flatification we may assume $f\colon X \to Z$ is flat. Now suppose D is a divisor on X, lying over some curve C on Z. We have $f^*C = \sum E_i$ with $E_0 = D$. Let C_j be the curves lying over C in Z', then if $E_{i,j}$ are the divisors lying over E_i, for some fixed i, they are in one-to-one correspondence with the C_j. We have $g'^*f^*C = \sum r_{i,j}E_{i,j} = \sum_j r_i \sum_i E_j$ and thus none of the $r_{i,j}$, in particular the $r_{0,j}$ are divisible by p. Moreover the $E_{0,j} \to E_0$ must be separable since the $C_j \to C$ are.

The same holds after taking a higher birational model of X, and thus $X' \to X$ is divisorially tamely ramified as claimed.

□

In practice we deal exclusively with tame conic bundles arising in the following fashion.

Lemma 3.2.6. *Suppose that (X, Δ) is klt and LCY, equipped with a Mori fibre space structure over a surface Z and the horizontal components of Δ have coefficients bounded below by δ. Then if X is defined over a field of characteristic $p > \frac{2}{\delta}$, $f\colon (X, \Delta) \to Z$ is a tame conic bundle.*

Proof. Since $\delta < 1$, the characteristic is larger than 2 and the general fibre is necessarily a smooth rational curve, in particular X is a conic bundle. Let G be the generic fibre, so that (G, Δ_G) is klt and G is also smooth rational curve. Then if D is some horizontal component of Δ the degree of $f\colon D \to Z$ is precisely the degree of $D|_G$. However $\deg \delta D|_G < \deg \Delta|_G = -2$ and thus $\deg D < p$. Replacing D by its normalisation, D' does not change the degree, so $D' \to Z$ has degree $< p$ and thus is divisorially tamely ramified. □

Remark 3.2.7. *One might be tempted to ask if this bound could be further improved for ϵ-klt pairs, (X, Δ). In this case we have (G, Δ_G) is ϵ-klt and so one might attempt to use a bound of the form $p > \frac{1-\epsilon}{\delta}$ to prevent any component of Δ mapping inseparably onto the base. It does not seem however that such a bound would ensure that every component is divisorially tamely ramified and there may be wild ramification away from the general fibre.*

Theorem 3.2.8. *Let $f\colon (X, \Delta) \to Z$ be a sub ϵ-klt, tame conic bundle. Then for some choice of $M \sim_{\mathbb{Q}} M_Z$ we have $(Z, D_Z + M)$ sub ϵ-klt. If in fact $\Delta \geq 0$, we may take D_Z, M to be effective also.*

Remark 3.2.9. *The implicit condition that (X, Δ) is a threefold pair is necessary only in that it assures the existence of log resolutions. This result holds in dimension d so long as the existence of log resolutions of singularities holds in dimensions $d, d-1$.*

We will prove this in several steps. First we consider the case that Δ^h, the horizontal part of Δ, is a union of sections of f. In this setting we have an even stronger result. After moving to a higher birational model, we have that (Z, D_Z) is klt and M_Z is semiample.

Lemma 3.2.10. *Suppose that $f\colon (X, \Delta) \to Z$ is a sub ϵ-klt conic bundle with Δ^h effective and with support that is generically a union of sections of f, then there is $\pi\colon Z' \to Z$ a birational morphism with $(Z', D_{Z'})$ sub ϵ-klt and $M_{Z'}$ semiample. In particular for some choice of $M \sim M_{Z'}$ we have $(Z, D_Z + \pi_* M)$ sub ϵ-klt.*

Proof. This result is well known and essentially comes from [PS09]. Details specific to positive characteristic can be found in [DH16, Section 4], [Wit18b, Lemma 3.1] and [CTX13, Lemma 6.7]

We sketch, some key points of the proof.

Since generically $X \to Z$ is a $\mathbb{P}^1$ bundle and the horizontal part of Δ is a union of sections, we induce a rational map $\phi\colon Z \dashrightarrow \overline{\mathcal{M}}_{0,n}$, the moduli space of n-pointed stable curves of genus 0. By taking an appropriate resolution we may suppose that (X, Δ) is log smooth, Z is smooth and ϕ is defined everywhere on Z. Blowing down certain divisors on the universal family over $\overline{\mathcal{M}}_{0,n}$ and pulling back to Z we may further assume that $X \to Z$ factors through a $\mathbb{P}^1$ bundle over Z via a birational morphisms.

Then working locally over each point of codimension 1 and applying 2 dimensional inversion of adjunction, we see that in fact D_Z is determined by the vertical part of Δ, indeed $\Delta^V = f^* D_Z$, and that M_Z is the pullback of an ample divisor on $\overline{\mathcal{M}}_{0,n}$ by ϕ. In particular M_Z is semiample and D_Z takes coefficients in the same set as Δ^v and therefore they are bounded above by $1 - \epsilon$.

From the following lemma, we see that in fact we may further suppose that (Z, D_Z) is log smooth. Since if $\pi\colon (Z', \Delta') \to Z$ is a log resolution of (Z, D_Z) we have $K_{Z'} + \Delta' =$

$\pi^*(K_Z+D_Z)$, $\pi^*M_Z = M_{Z'}$ and $K_{Z'}+D_{Z'}+M_{Z'} = \pi^*(K_Z+D_Z+M_Z) = K_{Z'}+\Delta'+M_{Z'}$, giving $D_{Z'} = \Delta'$ as required. In particular then **??** gives that $(Z, D_Z + M_Z)$ is sub ϵ-klt. □

Lemma 3.2.11. *Suppose that Z is as given above and $Z' \to Z$ is the birational model found in the proof with $M_{Z'}$ semiample. Suppose further that Y is a normal variety admitting a birational morphism $\pi\colon Y \to Z'$. If M_Y is the moduli part coming from the induced conic bundle $X_Y \to Y$ then $\pi^*M_{Z'} = M_Y$.*

Proof. Let $\phi\colon Z' \to \overline{\mathcal{M}}_{0,n}$ and $\chi\colon Y \dashrightarrow \overline{\mathcal{M}}_{0,n}$ be the rational maps induced by the base changes of $X \to Z$. By assumption ϕ is a morphism.

Although χ is a priori defined only on some open set, it must factor through ϕ whenever it is defined, and hence extends to a full morphism $\chi = \phi \circ \pi$.

Write then that $M_{Z'} = \phi^*A$ and $M_Y = \chi^*A'$. A more careful study of the proof of the previous result would give $A = A'$ and the result follows. However for simplicity one can also note that $M_{Z'} = \pi_*M_Y = \pi_*\chi^*A' = \phi^*A'$, so that $M_Y = \pi^*\phi^*A' = \pi^*M_{Z'}$. □

We now reduce from the general case of **??** to the special case of **??** to prove the theorem. This requires the following lemma, due essentially to Ambro.

Lemma 3.2.12. [Amb99, Theorem 3.2] *Suppose that $f\colon (X,\Delta) \to Z$ is a tame conic bundle. Let $g\colon Z' \to Z$ be a finite, divisorially tamely ramified morphism of normal varieties and $(X',\Delta') \to Z'$ the induced fibration. Then $(X',\Delta') \to Z$ is tame and $g^*(K_Z + D_Z) = K_{Z'} + D_{Z'}$ for $D_{Z'}$ the induced discriminant divisor of $(X',\Delta') \to Z'$.*

Proof. By **??**, $(X',\Delta') \to Z'$ is tame and hence $D_{Z'}$ is well defined by the discussion proceeding **??**.

It remains to show that $g^*(K_Z + D_Z) = K_{Z'} + D_{Z'}$. To see this fix Q a prime of Z' and write r_Q for the degree of the induced map onto some P a prime of Z.

From the proof of **??** we see that if $K_{Z'} + B = g^*(K_Z + D_Z)$ then $1 - \mathrm{Coeff}_Q(B) = r_Q(\mathrm{Coeff}_P(D_Z) - 1)$. In particular then it suffices to show that $d_Q = r_Q d_P$. We consider two cases.

Suppose that $c \leq d_P$. Then we have $(X, \Delta + cf^*P)$ log canonical over P. Hence $(X', \Delta' + g'^*f^*P = \Delta + cf'^*g^*P)$ is also log canonical by the **??**. But $f'^*g^*P \geq f'^*r_Q Q$ so it must be that $d_Q \geq r_Q c$. Hence in fact $d_Q \geq r_Q d_P$.

Conversely if $c \geq d_P$ then,$(X, \Delta + cf^*P)$ is not log canonical over P. In particular replacing X with a suitable birational model $X'' \to X$ we suppose that there is some prime E of X with $f_E = P$ and $\mathrm{Coeff}_E(\Delta + cf^*P) < -1$. Similarly there is E' on X' with $g'(E') = E$ and $f'(E') = Q$ which also has $\mathrm{Coeff}_E(\Delta' + cg'^*f^*P) < -1$ but $\mathrm{Coeff}_E(cg'^*f^*P) = \mathrm{Coeff}_E(cf^*r_Q P)$ and hence $c \geq rd_Q$. Thus we have the equality $d_Q = r_Q d_Q$. □

Note that in the setup above $g^*(K_Z + D_Z + M_Z) = K_{Z'} + D_{Z'} + M_{Z'}$ so we must have that $M_{Z'} = g^* M_Z$.

Lemma 3.2.13. *Suppose that $f\colon X \to Z$ is a tame conic bundle. Then there is a finite, divisorially tamely ramified morphism $g\colon Z' \to Z$ with $g^*(K_Z + D_Z + M_Z) = K_{Z'} + D_{Z'} + M_{Z'}$ and a birational morphism $h\colon Z'' \to Z'$ such that $M_{Z''}$ is semiample.*

Proof. Let D be any horizontal component of Δ which is not a section of f then f restricts to a divisorially tamely ramified morphism $D \to Z$. After replacing D with its normalisation and Stein factorising, we may suppose that $D \to Z$ is finite with D normal. Taking the fibre product of $X \to Z$ with the normalisation $\tilde{D}$ of D we find $X' \to \tilde{D}$ satisfying the initial conditions but with the one component of Δ is now generically a section.

In this fashion, we eventually get to $Z' \to Z$ with $g^*(K_Z + D_Z + M_Z) = K_{Z'} + D_{Z'} + M_{Z'}$ and all the horizontal components of Δ being generically sections. Hence we may apply **??** to give the result. ☐

Proof of **??**. Take $f\colon (X, \Delta) \to Z$ as given. Then we have $g\colon Z' \to Z$ and $h\colon Z'' \to Z$ as above. Write d for the degree of g. Fix $B_{Z''} \sim M_{Z''}$ making $(Z'', D_{Z''} + B_{Z''})$ sub klt. Write $B_Z = \frac{1}{d} g_* h_* B_{Z''}$. It is sufficient to show that $(Z, D_Z + B_Z)$ is sub ϵ-klt since $B_Z \sim M_Z$ is always effective and $D_Z \geq 0$ whenever Δ is.

Let $Y \to Z$ be a log resolution of $(Z, D_Z + B_Z)$ and take Y', Y'' appropriate fibre products to form the following diagram.

$$\begin{array}{ccc} Y'' & \xrightarrow{\pi''} & Z'' \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ Y' & \xrightarrow{\pi'} & Z' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{\pi} & Z \end{array}$$

We have that $M_{Y''} = \pi''^* M_{Z''}$, so write $B_{Y''} = \pi''^* B_{Z''}$ and $\frac{1}{d} g'_* h'_* B_{Y''} = B_Y$. Then we must have that $\pi_* B_Y = B_Z$ and $K_Y + D_Y + B_Y \sim \pi^*(K_Z + D_Z + B_Z)$. Note further that $\pi^* B_Z$ and B_Y differ only over the exceptional locus, hence B_Y has SNC support. Indeed $D_Y + B_Y$ has SNC support. Further since $(Y'', D_{Y''} + B_{Y''})$ is sub ϵ-klt and $g'_* h'_*(D_{Y''} + B_{Y''}) = d(D_Y + B_Y)$ it must be that $D_Y + B_Y$ have coefficients strictly less than $1 - \epsilon$, thus $(Y, D_Y + B_Y)$ is sub ϵ-klt and therefore so is $(Z, D_Z + B_Z)$. ☐

3.2.1 Generic smoothness

We will also need to consider the pullbacks of very ample divisors on the base of a suitably smooth conic bundle. This is done to obtain an adjunction result which is required in the next section. We work here under the assumption the ground field is closed of positive characteristic $p > 2$.

Lemma 3.2.14. *Let $(X, \Delta) \to Z$ be a regular conic bundle. Then there is some, possibly reducible, curve C on Z such that for any $P \in Z$ the fibre, F_P, over P is determined as follows:*

1. *If $P \in Z \setminus C$ then F_p is a smooth rational curve.*
2. *If $P \in C \setminus Sing(C)$ then F_p is a the union of two rational curves meeting transversally.*
3. *If $P \in Sing(C)$ then F_p is a non-reduced rational curve.*

Further if H is a smooth curve meeting C transversely away from $Sing(C)$ then f^*H is smooth.

Proof. This is essentially [Sar83, Proposition 1.8]. We sketch the proof as our statement is slightly different.

Since X is smooth $-K_X$ is relatively ample and defines an embedding into a $\mathbb{P}^2$ bundle over Z. Fix any point P in X then in some neighbourhood U around P, X_U is given inside $\mathbb{P}^2 \times U$ by the vanishing of $x^t Q x$. Here Q is a diagonalisable 3×3 matrix taking coefficients in $\kappa[U]$, unique up to invertible linear transformation, so we may take C to be the divisor on which the rank of Q is less than 3. That Q has rank 3 on some open set follows from smoothness of the generic fibre.

Then the singular points of C are precisely the locus on which Q has rank less than 2. By taking a diagonalisation of Q we may write X_U as the vanishing of $\sum A_i x_i^2$ for some $A_i \in \kappa[U]$ and we obtain the classification of fibres by consideration of the rank.

Suppose then H is a smooth curve as given. Away from C, f^*H is clearly smooth, so it suffices to consider the intersection with C, however we can see it is smooth here by computing the Jacobian using the local description of X given above. □

Theorem 3.2.15 (Embedded resolution of surface singularities)**.** [Cut09, Theorem 1.2] *Suppose that V is a non-singular variety over an algebraically closed field of dimension 3, S a reduced surface in V and E a simple normal crossings divisor on V then there is a sequence of blowups $\pi \colon V_n \to V_{n-1} \to ...V$ such that the strict transform S_n of S to V_n is smooth. Further each blowup is the blowup of a non-singular curve or a point and the blown up subvariety is contained in the locus of V_i on which the preimage of $S + E$ is not log smooth.*

Corollary 3.2.16. *Suppose $(X, \Delta) \to Z$ is a regular, tame conic bundle and we fix a very ample linear system $|A|$ on Z. Then there is a log resolution $(X', \Delta') \to (X, \Delta)$ such that for any sufficiently general element $H \in |A|$, its pullback G' to X' has $(X', G' + E)$ log smooth for E the reduced exceptional divisor of π.*

Proof. By the previous theorem we may find birational morphism $\pi\colon X' \to X$ which is a log resolution of (X, Δ) factoring as blowups $X' = X_n \to X_{n-1} \to ....X_0 = X$ of smooth subvarieties contained in the non-log smooth locus of each step.

We show first a general G' is smooth. At each stage we blow-up smooth curves V_i in the non-log smooth locus. Let G_i be the pullback of H to X_i, suppose for induction it is smooth. That G_0 is smooth is the content of **??** and so the base case of the induction argument holds.

We may assume that $f_{i,*}V_i = V_{Z,i}$ is a curve for $f_i\colon X_i \to X \to Z$ else a general H avoids it and so a general G_{i+1} is smooth also. Note that each vertical component of Δ is log smooth near the generic point of their image, since X is a regular conic bundle, so V_i must be contained in the strict transform of some horizontal component of Δ. Since V_i is not contracted, it follows that $V_i \to V_{Z,i}$ is separable as (X, Δ, Z) is tame. Thus as a general H meets $V_{Z,i}$ transversely, a general G_i meets V_i transversely and hence a general G_{i+1} is smooth. By induction then $G' = G_n$ is smooth.

Suppose that V is a curve contained in the locus on which π^{-1} is not an isomorphism that is not contracted by f. Then for a general point P of V, we claim that the fibre over P is log smooth. As before we argue by induction, the the base case trivially true. Suppose then that we blowup a curve V_i lying over V on X and V_Z on Z. Then V_i must meet the fibre over P transversally. Indeed $V_i \to V \to V_Z$ is separable, as above, forcing $V_i \to V$ to be separable also. But then V_i meets a general fibre transversally as claimed.

Suppose now that E is an integral exceptional divisor of $X' \to X$. Let $V = \pi_* E$, then as before general G meets V transversely if V is a curve, or not at all otherwise. Suppose V is a curve, then for a general point P of V, the fibre over P is a system of log smooth curves. Finally then the intersection of a general G' and E is a scheme of pure dimension 1 contained in the disjoint union of such systems of log smooth curves, in particular it is log smooth.

Suppose then we fix two exceptional divisors E_1, E_2 meeting at a curve V. Again we suppose that V is not contracted by $f' = f \circ \pi$. Write $\pi_* V = V_X$ and $f'_* V = V_Z$. Then $V_X \to V_Z$ is separable as before and for a general G' meeting V transversely, the intersection of G with $\pi^* V'$ is a log smooth system of rational curves, and then $G.V \subseteq G.\pi^* V_X$ is log smooth, or equally it is finitely many points with multiplicity 1. ☐

Theorem 3.2.17. *Let $(X, \Delta) \to Z$ be a regular, tame conic bundle and $|A|$ a very ample linear system on Z. Then there is a log resolution $(X', \Delta') \to (X, \Delta)$ such that for a general $H \in |A|$, the pullback G' to X' is smooth with $(X', \Delta' + G')$ log smooth.*

Proof. Write E for the reduced exceptional divisor. For a general $H \in |A|$ we let $G = f^* H$ be the pullback to X. We then take X' as in **??**.

Clearly a general G' avoids the intersection of any 3 components of $\mathrm{Supp}(\Delta') + E$, and from above $(X', G' + E)$ is log smooth. Suppose D is a vertical component of Δ. Then either G can be assumed to avoid it, or to meet it at a smooth fibre. By the usual arguments, since the only non-contracted curves we blow up map separably onto their image, G' meets D' the strict transform of D on X' along a log smooth locus. Further this locus meets any exceptional divisor either transversally or not at all. Now suppose D_2 is any other component of $\mathrm{Supp}(\Delta') + E$ which does not dominate Z. Then if either $D_2.D'$ has dimension less than 1 or is contracted over Z then a general G' avoids it, so suppose otherwise. In which case D_2 must be exceptional over X with image $V \subseteq D$ on X. However $D_2.D'$ is just the strict transform of V inside D' and, for a general G', $G'.D_2.D$ is log smooth as required.

It remains then to consider the horizontal components of Δ. Let D be any such component and D' its strict transform. Since (X, Δ, Z) is tame, so is (X', Δ', Z). In particular then $D' \to Z$ is divisorially tamely ramified and so residually separated over Z away from finitely many points of Z. Hence by Bertini's Theorem, **??**, the pullback of a general H, which is just the intersection of a general G' with D' is smooth. Further as $D' \to Z$ is divisorially tamely ramified, if V is any curve on D' not contracted over Z a general $G'|_{D'}$ meets it transversally. Hence for any other component D_2 of $\mathrm{Supp}(\Delta') + E$ we have $(X', D' + D_2 + G')$ log smooth for a general G' and the result follows. □

Corollary 3.2.18. *Suppose (X, Δ, Z) is a terminal, sub ϵ-klt, tame conic bundle. Take a general very ample H on Z, with $G = f^*H$, then $(G, \Delta|_G = \Delta_G)$ is sub ϵ-klt.*

Proof. Throwing away finitely many points of Z we may freely suppose that the conic bundle is regular.

By the previous theorem there is a log resolution $\pi\colon (X', \Delta') \to (X, \Delta)$ with $(X', \Delta' + G')$ smooth. Write $\pi_G\colon G' \to G$ for the restricted map. Then $(K_{X'} + \Delta' + G')|_{G'} = \pi_G^*(K_G + \Delta_G) = K_{G'} + \Delta'|_G$. However $\Delta'|_G$ is log smooth with coefficients less than $1 - \epsilon$ by construction, and hence (G, Δ_G) is ϵ-klt by assumption. □

3.3 F-Split Mori Fibre Spaces

The aim of this section is to prove the following theorem.

Theorem 3.3.1. *For a field κ of positive characteristic we let S_κ be the set of (X, Δ), ϵ-LCY threefold pairs with X terminal, globally F-split and rationally chain connected over κ. We further require that (X, Δ) admits a K_X Mori fibration $f\colon (X, \Delta) \to Z$ where either*

1. *Z is a smooth rational curve, there is H on Z very ample of degree 1 and a general fibre G of $X \to Z$ is smooth.*

or

2. *$p > 2$ and $(X, \Delta) \to Z$ is a tame, terminal conic bundle such that there is a very ample linear system $|A|$ on Z with $A^2 \leq c$. In which case G the pullback of a sufficiently general $H \in |A|$ is smooth with $(G, \Delta_G = \Delta|_G)$ ϵ-klt by* **??**.

Then the set of base varieties

$$S' = \{X \text{ such that } \exists \Delta \text{ with } (X, \Delta) \in S_\kappa \text{ for algebraically closed } \kappa\}$$

is birationally bounded over $\mathbb{Z}$.

Remark 3.3.2. *In practice this will be applied to pairs over fields of characteristic* $p > 7, \frac{2}{\delta}$ *with boundary coefficients bounded below by* δ. *The constraints on* p *come from* **??** *and* **??**, *via* **??**.

This chapter is devoted to the proof, but the outline is as follows. We fix a general, very ample divisor H on the base and write $G = f^*H$. Then argue that $A = -mK_X + nG$ is ample, for m, n not depending on X, Δ or G. This is done by bounding the intersection of K_X with curves not contracted by f and generating an extremal ray in the cone of curves. We then show that in fact we may choose these m, n such that A defines a birational map, by lifting sections from G using appropriate boundedness results in lower dimensions. The F-split assumption is used to lift sections from G with Lemma 2.1.33, it will also be needed to apply Definition 2.1.2 by ensuring that the bases Z are suitably bounded.

If, for some $t > 0$, the non-klt locus of $(X, (1+t)\Delta)$ is contracted then since $(K_X + (1+t)\Delta) \sim -tK_X$ it follows that every $-K_X$ negative extremal ray is generated by a curve γ with $K_X.\gamma \leq \frac{6}{t}$. In particular as we have $G.C \geq 1$ for any $-K_X$ negative curve C it must be that $-K_X + \frac{7}{t}G$ is ample. Clearly for any $(X, \Delta) \to Z$ there is such a t, however we wish to find one independent of the pair. For this we may use a result due to Jiang, the original proof is a-priori for characteristic 0, but the proof is arithmetic in nature and holds in arbitrary characteristic.

Theorem 3.3.3. [Jia18, Theorem 5.1] *Fix a positive integer* m *and* $\epsilon > 0$ *a real number. Then there is some* λ *depending only on* m, ϵ *satisfying the following property.*

Take (T, B) *any smooth, projective* ϵ*-klt surface. Write* $B = \sum b_i B_i$ *and suppose* $K_T + B \equiv N - A$ *for* N *nef and* A *ample. If* $B.N, \sum b_i, B^2 \leq m$ *then* $(T, (1+\lambda)B)$ *is klt.*

First we show that results of this form lift to characterisations of the non-klt locus of $(X, (1+t)\Delta)$, then show how the result above may be applied here.

Lemma 3.3.4. *We use the notation of Definition 2.1.2. Suppose* Z *is a surface and there is* t *such that* $(G, (1+t)\Delta_G)$ *is klt. Then every curve in the non-klt locus of* $(X, (1+t)\Delta)$ *is contracted by* f.

Proof. Let $\pi \colon X' \to X$ be a log resolution of $(X, \Delta + G)$ with $K_{X'} + \Delta' = \pi^*(K_X + \Delta)$, then $(X', \Delta' + G')$ is log smooth and Δ' and G have no common components, where G' is the pullback of G. Now $X' \to X$ must also be a log resolution of $(X, (1+t)\Delta)$, and hence if we write $K_{X'} + B = \pi^*(K_X + (1+t)\Delta)$ then it is also true that $(X', B + G')$ is log smooth and that B and G' have no common components. Hence $(G', B|_{G'})$ is sub klt by assumption and in particular it has coefficients strictly less than 1.

Suppose Z is a non-klt center of $(X,(1+t)\Delta)$ and E is a prime divisor lying over Z inside X'. Then E has coefficients strictly larger than 1 in B. Since $(X',B+G')$ is log smooth, it must be that $E|_{G'}$ is an integral divisor and it is trivial if and only if E and G' do not meet. But then $E|_{G'}=\lfloor E|_{G'}\rfloor=0$ and so E does not meet G'. Hence neither does H meet $f_*\pi_*E=f_*Z$. In particular if C is a curve in the non-klt locus, then there is an ample divisor H on Z not meeting f_*C. This is possible only if f_*C is a point. □

Lemma 3.3.5. *Using the notation of Definition 2.1.2 suppose that Z is a curve and write Y for the generic fibre of $f\colon X\to Z$. If there is t such that $(Y,(1+t)\Delta_Y)$ is klt, then every curve in the non-klt locus of $(X,(1+t)\Delta)$ is contracted by f.*

Proof. This follows essentially as above. Take a log resolution $\pi\colon(X',\Delta')\to(X,\Delta)$. Write Y' for the generic fibre of $X'\to Z$. Then $(Y',\Delta'|_{Y'})\to(Y,\Delta_Y)$ is a log resolution. Again write $K_{X'}+B=\pi^*(K_X+(1+t)\Delta)$. Then again if B has a component D with coefficient at least 1 then D cannot dominate Z, else it would pull back to G' to give a contradiction. Hence the non-klt locus of $(X,(1+t)\Delta)$ must be contracted as claimed. □

Lemma 3.3.6. *Using the notation of the previous lemmas. There is some λ independent of (X,Δ) and G for which the non-klt locus of $(X,(1+t)\Delta)$ is contracted for all $t\leq\lambda$.*

Proof. We consider two cases.

Suppose first Z is a curve, so the generic fibre Y is a regular del Pezzo surface and (G,Δ_G) is ϵ-klt LCY. Then, by the work of Tanaka [Tan19, Corollary 4.8], $(-K_G)^2\leq 9$. We write $\Delta_G=\sum\lambda_iD_i$ and since G is regular we have $D_i.K_G\geq 1$. Hence $\sum\lambda_i\leq\Delta_G.(-K_G)\leq 9$ and $\Delta_G^2=(-K_G)^2\leq 9$. We conclude the result holds by **??** with $N=-K_G$ and $A=-2K_G$.

Suppose then that Z is a surface. Then by **??**G is a smooth surface, geometrically over a general very ample divisor H on Z. Further by **??**, (G,Δ_G) is ϵ-klt and by assumption $K_G+\Delta_G\sim kF$ where F is the general fibre over H and $H^2=k\leq c$. Finally note that $\Delta_G^V\sim_{f,\mathbb{Q}}0$.

We may write $\Delta_G=\sum\lambda_iD_i+\sum\mu_iF_i$ where F_i are fibres over H and D_i dominate H. Since F_i is a fibre and G is smooth, each F_i is reduced by the genus formula and contains at most 2 components since $-K_X.F_i=-2$. Further $\Delta_G.F=(-K_G).F=2$ and hence $\Delta_G^2=(-K_G+kF)^2=(-K_G)^2-2kK_G.F+(kF)^2\leq(-K_G)^2+4c$ which in turn is bounded above by $8+4c$ due to [`?buadescu2001algebraic`, Proposition 11.19], since G is a smooth geometrically ruled surface.

It remains then to show that the sum of the coefficients of Δ_G is bounded. Note that $\sum\lambda_i\leq\sum\lambda_iD_i.F=\Delta_G.F=2$. We therefore need only bound $\sum\mu_i$.

Suppose for contradiction that $w=\sum\mu_i>3+k$. Let $B=\sum\lambda_iD_i+(1-\frac{3+k}{w})\sum\mu_iF_i\sim -K_G-(F^1+F^2+F^3)$, for general fibres F^i.

Then (G,B) is klt and so by **??**, $D=F^1+F^2+F^3$ has 2 connected components, a clear contradiction.

Therefore we may choose A small and ample with $A.\Delta_G < c$ and write $N = kF + A$ to satisfy the conditions of **??**. The result then follows as $\Delta_G.N = kF.\Delta_G + A.B \leq 3c$ is still bounded. □

Corollary 3.3.7. *There is some n such that for any $(X, \Delta) \to Z$ and G as in **??** we have $-K_X + nG$ is ample.*

Proof. Take any $n \geq \frac{7}{\lambda}$ for λ as in the previous lemma. Then any curve, C, on X is either contracted by $X \to Z$, in which case $-K_X.C > 0 = G.C$. Else C is not contracted and we may apply the Nlc Cone Theorem, **??**, to $(X, (1+\lambda)\Delta)$. It follows that C is in the span of curves Γ_i with $(-K_X + (1+\lambda)\Delta).\Gamma_i = -\lambda K_X.\Gamma_i \geq -6$. In either case, since G is Cartier, $n > \frac{\lambda}{7}$ ensures $(-K_X + nG).C > 0$. □

Theorem 3.3.8. *Let $(X, \Delta) \to Z$ and G be as in Definition 2.1.2. Then there is t not depending on the pair (X, Δ) nor on G with $-3K_X + tG$ ample and defining a birational map.*

Proof. Consider first the case that $\dim Z = 1$. Then G is a smooth del Pezzo surface, so $-3K_X$ is very ample Globally generated? - Tanaka. Let G_1, G_2 be other general fibres and consider

$$0 \to \mathcal{O}_X(-3K_X+kG-G_1-G_2) \to \mathcal{O}_X(-3K_X+kG) \to \mathcal{O}_{G_1}(-3K_{G_1}) \oplus \mathcal{O}_{G_2}(-3K_{G_2}) \to 0.$$

Since X is globally F-split $H^i(X, A) = 0$ for all $i > 0$ and A ample by Lemma 2.1.33. In particular then $H^1(X, \mathcal{O}_X(-3K_X + kG - G_1 - G_2))$ vanishes when $k \geq 3n+2$ for n as given by the proceeding corollary. Therefore we may lift sections of $-3K_{G_i}$ to see that $-3K_X + kG$ defines a birational map for any $k \geq 3n+2$.

Suppose instead that $\dim Z = 2$, so G is a conic bundle. Choose a general $H' \sim H$ on Z and let G' be its pullback. Consider $A_k = (-K_X + kG)|_{G'} = (-k_{G'} + (k-1)dF)$ for $d \geq 1$, where F is the general fibre of $G' \to H'$. Then A_k is ample for $k > n$ and is Cartier since G is smooth. In particular by the Fujita conjecture for smooth surfaces [Ter99, Corollary 2.5], $K_{G'} + 4A_k$ is very ample. Choosing suitable k, k' we may write $K_{G'} + 4A_k = -3K_{G'} + 4(k-1)dF = (-3K_X + k'G)|_{G'}$. Consider now

$$0 \to \mathcal{O}_X(-3K_X + (k'-1)G) \to \mathcal{O}_X(-3K_X + k'G) \to \mathcal{O}_{G'}(-3K_{G'} + 4(k-1)dF) \to 0.$$

Again the higher cohomology of $-3K_X + (k'-1)G$ vanishes and we may lift sections to $H^0(X, \mathcal{O}_X(-3K_X + k'G))$ from general fibres. In particular $-3K_X + k'G$ separates points on a general G' so $-3K_X + (k'+1)G$ separates general points and thus defines a birational map.

We may then pick some suitably large t for which the result holds as k, k' were chosen independently of $(X, \Delta) \to Z$ and G, G_1, G_2. □

Lemma 3.3.9. *Let $(X, \Delta) \to Z, S$ and G be as in Definition 2.1.2 and t as in **??**. Then there is some constant C with $(-3K_X + tG)^3 \leq C$ and $(X, \Delta) \in S$.*

Proof. The anticanonical volumes $\mathrm{Vol}(X, -K_X)$ are bounded by some V by **??** which is proved in the next section.

Suppose first $\dim Z = 1$. Then $\mathrm{Vol}(G, -K_G) = (-K_G)^2 \leq 9$ and so by Lemma 3.4

$$\mathrm{Vol}(X, -3K_X + nG) \leq \mathrm{Vol}(X, -3K_X) + 3t\mathrm{Vol}(G, -3K_G) \leq 27(V + 9t)$$

as required.

Suppose instead then that $\dim Z = 2$. So G is a conic bundle over some H on Z with $H^2 \leq c$. Hence we get

$$\mathrm{Vol}(G, (-3K_X + tG)|_G) = (-3K_G + (t+1)H^2F)^2 = 9K_G^2 - 2(t+1)H^2(K_G.F)$$

where F is a general fibre of $G \to H$. Hence F is a smooth rational curve and $K_G.F = -2$ and $\mathrm{Vol}(G, (-3K_X + tG)|_G) \leq 72 + 4(t+1)c$. Then as before we may apply **??** to get

$$\mathrm{Vol}(X, -3K_X + tG) \leq \mathrm{Vol}(X, -3K_X) + 3n\mathrm{Vol}(G, (-3K_X + tG)|_G)$$

and boundedness follows. □

Proof of Definition 2.1.2. Suppose $(X, \Delta) \in S$. Then $A = -3K_X + tG$ is birational with bounded volume by the preceding results. Thus S' is birationally bounded by **??**. □

3.4 Weak BAB for Mori Fibre Spaces

This section is devoted to providing a bound on the volume of $-K_X$ under suitable conditions. Namely we show that the claim holds if X belongs to a suitable family of ϵ-LCY Mori fibre spaces whose bases are bounded. We consider first the case that X that is a tame conic bundle over a surface.

Theorem 3.4.1. *Pick $\epsilon, c > 0$. Then there is $V(\epsilon, c)$ such that if $f\colon (X, \Delta) \to S$ is any projective, tame conic bundle over any closed field of characteristic $p > 5$, (X, Δ) is ϵ-klt and S admits a very ample divisor H with $H^2 \leq c$, then $Vol(-K_X) \leq V(\epsilon, c)$.*

We may further assume that H and $G = f^{-1}H$ are smooth. Moreover H may be taken so that $(G, \Delta|_G)$ is ϵ-klt also by **??**.

If $\mathrm{Vol}(-K_X) = 0$ the result is trivially true, so we may suppose that $-K_X$ is big. In particular we may write $-K_X \sim A + E$ where A is ample and $E \geq 0$. Note that

$$-K_X - (1-\delta)\Delta \sim -\delta K_X \sim \delta A + \delta E$$

for any $0 < \delta < 1$. Choose δ such that $(X, (1-\delta)\Delta + \delta E)$ and $(G, (1-\delta)\Delta|_G + \delta E|_G)$ are ϵ-klt and write $B = (1-\delta)\Delta + \delta E$. Then (X, B) is ϵ-log Fano by construction. The proof follows essentially as in characteristic zero, which can be found in [Jia14], but we include a full proof for completeness as some details are modified.

Lemma 3.4.2. [Jia14, Lemma 6.5] *With notation as above, Vol*$(-K_X|_G) \leq \frac{8(c+2)}{\epsilon}$.

Proof. Suppose for contradiction $\mathrm{Vol}(-K_X|_G) > \frac{8(c+2)}{\epsilon}$ and choose r rational with $\mathrm{Vol}(-K_X|_G) > 4r > \frac{8(c+2)}{\epsilon}$.

Write F for the general fibre of $G \to H$. Then $G|_G = H^2F = kF$ and for suitably divisible m and any n we have the following short exact sequence.

$$0 \to \mathcal{O}_G(-mK_X|_G - nF) \to \mathcal{O}_G(-mK_X|_G - (n-1)F) \to \mathcal{O}_F(-mK_F) \to 0$$

In particular then $h^0(G, -mK_X|_G - nF) \geq h^0(G, -mK_X|_G - (n-1)F) - h^0(F, -mK_F)$. Hence by induction we have $h^0(G, -mK_X|_G - nF) \geq h^0(G, -mK_X|_G) - n \cdot h^0(F, -mK_F)$.

Note however that, letting $n = mr$ we have

$$\lim_{m\to\infty} \frac{2}{m^2}(h^0(G, -mK_X|_G) - n \cdot h^0(F, -mK_F)) = \mathrm{Vol}(-K_X|_G) - 2r\mathrm{Vol}(-K_F) > 0$$

since F is a smooth rational curve. Hence $-mK_X|_G - mrF$ admits a section for m sufficiently large and divisible. Choose an effective $D \sim_{\mathbb{Q}} -K_X|_G - rF$.

Consider now

$$(G, (1 - \frac{k+2}{r})B|_G + \frac{k+2}{r}D + F_1 + F_2)$$

for two general fibres F_1, F_2. This has

$$\begin{aligned} & -K_G + (1 - \frac{k+2}{r})B|_G + \frac{k+2}{r}D + F_1 + F_2 \\ \sim & -(K_X|_G + kF) + \frac{k+2}{r})B|_G + \frac{k+2}{r}(-K_X|_G - rF) + F_1 + F_2 \\ \sim & -(1 - \frac{k+2}{r})(K_X + B)|_G \end{aligned}$$

and hence we may apply the Connectedness Lemma for surfaces, Theorem 2.2.12, to see that its non-klt locus is connected. Note that we have $r > c + 2 \geq k + 2$ and so as $-(K_X + B)$ is ample, this pair satisfies the assumptions of the Connectedness Lemma.

Since both F_1 and F_2 are contained in the non-klt locus, there must be a non-klt center W dominating H. Thus it follows that $(F, (1 - \frac{k+2}{r})B|_F + \frac{k+2}{r}D|_F)$ is non-klt. However $(F, (1-\frac{k+2}{r})B|_F)$ is ϵ-klt so we must have $\deg(\frac{k+2}{r}D|_F) \geq \epsilon$. Finally since $D|_F \sim -K_X|_F = -K_F$ we have $\deg(D|_F) = 2$ and hence $\frac{2(c+2)}{r} \geq \frac{2(k+2)}{r} \geq \epsilon$, contradicting the choice of r. □

Proof of ??. Take $V(\epsilon, c) = \frac{144(c+2)}{\epsilon^2}$ suppose for contradiction that $\mathrm{Vol}(-K_X) > \frac{144(c+2)}{\epsilon^2}$. Choose t with $\mathrm{Vol}(-K_X) > t \cdot \frac{24(c+2)}{\epsilon} > \frac{144(c+2)}{\epsilon^2}$ and consider the following short exact sequence.

$$0 \to \mathcal{O}_X(-mK_X - nG) \to \mathcal{O}_X(-mK_X + (n-1)G) \to \mathcal{O}_G(-mK_X|_G - (n-1)G) \to 0$$

Arguing as before we see that $h^0(X, -mK_X - tmG)$ grows like $\frac{r}{6}m^3$ with $r \geq \mathrm{Vol}(-K_X) - 3t\mathrm{Vol}(-K_X|_G) > 0$ by the previous lemma. In particular we may find $D \sim_{\mathbb{Q}} -K_X + tG$.

Let $\pi\colon Y \to X$ be a log resolution of $(X, (1-\frac{3}{t})B + \frac{3}{t}D)$. We may write $K_Y + \Delta_Y + E = \pi^*(K_X + (1-\frac{3}{t})B + \frac{3}{t}D)$ where (Y, Δ_Y) is klt and E is supported on the non-klt places of $(X, (1-\frac{3}{t})B + \frac{3}{t}D)$.

As shown by Tanaka in [Tan17, Theorem 1], since $|L| = \pi^* f^*|H|$ is base point free there is some m with $(Y, \Delta_Y + \frac{1}{m}(L_1 + L_2 + L_3))$ still klt for every choice of $L_i \in |L|$. In particular, fixing some general $z \in Z$ we may take $H_i \in |H|$ meeting Z for $1 \leq i \leq 2m$ such that for any $I \subseteq \{0, 1, ..., 2m\}$ with $|I| = 3$ the following hold:

- $(Y, \Delta_Y + \sum_{i \in I} \frac{1}{m}\pi^* f^* H_i)$ is klt;
- $\bigcap_{i \in I} H_i = z$.

Thus we must have

$$\mathrm{Nklt}(X, (1-\frac{3}{t})B + \frac{3}{t}D) = \mathrm{Nklt}(X, (1-\frac{3}{t})B + \frac{3}{t}D + \frac{1}{m}f^* H_i)$$

for each i.

Let F be the fibre over z and $G_1 = \sum_{i=1}^{2m} \frac{1}{m} H_i$. Then clearly $\mathrm{mult}_F(G_1) \geq 2$ and hence (X, G) cannot be klt at F. By construction we have

$$\mathrm{Nklt}(X, (X, (1-\frac{3}{t})B + \frac{3}{t}D)) \cup F = \mathrm{Nklt}(X, (X, (1-\frac{3}{t})B + \frac{3}{t}D + G_1)).$$

Similarly we may further take $G_2 \sim f^* H$ not containing F such that

$$\mathrm{Nklt}(X, (X, (1-\frac{3}{t})B + \frac{3}{t}D) + G_1 + G_2) = \mathrm{Nklt}(X, (X, (1-\frac{3}{t})B + \frac{3}{t}D + G_1)).$$

Now $-(K_X + (1-\frac{3}{t})B + \frac{3}{t}D + G_1 + G_2) \sim (1-\frac{3}{t})(K_X + B)$ is ample, so we may apply Theorem 2.2.12 to see there is a curve in the non-klt locus of $(X, (1-\frac{3}{t})B + \frac{3}{t}D)$ meeting F. In particular then the non-klt locus dominates S. Hence we must also have that $(F, (1-\frac{3}{t})B|_F + \frac{3}{t}D|_F)$ is not-klt for the generic fibre F, however $(F, B|_F)$ is ϵ-klt and F is a smooth rational curve. Therefore by degree considerations, since $-K_X|_F \sim D|_F$ we must have $t \leq \frac{6}{\epsilon}$, contradicting our choice of t. $\square$

Theorem 3.4.3 (Ambro-Jiang Conjecture for surfaces)**.** [Jia14, Theorem 2.8] *Fix* $0 < \epsilon < 1$. *There is a number* $\mu(\epsilon)$ *depending only on* ϵ *such that for any surface* S *over any closed field k, if* S *has a boundary* B *with* (S, B) ϵ*-klt weak log Fano then*

$$\inf\{ulct(S, B; G) \text{ where } G \sim_{\mathbb{Q}} -(K_S + B) \text{ and } G + B \geq 0\} \geq \mu(\epsilon)$$

Here $ulct(S, B; G) = \sup\{t\colon (S, B + tG)$ is lc and $0 \leq t \leq 1\}$ and in particular it is at most the usual lct, if G is effective.

Though the proof is given for characteristic zero, it is essentially an arithmetic proof that the result holds for $\mathbb{P}^2$ and $\mathbb{F}_n$ for $n \leq \frac{2}{\epsilon}$. The arguments of the proof work over any algebraically closed field and as the bound is given explicitly in terms of ϵ it is independent of the base field.

By applying this result to a general fibre of a Mori fibration over a curve we obtain the desired boundedness result.

Theorem 3.4.4. *Pick $\epsilon > 0$. Suppose that $f\colon X \to \mathbb{P}^1$ is a terminal threefold Mori fibre space with smooth generic fibre over a closed field of characteristic $p > 0$. If there is a pair (X, Δ) which is ϵ-LCY then Vol$(-K_X) \leq W(\epsilon)$ for some $W(\epsilon)$ depending only on ϵ.*

Proof. By **??**, there is some $t(\epsilon) \geq 1$ depending only on ϵ with $-K_X + tF$ ample, where F is a general fibre.

Let $\mu = \mu(1)$ as given in **??** and take $W(\epsilon) = \frac{27(t(\epsilon)+2)}{\mu}$. Suppose for contradiction $\mathrm{Vol}(-K_X) > W(\epsilon)$ and choose s rational with $\mathrm{Vol}(-K_X) > 27s > W(\epsilon)$. Clearly $s > \frac{(t(\epsilon)+2)}{\mu} > t(\epsilon) + 2$.

For any n and for sufficiently divisible m, we have the following short exact sequence.

$$0 \to \mathcal{O}_X(-mK_X - nF) \to \mathcal{O}_X(-mK_X - (n-1)F) \to \mathcal{O}_F(-mK_F) \to 0.$$

This gives $h^0(X, -mK_X - nF) \geq h^0(X, -mK_X) - nh^0(F, -mK_F)$ and subsequently

$$\lim_{m\to\infty} \frac{6}{m^3}(h^0(X, -mK_X) - smh^0(F, -mK_F) = \mathrm{Vol}(-K_X) - 3s\mathrm{Vol}(-K_F).$$

Since F is a smooth del Pezzo surface we have $\mathrm{Vol}(-K_F) \leq 9$. So by construction $-mK_X - smF$ is effective for large, divisible m.

Choose $D \geq 0$ with $D \sim_{\mathbb{Q}} -K_X - sF$ and consider $(X, \frac{t(\epsilon)+2}{s}D + F_1 + F_2)$ for F_1, F_2 general fibres. By construction we have

$$\begin{aligned}-(K_X + \frac{t(\epsilon)+2}{s}D + F_1 + F_2) &\sim -(K_X - \frac{t(\epsilon)+2}{s}K_X - t(\epsilon)F) \\ &\sim (1 - \frac{t(\epsilon)+2}{s})(-K_X + tF) + \frac{t(\epsilon)(t(\epsilon)+2)}{s}F\end{aligned}$$

which is ample since F is nef and $-K_X + t(\epsilon)F$ is ample. Then Theorem 2.2.12 gives that the non-klt locus is connected, and clearly contains F_1, F_2, so it must contain a non-klt center W which dominates $\mathbb{P}^1$. Thus it must be that $(F, \frac{t+2}{s}D|_F)$ is not klt. However F is smooth, and equivalently terminal, with $-K_F \sim D|_F$ ample, so by **??** it follows that $\frac{t(\epsilon)+2}{s} \geq lct(F, 0; D|_F) \geq \mu = \mu(1)$. Thus we have $s \leq \frac{t(\epsilon)+2}{\mu}$ contradicting our choice of s and proving the result. □

3.5 Birational Boundedness

We are now ready to prove the main theorems using the results of the previous sections.

Lemma 3.5.1. *Suppose that* (X, Δ) *is an* ϵ*-klt LCY pair in characteristic* $p > 5$*, with* $\Delta \neq 0$ *and* X *both rationally chain connected and* F*-split. Then there is a birational map* $\pi \colon X \dashrightarrow X'$ *such that* X' *has a Mori fibre space structure* $X' \to Z$ *and* $\Delta' = \pi_*\Delta$ *on* X' *making* (X', Δ') *klt and LCY. Further both* X' *and* Z *are rationally chain connected and* F*-split and if* X *is terminal, so is* X'*.*

Proof. Since (X, Δ) is klt so is $(X, 0)$ and hence we may run a terminating K_X MMP $X = X_0 \dashrightarrow X_1 \dashrightarrow ... \dashrightarrow X_n = X'$. At each step $X_i \dashrightarrow X_{i+1}$ we may pushforward Δ_i to Δ_{i+1}, which is still klt since $K_X + \Delta \equiv 0$. Similarly since X_i is F-split and rationally chain connected, so is X_{i+1} as these are preserved under birational maps of normal varieties. Since K_X cannot be pseudo-effective, X' has a Mori fibre space structure $X' \to Z$, where Z is also rationally chain connected and F-split. If X is terminal we may run a K_X MMP terminating at a terminal variety, hence X' is terminal also. □

Proof of **??**. Take any $(X, \Delta) \in S$ and replace it by a Mori fibre space $(X', \Delta') \to Z$ by **??**. Then Z is F-split and rationally chain connected. If Z is a surface then $p > \frac{2}{\delta}$ ensures that $(X', \Delta') \to Z$ is a tame conic bundle by **??**. In particular Z admits a boundary Δ_Z such that (Z, Δ_Z) is ϵ-LCY by **??**. Hence by BAB for surfaces, **??**, there is $|A|$ a very ample linear system on Z with $A^2 \leq c$ for some c independent of X, Δ, Z.

On the other hand, if Z is a curve then it is a smooth rational curve and $p > 7$ gives that the general fibre of $X \to Z$ is smooth by **??**. Let then $S'_{\delta,\epsilon,V}$ be set of such Mori fibre space $(X', \Delta') \to Z$ with Z not a point and $\mathrm{Vol}(-K_X) \leq V(\epsilon, c)$. By Definition 2.1.2 this is birationally bounded. □

Proof of **??**. Take $(X, \Delta) \in T_{\delta,\epsilon}$ and let $X \to Z$ be the associated Mori Fibre Space structure. If Z is a curve then we conclude that $\mathrm{Vol}(-K_X)$ is bounded by **??** in light of **??**. If instead Z is a surface then the set of possible such Z is bounded by **??** and **??** as above. Hence we conclude the claim by **??**.

□

Chapter 4

Abundance

The key focus of this section is to show the validity of the abundance conjecture for mixed characteristic threefolds. It contains the main results of [BBS21] and the work was completed in collaboration with F. Bernasconi and I. Brivio.

We work under the assumption that the residue fields of closed points of R have characteristic $p \neq 2, 3$ or 5 and that T has a point of positive characteristic.

Theorem 4.0.1 (**??**). *Suppose that $(X, B)/T$ is a klt R-pair of dimension 3 with positive dimensional image containing a positive characteristic point. If $K_X + B$ is nef, then it is semiample.*

A well-known and immediate consequence of abundance is the finite generation of the canonical ring.

Theorem 4.0.2. *Suppose that $(X, B)/T$ is an R-pair of dimension 3 with $\mathbb{Q}$-boundary where T is positive dimensional and contains a positive characteristic point. Then the canonical $\mathcal{O}_T$-algebra*

$$R(\pi, \Delta) := \bigoplus_{m \in \mathbb{N}} \pi_* \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)$$

is finitely generated.

In characteristic 0, finite generation of the canonical ring follows from finite generation in the log general type case ([BCHM10]) and by a result of Fujino and Mori [FM00, Theorem 5.2]. However, their result requires a canonical bundle formula which is not available in the positive or mixed characteristic settings.

Theorem 4.0.3 (**??**). *Let (X, B) be a three-dimensional klt R-pair. Suppose that the following conditions are satisfied:*

(1) $(X, X_k + B)$ is plt with X_k integral and normal;

(2) if V *is a non-canonical centre of* $(X, B + X_k)$ *contained in* $\mathbf{B}_{-}(K_X + B)$*, then* $\dim(V_k) = \dim(V) - 1$.

Suppose further that at least one of the following holds:

1. $\kappa(K_{X_k} + B_k) \neq 1$*; or*
2. B_k *is big over* $\operatorname{Proj}(K_{X_k} + B_k)$

Then there is $m_0 \in \mathbb{N}$ *such that*

$$h^0(X_K, m(K_{X_K} + B_K)) = h^0(X_k, m(K_{X_k} + B_k))$$

for all $m \in m_0\mathbb{N}$.

4.1 Preliminaries

In this section we fix S to be an excellent Noetherian base scheme.

4.1.1 Algebraic spaces

We refer to [Sta, Tag 0ELT] for the definition of algebraic spaces and their general theory. We record here a few key results to be used later. First, Stein factorisation exists for algebraic spaces.

Theorem 4.1.1 (Stein factorisation, [Sta, Tag 0A1B])**.** *Let* S *be a scheme and* $f\colon X \to Y$ *be a proper morphism of Noetherian algebraic spaces over* S*. Then there is a morphism* $f'\colon X \to Y'$*, together with a finite morphism* $\pi\colon Y' \to Y$*, factorising* f *into* $f = \pi \circ f'$ *such that*

- f' *is proper and surjective;*
- $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$*;*
- $Y' = \underline{\operatorname{Spec}}_Y(f_*\mathcal{O}_X)$*;*
- *and* Y' *is the normalisation of* Y *in* X*.*

We call $f = \pi \circ f'$ *the Stein factorisation of* f*.*

In particular if X is normal in **??**, then so is Y'. Moreover if X, Y are schemes then this agrees with the usual notion of Stein factorisation. We also have the following descent result for proper contractions of algebraic spaces.

Lemma 4.1.2. *Let $f\colon W \to X$ and $g\colon W \to Y$ be projective contractions of Noetherian integral normal algebraic spaces over S. Suppose that every proper curve $C \subset W$ contracted by f is contracted by g. Then there is a unique contraction $h\colon X \to Y$ with $g = h \circ f$.*

Proof. First, note that any $h\colon X \to Y$ such that $g = h \circ f$ is necessarily a contraction. Consider $(g \times_S f)\colon W \to X \times_S Y$ and let $\phi\colon W \to \Gamma$ be the contraction part of its Stein factorisation. Thus Γ is an integral, normal algebraic space which is proper over S. If $\gamma\colon \Gamma \to X$ is the induced morphism, it is then enough to show that γ is an isomorphism.

Let $x \in X$ be any point, and let $F := \gamma^{-1}(x)$. Then $\phi^{-1}(F) = f^{-1}(x)$ is contracted by g, hence by ϕ, so γ is quasi-finite.

Let $\xi \in X$ be the generic point. As f is a contraction, we have $H^0(W_\xi, \mathcal{O}_{W_\xi}) = \kappa(\xi)$. As ϕ is a contraction and Stein factorisation commutes with flat base-change, we have that $\phi_\xi\colon W_\xi \to \Gamma_\xi$ is a contraction as well, thus $H^0(\Gamma_\xi, \mathcal{O}_{\Gamma_\xi}) = H^0(\Gamma_\xi, \phi_{\xi,*}\mathcal{O}_{W_\xi}) = \kappa(\xi)$. By [Sta, Tag 0AYI] we then have that γ is a contraction, and by [Sta, Tag 082I] we conclude it is an isomorphism. □

Remark 4.1.3. *The notion of an integral algebraic space,* [Sta, Tag 0AD3]*, is somewhat subtle. However we will only ever apply* **??** *in the case where W, X are integral schemes, in which case W, X, Y are also integral as algebraic spaces.*

It will prove useful to know that proper algebraic spaces are schemes on a big open set.

Lemma 4.1.4. *Let S be a Noetherian scheme and X be a proper algebraic space over S. Then there is a big open immersion of a scheme $U \to X$. If X is normal, we can choose U to be regular.*

Proof. By [Sta, Tag 0ADD], for each codimension 1 point $P \in X$ there is an open subspace U_P containing P which is a scheme. Take the open subspace $U = \bigcup_{\operatorname{codim}_X(P)=1} U_P$, of X. By [Sta, Tag 01JJ] we observe that in fact U is a scheme. Note that U is a sheaf on the Zariski topology since by definition it is a sheaf on the finer fppf topology, [Sta, Tag 025Y].

If X is normal, then so too are the U_P, in particular after shrinking them as needed we may suppose that each U_P is regular and thus that U is regular. □

4.1.2 Semiample and EWM line bundles

In this subsection we recall some basic results about semiample and EWM line bundles we will need later on.

Definition 4.1.5. *Let $\varphi\colon X \to S$ be a proper morphism. A line bundle L on X is said to be semiample over S if there exists $m > 0$ such that $L^{\otimes m}$ is globally generated over S,* i.e. *the natural morphism $\varphi^*(\varphi_* L^{\otimes m}) \to L^{\otimes m}$ is surjective.*

Theorem 4.1.6. *Let X be a normal projective S-scheme and let L be a line bundle on X. Then the following are equivalent.*

1. *L is semiample over S;*
2. *there is a contraction $f\colon X \to Z/S$ such that f is the S-morphism induced by $|L^m/S|$ for all sufficiently divisible m;*
3. *There is a contraction $f\colon X \to Z/S$ such that $L \sim_{\mathbb{Q}} f^*A$ for A ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Z.*

Proof. The direction (1) $\implies$ (2) $\implies$ (3) is the content of [Laz04, Theorem 2.1.26]. That (3) $\implies$ (1) follows straight from the definition of ample. □

The morphism f is the same in both (a) and (b) of **??** is called the *semiample contraction of L*.

Definition 4.1.7. *Let $\varphi\colon X \to S$ be a proper morphism of schemes. A nef line bundle L on X is said to be* EWM over S *if there exists a proper S-morphism $f\colon X \to Y$ to an algebraic space Y proper over S such that an integral closed subscheme $V \subset X$ is contracted (that is,* $\dim(V) < \dim(f(V))$*) if and only if $L|_V$ is not big.*

By **??**, we can suppose f is a contraction and we call this the *EWM contraction* associated to L, which is unique up to isomorphism by **??**.

The definition of semiample (resp. EWM) extends naturally to $\mathbb{Q}$-Cartier divisors (resp. $\mathbb{R}$-Cartier divisors). We say that an $\mathbb{R}$-Cartier divisor D is semiample if there exist $r_i > 0$ and L_i semiample Cartier divisors such that $D \sim_{\mathbb{R}} \sum_i r_i L_i$. A natural extension of condition (c) in **??** is that D is semiample if and only if there is a morphism $f\colon X \to Z$ of S-schemes such that $D \sim_{\mathbb{R}} f^*A$, where A is an ample $\mathbb{R}$-divisor over S. Note that any semiample $\mathbb{R}$-Cartier divisor is EWM.

4.1.2.1 Semiampleness Criteria

We recall the Keel-Witaszek Theorem, which will be a crucial tool in the proof of abundance.

Theorem 4.1.8. [Wit20, Theorem 6.1], [BMP$^+$20, Theorem 2.44] *Let L be a nef line bundle on a scheme X projective over an excellent Noetherian base scheme S. Then L is semiample (resp. EWM) over S if and only if both $L|_{\mathbb{E}(L)}$ and $L|_{X_{\mathbb{Q}}}$ are so.*

We will need the following descent result on semiampleness for normal schemes.

Lemma 4.1.9. *Let $f\colon X \to Y$ be a proper surjective morphism of integral, excellent schemes over S. Suppose that Y is normal and L is a line bundle on Y such that f^*L is semiample over S. Then L is semiample over S.*

Proof. The proof is similar to [Kee99, Lemma 2.10]. We may freely assume that X is normal. Let $X \xrightarrow{\varphi} Z \xrightarrow{\psi} Y$ be the Stein factorisation of f, where φ is a contraction and ψ is a finite map. We first show that ψ^*L is semiample. Take $m > 0$ such that f^*L^m is base point free. By the projection formula $H^0(X, f^*L^m) = H^0(Z, \psi^*L^m)$ and so ψ^*L^m is base point free.

We can thus assume that f is a finite morphism of degree d. By [Sta, Tag 0BD3], there exists a norm function $\mathrm{Norm}_f \colon f_*\mathcal{O}_X \to \mathcal{O}_Y$ of degree d for f which induces a group homomorphim $\mathrm{Norm}_f \colon \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ by [Sta, Tag 0BCY]. Take $m > 0$ such that f^*L^m induces the semiample contraction, and let $y \in Y$ be a point. Then there is a section $s \colon \mathcal{O}_X \to f^*L^m$ not vanishing at any of the points in $f^{-1}(y)$. By [Sta, Tag 0BCY] and [Sta, Tag 0BCZ] we then construct a section $\mathrm{Norm}_f(s) \colon \mathcal{O}_Y \to L^{md}$ not vanishing at y, concluding.

□

We will need a similar, but slightly weaker result for algebraic spaces. First we make the following observation.

Lemma 4.1.10. *Let $f \colon Y \to X$ be a contraction of integral normal proper S-schemes. Let L be a line bundle on X nef over S. Let $V \subset X$ (resp. $V' \subset Y$) be an integral closed subscheme. Suppose $f(V') = V$. Then $f^*L|_{V'}$ is big over S if and only if $L|_V$ is big over S and* $\dim(V) = \dim(V')$.

Proof. Let d be the dimension of V'. Since f^*L is nef, it is big on V' if and only if $(f^*L)^d \cdot V' > 0$. Hence by the projection formula ([Kol96, Proposition VI.2.11]) it is big on V' if and only if $L^d \cdot V > 0$. In turn this occurs if and only if $\dim(V) = d$ and L is big on V. □

Lemma 4.1.11. *Let S be an excellent Noetherian scheme and suppose $f \colon Y \to X$ is a contraction of integral normal projective S-schemes. A line bundle L on X is EWM if and only if f^*L is so.*

Proof. Suppose first that L is EWM and let $g \colon X \to Z$ be the associated EWM contraction. We claim that $h = g \circ f$ contracts an integral subscheme V of Y if and only if $f^*L|_V$ is not big. By **??**, $f^*L|_V$ is not big if and only $\dim(f(V)) < \dim(V)$ or $L|_{f(V)}$ is not big, concluding.

Now suppose that f^*L is EWM. Let $g \colon Y \to Z$ be the associated EWM contraction. By **??** there exists a morphism $h \colon X \to Z$ with $g = h \circ f$. Take $V \subset X$ integral of dimension d. We can choose an integral V' lying over V of dimension d by cutting $f^{-1}(V)$ with hyperplanes and taking a dominant component. By **??** we see that L is not big on V if and only if V' is contracted by h, concluding. □

Remark 4.1.12. *Clearly, if L is an EWM line bundle on X and T is any integral closed subscheme, then $L|_T$ is EWM.*

4.1.2.2 Semiample line bundles over DVRs

We now specialize to the case in which $X \to R$ is a family of normal projective varieties over a DVR and we study how the spaces of global sections of L behave in family. Given a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on a normal variety X over a field k, we denote by $\kappa(L)$ its Iitaka dimension (see [Laz04, Definition 2.1.3]).

Lemma 4.1.13. *Let R be a DVR and let $\pi\colon X \to R$ be a flat projective morphism. Let L be a $\mathbb{Q}$-Cartier divisor on X, semiample over R. Then $\kappa(L_k) = \kappa(L_K)$.*

Proof. Let $f\colon X \to Z$ be the semiample contraction of L over R, let $\delta\colon Z \to \operatorname{Spec}(R)$ be the structure morphism, and note that δ is flat, hence equi-dimensional. Let d be the dimension of the fibers of δ, and let A be an ample $\mathbb{Q}$-divisor on Z such that $L \sim_{\mathbb{Q}} f^*A$. By the projection formula ([Sta, Tag 01E8]) and asymptotic Riemann-Roch ([Kol96, Theorem VI.2.15]), for each $t \in \operatorname{Spec}(R)$ we have

$$\begin{aligned} h^0(X_t, mL_t) &= h^0(Z_t, f_{t,*}\mathcal{O}_{X_t} \otimes \mathcal{O}_{Z_t}(mA_t)) \\ &= \operatorname{rk}(f_{t,*}\mathcal{O}_{X_t}) \frac{(mA_t)^d}{d!} + O(m^{d-1}) \end{aligned}$$

for all $m > 0$ sufficiently divisible. Thus we conclude $\kappa(L_t) = d$ for each $t \in \operatorname{Spec}(R)$. $\square$

Lemma 4.1.14. *Let R be a DVR and let $\pi\colon X \to R$ be a projective, normal, integral R-scheme such that X_k is normal. Let L be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, semiample over R and let $f\colon X \to Z$ be the semiample contraction induced by L. Then the following are equivalent:*

(1) $f_{k,*}\mathcal{O}_{X_k} = \mathcal{O}_{Z_k}$;

(2) $h^0(X_k, mL_k) = h^0(X_K, mL_K)$ for all $m \geq 0$ sufficiently divisible.

Proof. Let A be an ample $\mathbb{Q}$-divisor on Z such that $L \sim_{\mathbb{Q}} f^*A$. By the projection formula we have

$$h^0(X_t, mL_t) = h^0(Z_t, f_{t,*}\mathcal{O}_{X_t} \otimes \mathcal{O}_{Z_t}(mA_t)) \tag{4.1}$$

for all sufficiently divisible m and all $t \in \operatorname{Spec}(R)$. By flat base change we have $f_{K,*}\mathcal{O}_{X_K} = \mathcal{O}_{Z_K}$.

(1) $\Rightarrow$ (2). Suppose that $f_{k,*}\mathcal{O}_{X_k} = \mathcal{O}_{Z_k}$. Then the right hand side of Equation (**??**) coincides with $\chi(Z_t, mA_t)$ when $m \gg 0$ by Serre vanishing. Hence we conclude by invariance of the Euler characteristic in a flat family.

(2) $\Rightarrow$ (1). By Grauert's theorem ([Har77, Corollary III.12.9]) the natural restriction map $H^0(X, \mathcal{O}_X(mL)) \to H^0(X_k, \mathcal{O}_{X_k}(mL_k))$ is surjective for all $m \geq 0$ sufficiently divisible. Hence f_k is the semiample contraction of L_k by **??**, in particular $f_{k,*}\mathcal{O}_{X_k} = \mathcal{O}_{Z_k}$. $\square$

Remark 4.1.15. *Suppose that Z_k is normal in **??** and let $X_k \to Y_k \xrightarrow{g} Z_k$ be the Stein factorisation of f_k. If k is a field of characteristic 0 then g is birational and finite, hence an isomorphism.*

On the other hand if k is a positive characteristic field then g may be a non-trivial purely inseparable morphism of normal varieties. This an obstruction to lifting sections of mL_k (see [Bri20] *for an explicit construction with $L = K_X + B$). For this reason, a crucial step in* **??** *will be showing $f_{k,*}\mathcal{O}_{X_k} = \mathcal{O}_{Z_k}$ for the semiample contraction of the canonical divisor.*

4.1.3 MMP in families

We fix R to be an excellent DVR with residue field k of characteristic $p > 5$. We collect some results on the MMP in families over R that we will use in **??**. In particular we study the behaviour of the diminished base locus $\mathbf{B}_-(K_X + \Delta)$ under the steps of the MMP.

Definition 4.1.16. *If $X \to S$ is a projective morphism and D is a $\mathbb{Q}$-Cartier divisor on X, the* diminished locus of D over S *is*

$$\mathbf{B}_-(D/S) = \bigcup_{A\,\mathbb{Q}\text{-divisor ample } /S} \mathbf{B}(D + A/S).$$

If S is clear from the context, we will simply write $\mathbf{B}_-(D)$.

Lemma 4.1.17. *Let $(X, \Delta)/T$ be a klt R-pair. Let $f\colon X \dashrightarrow Y$ be a step of a $(K_X + \Delta)$-MMP over T and write $\Delta_Y = f_*\Delta$. Let*

$$\begin{array}{ccc} & W & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & \overset{f}{\dashrightarrow} & Y \end{array}$$

be a resolution of indeterminacies of f. Then $q^{-1}\mathbf{B}_-(K_Y + \Delta_Y) \subset p^{-1}\mathbf{B}_-(K_X + \Delta)$.

Proof. By the negativity lemma, we deduce $p^*(K_X + \Delta) = q^*(K_Y + \Delta_Y) + G$, where $G \geq 0$ and therefore we clearly have the following containment of stable base loci: $q^{-1}\mathbf{SB}(K_Y + \Delta_Y) \subset p^{-1}\mathbf{SB}(K_X + \Delta)$. Similarly, note that for every sufficiently small ample A on X, a $(K_X + \Delta)$-MMP step is a $(K_X + \Delta + A)$-MMP step. As A is ample and f birational, we can write $f_*A \sim_{\mathbb{Q}} H + E$, where H is ample and E effective. Therefore $q^{-1}\mathbf{SB}(K_Y + \Delta_Y + \frac{1}{n}H) \subset q^{-1}\mathbf{SB}(K_Y + \Delta_Y + \frac{1}{n}f_*A) \subset p^{-1}\mathbf{SB}(K_X + \Delta + \frac{1}{n}A)$. As $\mathbf{B}_-(K_Y + \Delta_Y) = \bigcup_{n \geq 0} \mathbf{SB}(K_Y + \Delta_Y + \frac{1}{n}H)$ by [ELM+06, Proposition 1.19] we conclude. □

We recall that, given a log pair (X, Δ), a *non-canonical centre* V of (X, Δ) is the centre of a divisorial valuation E with discrepancy $a(E, X, \Delta) < 0$. The following is a generalisation of [HMX18, Lemma 3.1] for arithmetic and positive characteristic threefolds.

Proposition 4.1.18. *Let R be an excellent DVR with residue field k of characteristic > 5. Let $X \to \mathrm{Spec}(R)$ be a projective contraction and suppose that (X, B) is a $\mathbb{Q}$-factorial klt threefold pair with $\mathbb{Q}$-boundary. Suppose the following conditions are satisfied:*

(1) $(X, B + X_k)$ is plt with X_k integral;

(2) if V is a non-canonical centre of $(X, B + X_k)$ contained in $\mathbf{B}_-(K_X + B)$, then $\dim(V_k) = \dim(V) - 1$.

Let $f\colon X \dashrightarrow Y$ be a step of a $(K_X + B)$-MMP over R. Then:

1. *If f is a contraction of fibre type, then so is f_k;*
2. *if f is birational, then:*
   (i) f is a divisorial contraction;
   (ii) if $\Gamma := f_*B$, then conditions (1) and (2) also hold for (Y, Γ).

In particular, if f is a projective birational morphism then $h^0(X_t, m(K_{X_t} + B_t)) = h^0(Y_t, m(K_{Y_t} + \Gamma_t))$ for all $t \in \mathrm{Spec}(R)$ and all $m \geq 0$ sufficiently divisible.

Proof. If f is a contraction of fibre type, hence f_k is not birational by upper semi-continuity of the dimension of the fibres for proper morphisms ([Sta, Tag 0D4Q]).

From now on, we assume that f is birational. Suppose for contradiction that f is a flip and consider the following diagram:

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & Y \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle g^+} & \\ & Z, & \end{array}$$

where g is a $(K_X + B)$-flipping contraction. Note that Y_k is irreducible since f does not extract divisors, thus f_k is birational. As $(X, B + X_k)$ is plt, so is $(Y, \Gamma + Y_k)$ hence both X_k and Y_k are normal by Corollary 2.2.27.

We now derive the contradiction. Since f is a flip, there exists a prime divisor D on Y_k such that its centre P on X_k is a closed point. Since f_k is not an isomorphism at N we have

$$a(D; X_k, B_k) < a(D; Y_k, \Gamma_k) \leq 0$$

by Lemma 2.1.15. Hence P is a non-canonical centre of (X_k, B_k). Note that $P \subseteq \mathrm{Exc}(g) \subseteq \mathbf{B}_-(K_X + B)$ since D is exceptional over Z_k. Moreover P is also a non-canonical centre of $(X, B + X_k)$ as

$$0 > \mathrm{TDisc}(P, X_k, B_k) \geq \mathrm{Disc}(P, X, X_k + B),$$

by easy adjunction ([Kol13, Lemma 4.8]). So P is an isolated non-canonical centre of $(X, X_k + B)$ contained in $\mathbf{B}_-(K_X + B)$, thus contradicting (2).

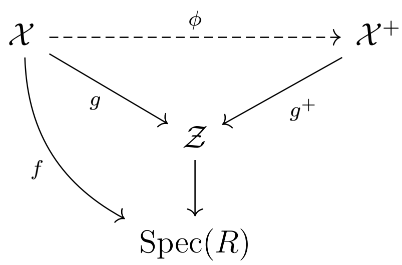
Thus f, and therefore f_k, is a divisorial birational projective contraction. Condition (1) holds on $(Y, \Gamma + Y_k)$ immediately, so it remains to check condition (2).

Suppose that V is a non-canonical centre of $(Y, \Gamma + Y_k)$ and take a model Z dominating X and Y, and containing an exceptional divisor E such that $V = \mathrm{centre}_Y(E)$ and $a(E, Y, \Gamma + Y_k) < 0$. Then by Lemma 2.1.15 it must be that $a(E, X, X_k + B) \leq a(E, Y, Y_k + \Gamma) < 0$, hence the image, W, of E on X is a non-canonical centre of $(X, B + X_k)$. By **??** if $V \subseteq \mathbf{B}_-(K_Y + \Gamma)$ then we have $W \subseteq \mathbf{B}_-(K_X + B)$ as well. In which case W is horizontal and hence so is V, therefore (2) holds as claimed.

Since a (K_X+B)-MMP over R is a (K_X+X_k+B)-MMP, we have that the map $(X_k, B_k) \to (Y_k, \Gamma_k)$ is a $(K_{X_k} + B_k)$-negative birational contraction and thus $h^0(X_t, m(K_{X_t} + B_t)) = h^0(Y_t, m(K_{Y_t} + \Gamma_t))$ for all $t \in \mathrm{Spec}(R)$ and all $m \geq 0$ sufficiently divisible by **??**. □

To explain the conditions we need to impose on the non-canonical locus of the family, we revisit an example due to Kawamata (see [Kaw99, Example 4.3]).

Example 4.1.19. *Let R be an excellent DVR and consider the following diagram of R-flat families:*



where

1. *$\mathcal{X}$ is a terminal threefold and the central fibre X_0 is klt with a singular point p;*
2. *g is an extremal $K_{\mathcal{X}}$-negative flipping contraction;*
3. *$\mathcal{X}^+$ is regular.*

A local model is given by the Francia flip explained in [Kaw99]. *As explained by Kawamata, one can construct such a situation and the map $f_*\mathcal{O}_{\mathcal{X}}(mK_{\mathcal{X}}) \to f_*\mathcal{O}_{X_0}(mK_{X_0})$ is not surjective.*

Note that this situation is excluded by condition (2) of **??** *and* **??**. *Indeed $\boldsymbol{B}_-(X_k, K_{X_k})$ clearly contains the flipped locus of g, which must contain the non-canonical singular points p of X_k. As $\mathcal{X}$ is terminal, p is not the restriction of a horizontal non-canonical centre of $\mathcal{X}$.*

4.2 Abundance for mixed characteristic threefolds

Given a klt pair (X, Δ) with a projective R-morphism $f\colon X \to T$ so that $K_X + \Delta$ is f-nef, then the abundance conjecture asserts that $K_X + \Delta$ is f-semiample. In the case where (X, Δ) is a klt threefold pair and $K_X + \Delta$ (or even just Δ) is big this is immediate by Theorem 2.2.6. We address the remaining cases in this section.

The starting point of our proof is the abundance theorem for surfaces over excellent bases, which we now recall.

Theorem 4.2.1. *Let $\pi\colon (S, B)/T$ be a klt R-pair of dimension 2. If $K_S + B$ is a π-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then it is π-semiample.*

Proof. If T is a field then this is [FT12, Theorem 1.2] for perfect fields and [Tan20] for imperfect fields. Suppose from now on that $\dim(T) > 0$. If $K_S + B$ is big over T then this follows immediately from the base-point-free theorem ([Tan18b, Theorem 4.2]) with $D = 2(K_S + B)$. Hence we may suppose that $\dim(T) = 1$ and $K_S + B$ is not big. In this case we have $(K_S + B)|_{S_{K(T)}} \sim_{\mathbb{Q}} 0$ by the abundance theorem for curves ([BMP+20, Lemma 9.22]) and the result follows by **??**. □

The following is [CT20, Lemma 2.17]. We include the proof for completeness as the result is used often.

Lemma 4.2.2. *Let $f\colon X \to Y$ be a contraction of integral, normal and excellent schemes. Suppose L is an f-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor with $L|_{X_{K(Y)}} \sim_{\mathbb{Q}} 0$. If Y is $\mathbb{Q}$-factorial and f is equi-dimensional then $L \sim_{Y,\mathbb{Q}} 0$.*

Proof. Since $L|_{X_{K(Y)}} \sim_{\mathbb{Q}} 0$ we may write $L \sim_{Y,\mathbb{Q}} D \geq 0$ such that $D|_{X_{K(Y)}} = 0$. If C is any component of D then $f(C)$ is a prime divisor, since f is equi-dimensional. Thus, since Y is $\mathbb{Q}$-factorial, it is enough to know that $L \sim_{\mathbb{Q},Y} 0$ after localisation about any codimension one point of Y. In particular we may suppose that $Y = \mathrm{Spec}(R)$ for some DVR R with closed point P.

Let $\{G_i\}_{i=1}^n$ be the irreducible components of the special fibre $F = f^*P$, so that by construction $D = \sum_{i=1}^n a_i G_i$ for certain $a_i \geq 0$.

We introduce $r := \min\{t \mid D - tF \leq 0\}$. We are left to show that $D - rF = 0$. If not, up to rearranging the order of G_i, we have $D - rF = -\sum_{i=2}^n l_i G_i \equiv_Y 0$, with $l_2 > 0, l_i \geq 0$ and G_1 meeting G_2. Note that $(rF - D)$ is effective curve not containing G_1 but intersecting it. Hence there must be a curve C on G_1 with $(rF - D) \cdot C > 0$, but $rF - D \sim_T -D$ and D is nef, a contradiction. Therefore $D - rF = 0$ as claimed. □

The following gives a sufficient condition for a nef divisor to be EWM together with a very controlled version of resolution of indeterminacy of an EWM morphism (cf. [BMP+20, Lemma 9.25]).

Lemma 4.2.3. *Let $X \to T$ be a projective contraction of normal, integral, quasi-projective R-schemes. Let L be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, nef over T such that $L|_{X_{K(T)}}$ and $L|_{X_{\mathbb{Q}}}$ are semiample. Assume* $\dim(X) \leq 3$ *and L is not big. Then L is EWM and there is a* *commutative diagram of proper algebraic spaces over T:*

$$\begin{array}{ccc} W & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\pi} & Z, \end{array}$$

such that

1. *f is the EWM contraction associated to L;*
2. *ϕ and π are proper birational contraction;*
3. *g is equi-dimensional, W is a T-projective scheme, and Y is a T-projective regular scheme of dimension ≤ 2;*
4. *g agrees with the map induced by ϕ^*L over the generic point of Z;*
5. *there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that $\phi^*L \sim_{\mathbb{Q}} g^*D$.*

Proof. Note that if $\dim(T) = 0$ there is nothing to prove, hence we can assume $\dim(T) \geq 1$. By [BMP+20, Lemma 9.24] and its proof we can find a diagram of schemes over T:

$$\begin{array}{ccc} W & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle g} & & \\ Y & & , \end{array}$$

such that ϕ is birational and there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that (c)-(e) hold. By **??** and **??** it is sufficient to show that D is EWM to conclude. If $\dim(Y) \leq 1$, the result is trivial and if $\dim(Y) = 2$, we apply [BMP+20, Lemma 2.48]. $\square$

If f is equi-dimensional, it is possible to prove a suitable semiampleness result.

Proposition 4.2.4. *Let $X \to T$ be a projective contraction of normal quasi-projective schemes over R, where $\dim(X) \leq 3$. Let L be an EWM $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that its associated EWM contraction $f\colon X \to Z$ is equi-dimensional. If $L|_{X_{K(T)}}$ and $L|_{X_{\mathbb{Q}}}$ are semiample, then L is semiample.*

Proof. Without loss of generality we may assume $\dim(T) \geq 1$. If L is big and f is equi-dimensional, then L is necessarily ample and we conclude. We can thus suppose L is not big. We can then apply **??** and thus there exists a commutative diagram of proper algebraic spaces over T

$$\begin{array}{ccc} W & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\pi} & Z, \end{array}$$

such that the following hold:

1. W, X, Y are normal T-projective schemes and $\dim(Y) \leq 2$;
2. the vertical maps f and g are equi-dimensional;
3. the horizontal maps ϕ and π are proper and birational;
4. there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that $\phi^*L \sim_{\mathbb{Q}} g^*D$.

Since Z is normal, there is an open immersion of a regular scheme $U \to Z$ containing every codimension 1 point of Z by **??**. By (b), $X_U \to U$ satisfies the assumptions of **??** and thus $L|_{X_U} \sim_{U,\mathbb{Q}} 0$.

If $\dim(Z) = 1$, then we conclude immediately, so we can suppose Z is a surface. Therefore $Z \setminus U$ consists of finitely many points. Then we may choose S to be a general hyperplane on X such that S meets each fibre over $Z \setminus U$ at only finitely many points.

Note that $L|_S$ is clearly big and moreover if C is any curve on S with $L \cdot C = 0$, then C must be contracted by $X \to Z$ as f is the EWM contraction associated to L. In particular C is contained in some fibre of f and by construction C is not contained in a fibre over $Z \setminus U$, as S contains no such curves. Thus in fact $C \subseteq X_U$. Therefore $\mathbb{E}(L|_S) \subseteq X_U$ and so $S|_{\mathbb{E}(L|_S)}$ is semiample since $L|_{X_U}$ is. As $L|_{S_{\mathbb{Q}}}$ is semiample by assumption, we conclude that $L|_S$ is semiample by [Wit20, Theorem 6.1].

Let S' be the strict transform of the surface S on W, which must dominate Y. Let $\phi_{S'}, g_{S'}$ be the restrictions of ϕ, g to S'. Then $(\phi^*L)|_{S'} = \phi^*_{S'}(L|_S) = g^*_{S'}D$ and since $L|_S$ is semiample and Y is normal, we must have that D is semiample by **??**. In turn this implies that L is semiample as $\phi^*L = g^*D$. □

The following is a useful MMP technique to reduce to the case of equi-dimensional morphisms.

Proposition 4.2.5. *Let $(X, B)/T$ be a $\mathbb{Q}$-factorial klt threefold R-pair. Suppose that*

1. *$K_X + B$ is a nef EWM $\mathbb{Q}$-divisor over T with $h \colon X \to Z$ be the associated EWM contraction;*
2. *Z has dimension 2.*

Then there exists a $(K_X + B)$-trivial birational contraction $(X, B) \dashrightarrow (X', B')$ over Z such that $X' \to Z$ is equi-dimensional.

Proof. Let $z \in Z$ be a closed point such that the fibre $h^{-1}(z)$ is not one-dimensional. By upper semi-continuity of fibre dimensions for proper morphisms ([Sta, Tag 0D4Q]) $h^{-1}(z)$ must contain an irreducible divisor F.

Take $t > 0$ with $(X, B + tF)$ klt and run a $(K_X + B + tF)$-MMP over T. We now show that this is an MMP over Z as well. Let C be a curve generating an extremal

(K_X+B+tF)-negative ray. As (K_X+B) is nef over T, then $F \cdot C < 0$. Therefore $C \subseteq F$ and since F is contracted by h to a point, so too is C. By definition $X \to Z$ contracts only (K_X+B)-trivial curves. From this we can conclude that the (K_X+B+tF)-MMP over T is also a (K_X+B+tF)-MMP over Z by **??**.

Since this is an MMP of a pseudo-effective klt pair over T it terminates by Theorem 2.2.6. In fact we claim it terminates when the strict transform of F is contracted.

If $X \dashrightarrow X'$ does not contract F then its transform on X' remains the divisorial part of a fibre, so to establish this claim it is sufficient to show that such divisorial part is never nef. By abundance (**??**) on the generic fibre $(X'_{K(Z)}, B'_{K(Z)})$ we can apply **??** to find a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\phi} & X' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{\pi} & Z, \end{array}$$

where g is equi-dimensional, Y is a regular projective surface over T and ϕ, π are proper birational. Let F' be the strict transform of F on W. Then $g(F') = \gamma$ must be an irreducible curve by equi-dimensionality and $\pi_*(\gamma) = z$. Choose a general curve C in F' such that $g(C) = \gamma$. Since is D big, we write $D \sim_{\mathbb{R}} A + E$, for A ample and E effective by Kodaira's lemma. By Bertini theorems ([BMP+20, Theorem 2.15]) we can choose a general $H \sim_{\mathbb{R}} A$ meeting γ transversally. Then $g^*H \cap C$ is a finite set of points and g^*H is not contracted by ϕ as H is general. We have $K_X + B \sim_{\mathbb{R}} \phi_* g^*(A+E) \sim_{\mathbb{R}} \phi_* g^* H + S$ where $S \geq 0$. As C is general in F' we have

$$\phi_* g^* H \cdot \phi_* C > 0 \text{ and } (K_X + B) \cdot \phi_* C = g^* D \cdot C = D \cdot \gamma = 0 \text{ as } \pi_* \gamma = z,$$

so we have $S \cdot C < 0$. Since C is general in F' we must have that F is contained in the support of S and $F \cdot \phi_* C < 0$.

Since there are only finitely many closed points $z \in Z$ for which the fibres are not one dimensional, we can repeat the above process a finite number of times and we terminate with a crepant model (X', B') which is equi-dimensional over Z. □

We are now ready to prove the abundance theorem for klt threefolds over a positive-dimensional base which is not of pure characteristic 0.

Theorem 4.2.6. *Let $(X,B)/T$ be a $\mathbb{Q}$-factorial klt threefold R-pair which contains a point of positive characteristic. If $K_X + B$ is π-nef, then it is π-semiample.*

Proof. By Stein factorisation we can assume π to be a contraction of normal schemes, so $\dim(T) \geq 1$. If X is not $\mathbb{Q}$-factorial, then we may freely replace it with a $\mathbb{Q}$-factorialisation by Theorem 2.2.9 and Lemma 2.2.29. Moreover by Proposition 2.2.34 we can suppose that Δ is a $\mathbb{Q}$-boundary (we reduce to this case where we can apply the results of [Wit20] which apply only to $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors). As the dimension $X_{k(T)}$ is at most 2, we conclude by [BMP+20, Lemma 9.22] and **??** that $\kappa(K_{X_{k(T)}} + B_{k(T)}) \geq 0$. We now divide the proof according to the value of $\kappa(K_{X_{k(T)}} + B_{k(T)}) + \dim(T)$.

Case 1. $\kappa(K_{X_{k(T)}} + B_{k(T)}) + \dim(T) = 3.$

In this case, $K_X + B$ is big and we can conclude by applying the basepoint free theorem (Theorem 2.2.6) to $L := 2(K_X + B)$.

Case 2. $\kappa(K_{X_{k(T)}} + B_{k(T)}) + \dim(T) = 2.$

By **??**, $K_{X_{k(T)}} + B_{k(T)}$ and $K_{X_{\mathbb{Q}}} + B_{\mathbb{Q}}$ are semiample $\mathbb{Q}$-divisors. As $K_X + B$ is not big, by **??** then $K_X + B$ is EWM and we denote by $f\colon X \to Z$ the associated EWM contraction. By **??** and Lemma 2.2.29 we may replace X so that $X \to Z$ is equi-dimensional. We then apply **??** to deduce that $K_X + \Delta$ is π-semiample.

Case 3. $\kappa(K_{X_{k(T)}} + B_{k(T)}) + \dim(T) = 1.$

The hypothesis $\dim(T) \geq 1$ implies $\kappa(K_{X_{k(T)}} + B_{k(T)}) = 0$. Then $\pi\colon X \to T$ is flat, since T is a Dedekind scheme and X is integral by [Har77, Proposition 9.7]. Since $K_{X_{k(T)}} + B_{k(T)}$ is semiample by **??**, we conclude $K_X + B$ is semiample by **??**. □

Remark 4.2.7. *While in this section we worked on threefolds over mixed characteristic rings whose residue fields have characteristic different from* 2, 3 *and* 5*, this is just due to the current state of the art on the MMP. The arguments in the section for* $\kappa(K_{X_{k(T)}} + B_{k(T)}) + \dim(T) \leq 2$ *work as long as the MMP results are known to hold. In particular, abundance holds for mixed characteristic threefolds over a Dedekind domain with residue characteristics different from* 2, 3 *by* [XX22].

4.3 Applications to invariance of plurigenera

In this section, R will always be an excellent DVR with residue field k of characteristic $p > 5$ and fraction field K.

The purpose of this section is to generalise the asymptotic invariance of plurigenera proven in [EH21, Theorem 3.1] to families of *non-log-smooth* surface pairs, as well as DVRs with non-perfect residue field. Similar results in characteristic zero are proven in [HMX13, HMX18]. The first case we discuss is the asymptotic invariance for families of good minimal models.

Theorem 4.3.1. *Let* (X, B) *be a three-dimensional* R*-pair with* $\mathbb{Q}$*-boundary. Assume that* $(X, B + X_k)$ *is plt and* $K_X + B$ *is semiample over* R*. Suppose one of the following holds:*

1. $\kappa(K_{X_k} + B_k) \neq 1$*; or*

2. B_k *big over* $\operatorname{Proj} R(K_{X_k} + B_k)$*.*

Then there exists an $m_0 \in \mathbb{N}$ *such that*

$$h^0(X_K, m(K_{X_K} + B_K)) = h^0(X_k, m(K_{X_k} + B_k))$$

for all $m \in m_0\mathbb{N}$.

We start by showing the normality of the central fibre of the image of the $(K_X + B)$ semiample contraction.

Proposition 4.3.2. *Let* (X, B) *be a three-dimensional klt R-pair with* $\mathbb{Q}$*-boundary. Suppose that* $(X, B + X_k)$ *is plt. If* $f: X \to Z$ *is a birational morphism over* R *such that* $-(K_X + B)$ *is* f*-nef, then* Z_k *is normal and* $f_{k,*}\mathcal{O}_{X_k} = \mathcal{O}_{Z_k}$.

Proof. By Corollary 2.2.27 the central fibre X_k is normal. As f is birational over R, so is f_k and thus $-(K_{X_k} + B_k)$ is f_k-big and f_k-nef. We conclude by Lemma 2.2.24. □

The previous is useful for small $(K_X + B)$-trivial birational morphisms, and in particular to reduce the non $\mathbb{Q}$-factorial case to the $\mathbb{Q}$-factorial one.

Lemma 4.3.3. *Let* $Y \to \mathrm{Spec}(R)$ *be a projective contraction such that* $(Y, Y_k + \Delta)$ *is a plt threefold R-pair. Let* $f: Y \to X$ *be a* $(K_Y + \Delta)$*-trivial small birational contraction over* R *with* $B = \pi_*\Delta$*. Then*

$$h^0(X_k, m(K_{X_k} + B_k)) = h^0(Y_k, m(K_{Y_k} + \Delta_k))$$

for all m *sufficiently divisible.*

Proof. As f is small, the central fibre Y_k is irreducible. By the basepoint free theorem Theorem 2.2.6, $K_Y + Y_k + \Delta \sim_{\mathbb{Q}} f^*(K_X + X_k + B)$. Then by **??**, Y_k and X_k are both normal. As f is the semiample contraction associated to $K_Y + \Delta$ over X and it is birational, we conclude by **??**. □

We now discuss the delicate case of invariance for plurigenera where the Kodaira dimension is one and the boundary is big.

Proposition 4.3.4. *Let* $(X, B)/R$ *be a* $\mathbb{Q}$*-factorial klt R-pair with* $\mathbb{Q}$*-boundary of dimension* 3 *such that* $(X, B + X_k)$ *is plt. Suppose that*

1. $K_X + B$ *is semiample and let* $f: X \to Z$ *its Iitaka fibration over* R;
2. $\kappa(K_{X_k} + B_k) = 1$;
3. B_k *is big over* Z_k.

Then there exists an $m_0 \in \mathbb{N}$ *such that*

$$h^0(X_K, m(K_{X_K} + B_K)) = h^0(X_k, m(K_{X_k} + B_k))$$

for all $m \in m_0\mathbb{N}$.

Proof. As K_X is not pseudoeffective over X, we now run a K_X-MMP over Z with scaling of A which terminates by Theorem 2.2.6 with a Mori fibre space $X \to Z'/Z$ since B_k is big over Z_k. Since each step is $(K_X + B)$-trivial and does not contract X_k, $(X, B + X_k)$ remains plt and so X_k stays irreducible and normal by Corollary 2.2.27.

Consider a step of this MMP $\phi\colon X \dashrightarrow X'$ and let $B' := \phi_* B$. We have

$$\begin{array}{ccccc} X & & \overset{\phi}{\dashrightarrow} & & X' \\ & {\scriptstyle g}\searrow & & \swarrow{\scriptstyle h} & \\ & & Y & & \\ & & \downarrow & & \\ & & Z, & & \end{array}$$

where h may either be an isomorphism or a small birational contraction. As g is (K_X+B)-trivial, by **??** we have that Y_k is irreducible and normal as well. Let now (Y, Ξ) be the induced pair on Y: we then have

$$h^0(X_k, m(K_{X_k} + B_k)) = h^0(Y_k, m(K_{Y_k} + \Xi_k)) = h^0(X'_k, m(K_{X'_k} + B'_k))$$

where the first equality also follows from **??** and **??** and the latter is **??**. Also the sections of $m(K_X + B)$ are preserved by this MMP for large divisible $m > 0$.

Hence we can now suppose X admits a Mori fibre space $X \to Z'/Z$, with B_k big over Z_k.

We claim that $Z' = Z$. Indeed, suppose for contradiction that there exists a divisor D on Z' which is contracted by $Z' \to Z$. Since $\dim Z = 2$, D must be contained in Z'_k. But then $f^{-1}D$ is a surface inside X_k, which is irreducible by assumption. It cannot be that X_k is contracted to a point over Z, thus no such D exists and we have $Z = Z'$.

In particular $-K_{X'}$ is ample over Z and hence by Lemma 2.2.24 we have that $f_{k,*}\mathcal{O}_{X_k} = \mathcal{O}_{Z_k}$ and the result follows from **??**. □

We can now prove the asymptotic invariance of plurigenera in a family of minimal models.

Proof of **??**. By **??** we can suppose X is $\mathbb{Q}$-factorial. We have $\kappa(K_{X_k} + B_k) = \kappa(K_{X_K} + B_K) = \kappa$, since the Iitaka dimension is deformation invariant for semiample line bundles by **??**. Let $f\colon X \to Z/\operatorname{Spec}(R)$ be the relative Iitaka fibration, so that $K_X + B \sim_{\mathbb{Q}} f^*A$ for some ample $\mathbb{Q}$-divisor. We now divide in various cases.

If $\kappa = 0$, then $K_X + B \sim_{\mathbb{Q}} 0$ and hence we conclude by **??**. If $\kappa = 1$, then this is **??**. Finally in the case $\kappa = 2$ we conclude by **??** and **??**. □

Putting these results together with **??** and the abundance theorem **??** we deduce an asymptotic invariance result for plurigenera on suitable families.

Imposing the conditions of **??**, we are able to prove the invariance of plurigenera for families of klt surfaces from **??**.

Theorem 4.3.5. *Suppose that (X, B) is a three dimensional klt R-pair with $\mathbb{Q}$-boundary. Suppose that all of the following are satisfied:*

(1) $(X, X_k + B)$ *is plt with* X_k *integral and normal;*

(2) if V *is a non-canonical centre of* $(X, B + X_k)$ *contained in* $\mathbf{B}_-(K_X + B)$*, then* $\dim(V_k) = \dim(V) - 1$.

Suppose further that at least one of the following holds:

1. $\kappa(K_{X_k} + B_k) \neq 1$*; or*
2. B_k *is big over* $\mathrm{Proj}(K_{X_k} + B_k)$

Then there is $m_0 \in \mathbb{N}$ *such that*

$$h^0(X_K, m(K_{X_K} + B_K)) = h^0(X_k, m(K_{X_k} + B_k))$$

for all $m \in m_0\mathbb{N}$.

Proof. By **??** we can suppose X is $\mathbb{Q}$-factorial. We may run a $(K_X + B)$-MMP over R which terminates by Theorem 2.2.6. We call (Y, Γ) the end-product of this MMP. Since (X, B) satisfies conditions (1)-(2) of **??** we deduce $h^0(X_k, m(K_{X_k}+B_k)) = h^0(Y_k, m(K_{Y_k}+\Gamma_k))$ for all sufficiently divisible m. In the case where $\kappa(K_{X_k} + B_k) = 1$, the condition that B_k is big over $\mathrm{Proj}(K_{X_k} + B_k)$ is also preserved by the MMP.

If $K_X + B$ is pseudo-effective then $K_Y + \Gamma$ is nef over R. Therefore, by **??**, $K_X + B$ is semiample and the result then follows from **??**. If $K_X + B$ is not pseudoeffective over R, then there is a Mori fibre space structure $(Y, \Gamma) \to Z$. This ensures that neither $K_Y + \Gamma$ nor $K_{Y_k} + \Gamma_k$ are pseudo-effective and thus the result holds trivially. □

Remark 4.3.6. *The $p > 5$ assumption is essential to the adjunction type results used in* **??**. *Even if the MMP was known in lower characteristic, our arguments in this section would not extend immediately.*

Chapter 5

Finiteness of Minimal Models

This chapter addresses remaining questions around Mori Fibrations in mixed characteristic. It comprises the bulk of [Sti21b].

All the results besides termination also apply to pairs of pure characteristic. Much of the content is the same as [Sti21b]. Here R will always be an excellent ring with dualising complex. We also require that the residue fields of closed points are all of characteristic $p > 5$ or $p = 0$, but this is a limitation only of current MMP results in mixed characteristic.

We keep the notation of Definition 2.1.2, however in this chapter we will always work over a base T that is positive dimensional. This is mostly out of an abundance of caution. The results of [DW19a] should be sufficient to carry out the arguments needed for **??** over an F-finite field. Care would also need to be taken with **??** in this setting, since termination of an MMP with scaling is needed for **??** and **??**. This should follow from **??**, however. Over a perfect field of positive characteristic this theorem is already known due to [Das20] and in characteristic 0 due to [SC11]. The result is also known in higher dimensions over characteristic fields by [BCHM10].

First it is shown that in fact the threefold MMP over a positive dimensional base always terminates, extending the termination result of [BMP+20] to pairs which are not pseudo-effective.

Theorem 5.0.1 (**??**)**.** *Let $f : (X, \Delta) \to T$ be a threefold dlt pair over R, then any $K_X + \Delta$ MMP terminates.*

Next, it is shown that any two threefold Mori fibres spaces which are the output of the same MMP are related by Sarkisov links.

Theorem 5.0.2 (**??**)**.** *Fix an integral quasi-projective scheme T over R. Let $g_1 : Y_1 \to Z_1$ and $g_2 : Y_2 \to Z_2$ be two Sarkisov related, klt Mori fibre spaces of dimension 3, projective T. If the Y_i have positive dimension image in T, then they are connected by Sarkisov links.*

The proof of this second theorem follows closely the work of [HM09]. The main technical work comes in proving a suitable version of Finiteness of Minimal Models.

Theorem 5.0.3 (**??**). *Let X be an integral, normal threefold over R equipped with a projective morphism $X \to T$, where T is quasi-projective over R and the image of X in T is positive dimensional. Let A be an ample $\mathbb{Q}$-Cartier divisor and C be a rational polytope inside $\mathcal{L}_A(V)$. Suppose there is a boundary $A + B \in \mathcal{L}_A(V)$ such that $(X, A+B)/T$ is a klt R-pair. Then the following hold:*

1. *There are finitely many birational contractions $\phi_i : X \dashrightarrow Y_i$ such that*
$$\mathcal{E}(C) = \bigcup \mathcal{W}_i = \mathcal{W}_{\phi_i}(C)$$
where each $\mathcal{W}_i$ is a rational polytope. Moreover if $\phi : X \to Y$ is a wlc model for any choice of $\Delta \in \mathcal{E}(C)$ then $\phi = \phi_i$ for some i, up to composition with an isomorphism.
2. *There are finitely many rational maps $\psi_j : X \dashrightarrow Z_j$ which partition $\mathcal{E}(C)$ into subsets $\mathcal{A}_{\psi_j}(C) = \mathcal{A}_i$.*
3. *For each W_i there is a j such that we can find a morphism $f_{i,j} : Y_i \to Z_j$ and $W_i \subseteq \overline{A_j}$.*
4. *$\mathcal{E}(C)$ is a rational polytope and A_j is a union of the interiors of finitely many rational polytopes.*

In fact these results hold for a slightly more general class of singularities - rlt pairs, which are essentially pairs which are replaceable by linearly equivalent klt pairs locally over the base. This generalisation is necessary due to the lack of appropriate Bertini type theorems over a general ring. Even if one starts with Mori Fibre Spaces coming from a klt MMP, the Sarkisov links may involve rlt pairs. A full definition of rlt is given in **??** and a description of Sarkisov links in **??**.

5.1 Termination

In this section we study termination for threefold pairs over positive dimensional bases. In this setting we will show that every $K_X + \Delta$ MMP terminates for a dlt pair $(X, \Delta)/T$. As always in this chapter, we consider only positive dimensional bases. If $X \to T$ is projective and $U \subseteq T$ is an open set we will write $X_U = X \times_T U$ and $\Delta_U = \Delta|_{X_U}$

Termination for pseudo-effective pairs in this setting is assured by the following theorem, together with non-vanishing on the generic fibre.

Theorem 5.1.1. [BMP+20, Proposition 9.20] *Let $(X, \Delta)/T$ be a threefold dlt R-pair. Suppose that*
$$(X, \Delta) =: (X_0, \Delta_0) \dashrightarrow (X_1, \Delta_1) \dashrightarrow$$
is a sequence of $(K_X + \Delta)$ flips. Then neither the flipped nor the flipping locus are contained the support of Δ_n for all sufficiently large n.

We rely heavily on **??**. The key remaining argument is if $X \to T$ is a klt pair then there is an open set on which every contraction is horizontal. We prove this by reducing to the case that (X, Δ) is terminal. In mixed and positive characteristic this then follows from the liftablity of -1 curves, see [KU85]. This argument does not work in purely positive characteristic but provides motivation for our approach. Instead we adapt a termination argument for terminal pairs, largely due to Shokurov [Sho86].

We will also need the following construction, essentially due to [Mum61].

Lemma 5.1.2. *Let $\pi : X \to Y$ be a projective contraction from a regular scheme to a normal scheme, both of dimension 2. Let $E_1, ..., E_n$ be the exceptional curves. Choose a divisor D on Y and write D' for the strict transform of D. Then there are unique $m_i \geq 0$ with $D' + \sum m_i E_i \equiv_Y 0$. If D is $\mathbb{Q}$-Cartier then we have $\pi^* D = D' + \sum m_i E_i$.*

Proof. By [Kol13, Theorem 10.1], the intersection form $[E_i.E_j]$ is negative definite. Hence there is a unique choice of m_i with $D' + \sum m_i E_i \equiv_Y 0$. It remains to show that $m_i \geq 0$. By [Kol13, Lemma 10.2] there is $E = \sum r_i E_i$ effective on X with $-E$ ample over Y. Then $E.E_i < 0$ for each i ensures that $r_i > 0$ for all i.

Now suppose for contradiction that $m_k < 0$ for some k. Then we may suppose that m_k/r_k is minimal, otherwise if m_j/r_j is minimal we just replace k with j as we must still have $m_j < 0$. We must have, for every j, that $D'.E_j \geq 0$ as it does not contain any E_j and thus as $D' \equiv_Y -\sum m_i E_i$ we have

$$0 \geq (\sum_i m_i E_i).E_j = \sum_I \frac{m_i}{r_i}(r_i E_i.E_j) \geq \frac{m_k}{r_k} \sum_i (r_i E_i.E_j) > 0$$

This is a contradiction and hence in fact $m_i \geq 0$ for each i. That this agrees with the pullback when D is $\mathbb{Q}$-Cartier is immediate from uniqueness.

□

Lemma 5.1.3. *Let X be an $\mathbb{Q}$-factorial scheme together with a projective morphism $f : X \to Y$ with geometrically connected fibres to an excellent normal scheme of dimension 2. Suppose V is a closed subscheme of X with $f(V)$ contained in a divisor D. Then there is a divisor D' on X lying over D, numerically trivial over Y and containing V.*

Proof. Let $\pi \colon Y' \to Y$ be a resolution of Y and X' be the normalisation of the dominant component of the fibre product $X \times_Y Y'$. From above we have F on Y lying over D with $F \equiv_Y 0$. We have induced maps $g \colon X' \to Y'$ and $\phi \colon X' \to X$. Now $g^* F$ is numerically trivial over Y, and hence over X. Thus as X is $\mathbb{Q}$-factorial there is D' with $\phi^* D' = F$. It is clear from the construction that $f_* D' = \pi_* F = D$. Suppose that C is a curve lying over D, then we must have $D'.C = 0$. If C is not contained in D' then since f has connected fibres we may suppose that D' meets C, up to replacing C with another curve in the same fibre, but then $D'.C > 0$, a contradiction. Hence D' contains every curve, and hence every fibre, over D. In particular it contains V.

□

Definition 5.1.4. *Let X be a terminal threefold log pair quasi-projective over R. We define the difficulty*

$$d(X) = \#\{E\colon a(E, X) < 1\}$$

this is finite by [KM98, Proposition 2.36]*, since log resolutions exist by* [BMP+20, Proposition 2.12].

Clearly if $Y \hookrightarrow X$ is an open immersion then $d(Y) \leq d(X)$ since every valuation with centre on Y is also a valuation with centre on X. If $X \dashrightarrow X'$ is a K_X flip then $d(X') \leq d(X)$ by Lemma 2.1.15 We claim in fact this inequality is strict.

Lemma 5.1.5. *(see* [KM98, Lemma 6.21]*)*

Let X/T be a terminal threefold R-pair and $X \dashrightarrow X'$ a K_X flip, then $d(X') < d(X)$.

Proof. It suffices to find a divisor E with $a(E, X) < 1$ and $a(E, X') \geq 1$. Let C' be an irreducible component of the flipped curve. Then X' is terminal, so it is smooth at the generic point P of C by [Kol13, Corollary 2.30]. Let $Y \to X$ be the blowup of C' and E the dominant component of the exceptional divisor. By localising at P we see that $a(E, X') = 1$, since this is the blowup of a smooth point on a surface.

Let C be the centre of E on X. Then C is a component of the flipping curve and so we have $a(E, X) < a(E, X')$ by [KM98, Lemma 3.38] concluding the proof. □

Theorem 5.1.6. *Let $(X, \Delta)/T$ be a terminal threefold R-pair. Then there is an open set $U \subseteq T$ such that every $K_{X_U} + \Delta_U$ negative contraction is a horizontal divisorial contraction.*

Proof. Write $\Delta = \sum_1^n a_k D_k$, we argue by induction on n. Suppose first that $n = 0$ and for contradiction there is no such U. Thus we have a sequence of non-empty open sets $U_i \subseteq U_{i-1}$ such that there is a $K_{X_{U_i}}$ negative extremal ray L_i supported away from U_{i+1}. We write $X_i = X \times U_i$.

If L_i induces a divisorial contraction $f_i\colon X_i \to X_i'$ then $\rho(X_{i+1}) \leq \rho(X_i') < \rho(X_i)$ since f_i is an isomorphism over U_{i+1}. Similarly if L_i induces a flip $f_i\colon X_i \dashrightarrow X_i'$ then $d(X_{i+1}) \leq d(X_i') < d(X_i)$. Since both are positive integers there can be only finitely many such U_i, a contradiction.

Now suppose $n > 0$. Let $\Delta^{n-1} = \sum_1^{n-1} a_i D_i$ then by induction there is an open set $U \subseteq T$ such that every $K_{X_U} + \Delta_U^{n-1}$ negative contraction is a horizontal divisorial contraction. If D_n is not horizontal, we can shrink U so it doesn't meet the image of D_n and the result follows immediately. This gives the result if $\dim T = 3$.

Otherwise let S be the normalisation of D_n. If $\dim T = 2$ then there is an open set V of T on which $S_V \to T$ is finite, and hence of relative Picard rank 0. In particular S_V contains no curves. If $\dim T = 1$ then by [Tan18b, Lemma 2.13] there is an open set V of T such that S_V has relative Picard rank 1.

In either case, replace U with $U \cap V$, then X, S with X_U, S_U and Δ with $\Delta|_{X_U}$. It suffices to show that every extremal $K_X + \Delta$ negative contraction is a horizontal divisorial contraction. Suppose for contradiction L is an extremal ray inducing one that is not. We must have $D_n.L < 0$ from our choice of U. Thus induced contraction restricts to a nontrivial birational morphism $S \to S'$ say. However S has Picard rank at most 1, so the only possibility is this map contracts S entirely. In particular this defines a horizontal divisorial contraction, a contradiction. The claim follows.

□

We can extend this immediately to klt pairs.

Theorem 5.1.7. *Let $(X, \Delta)/T$ be a terminal threefold R-pair. Then there is an open set $U \subseteq T$ such that every $K_{X_U} + \Delta_U$ negative contraction is a horizontal divisorial contraction.*

Proof. Let $\pi\colon (Y, \Delta_Y) \to (X, \Delta)$ be a terminalisation, which exists by [BMP$^+$20, Proposition 9.17]. Then by **??** there is an open set $U \subseteq T$ over which every $K_{Y_U} + \Delta_{Y_U}$ negative contraction is divisorial. We claim the same holds for $K_{X_U} + \Delta_U$ negative contractions.

Indeed if $f\colon X_U \to Z$ is any such contraction then $K_{Y_U} + \Delta_{Y_U}$ is not nef over Z. In particular we get a contraction $g\colon Y_U \to Z$, which is necessarily a horizontal divisorial contraction. In particular g is not an isomorphism over the generic point ν of T. However then neither can f be, else $K_{Y_\nu} + \Delta_{Y_\nu}$ would be nef over Z_ν. Thus f is a horizontal divisorial contraction as claimed.

□

Corollary 5.1.8. *Let $f : (X, \Delta) \to T$ be a $\mathbb{Q}$-factorial threefold dlt pair over R, then any $K_X + \Delta$ MMP terminates.*

Proof. It is enough to show there is no infinite sequence of flips. Note that **??** ensures that the flipping and flipped curves are eventually disjoint from $\lfloor \Delta \rfloor$. Therefore, replacing Δ with $\Delta - \lfloor \Delta \rfloor$, we may assume (X, Δ) is klt.

By **??**, there is always some divisor D on T such that all the flips take place over D. If T is $\mathbb{Q}$-factorial then $(X, \Delta' = \Delta + tf^*D)$ is klt for small $t > 0$ and a $K_X + \Delta$ MMP is also a $K_X + \Delta'$ MMP. Since all the flips are contained in the support of Δ' the sequence must terminate. Otherwise we must have $\dim T = 2$ so we use **??** in place of pulling back D and conclude exactly as above.

□

5.2 Relatively Log Terminal Pairs

Here we introduce relatively log terminal pairs, which are essentially pairs which are replaceable by a klt pair locally over the base, and verify that the main results of the MMP extends to this setting. A suitable Bertini type theorem is also established. In this section T will always be positive dimensional, in any case the results would be superfluous if T were the spectrum of a field.

Definition 5.2.1. *We say an R-pair $(X, \Delta)/T$ is relatively log terminal (rlt) (resp. relatively log canonical (rlc)) if there is a finite open cover U_i of T such that on each $X_i = U_i \times X$ we have $(K_X + \Delta)|_{U_i} \sim_{\mathbb{R}} K_{X_i} + \Delta_i$ where (X_i, Δ_i) is a klt (resp. lc) pair. In this case we say that (X, Δ) is witnessed by (X_i, Δ_i). We also sometimes say Δ is witnessed over U_i.*

If $S \subseteq WDiv(X)$ then we say (X, Δ) is rlt (resp. rlc) with witnesses in S if $\Delta_i \in S|_{U_i}$ for each i for some choice of witnesses.

Remark 5.2.2. *T is always quasi-compact so this is equivalent to asking for $K_X + \Delta \sim K_{X_p} + \Delta_p$ with (X_p, Δ_p) klt for each $p \in T$ where $X_p = X \times T_p$ for T_p the localisation at p.*

Being rlt can be quite a sensitive condition. In particular it's not true that if $B \leq B'$ and (X, B') is rlt that (X, B) must be rlt. For example, for any choice of B and sufficiently ample H, on X klt and $\mathbb{Q}$-factorial, we have that $(X, B + H)$ is rlt, though B might not be. It fits well in the context of polytopes however as if B_i are rlt then so is $\sum_1^n \lambda_i B_i$ for any choices of $\lambda_i \geq 0$ with $\sum \lambda_i \leq 1$.

The pseudo-effective cone is the closure of the big cone, and D is big if and only if its pullback to the generic fibre of $X \to T$ is. Hence if U_i is any open cover of T, then D is pseudo-effective if and only if $D|_{U_i}$ is. In particular an rlc pair is pseudo-effective (resp. big) if and only if its witnesses are.

The definitions of various birational models for klt or lc pairs in Definition 2.2.28 extend naturally to the rlt case.

Definition 5.2.3. *Let $\phi : X \dashrightarrow Y$ be a rational map. If U_i is an open cover of T we write $\phi_i : X_i \dashrightarrow Y_i = Y \times U_i$. If (X, Δ) is a pseudo-effective rlc pair witnessed by (X_i, Δ_i) then ϕ is a weak log canonical (wlc) model of (X, Δ) if ϕ_i is an (X, Δ_i) wlc model for each i. Equally if (X, Δ) is rlt then ϕ is a log terminal model of (X, Δ) if and only if each ϕ_i is a log terminal model of (X_i, Δ_i).*

By Lemma 2.2.30 these definitions are independent of the choice of witnesses. In particular if (X, Δ) is lc then the definition of wlc models agrees with usual one, equally if it is klt then the definition of log terminal model is unchanged.

Remark 5.2.4. *The usual definition of ample model works here with no modification, it is equivalent to asking for it to be an ample model for the witnesses.*

We will need the following Bertini type result which provides one of the key motivations for the introduction of rlt pairs.

Lemma 5.2.5. *Let $(X, \Delta)/T$ be an rlt R-pair. Take $A \geq 0$ big and nef, then $(X, \Delta + A)$ is rlt. Moreover if D is a divisor on X sharing no components with the augmented base locus $\mathbf{B}_+(A)$ nor any witness of (X, Δ) then we may assume no witness of $(X, \Delta + A)$ shares a component with D.*

Proof. Write $A \sim A' + E$ for A' ample and $E \geq 0$. We may assume E is arbitrarily small, by writing $A \sim \delta A' + (1-\delta)A = \delta E$ and replacing A' with $\delta A' + (1-\delta)A$. Thus we may suppose $(X, \Delta + E)$ is rlt such that no witnesses shares a component with D and reduce to the case A is ample.

Pick a point $P \in T$ and localise. Write $X_P = X \times T_P$, Δ_P for the witness over P and D_P for the restriction of D. Let $\pi : Y \to X_P$ be a log resolution of $(X_P, \Delta_P + D)$. Let $D' = \mathrm{Supp}(\pi_*^{-1} D)$ and take $-E$ effective, exceptional and anti-ample over X_P. So $A' = \pi^* A_P - E$ is ample. Write $K_Y + \Delta' = \pi^*(K_{X_P} + \Delta_P)$.

By [BMP+20, Theorem 2.11] we can choose $A' \geq 0$ with $(Y, \Delta' + A' + E)$ klt and $(Y, \Delta' + A' + E + D')$ lc. In particular this choice of A' cannot share a component with D'. Now $(X_P, \Delta_P + \pi_* A')$ is klt and $\pi_* A'$ shares no components with D. Then this pair lifts to klt pair over some neighbourhood of P. The result follows by quasi-compactness. □

The MMP for these pairs lifts naturally from the klt case. We work in the setting of [BMP+20], however the rlt (resp. rlc) case always follows from corresponding results for klt (resp. lc) pairs.

Theorem 5.2.6 (rlc Cone Theorem)**.** *Let $(X, \Delta)/T$ be an rlc $\mathbb{Q}$-factorial threefold pair R-pair with $\mathbb{R}$ boundary. Then there is a countable collection of curves $\{C_i\}$ on X such that:*

1.
$$\overline{NE}(X/T) = \overline{NE}(X/T)_{K_Y + \Delta \geq 0} + \sum_i \mathbb{R}[C_i]$$

2. *The rays C_i do not accumulate in $(K_Y + \Delta)_{<0}$.*

3. *There is an integer M such that for each i there is d_{C_i} with*
$$0 < -(K_X + \Delta).C_i \leq M d_{c_i}$$
and d_{C_i} divides $L \cdot_k C_i$ for every Cartier divisor L on X.

Proof. For ease of notation we will often view cycles on X_i as cycles on X without renaming.

Suppose that (X, Δ) has witnesses $(X_i = X \times U_i, \Delta_i)$ for some open cover U_i of T. Then U_i is still quasi-projective over R and the Cone Theorem holds for each (X_i, Δ_i). Let $\gamma_{i,j}$

be the $K_{X_i} + \Delta_i$ negative extremal curves. These are also $K_X + \Delta$ negative, though they need not be extremal on X.

Suppose now that R is a $K_X + \Delta$ negative extremal ray. Let $r \in R$ be a non-zero cycle. Then r is the limit of some effective cycles r^k. We write r_i^k for the part of r supported over U_i. Then $r_i = \lim r_i^k$ is still pseudo-effective, moreover $r - r_i = \lim r^k - r_i^k$ is also. Since R is extremal we must have for each i that either $r_i = 0$ or $r = t_i r_i$ for some $t_i > 0$. There must be some i with $r_i \neq 0$, else we would have $r = 0$. However r_i then generates an extremal $K_{X_i} + \Delta_i$ negative ray, hence $r = t_i r_i = t\gamma_{i,j}$ for some j and some $t > 0$. Thus the $\gamma_{i,j}$ generate all the $K_X + \Delta$ negative extremal rays. (1) and (3) follow immediately by Theorem 2.2.12. Since there are finitely many U_i if the rays accumulated on X we could chose a subsequence consisting of extremal rays coming from some X_i which would then accumulate on X_i, thus 2 also holds. □

Theorem 5.2.7 (rlt Basepoint Free Theorem). *Let $(X, \Delta)/T$ be a $\mathbb{Q}$-factorial threefold rlt R-pair with $\mathbb{R}$-boundary. Let L be a nef Cartier divisor over T such that $L - (K_X + \Delta)$ is big and nef over T. Then L is semiample.*

Proof. This is immediate from the klt case, [BMP+20][Theorem 9.26], since semi-ampleness is local on the base and if $L - (K_X + \Delta)$ is big and nef over T then $L_{X_i} - (K_{X_i} + \Delta_i)$ is big and nef over U_i for each i. □

Theorem 5.2.8 (Existence of rlt flips). *Let $(X, \Delta)/T$ be a threefold rlt R-pair with $\mathbb{R}$-boundary. Suppose $X \to Y$ is a flipping contraction over T then the flip $X \dashrightarrow X^+$ exists.*

Proof. Let $\phi : X \to Y$ be a flipping contraction for an rlt pair (X, Δ). Suppose (X, Δ) is witnessed by (X_i, Δ_i) and let $\phi_i : X_i \to Y_i$ be the induced morphism U_i. Then ϕ_i is either still a flipping contraction or an isomorphism. If ϕ_i is a flipping contraction, then the existence of flip X_i^+ is ensured by [BMP+20][Theorem 9.12], otherwise we take simply take $X_i^+ = X_i$. Hence we have a suitable X_i^+ for each i. Since flips are unique these X_i^+ glue to a variety X^+ over T such that $X \dashrightarrow X^+$ is the required flip. □

Theorem 5.2.9 (Termination of rlt flips). *Let $(X, \Delta)/T$ be a threefold rlt R-pair with $\mathbb{R}$-boundary, then any sequence of $(K_X + \Delta)$ flips terminates.*

Proof. Suppose first $K_X + \Delta$ is pseudo-effective. Let $f^i : X^i \to X^{i+1}$ be a sequence of flips from $X = X^0$ of an rlt pair $(K_X + \Delta)$. Then $(K_X + \Delta)$ is witnessed over some finite open cover U_j and the restriction $f_j^i : X_j^i \to X_j^{i+1}$ is a sequence of flips for the klt pair $(K_{X_j} + \Delta_j)$ for each j. In particular for fixed j the sequence eventually terminates by Corollary **??**, but then as there are finitely many j, the global sequence f^i also terminates. □

Theorem 5.2.10 (MMP for rlt pairs). *Let $(X, \Delta)/T$ be a threefold rlt R-pair with $\mathbb{R}$-boundary, then we can run a $K_X + \Delta$ MMP. If $K_X + \Delta$ is pseudo-effective then this terminates with a good log terminal model, otherwise it ends in a Mori fibre space.*

Proof. Existence of the claimed MMPs and their termination is immediate from the above results. Suppose then $\phi : X \dashrightarrow Y$ is a log terminal model, since semiampleness is checked locally over the base we can assume that (X,Δ) is klt. Then $K_Y+\Delta_Y$ is a good log terminal model by **??**.

□

5.3 RLT Polytopes

In this section we introduce rlt versions of Shokurov polytopes and provide some key technical results for their usage in the proof of Finiteness of Minimal Models. In particular we show that $\mathcal{RL}_A(V)$ is in fact a rational polytope. In this section, as in all subsequent ones, R will always be an excellent ring with dualising complex, T will be a positive dimensional, quasi-projective R scheme and X will always be an integral scheme projective and surjective over T. All pairs will be considered as R pairs over T.

Definition 5.3.1. *Fix a $\mathbb{Q}$-divisor $A \geq 0$. Let V be a finite dimensional, rational affine subspace of $WDiv_{\mathbb{R}}(X)$ containing no components of A. Such V is called a coefficient space (for A).*

We have the following.

$$V_A = \{A+B : B \in V\}$$
$$\mathcal{L}_A(V) = \{\Delta = A+B \in V_A : (X,\Delta)/T \text{ is an lc pair}\}$$
$$\mathcal{RL}_A(V) = \{\Delta = A+B \in V_A : (X,\Delta)/T \text{ is an rlc pair with witnesses in } V_A\}$$

We call a polytope C inside $\mathcal{RL}_A(V)$ rlt if it is rational and contains only boundaries of rlt pairs.

If $C \subseteq \mathcal{RL}_A(V)$ is a rational polytope then we have

$$\mathcal{E}(C) = \{\Delta \in C : K_X+\Delta \text{ is pseudoeffective}\}$$
$$\mathcal{N}(C) = \{\Delta \in C : K_X+\Delta \text{ is nef}\}$$

Given a birational contraction $\phi : X \dashrightarrow Y$ we also define

$$\mathcal{W}_\phi(C) = \{\Delta \in \mathcal{E}(C) : \phi \text{ is a weak log canonical (wlc) model of } (X,\Delta)\}$$

and given a rational map $\psi : X \dashrightarrow Z$

$$\mathcal{A}_\phi(C) = \{\Delta \in \mathcal{E}(C) : \phi \text{ is the ample model of } (X,\Delta)\}$$

Remark 5.3.2. *As defined above, $\mathcal{RL}_A(V)$ is non-empty only when (X,A) is log canonical. We might wish to allow (X,A) to be rlc with fixed witnesses instead. This quickly becomes non-trivial because of the overlap of sets in the corresponding open cover.*

If we're interested in a pair $(X,A+B)$ where (X,B) is rlt and A is big and nef then for suitably small $t>0$, and some coefficient space V, we always have that $(X,tA+(1-t)A+B)$ is rlt with coefficients in $\mathcal{RL}_{tA}(V)$ by **??**. *Moreover if we have finitely many such pairs, we can find t,V suitable for all of them. This is normally enough in practice.*

We consider $X \to T$ to be part of the definition of X and omit any mention of T from the notation for rlt polytopes.

Lemma 5.3.3. *Take $A \geq 0$ and let V be a coefficient space. Let $C \subseteq \mathcal{RL}_A(V)$ be a rational polytope. Then there is an open cover U_i such that every $\Delta \in C$ is witnessed over U_i. If C is an rlt polytope then we may choose U_i such that every witness is klt.*

Proof. We can take the vertices D_i of C. Then take witnesses $(X_{i,j}, B_{i,j})$ of D_i. Since there are finitely many D_i, we can assume that for all i we have $X_{i,j} = X_j$ for some X_j not depending on i, after taking intersections of combinations of the $X_{i,j}$ and renumbering as necessary. Now C is the convex hull of the D_i and $\Delta = \sum \lambda_i D_i$ has witnesses $\Delta_j = \sum \lambda_i B_{i,j}$ as required. □

Note that if C is not an rlt polytope and $\Delta \in C$ is an rlt boundary, it might be that the above lemma gives only log canonical witnesses on each U_i.

We will essentially only ever work with rational polytopes containing a klt boundary. Since the questions are always local we can normally assume these polytopes are simplices. By the following lemma, it is then enough to work with rlt polytopes.

Lemma 5.3.4. *Suppose A is ample, V is a coefficient space and that $C \subseteq \mathcal{RL}_A(V)$ is a rational simplex. If there is some boundary $B_0 \in \mathcal{RL}_A(V)$ with (X, B_0) rlt, then there is an affine bijection $f : C \to C'$, where C' is an rlt polytope inside $\mathcal{RL}_{A/2}(W)$ for some coefficient space W. Further f, f^{-1} preserve rationality and $\mathbb{Q}$-linear equivalence.*

Proof. To show a rational polytope $C' \subseteq \mathcal{RL}_{A'}(V')$ is rlt it is enough to show that every vertex boundary B_i of C' is rlt with witnesses in V'.

Indeed if this is the case then for $B \in C'$ we have $B = \sum \lambda_i B_i$ for $\lambda_i \geq 0$ with $\sum \lambda_i = 1$. Let U_j be an open cover such that each B_i is witnessed by $(X_j, B_{i,j})$, then $B|_{X_j} \sim \sum \lambda_i B_{i,j}$, so (X, B) must be rlt as claimed.

Write the vertices of C as $B_i = A + \Delta_i$ for $i > 0$ and let be $B_0 = A + \Delta_0 \in \mathcal{RL}_A(V)$ be the rlt boundary. Now choose $\Gamma_i = (1 - t_i)\Delta_i + t_i\Delta_0$ for t_i rational and sufficiently small that $\frac{A}{2} + t_i(\Delta_i - \Delta_0)$ is ample. By construction $(X, A + \Gamma_i)$ is rlt.

Further choose $H_i \sim_{\mathbb{Q}} \frac{A}{2} + t_i(\Delta_i - \Delta_0)$ effective and sharing no support with A. Then by construction

$$A + \Delta_i \sim_{\mathbb{Q}} \frac{A}{2} + \Gamma_i + H_i = D_i$$

and (X, D_i) is rlt by **??**. Reselecting H_i if needed we may suppose that D_i is not in the span of $\{D_j : i \neq j\}$ for each i. This can always be done since the H_i are all ample.

Let W be a coefficient space containing the components of Δ_i, H_i such that each (X, D_i) is rlt with witnesses in W. Now let C' be the convex hull of the D_i, so that C' is an rlt polytope inside $\mathcal{RL}_A(W)$.

Since C is a simplex, by assumption, we can write any $B \in C$ uniquely as $B = \sum \lambda_i B_i$ where $\lambda_i \geq 0$ and $\sum \lambda_i = 1$. Therefore, we can define a bijective affine map $f : C \to C'$ by sending $B_i = A + \Delta_i \to D_i$ and then writing $f(B) = \sum \lambda_i D_i$.

Clearly B is rational if and only if $\lambda_i \in \mathbb{Q}$, which happens if and only if $f(B) = \sum \lambda_i D_i$ is rational. So f, f^{-1} preserve rationality. Equally as $B_i \sim_{\mathbb{Q}} D_i$ we must have $B \sim_{\mathbb{Q}} f(B)$, and the same holds for f^{-1}.

□

Remark 5.3.5. *With the notation of* **??**, *if $S \subseteq C$ is a rational polytope then $f(S)$ is also a rational polytope since f is affine and preserves rationality. The converse is also true since f^{-1} is also still affine and $f^{-1}f(S) = S$ as f is a bijection.*

Given a general rlc polytope we can always take a rational triangulation and define a piecewise affine bijection, f, by using the above procedure on each simplex. However, this does not in general preserve convexity, so it easier in practice to work locally on the polytope and assume it is a simplex. Alternatively, this could be remedied by working with C', the convex hull of $f(C)$, since this must still be an rlt polytope. Then $f \colon C \to C'$ is no longer a bijection, but it is still preserves rationality and $\mathbb{Q}$-linear equivalence so would suffice for applications.

Definition 5.3.6. *Take $S, S' \subseteq \mathcal{RL}_A(V)$. We say $S \sim_{\mathbb{R}} S'$ if for every $\Delta \in S$ there is $\Delta' \in S'$ with $\Delta \sim_{\mathbb{R}} \Delta'$ and vice versa. The linear closure of S is given by*

$$S^* = \bigcup_{S' \sim S} S' = \{\Delta \in \mathcal{RL}_A(V) \text{ such that } \exists \Delta' \in S \text{ with } \Delta \sim_{\mathbb{R}} \Delta'\}$$

.

Lemma 5.3.7. *Let V be a finite dimensional, rational affine subspace of $WDiv_{\mathbb{R}}(X)$ and fix $A \geq 0$. Take $S \subseteq \mathcal{RL}_A(V)$ a rational polytope. Then the linear closure, S^* is also a rational polytope.*

Proof. By translating by $-A$ we can view S as a subset of V. Similarly, after a translation by say D of V we can suppose that V is a vector space. After these transformations we have that $S^* = \{B + E \text{ such that } B \in S, E \sim 0 \text{ and } B + E - D \geq 0\}$.

Let $N = \{E \in V : E \sim_{\mathbb{R}} 0\}$ and take $\phi : V \to W = V/N \subseteq \operatorname{Pic}(X) \otimes \mathbb{R}$, then $\phi(S) = \phi(S^*)$ is a rational polytope in W and its preimage $S+N$ is still cut out by finitely rational half spaces, but is no longer compact. Hence we must have that $S^* = (S+N) \cap (\Delta \geq D)$ is cut out by finitely many rational half spaces.

However for each point $B \in S$, the set $\{B\}^* = \{B + E \geq D \text{ such that } E \sim_{\mathbb{R}} 0\}$ is bounded, since the $E \in N$ such that $B+E \geq D$ are bounded by the coefficients of B and D. Since S is closed and bounded however we must have that S^* is bounded too. □

In particular $\mathcal{RL}_A(V)$ is a rational polytope over a local ring, since it is the linear closure of $\mathcal{L}_A(V)$. To lift from the local case, we essentially find an open cover of T which witnesses $\mathcal{RL}_A(V)$.

Theorem 5.3.8. *Let V be a finite dimensional, rational affine subspace of $WDiv_{\mathbb{R}}(X)$ and fix $A \geq 0$. Then $\mathcal{RL}_A(V)$ is a rational polytope.*

Proof. For W an affine subspace, let $\hat{W} = \{w - w' \text{ such that } w, w' \in W\}$.

Take a point $p \in T$, and consider $X_p = X \times T_p \to T_p$. Let A_p, V_p be the restrictions of A, V to X_p and let D_i be the vertices of $\mathcal{L}_{A_p}(V_p)$, then there are open sets U_i around p such that $(X \times U_i, D_i)$ are lc when D_i is extended over U_i. Moreover we may freely assume that there are no vertical components of V which meet $U_p = \bigcap U_i$ but are not supported over p, thus ensuring for E in $\hat{V}|_{X_{U_p}}$ where $X_{U_p} = X \times U_p$, we have $E \sim_{\mathbb{R}} 0$ if and only if $E|_{X_p} \sim_{\mathbb{R}} 0$. By compactness of T there are finitely many p_j such that $U_j = U_{p_j}$ is an open cover of T.

A pair (X, Δ) is rlc with witnesses in V if and only if it is witnessed over U_j. Indeed if it is rlc, then we must be able to find B_j such (X_{p_j}, B_j) is lc and $B_j \sim_{\mathbb{R}} \Delta$. By construction however B_j extends to an lc pair $(X_j = X \times U_{p_j}, B_j)$. Then (X, Δ) is witnessed by (X_j, B_j) as required.

Consider $\mathcal{RL}_A(V)$, by the previous paragraph we may take an open cover U_i such that every pair $(X, B) \in \mathcal{RL}_A(V)$ is witnessed by pairs $(X_i = X \times U_i, B_i)$. Let $C_i = \mathcal{L}_{A_i}(V_i)^*$ where A_i, V_i are the restrictions of A, V to X_i and write $S_i = \{\Delta \in V : \Delta|_{X_i} \in C_i\}$, then $\mathcal{RL}_A(V) = \bigcap S_i$ is a rational polytope since each C_i is and there are no divisors $D \neq 0$ with $D|_{X_i} \neq 0$ for every i. □

In particular then $\mathcal{RL}_A(V)$ is closed. Moreover since it is a polytope, if (X, Δ_i) is a sequence of rlc pairs with $\Delta_i \to \Delta$, then the witnesses of Δ may be chosen to be the limit of witnesses of Δ_i.

5.4 Finiteness of Log Terminal Models

In this section we prove our Finiteness of Minimal Models result. Here R will always be an excellent ring with dualising complex, T will be a quasi-projective R scheme and X will always be an integral scheme projective over T. All pairs will be considered as R pairs over T. We assume throughout that X is a scheme of dimension 3, though the claims and proofs hold in any generality in which subsection 2.2.1 hold.

Lemma 5.4.1. *Fix a $\mathbb{Q}$-divisor $A \geq 0$ and let $C \subseteq \mathcal{L}_A(V)$ be a rational polytope. Then $\mathcal{N}(C) = \{\Delta \in C \text{ such that } K_X + \Delta \text{ is nef}\}$ is also a rational polytope.*

Proof. Let B_i be the vertices of C. If $B \in C$ then $B = \sum \lambda_i B_i$ for $1 \geq \lambda_i \geq 0$ so $(K_X + B).C < 0$ ensures $(K_X + B_i).C < 0$ for some i. In particular if $R_{i,j}$ are the $K_X + B_i$ negative extremal rays then $K_X + B$ is nef if and only if $(K_X + B).R_{i,j} \geq 0$ for all i, j. Indeed, suppose that we have such a $K_X + B$ and that R is a $K_X + B$ negative extremal ray, then $(K_X + B_i).R < 0$ for some i and so $R = R_{i,j}$ for some j, a contradiction. Then the condition $(K_X + B).R_{i,j} \geq 0$ defines a rational polytope by [BMP+20, Proposition 9.31]. □

Since this result does not require A to be ample, we may often avoid the use of Bertini type theorems, [BCHM10, Lemma 3.7.3] in particular, to substitute a big divisor for an ample one. Versions of these results are available for rlt polytopes but making use of them requires extra back and forth between the klt and rlt case.

Lemma 5.4.2. *Let $\phi : X \dashrightarrow Y$ be a birational contraction and fix $A \geq 0$. Let $C \subseteq \mathcal{RL}_A(V)$ be an rlt polytope, then $\mathcal{W}_\phi(C)$ is a rational polytope.*

Proof. We can choose a finite open cover, U_i such that C is witnessed by klt pairs over U_i. On X_i we can write $N_i = \{E \sim_\mathbb{R} 0\} \subseteq V_i = V|_{X_i}$, $C_i = C|_{X_i}$ and consider the induced map $\phi_i : X_i \to Y_i$. Now let $C_i' = \mathcal{L}_{A_i}(V_i) \cap C_i^*$. After perhaps shrinking C_i' we may suppose it is a klt polytope and $C_i \subseteq (C')_i^*$. Thus $\mathcal{W}_{\phi_i}(C_i')$ is a rational polytope by [BCHM10, Corollary 3.11.2] with [BCHM10, Theorem 3.11.1] and [BCHM10, Lemma 3.7.4] replaced by **??**.

Therefore $\mathcal{W}_i = \mathcal{W}_{\phi_i}(C_i) = \mathcal{W}_{\phi_i}(C_i')^* \cap C_i$ is also a rational polytope. For each $\mathcal{W}_i$ we have a rational polytope $\hat{\mathcal{W}}_i = \{\Delta \in C : \Delta|_{X_i} \in \mathcal{W}_i\} \subseteq C$. The intersection of these polytopes is precisely $\mathcal{W}_\phi(C)$. □

Lemma 5.4.3. *Let $\phi : X \dashrightarrow Y$ be a birational contraction and fix $A \geq 0$. Let $C \subseteq \mathcal{RL}_A(V)$ be an rlt polytope and $F \subseteq \mathcal{W}_\phi(C)$ be a face, possibly with $F = \mathcal{W}_\phi(C)$. Suppose $f : X \dashrightarrow Z$ is an ample model for some B in the interior of F. Then there is a factorisation $f = g \circ \phi$ for some morphism $g : Y \to Z$, and moreover f is an ample model for every boundary in the interior of F.*

Proof. Since ϕ is a wlc model for B we have an induced map $g : Y \to Z'$ by **??**. However then $g \circ \phi$ is an ample model for (X, B), so after post-composition with an isomorphism we may suppose $Z = Z'$ and $f = g \circ \phi$. Suppose $B' \in \mathcal{W}_\phi(C)$ then f is an ample model for (X, B') if and only g is an ample model for $(Y, \phi_* B')$. Since $K_Y + \phi_* B$' is semiample g is an ample model if and only if the curves contracted by g are precisely those Γ with $(K_Y + \phi_* B').\Gamma = 0$.

Suppose then B' is in the interior of F. Consider $B_t = tB + (1-t)B'$, so that

$$K_Y + \phi_* B_t = t(K_Y + \phi_* B) + (1-t)(K_Y + \phi_* B').$$

Then if $(K_Y + \phi_* B').\Gamma \neq 0$ and $(K_Y + \phi_* B).\Gamma = 0$ it must be that $(K_Y + \phi_* B_t).\Gamma < 0$ for all $t < 0$. However for small t we have $B_t \in F$, a contradiction. By symmetry, we see that Γ is contracted by g if and only if $(K_Y + \phi_* B').\Gamma = 0$, so f is an ample model for $K_X + \phi_* B'$ also. □

Theorem 5.4.4. [BMP+20, Theorem 9.33] *Suppose that X is $\mathbb{Q}$-factorial and let C be a klt polytope in $\mathcal{L}_A(V)$ for $A \geq 0$ big. There is a finite collection of log terminal models $\phi_i : X \dashrightarrow Y_i$ such that every $B \in \mathcal{E}(C)$ has some j with ϕ_j a log terminal model of (X, B).*

Corollary 5.4.5. *Suppose that X is $\mathbb{Q}$-factorial and let C be a klt polytope with A big. Suppose that every $B \in C$ has components which span $NS(X)$, then there are finitely many birational maps $\phi_i : X \dashrightarrow Y_i$ such that for any $B \in \mathcal{E}(C)$ if $\phi : X \dashrightarrow Y$ is a wlc model then $\phi_i = f \circ \phi$ for some i and some isomorphism $f : Y \to Y_i$.*

Proof. After possibly expanding V, we can take $C' \subseteq \mathcal{L}_{\frac{A}{2}}(V)$ a klt polytope with $C \subseteq C'$ such that for any $B \in C$ if D is a component of B then $B + tD$ is in C' for any $|t| < \epsilon$, for some $\epsilon > 0$ depending only on B. This can be done by taking C' to be the convex hull of small perturbations of the vertices of C.

By the previous theorem there are finitely many birational maps $\phi_i : X \dashrightarrow Y_i$ such that for every $B \in \mathcal{E}(C')$ there is some ϕ_i a log terminal model of (X, Δ).

Further are then finitely many morphisms $f_{i,j} : Y_i \to Z_j$ such that $\psi_{i,j} = f_{i,j} \circ \phi_i$ are ample models such that $B \in \mathcal{E}(C')$ some $\psi_{i,j}$ is the (unique) ample model of (X, B). This is because the $f_{i,j}$ correspond to faces of the rational polytope $\mathcal{W}_{\phi_i}(C')$ by **??**.

Now pick $\Delta \in C$. Let $\psi : X \dashrightarrow Y$ be a wlc model for Δ. We can take D in the span of the components of B such that ϕ is $B + D$ negative and $\phi_* D$ is ample. By shrinking D, we can suppose that $B + D \in C'$. Thus we have that ψ is the ample model of some $B + D \in \mathcal{W}_\psi(C')$. Now take a log terminal model of $B + D$ of the form ϕ_i for some i,. By uniqueness of the ample model, up to post-composition with an isomorphism, we have $\psi = f_{i,j} \circ \phi_i = \psi_{i,j}$ for some j. Thus the family of models $\{\psi_{i,j}\}$ give the required maps.

□

Theorem 5.4.6. *Let A be a big $\mathbb{Q}$-divisor and chose V a coefficient space. Take C be an rlt polytope inside $\mathcal{RL}_A(V)$, then*

1. *There are finitely many birational maps $\phi_j : X \dashrightarrow Y_j$ such that for any $B \in \mathcal{E}(C)$ if $\phi : X \dashrightarrow Y$ is a wlc model then $\phi_j = f \circ \phi$ for some j and some isomorphism $f : Y \to Y_j$.*

2. *There are finitely many rational maps $\psi_k : X \dashrightarrow Z_k$ such that if $\psi : X \dashrightarrow Z$ is an ample model for some $B \in \mathcal{E}(C)$ then there is an isomorphism $f : Z \to Z_k$ with $\psi_k = f \circ \psi_k$.*

Proof. We prove 1., 2. follows immediately as ample models correspond to the interiors of faces of the $\mathcal{W}_{\phi_i}(C)$ by **??**.

Equally, it is enough to show this in the case that C is a klt polytope. Indeed suppose it holds for klt polytopes. Then take an open cover U_i of T witnessing C. For each i we may take a klt polytope C'_i with $C'_i \sim C_i = C|_{U_i}$. Given a wlc map $\phi : X \dashrightarrow Z$ for $B \in \mathcal{E}(C)$, we can let ϕ_i be the induced map on X_i which is a wlc model for some $B_i \in C'_i$. In particular for fixed i there are finitely many $\phi_{i,j}$ such that for any B and ϕ

we have $f_i \circ \phi_i = \phi_{i,j}$ for some j and f_i. As U_i is a finite cover there are finitely many $\phi_{i,j}$ indexed over i, j.

If we have another map $\Phi : X \dashleftarrow Z'$ with isomorphisms g_i such that $f_i \circ \phi_i = \phi_{i,j} = g_i \circ \Phi_i$, then $h_i = g_i \circ f_i^{-1}$ glues to an isomorphism $Z' \to Z$ over T. Thus there are only finitely many wlc models up to isomorphism.

Suppose then that C is a klt polytope.

Let $\pi : Y \to X$ be a log resolution of the support of V. Then for any Δ in C we have $\pi^*(K_X + \Delta) + E = (K_Y + \Delta')$ where $E \geq$ is exceptional and shares no components with Δ' and (Y, Δ') is klt. Sending $\Delta \to \Delta'$ as above we can find a new polytope C' on which it is sufficient to check the result holds. By replacing C with C', A with $\pi^* A$, X with Y and V with a suitable space, we may suppose that X is regular, though it may no longer be the case that A shares no support with V.

Let H_k be ample divisors spanning $NS(X)$ and sharing no components with A or V. Let $H = \sum H_k$. Note that for any open U in T we still have the components of $H|_{X_U}$ span $NS(X_U)$, since $NS(X)$ surjects on $NS(X_U)$ by $\mathbb{Q}$-factoriality of X.

After shrinking H we may take some $A', E \geq 0$ and a small $t > 0$ such that:

- $E \geq 0$ shares no components with H;
- $A - E$ is ample;
- $t(A - E) - H \simeq A' > 0$ is ample and shares no components with V, H or E;
- $\{A + H + B + tE \colon A + B \in C\}$ is a klt polytope; and
- $C' = \{A' + (1-t)A + H + B + tE \colon A + B \in C\}$ is an rlt polytope.

That we can choose C' to be rlt follows from **??**. Note that $A' + (1-t)A + H + B + tE \simeq A + B$ by construction. Thus it suffices to check the result for C' since $C' \subseteq \mathcal{L}_H(W)$ for some coefficient space W. As above, by taking an open cover, we may in fact assume that C' is klt. But then the result follows by **??**, since the components of H span $NS(X)$ by construction.

□

Theorem 5.4.7. *Let A be an ample $\mathbb{Q}$-Cartier divisor and C be a rational polytope inside $\mathcal{RL}_A(V)$. Suppose there is a boundary $A + B \in \mathcal{RL}_A(V)$ such that $(X, A + B)$ is rlt with witnesses in V_A. Then the following hold:*

1. *There are finitely many birational contractions $\phi_i : X \dashrightarrow Y_i$ such that*
$$\mathcal{E}(C) = \bigcup \mathcal{W}_i = \mathcal{W}_{\phi_i}(C)$$
where each $\mathcal{W}_i$ is a rational polytope. Moreover if $\phi : X \to Y$ is a wlc model for any choice of $\Delta \in \mathcal{E}(C)$ then $\phi = \phi_i$ for some i, up to composition with an isomorphism.

2. *There are finitely many rational maps $\psi_j : X \dashrightarrow Z_j$ which partition $\mathcal{E}(C)$ into subsets $\mathcal{A}_{\psi_j}(C) = \mathcal{A}_i$.*

3. *For each W_i there is a j such that we can find a morphism $f_{i,j} : Y_i \to Z_j$ and $W_i \subseteq \overline{A_j}$.*

4. *$\mathcal{E}(C)$ is a rational polytope and A_j is a union of the interiors of finitely many rational polytopes.*

If C is an rlt polytope then A big suffices.

Proof. Since the convexity condition of every sub-polytope in the theorem statement is clear, it is enough to show that the result holds for every simplex in a rational triangulation of C. Thus after extending V and changing A as needed we may suppose:

- C is a simplex;
- C is an rlt polytope by **??**;
- $\mathcal{E}(C)$ is covered by $\mathcal{W}_{\phi_i}(C)$ and has a decomposition into disjoint sets $\mathcal{A}_{\psi_j}(C)$ for some collection of birational contractions ϕ_i and rational maps ψ_j by **??**; and
- There are only finitely many ϕ_i and ψ_j by **??**.

Take one of the wlc models $\phi_i : X \dashrightarrow Y_i$, then just as in Lemma **??**, if Δ, Δ' are in the same face of $\mathcal{W}_i$ then they have the same ample model. In particular then let $\psi_j : X \dashrightarrow Z_j$ be the ample model corresponding to the interior of $\mathcal{W}_i$, then we have a morphism $f_{i,j} : Y_i \to Z_j$ and $W_i \subseteq \overline{A_j}$ as required.

Similarly by **??** we have that $A_j \cap \mathcal{W}_i$ is a union of the interiors of some faces of $\mathcal{W}_i$. Since there are finitely many $\mathcal{W}_i$ and they cover $\mathcal{E}(C)$ the result follows. □

Remark 5.4.8. *In practice since we can always extend V and C it is enough to know that (X, A) is klt, rather than needing an rlt pair $(X, A+B)$. Similarly if X is klt, we can always find $t > 0$ such that (X, tA) is klt. Then if $(X, A+B) = (X, tA+(1-t)A+B)$ is rlc with coefficients in V_A it is also rlc with witnesses in V'_{tA} for some coefficient space V'. By choosing V' such that all the vertices of C are rlc with witness in V'_{tA}, we see that it is enough to suppose that X is klt.*

5.5 Geography of Ample Models

We keep the notation of the previous section, though we denote the closure of $\mathcal{A}_\phi(C)$ by $\mathcal{D}_\phi(C)$. As before R will always be an excellent ring with dualising complex, T will be a quasi-projective R scheme and all other schemes will be integral and admitting a projective, surjective morphism to T over R. All pairs will be considered as R pairs over T. Like in the previous sections, we will work with A ample throughout. As in the previous section, we work with schemes of dimension at most 3, however this is a limitation only of currently known MMP results.

We will say the span of a polytope C is

$$\mathrm{Span}(C) = \{\lambda(B - B') \text{ such that } B, B' \in C \text{ and } \lambda \in \mathbb{R}\}.$$

In a slight abuse of notation we say that $C \subseteq WDiv(X)$ spans $NS(X)$ if the span of C surjects onto $NS(X)$. Equivalently this means if D is a divisor and B is in the interior of C then for all sufficiently small $t > 0$ $B + tD \equiv D'_t$ for some $D'_t \in C$.

Lemma 5.5.1. *Let $X \to T$ be a $\mathbb{Q}$-factorial, klt threefold over R. Let $\phi : X \dashrightarrow Y$ be a wlc model of an rlc pair $(X, \Delta)/T$. Let $A \geq 0$ be an ample $\mathbb{Q}$-divisor and C be a polytope inside $\mathcal{L}_A(V)$. Then we have that $\mathcal{D}_\phi(C) := \overline{\mathcal{A}_\phi(C)} \subseteq \mathcal{W}_\phi(C)$ is a rational polytope, moreover if C spans $NS(X)$ and contains an open set around Δ then this inclusion is an equality.*

Proof. Suppose that $B \in \mathcal{A}_\phi(C)$. Then by [BCHM10, Theorem 3.6.5] we see that in fact ϕ is a wlc model for B and thus we have $\mathcal{A}_\phi(C) \subseteq \mathcal{W}_\phi(C)$. So $\overline{\mathcal{A}_\phi(C)}$ is a union of faces of $\mathcal{W}_\phi(C)$ by **??** and **??**. However $\mathcal{A}_\phi$ is convex inside $\mathcal{W}_\phi(C)$ so it must be that $\overline{\mathcal{A}_\phi(C)}$ is a face of $\mathcal{W}_\phi(C)$, and thus is a polytope.

Now suppose C spans $NS(X)$ and contains an open set around Δ. Let H be a general ample divisor on Y. Let W be a common resolution with maps $p : W \to X$, $q : W \to Y$. Then by assumption there is some $H' \equiv p_* q^* H$ with support contained in the support of Δ, and hence in the support of any B in the interior of $\mathcal{W}_\phi(C)$. Take such a B, then there is $\epsilon > 0$ with $(X, B + \epsilon H') \in C$, for any $\epsilon' \in ((0, \epsilon])$ ϕ is an ample model of $(X, B + \epsilon' H')$, such an ϵ exists since ϕ is necessarily H' non-negative. Thus $B + \epsilon' H' \in \mathcal{A}_\phi(C)$. But then we must have $\mathcal{W}_\phi(C) \subseteq \overline{\mathcal{A}_\phi(C)}$. □

Theorem 5.5.2. [HM09, Theorem 3.3] *Let C be a polytope inside $\mathcal{RL}_A(V)$, then there are finitely many maps $f_i : X \dashrightarrow Y_i$ over T with the following. properties.*

1. *$\{A_i = A_{f_i}(C)\}$ partition $\mathcal{E}(C)$. If f_i is birational then $\mathcal{D}_i = \mathcal{D}_{f_i}(C)$ is a rational polytope.*
2. *If $A_j \cap \mathcal{D}_i \neq \emptyset$ then there is a morphism $f_{i,j} : Y_i \to Y_j$ such that $f_i = f_{i,j} \circ f_j$.*

Moreover if C spans $NS(X)$ then we also have the following.

3. *Pick i such that a connected component, $\mathcal{D}$ of $\mathcal{D}_i$ meets the interior of C. Then the following are equivalent:*
   - $\dim \mathcal{D} = \dim C$.
   - $\mathrm{Span}(\mathcal{D}) = \mathrm{Span}(C)$.

- *If $B \in \mathcal{A}_i \cap \mathcal{D}$ then f_i is a log terminal model of (X, B).*
- *f_i is birational and X_i is $\mathbb{Q}$-factorial.*

4. *Suppose that $\mathcal{D}_i$ has the same span as C and B is a general point in $\mathcal{A}_j \cap \mathcal{D}_i$. If in fact B is in the interior of C then the relative picard number of Y_i/Y_j is the difference in dimension of $\mathcal{D}_i$ and $\mathcal{D}_j \cap \mathcal{D}_i$.*

This result is stated for $C = \mathcal{L}_A(V)$ in characteristic zero, but the proof goes through essentially verbatim in this setting.

For brevity we fix some notation, essentially due to Shokurov.

Definition 5.5.3. *Take a coefficient space V, an ample divisor A and then let C be a polytope inside some $\mathcal{RL}_A(V)$.*

Suppose that 3 and 4 of the previous lemma hold for C, then the triple (C, A, V) is a said to be a geography, when A and V are clear we sometimes just call C a geography. The dimension of (C, A, V) will be the dimension of C. The $\mathcal{D}_\phi$ are called classes. If C is a geography and $\dim \mathcal{D}_\phi = \dim C$ then $\mathcal{D}_\phi$ is said to be a country. The codimension 1 faces of countries are called borders, and a codimension 2 face is called a ridge. If (X, B) is a pair such that every country in C is induced by a log terminal model of (X, B) then (C, A, V) is a geography for (X, B).

?? then says that if (C, A, V) is a triple such that C spans $NS(X)$ then C is a geography. This combined with following will be the main method of producing geographies for the remainder of the section.

Lemma 5.5.4. *Let (C, A, V) be a geography. Take $W \subseteq V$ be a general coefficient space and let $W_A = \{A + B, B \in W\}$ then $C' = C \cap W_A$ is a geography.*

Proof. Index all of the faces of every polytope in the decomposition by $\mathcal{D}_i$ as F_j. Then for C' to be a geography it is enough to know that intersecting with W preserves the codimension of the F_j meeting W. For fixed j, however the choices of W such that either W does not meet F_j or $F'_j = F_j \cap W_A \subseteq C'$ has the same codimension as $F \subseteq C_A$ form an open set in the Grassmanian. Since there are finitely many faces the result holds for suitably general choice of W. □

Lemma 5.5.5. *Suppose V is a coefficient space which spans $NS(X)$. Let C be any polytope contained $\mathcal{RL}_A(V)$, then after perturbing the vertices by an arbitrarily small amount (C, A, V) is a geography.*

Proof. Since we can perturb the vertices of C we may suppose it is rational and contained in the interior of $\mathcal{RL}_A(V)$. Let W be the minimal coefficient space in V with $C \subseteq W_A \cap \mathcal{RL}_A(V)$. Since C is contained in the interior of $\mathcal{RL}_A(V)$, we can pick an rlt polytope C' which spans $NS(X)$ with $W_A \cap C' = C$. Then after a small perturbation of the vertices we may suppose that $W_A \cap C'$ is a geography, as required. □

Lemma 5.5.6. [HM09, Lemma 3.6] *Let $(X, \Delta)/T$ be an rlt threefold pair and $f\colon X \dashrightarrow Y$ a birational contraction of $\mathbb{Q}$-factorial projective T-schemes. Suppose that $B-\Delta$ is ample and f is an ample model for $K_X + B$. Then f is a log terminal model for (X, Δ).*

Lemma 5.5.7. *Suppose that $f_i : (X, \Delta) \to (Y_i, \Delta_i)$ for $i = 1, ..n$ are a finite collection of $\mathbb{Q}$-factorial Mori Fibre spaces obtained by running an MMP for a rlt threefold pair (X, Δ) with X regular. Then there is a geography (C, A, V) for (X, Δ) of dimension at most n such that every $\mathcal{D}_{f_i}$ is a country.*

Moreover if $g_i : Y_i \to Z_i$ are the Mori Fibrations and we write $h_i = g_i \circ f_i$. Then we may choose C such that $\mathcal{D}_{h_i}$ are borders of the $\mathcal{D}_{f_i}$ and their interiors are connected by a path through the border of $\mathcal{E}(C)$ contained entirely in the interior of C.

Proof. Pick A'_i ample on Z_i such that $g_i^* A'_i - (K_{Y_i} + \Delta_i)$ is ample.

We may choose H ample on X whose components span $NS(X)$ together with A ample both sufficiently small such that:

- $(X, H + A)$ is klt,
- the $A_i = g_i^* A'_i - (K_{Y_i} + \Delta_i + f_{i,*}(A + H))$ are ample,
- $(X, \Delta + A + H)$ is an rlt pair which is not pseudo-effective, and
- each f_i is $(K_X + \Delta + A + H)$ negative.

Further, we may pick A such that it avoids the exceptional loci of the f_i and shares no components with H.

By **??** we can take $B_i \sim f_i^* A_i$ such that each $(X, \Delta + H + A + B_i)$ is rlt. Moreover we can choose the B_i such that they share no components with A since the augmented base locus of B_i is precisely the exceptional locus of f_i. Thus the $(X, \Delta + B_i)$ all have witnesses in some W for which $(X, \Delta + H + A + B_i)$ have witnesses in W_{A+H}.

By construction, then, after adding the components of H to W we have $(X, \Delta + B_i + H + A) \in \mathcal{RL}_A(W)$, a geography. Further the f_i are wlc models of the $(X, \Delta + B_i + H + A)$ and the h_i are the ample models.

Let C be the convex hull of the $\Delta + B_i + H + A$ and $\Delta + H$. Since the components of H span NS(X), and the f_i are wlc models for boundaries in C, we can find boundaries in $\mathcal{RL}_A(W)$ for which the f_i is an ample model. Moreover we can find them arbitrarily close to C. Thus we can freely move the vertices of C an arbitrarily small amount such that it meets the interior of each of the $\mathcal{D}_{f_i}$ and their borders $\mathcal{D}_{h_i}$ while ensuring they are sufficiently general that C is a geography.

By construction, $C_{-\Delta}$ is contained in the ample cone and $\dim C \leq n$. In particular C is a geography for (X, Δ) by **??**. It remains to check that $\mathcal{D}_{h_i}$ are borders of the $\mathcal{D}_{f_i}$ and their interiors are connected by a path through the border of $\mathcal{E}(C)$ contained entirely in the interior of C.

Since C contains a vertex $D = \Delta + H \notin \mathcal{E}(C)$ such that C_{-D} is contained in the effective cone, it is enough to check that for each i the interior of $\mathcal{D}_{h_i}$ meets the interior of C, but this again is ensured by the construction. Thus we may take E_i, E_j in the interiors of $\mathcal{D}_{h_i}, \mathcal{D}_{h_j}$ respectively and both contained in the interior of C. Then the simplex formed by D, E_i, E_j meets the boundary of $\mathcal{E}(C)$ along a path connecting E_i and E_j, wholly contained in the interior of C.

□

Lemma 5.5.8. [HM09, Lemma 3.5] *Let (C, A, V) be a geography on X of dimension 2. Take two ample classes $\mathcal{D}_f$ and $\mathcal{D}_g$ corresponding to some maps $f : X \dashrightarrow Y$ and $g : X \dashrightarrow Z$. Suppose that $\mathcal{D}_f$ is a country and that they meet along a border $\mathcal{B}$ not contained in the boundary of C. Suppose further that $\rho(Y) \geq \rho(Z)$*

Let $h : Y \dashrightarrow Z$ be the map induced by $\mathcal{B}$. Take B an interior point of $\mathcal{B}$ and let $\Delta = f_*B$, then one of the following holds.

1. *$\rho(Y) = \rho(Z) + 1$ and h is a $K_Y + \Delta$ trivial morphism. Thus either*
   - *a) h is a divisorial contraction and $\mathcal{B} \neq \mathcal{D}_g$*
   - *b) h is a small contraction and $\mathcal{B} = \mathcal{D}_g$*
   - *c) h is a MFS and $\mathcal{B} = \mathcal{D}_g$ is contained in the boundary of $\mathcal{E}(C)$.*
2. *$\rho(W) = \rho(Y)$ and h is a $K_Y + \Delta$ flop and $\mathcal{B} \neq \mathcal{D}_g$ is not contained in the boundary of $\mathcal{E}(C)$.*

5.6 Sarkisov Program

Fix a positive dimensional quasi-projective R scheme, T. Suppose that $f : X \to Z$, $g : Y \to W$ are two Mori Fibre Spaces, projective and surjective over T. We say that they are Sarkisov related if they are both outputs of an MMP from the same $\mathbb{Q}$-factorial rlt pair. In particular we require X, Y to be $\mathbb{Q}$-factorial.

A Sarkisov link $s : X \dashrightarrow Y$ is one the following.

$$
\begin{array}{cccc}
I & II & III & IV \\
\begin{array}{ccc} X' & \dashrightarrow & Y \\ \downarrow & & \downarrow \\ X & & W \\ \downarrow & \swarrow & \\ Z & & \end{array}
&
\begin{array}{ccc} X' & \dashrightarrow & Y' \\ \downarrow & & \downarrow \\ X & & Y \\ \downarrow & & \downarrow \\ Z & = & W \end{array}
&
\begin{array}{ccc} X & \dashrightarrow & Y' \\ \downarrow & & \downarrow \\ Z & & Y \\ & \searrow & \downarrow \\ & & W \end{array}
&
\begin{array}{ccccc} X & & \dashrightarrow & & Y \\ \downarrow & & & & \downarrow \\ Z & & & & W \\ & \overset{p}{\searrow} & & \overset{q}{\swarrow} & \\ & & T & & \end{array}
\end{array}
$$

Such that the following holds:

- There is an rlt pair $(X, \Delta)/T$ or $(X', \Delta')/T$ as appropriate such that the horizontal map is a sequence of flops for this pair
- Every vertical morphism is a contraction
- If the target of a vertical morphism is X or Y then it is an extremal divisorial contraction
- Either p, q are both Mori Fibre Spaces (this is type IV_m) or they are both small contractions (type IV_s)

We realise these Sarkisov links inside two dimensional geographies as follows.

Fix $X \to T$ a threefold over R and a geography (C, A, V) on X of dimension 2.

Let Δ be a point in the boundary of $\mathcal{E}(C)$ but in the interior of C. Let $\mathcal{T}_1 = \mathcal{D}_{f_1}, ..., \mathcal{T}_k = \mathcal{D}_{f_k}$ be the countries which meet Δ. Let $\mathcal{B}_i$ be the borders $\mathcal{T}_i$ meeting Δ such that after reordering we have $\mathcal{B}_i = \mathcal{T}_i \cap \mathcal{T}_{i+1}$ for $1 \leq i \leq k-1$. Then $\mathcal{B}_0, \mathcal{B}_k$ are contained in the boundary of $\mathcal{E}(C)$. Let $g_i : X \to Z_i$ be the ample models associated to the interiors of $\mathcal{B}_i$

Relabel $\phi = f_0 : X \dashrightarrow Y$, $Z = Z_0$, $\psi = f_k \dashrightarrow W$ and $T = Z_k$. Then we have p, q with $p \circ \phi = g_0$ and $q \circ \psi = g_k$.

Theorem 5.6.1. [HM09, Theorem 3.7] *With notation as above, suppose B is any divisor on X with $\Delta - B$ ample. Then $q : Y \to Z$ and $q : W \to T$ are two threefold Mori Fibre spaces obtained by running (X, B) MMPs and they are connected by Sarkisov links.*

Theorem 5.6.2. *Fix an integral quasi-projective scheme T over R. Let $g_1 : Y_1 \to Z_1$ and $g_2 : Y_2 \to Z_2$ be two Sarkisov related, klt Mori fibre spaces of dimension 3, projective T. If the Y_i have positive dimension image in T, then they are connected by Sarkisov links.*

Proof. By assumption these Mori fibre spaces are outputs of an MMP for some pair klt $(X, \Delta)/T$. Replacing X with a suitable resolution, we may suppose that X is smooth and admits morphisms $f_i : X \to Y_i$. Let $h_i = g_i \circ f_i$ then by Lemma **??** there is a geography for (X, Δ) of dimension 2 such that the $\mathcal{D}_{f_i}(C)$ are countries and the interiors of the $\mathcal{D}_{h_i}$ are connected by a path along the boundary of $\mathcal{E}(C)$.

Each ridge in this path corresponds to a Sarkisov link by **??**. Thus following the path gives a (non-unique) decomposition of $f_2 \circ f_1^{-1} : Y_1 \dashrightarrow Y_2$ into Sarkisov links. Since $\mathcal{E}(C)$ is a rational polytope, there are finitely many links.

□

Chapter 6

The Augmented Base Locus in Mixed Characteristic

This chapter studies the stable and augmented base loci of nef divisors in mixed characteristic. Generally under the further assumption that the divisor is semiample in characteristic 0. This work is published in [Sti21a].

We give a characterisation of the augmented base locus in this setting.

Theorem 6.0.1 (**??**)**.** *Let X be a projective scheme over an excellent Noetherian base S with L a nef line bundle on X. Suppose that one of the following holds:*

1. *$S_{\mathbb{Q}}$ has dimension 0;*
2. *$L|_{X_{\mathbb{Q}}}$ is semiample;*

Then $\mathbf{B}_+(L) = \mathbb{E}(L)$.

We also extend the semiampleness result of [Wit20] to show that there is an equality of stable base loci when the characteristic 0 part is semiample.

Theorem 6.0.2 (**??**)**.** *Suppose that X is a projective scheme over an excellent Noetherian base with L a nef line bundle on X. Then $\mathbf{SB}(L) = \mathbf{SB}(L|_{\mathbb{E}(L)})$ so long as $L|_{X_{\mathbb{Q}}}$ is semiample.*

6.1 Preliminaries

We will work exclusively with line bundles. Since the schemes we work with need not be normal, line bundles are not the same as Cartier divisors, however we typically use the traditional notation for divisors as we still sometimes treat line bundles as Cartier divisors when appropriate. That is we write the tensor product of L, L' as $L+L'$, $L^{\otimes k}$ is often written kL and given $f: Y \to X$, then $f^*L = L|_Y$ is often written $\mathcal{O}_Y(L)$, including for $Y = X, f = id$.

Since the questions considered are local on the base, it suffices to work only with affine bases. In particular, for notational simplicity, $H^i(X, L)$ will often be used to denote the higher derived pushforwards of L by $X \to S$.

Definition 6.1.1. *Let L be a line bundle on a projective Noetherian scheme X over some Noetherian scheme S. Then base locus is given as*

$$\mathbf{B}(L) = \bigcap_{s \in H^0(X,L)} Z(s)_{red}$$

where $Z(s)$ is the zero set of s equipped with the obvious scheme structure. The stable base locus is then

$$\mathbf{SB}(L) = \bigcap_{m \geq 0} \mathbf{B}(mL).$$

Fix an ample line bundle A. The augmented base locus is given as

$$\mathbf{B}_+(L) = \bigcap_{m \geq 0} \mathbf{SB}(mL - A)$$

and is independent of the choice of A.

We could also write

$$\mathbf{B}_+(L) = \bigcap_{A \text{ ample},\ m \geq 0} \mathbf{SB}(mL - A)$$

for a definition that involves no choice of ample line bundle. By Noetherianity if we choose m sufficiently large and divisible then in fact $\mathbf{B}_+(L) = \mathbf{SB}(mL - A)$.

Definition 6.1.2. *Let L be a line bundle on a projective scheme X. The exceptional locus, $\mathbb{E}(L)$, is the union of integral subschemes on which L is not big.*

The previous two definitions are invariant under scaling by $n \in \mathbb{N}_{\geq 0}$ and line bundles will frequently be replaced with higher multiples.

Theorem 6.1.3. [Wit20]*[Theorem 1.10] Suppose that X is a projective scheme over an excellent Noetherian base S and L is a nef line bundle on X. Then if $L|_{X_{red}}$ and $L|_{X_{\mathbb{Q}}}$ are semiample so too is L.*

Theorem 6.1.4. [Kee03]*[Theorem 1.5] Let X be a projective scheme over a Noetherian ring, $\mathcal{A}$ an ample line bundle and $\mathcal{F}$ a coherent sheaf. Then there is some m_0 with*

$$H^i(X, \mathcal{F} \otimes \mathcal{A}^m \otimes \mathcal{N}) = 0$$

for all $i > 0, m \geq m_0$ and all nef line bundles $\mathcal{N}$.

Lemma 6.1.5. [CMM14]*[Lemma 2.2] Let X be an n-dimensional projective scheme over a field k and L a line bundle on X. For every coherent sheaf $\mathcal{F}$ on X, there is $C > 0$ such that $h^0(X, \mathcal{F} \otimes L^m) \leq Cm^n$ for every $m \geq 1$.*

Lemma 6.1.6. *Let X be a reduced projective scheme over a ring R and L, A line bundles on X with A ample. Then for large m and general $s \in H^0(X, mL-A)$ and any irreducible component Y of X with $L|_Y$ big we have $Y \not\subseteq Z(s)$.*

Proof. Let $f : X \to S$ be the structure morphism. Suppose for contradiction that $f_*\mathcal{O}_X(mL - A) \to f_*\mathcal{O}_Y(mL - A)$ is the zero map for infinitely many m.

Let W be the union of the other components of X so that we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_W \to \mathcal{O}_{Y \cap W} \to 0$$

where Y, W are given the reduced subscheme structure. For convenience we write $Z = Y \cap W$

Tensoring and pushing forwards we get

$$0 \to f_*\mathcal{O}_X(mL - A) \to f_*\mathcal{O}_Y(mL - A) \oplus f_*\mathcal{O}_W(mL - A) \to f_*\mathcal{O}_Z(mL - A)$$

In particular if $f_*\mathcal{O}_X(mL - A) \to f_*\mathcal{O}_Y(mL - A)$ is the zero map, we must have an injection $f_*\mathcal{O}_Y(mL - A) \hookrightarrow f_*\mathcal{O}_Z(mL - A)$. Let $V = f(Y)$ and $g = f|_Y : Y \to V$. Then we may view $\mathcal{O}_Y(mL - A), \mathcal{O}_Z(mL - A)$ as sheaves on Y, then there is a corresponding injection $g_*\mathcal{O}_Y(mL - A) \hookrightarrow g_*\mathcal{O}_Z(mL - A)$ since the pushforward is left exact. Since Y is irreducible so too is V and hence we may pull back to the generic point ν of V.

Now we have that Y_ν is a projective scheme over $K(V)$ of dimension say n. Equally Z_ν is a closed subscheme of Y_ν of dimension at most $n - 1$. We now find a contradiction by counting sections over $K(V)$.

On the one hand we have an injection

$$H^0(Y_\nu, \mathcal{O}_{Y_\nu}(mL - A)) \hookrightarrow H^0(Z_\nu, \mathcal{O}_{Z_\nu}(mL - A)),$$

which ensures that there is $C > 0$ such that $h^0(Y_\nu, \mathcal{O}_{Y_\nu}(mL - A)) \leq Cm^{n-1}$ for every $m \geq 1$ by **??**. On the other, $kL|_{Y_\nu}$ is big, and Y_ν is integral, thus $h^0(Y_\nu, \mathcal{O}_{Y_\nu}(mL - A))$ grows like m^n by [Bir17, Lemma 4.2]. This is a contradiction and the result follows. □

Remark 6.1.7. *When X is a reduced scheme and $X = X_1 \cup X_2$ (as topological spaces) for closed subschemes X_1, X_2 we have a short exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \to \mathcal{O}_{X_1 \cap X_2} \to 0$$

as used above. In particular if L is a line bundle on X with sections s_1, s_2 on X_1, X_2 respectively which agree on $X_1 \cap X_2$ then they glue to a section of L on X.

This is not the case when X is reducible. If X_j are given by ideal schemes I_j then it need not be the case that $I_1 \cap I_2 = 0$. However replacing I_1 with a higher power we may suppose that this is the case, see for instance [Sta, Tag 01YC]. *In particular we may always choose subscheme structures such that the short exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \to \mathcal{O}_{X_1 \cap X_2} \to 0$$

still holds. When we work with components of a reducible scheme we can always chose the subscheme structure in this fashion, and in particular we will always be able to glue appropriate sections.

Lemma 6.1.8. [EH06]*[Proposition IV-21] Let X be a scheme and $Z \subseteq X$ a subscheme with $Y \to X$ the blowup of X along Z. If $f : X' \to X$ is any morphism and we write $Z' = f^{-1}Z$, then the closure W of $\pi_{X'}^{-1}(X' \setminus Z')$ inside $X' \times_X Y$ is exactly the blowup of X' along Z'.*

Lemma 6.1.9. [Sta, Tag 0808] *Let X be a scheme. Let $I \subseteq \mathcal{O}_X$ be a quasi-coherent sheaf of ideals. If X is reduced, then the blowup X' of X along I is reduced.*

Together these tell us that 'the blowup of the reduction is the reduction of the blowup'. More precisely we have the following.

Lemma 6.1.10. *Let X be a scheme and Z a proper closed subscheme of X_{red}. Let $\pi : X' \to X$ be the blowup of X along Z, viewed as a subscheme of X. Let Y be the blowup of X_{red} along Z, then we have isomorphisms*

$$Y \simeq X' \times_X X_{red} \simeq X'_{red}$$

Proof. First we observe that $X' \times_X X_{red} \simeq X'_{red}$. Indeed if $f : Z \to X'$ is a morphism from a reduced scheme, then we have a composition $g = \pi \circ f : Z \to X$. And thus a unique induced morphism $Z \to X_{red}$. By definition this induces a unique morphism $Z \to X' \times_X X_{red}$ and hence $X' \times_X X_{red}$ satisfies the universal property of the reduced subscheme, ensuring that $X' \times_X X_{red} \simeq X'_{red}$.

Now by **??** we have that Y is the closure of $(X_{red} \setminus Z)$ inside $X' \times_X X_{red}$. However $X_{red} \setminus Z$ is a dense subscheme and so Y is precisely the reduced subscheme of $X' \times_X X_{red}$, but then in fact they are equal as $X' \times_X X_{red}$ is already reduced.

□

Lemma 6.1.11 (Elimination of Indeterminacy by blowups)**.** *Let $f : X \dashrightarrow Y$ be a rational map of S schemes associated to an S-linear system $|V| \subseteq H^0(X, L)$ without fixed part, then there is Z with maps $\phi_1 : Z \to X$, $\phi_2 : Z \to Y$ such that $\phi_1^* L = M + F$ for M a line bundle globally generated by $\phi_1^*|V|$. Here $F \geq 0$ is such that $\mathcal{O}_Y(-F)$ is a line bundle, $\phi_1(F) = \mathbf{B}|V|$ as reduced schemes and $\phi_2 = f \circ \phi_1$. Further we may construct $Z \to X$ as a blowup of X.*

Proof. Consider the following morphism of line bundles $V \otimes L^{-1} \to \mathcal{O}_X$ and let $\mathcal{I}$ be the image. Then $\mathcal{I} \otimes L$ is the image of $V \otimes \mathcal{O}_X \to L$, in particular the support of $\mathcal{I}$ is exactly $\mathbf{B}|V|$.

Let $\pi : Z \to X$ be the blowup of X along $\mathcal{I}$. We then have $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_Z = \mathcal{O}_Z(-F)$ for some F an effective Cartier divisor. Hence we have

$$\pi^*(V \otimes L) \twoheadrightarrow \mathcal{O}_Z(-F) \hookrightarrow \mathcal{O}_Z$$

where the first map is surjective by right exactness of the pullback functor. Tensoring by $\pi^* L$ then gives the following.

$$\pi^*(V \otimes \mathcal{O}_z) \twoheadrightarrow \pi^* L(-F) \hookrightarrow \pi^* L$$

In particular the line bundle in the middle, which we may write M is globally generated by sections indexed by $\pi^*|V|$ and we have $M = \pi^* L(-F)$ by construction. Clearly $\pi(F)$ is the support of $\mathcal{I}$, which is nothing but $\mathbf{B}|V|$. Since M is globally generated it defines a morphism $\phi_2 := \phi_{\pi^*|V|} : Z \to Y$ and as $\phi_1 := \pi$ is an isomorphism away from F the sections in $\pi^*|V|$ agree with those of $|V|$ on this locus. Hence $\phi_{\pi^*|V|}$ agrees with f here, that is $\phi_2 = f \circ \phi_1$ as required. □

Lemma 6.1.12. *Let H be a very ample divisor on X. Suppose that s_i are sections of H which induce a closed immersion $X \to \mathbb{P}^V$. Let V be the submodule generated by the s_i.*

Then for k sufficiently large we have that $V^{\otimes k} = H^0(X, H^k)$.

Proof. Thought of as a subscheme of $\mathbb{P}^V$, X is cut out by an ideal sheaf $\mathcal{I}$. Hence we have

$$0 \to \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^V}(k) \to \mathcal{O}_{\mathbb{P}^V}(k) \to H^k \to 0.$$

Since $H^1(\mathbb{P}^V, \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^V}(k)) = 0$ for large k, we get a surjection

$$H^0(\mathbb{P}^V, \mathcal{O}_{\mathbb{P}^V}(k)) \to H^0(X, H^k).$$

However, the image of this map is precisely $V^{\otimes k}$ since we have $H^0(\mathbb{P}^V, \mathcal{O}_{\mathbb{P}^V}(k)) = \bigotimes_1^k H^0(\mathbb{P}^V, \mathcal{O}_{\mathbb{P}^V}(1))$.

□

Remark 6.1.13. *They key point of this lemma is the following. Suppose we take $|V|$ as in* **??** *on X. Then we have a blowup $\phi_1^* \colon Z \to X$ such that $\phi_1^*|V|$ is basepoint free inside $H^0(Z, M)$. Take the induced morphism $\phi_2 \colon Z \to Y$ and let H be the very ample divisor on Y induced by $|V|$. Then we have $\phi_2^* H^0(Y, kH) \subseteq \phi_1^*|V|^{\otimes k}$ for $k >> 1$.*

This may not be true for $k = 1$, even without the resolution of indeterminacy. Consider for example $X = \mathbb{P}^1$ and $L = \mathcal{O}_X(4)$. If we take

$$|V| = < x^4, x^3y, y^3x, y^4 >$$

then we get an induced morphism $X \to \mathbb{P}^3$. The image, Y, is not projectively normal however, since $X \to Y$ is an isomorphism but $\dim |V| = 4$ *and* $\dim H^0(X, L) = 5$. *In this example $k = 3$ suffices.*

6.2 Stable Base Loci

In this section we will examine the stable base locus of line bundles which are semiample over $\mathbb{Q}$. This is then applied to the case of a big and nef line bundle restricted to its exceptional locus. We begin with an extension of [Wit20, Theorem 1.10]. The proof follows the same structure, however more care is needed to keep track of sections.

If L is a line bundle on X, semiample over $\mathbb{Q}$, we would like to claim that $\mathbf{SB}(L) = \mathbf{SB}(L|_{X_{red}})$. If L or $L|_{X_{red}}$ is semiample then this follows from [Wit20, Thereom 1.10]. We would then like to prove the general case by blowing up the base locus of $L|_{X_{red}}$ and reducing to the case that the line bundle is semiample on the reduction. Unfortunately if $Y \to X$ is a blowup then the pullback map $H^0(X, L) \to H^0(Y, \pi^*L)$ is, in general, neither injective nor surjective if X is not integral. It is the lack of surjectivity that causes the issues, since we ultimately wish to show the existence of sections on the original scheme.

Suppose for example X is the union of two normal projective schemes X_1, X_2. Then if $\pi : Y \to X$ is the blowup of X_2, the map factors through the closed immersion $X_1 \hookrightarrow X$. Of course if L is a line bundle on X then $H^0(X, L) \to H^0(X_1, L|_{X_1}) \simeq H^0(Y, \pi^*L)$ is typically not a surjection.

The idea in [Wit20, Thereom 1.10] is essentially to show that L is semiample by producing a candidate morphism via pushout. Then one can lift sections back to L by building them from suitable sections of $L|_{X_{red}}$ and $L|_{X_{\mathbb{Q}}}$, up to perhaps replacing the line bundle with a higher power. The key remedy then, is to show that if we blow up the base locus of $L|_{X_{red}}$ via $\pi : Y \to X$, we may build sections of π^*L on Y using only those coming from X_{red} and $X_{\mathbb{Q}}$.

Theorem 6.2.1. *Let S be an excellent, Noetherian scheme, take X a projective scheme over S and L a line bundle on X. Write $i : X_{red} \to X$ for the inclusion of the reduced scheme. Suppose that $L|_{X_{\mathbb{Q}}}$ is semiample. Then $\mathbf{SB}(L) = \mathbf{SB}(L|_{X_{red}})$.*

Proof. We always have $\mathbf{SB}(L|_{X_{red}}) \subseteq \mathbf{SB}(L)$ since we can pull back sections of L, so it suffices to show the converse. We may also freely localise on S and assume that it is an affine, Noetherian $\mathbb{Z}_{(p)}$ scheme. After replacing L with a sufficiently high mulitple, we assume that $\mathbf{SB}(L) = \mathbf{B}(L)$, $\mathbf{SB}(L|_{X_{red}}) = \mathbf{B}(L|_{X_{red}})$ and $\mathbf{SB}(L|_{X_{\mathbb{Q}}}) = \mathbf{B}(L|_{X_{\mathbb{Q}}})$ as reduced schemes.

Step 1: Blow-up the base locus.

Fix a generating set s_i of $H^0(X_{red}, L|_{X_{red}})$. By **??** the blowup $W \to X_{red}$ along a subscheme Z eliminates the indeterminacy of L_{red}, where $Z = \mathbf{B}(L|_{X_{red}}) = \mathbf{SB}(L|_{X_{red}})$ as reduced schemes. Let $\pi : Y \to X$ be the blowup along Z, viewed here a subscheme of X.

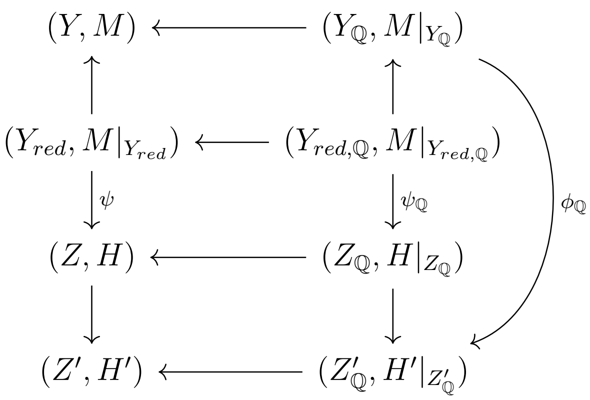
Then the reduction of Y is $Y_{red} \simeq W$ by **??**.

Let F be the exceptional divisor and $M = \pi^*L(-F)$. Note that since L is semiample on $X_{\mathbb{Q}}$, we have that $Y_{\mathbb{Q}} = X_{\mathbb{Q}}$ and $M|_{Y_{\mathbb{Q}}} = L|_{X_{\mathbb{Q}}}$ under this identification. We fix a generating set t_i of of $H^0(Y_{\mathbb{Q}}, M|_{Y_{\mathbb{Q}}})$, which induces a morphism $\phi_{\mathbb{Q}}\colon Y_{\mathbb{Q}} \to Z'_{\mathbb{Q}}$.

By definition the basis s_i of $H^0(X_{red}, L|_{X_{red}})$ now induces $\hat{s}_i$ in $H^0(Y_{red}, M|_{Y_{red}})$ which globally generate the line bundle. These sections induce a morphism $\psi : Y_{red} \to Z$ over S. Note that this may not be the same as the morphism induced by the full basepoint free linear system $H^0(Y_{red}, M|_{Y_{red}})$ since we need not have $H^0(Y_{red}, M|_{Y_{red}}) \simeq H^0(X_{red}, L|_{X_{red}})$ when X is not irreducible.

We then have an induced morphism $Z_{\mathbb{Q}} \to Z'_{\mathbb{Q}}$ which is a finite universal homeomorphism by [Sta, Tag 02OG]. We write $S = \pi^*_{red}H^0(X, L|_{X_{red}}) \subseteq H^0(Y_{red}, M|_{Y_{red}})$, which is generated by the $\hat{s}_i$ by construction.

Now by [?witaszek2020keel, Theorem 1.7, Corollary 4.20 and Lemma 2.20], there is a scheme Z', a universal homeomorphism $Z \to Z'$ and a line bundle H' on Z such that the following diagram commutes at the level of line bundles.



Step 2: Find compatible sections.

Since ψ is not induced by the full linear system on Y_{red}, it need not be the case that sections of $H^0(Z, H)$ pull back to sections inside the linear system $S \subset H^0(Y_{red}, M|_{Y_{red}})$ which defines ψ. By **??** however, we may replace M, L, S, H, H' with higher multiples so that $\psi^*H^0(Z, H) \subseteq S$. and $\phi^*_{\mathbb{Q}}H^0(Z'_{\mathbb{Q}}, H'|_{Z'_{\mathbb{Q}}}) \subseteq H^0(Y_{\mathbb{Q}}, M|_{Y_{\mathbb{Q}}})$. Taking further powers as needed, we may suppose also that H' is very ample.

We fix u_i a generating set for $H^0(Z', H')$, then let $v_i = u_i|_Z$ and $w_i = u_i|_{Z_{\mathbb{Q}}}$. By construction we have $\pi^*v_i \subseteq S$ so we can choose $x_i \in H^0(X_{red}, L|_{X_{red}})$ with $\pi^*x_i = \psi^*v_i$. Similarly we have $y_i \in H^0(X_{\mathbb{Q}}, L|_{X_{\mathbb{Q}}}) = H^0(X_{\mathbb{Q}}, M|_{X_{\mathbb{Q}}})$ with $\phi^*_{\mathbb{Q}}t_i = y_i$. Since the above diagram commutes we have the following identifications.

$$\pi^*v_i|_{Y_{red,\mathbb{Q}}} = \psi^*_{\mathbb{Q}}(u_i|_{Z_{red,\mathbb{Q}}}) = y_i|_{Y_{red,\mathbb{Q}}}$$

Since H' is very ample the y_i must generate a basepoint free linear system. Similarly the $\phi^* v_i$ are basepoint free on Y_{red}. Then as $\pi^{-1}\colon X \dashrightarrow Y$ is an isomorphism away from $\mathbf{SB}(L|_{X_{red}})$, the x_i are basepoint free away from it also.

Finally note that since $H^0(X_{red,\mathbb{Q}}, L|_{X_{red,\mathbb{Q}}}) \to H^0(Y_{red,\mathbb{Q}}, M|_{Y_{red,\mathbb{Q}}})$ is an isomorphism, we must have $x_i|_{X_{red,\mathbb{Q}}} = y_i|_{X_{red,\mathbb{Q}}}$.

Step 3: Glue sections on the original scheme.

By [?witaszek2020keel, Proposition 3.5], we have the following commutative diagram.

$$\begin{array}{ccc} H^0(X,L)^{\text{perf}} & \longrightarrow & H^0(X_{\mathbb{Q}}, L|_{X_{\mathbb{Q}}})^{\text{perf}} \\ \downarrow & & \downarrow \\ H^0(X_{red}, L|_{X_{red}})^{\text{perf}} & \longrightarrow & H^0(X_{red,\mathbb{Q}}, L|_{X_{red,\mathbb{Q}}})^{\text{perf}} \end{array}$$

Hence we can again replace L with a higher power, and x_i, y_i with the corresponding multiples, such that there are $r_i \in H^0(X,L)$ with $r_i|_{X_{red}} = x_i$ and $r_i|_{X_{\mathbb{Q}}} = y_i$. Once again then L is globally generated by the r_i away from $\mathbf{SB}(L|_{X_{red}})$, so we must have that $\mathbf{SB}(L) \subseteq \mathbf{SB}(L|_{X_{red}})$ as claimed. □

Remark 6.2.2. *In principle the condition that $L|_{X_{\mathbb{Q}}}$ is semiample is not completely necessary. The blowup of $\mathbf{B}(L|_{X_{red}})$, $\pi : Y \to X$ induces an injection*

$$H^0(X|_{red,\mathbb{Q}}, L_{X|_{red,\mathbb{Q}}}) \to H^0(Y|_{red,\mathbb{Q}}, L_{Y|_{red,\mathbb{Q}}})$$

which is sufficient to allow us to glue sections on the base. Much more care must be taken when replacing L with a higher power in this case, however.

This would extend the result to the case that $L|_{X_{\mathbb{Q}}}$ becomes basepoint free after we blowup the base locus of $L|_{X_{red}}$. However, it is not clear how this condition could be verified in practice.

We now consider the stable base locus of a big and nef line bundle on restriction to its exceptional locus, under the assumption that the characteristic 0 part of the line bundle is semiample.

Lemma 6.2.3. *Let L be a nef line bundle on X projective over an excellent Noetherian base S with and D an effective Cartier divisor such that $L(-D)$ is an ample line bundle. If $L|_{D_{\mathbb{Q}}}$ is semiample then*

$$\mathbf{SB}(L) = \mathbf{SB}(L|_D).$$

Proof. Clearly $\mathbf{SB}(L) \subseteq D$ as L is ample away from D and we have $\mathbf{SB}(L|_D) \subseteq \mathbf{SB}(L)$ by restriction. Consider the following short exact sequence.

$$0 \to \mathcal{O}_X(kL - mD) \to \mathcal{O}_X(kL) \to \mathcal{O}_{mD}(kL) \to 0$$

By **??**, we may choose $m >> 0$ such that

$$H^1(\mathcal{O}_X, kL - mD = mA + (k-m)L) = 0$$

for $k \geq m$. Then by **??** and the semiampleness assumption, we have $\mathbf{SB}(L|_D) = \mathbf{SB}(L|_{mD})$ and may pick $k >> m$ with $\mathbf{SB}(L|_D) = B(kL|_{mD})$ as reduced subschemes of X. In particular if P is any closed point of D, we may find a section of $kL|_{mD}$ avoiding it, and then lift this to a section of kL. Thus $\mathbf{SB}(L) \cap D \subseteq \mathbf{SB}(L|_D)$ and the result follows.

□

Lemma 6.2.4. *Suppose that X is a reduced projective scheme over an excellent Noetherian base. Suppose that L, A are line bundles with L nef and A ample. Take $Z = Z(s)$ for some section s of $L - A$. If $L|_{D_{\mathbb{Q}}}$ is semiample then $\mathbf{SB}(L) = \mathbf{SB}(L|_Z)$.*

Proof. Let Y_1 be the union of components of X contained in Z and Y_2 the union of those not contained in Z. If either are empty the result is clear so suppose otherwise. As in **??**, we give them a subscheme structure and replace L, A, s with higher powers to ensure we may glue appropriate sections.

Let $D = Z \cap Y_2$ and $L_2 = L|_{Y_2}$. By assumption D is a Cartier divisor on Y_2 with $D = (L - A)|_{Y_2}$. As above we have

$$0 \to \mathcal{O}_{Y_2}(kL_2 - mD) \to \mathcal{O}_{Y_2}(kL_2) \to \mathcal{O}_{mD}(kL_2) \to 0$$

and choosing $k > m >> 0$ this allows us to lift sections from $kL_2|_{mD}$. We then have $\mathbf{B}(kL|_{mZ}) = \mathbf{SB}(L|_{mZ}) = \mathbf{SB}(L|_Z) = \mathbf{B}(kL|_Z)$ for large enough k by **??**. Now, given any section t of $kL|_{mZ}$ we may restrict it to D and then lift it to t' a section of kL_2. By construction t' agrees with t on $D = Z \cap Y_2$, and since $Y_1 \subseteq Z$ it follows we may glue $t|_{Y_1}$ and t'. In particular then we must have $\mathbf{SB}(L) \cap Z = \mathbf{SB}(L|_Z)$, but since L is ample away from Z the result follows. □

Corollary 6.2.5. *Suppose that X is a projective scheme over an excellent Noetherian base with L a nef line bundle on X. Then $\mathbf{SB}(L) = \mathbf{SB}(L|_{\mathbb{E}(L)})$ so long as $L|_{X_{\mathbb{Q}}}$ is semiample.*

Proof. By Noetherian induction we may suppose that this holds on every proper closed subscheme. By **??** we may suppose that X is reduced and then we may also assume $\mathbb{E}(L) \neq X$, else the result is trivial. Let X' be the union of components on which L is big and X'' the union of those on which it is not.

Let A be an ample line bundle and s a general section of $mL - A$, then $Z = Z(s)$ must contain $\mathbb{E}(L)$. By **??** we have that $Z \neq X$, since s does not vanish on any component of X'. Since $\mathbb{E}(L|_Z) = \mathbb{E}(L) \cap Z = \mathbb{E}(L)$ we must have $\mathbf{SB}(L) = \mathbf{SB}(L|_Z) = \mathbf{SB}(L|_{\mathbb{E}(L)})$ by the induction hypothesis. □

6.3 Augmented Base Loci

This section considers the augmented base locus of a nef line bundle and its relation to the exceptional locus. This is done largely under the assumption that they are equal in characteristic 0, before showing this assumption is satisfied in two key cases.

Lemma 6.3.1. *Let X be a projective scheme, L a line bundle and A a very ample line bundle. Then for $m >> 0$ large and divisible we have that*

$$\mathbf{B}_+(L) = \mathbf{B}(mL - A).$$

Proof. Certainly we have n such that $\mathbf{B}_+(L) = \mathbf{SB}(nL - A)$ and thus also $\mathbf{B}_+(L) = \mathbf{B}(nkL - kA)$ for large divisible k. Conversely however $\mathbf{B}(nkL - A) \subseteq \mathbf{B}(nkL - kA)$ as A is very ample. Since $\mathbf{B}_+(L) \subseteq \mathbf{B}(nkL - A)$ by definition, taking $m = kn$ suffices. □

Lemma 6.3.2. *Let X be a projective scheme over an excellent Noetherian base with L a nef line bundle on X. If D is an effective Cartier divisor with $L(-D)$ an ample line bundle and $\mathbf{B}_+(L|_{kD}) = \mathbf{B}_+(L|_D)$ for all $k > 0$ then $\mathbf{B}_+(L) = \mathbf{B}_+(L|_D)$.*

Proof. Since $D = L - A$ we must have that $\mathbf{B}_+(L) \subseteq D$, and conversely $\mathbf{B}_+(L|_D) \subseteq \mathbf{B}_+(L)$ since we may always pullback sections. It suffices to show then that $\mathbf{B}_+(L) \subseteq \mathbf{B}_+(L|_D)$ and we need only check this on points inside D.

By taking multiples we may freely assume $L - D = 2A$ for A very ample. Consider the short exact sequence

$$0 \to \mathcal{O}_X(k(mL - D - A)) \to \mathcal{O}_X(kmL - kA) \to \mathcal{O}_{kD}(mkL - kA) \to 0.$$

We have that $H^1(X, kmL - kD - kA) = H^1(X, (k-1)mL + kA) = 0$ for $k >> 0$ which we now fix and for all $m > 0$.

In particular we may lift sections from $\mathcal{O}_{kD}(mkL - kA)$ for any $m > 0$. By assumption we have $\mathbf{B}_+(L|_{kD}) = \mathbf{B}_+(L|_D)$ and so we have that $\mathbf{B}_+(L|_{kD}) = \mathbf{B}((mkL - kA)_{kD})$ for sufficiently large and divisible m. Given this choice we may lift sections avoiding $\mathbf{B}((mkL - kA)_{kD})$ and thus $\mathbf{B}_+(L) \subseteq \mathbf{B}_+(L|_D)$. □

Lemma 6.3.3. *Let X be a projective scheme over an excellent Noetherian base with L a nef line bundle on X and A an ample line bundle. If $Z = Z(s)$ for some s a section of $mL - A$ and $\mathbf{B}_+(L|_{kZ}) = \mathbf{B}_+(L|_Z)$ for all $k \geq 0$ then $\mathbf{B}_+(L) = \mathbf{B}_+(L|_Z)$.*

Proof. As above we need only prove that $\mathbf{B}_+(L) \subseteq \mathbf{B}_+(L|_Z)$. Let Y_1 the union of components on which Z is non-zero and Y_2 the union of those on which it is not. From above we may assume that $Y_1 \neq \emptyset$ else $Z_{red} = X_{red}$ and the result follows. Let $D = Z|_{Y_1}$ and write $L|_{Y_1} = L', A|_{Y_1} = A'$. As in the proof of previous theorem, after possibly replacing L, D with a multiples, we may find k such that every section of $(mkL' - kA')|_{kD}$ lifts to one of $mkL' - kA'$.

Similarly for $n >> 0$ sufficiently divisible we have $\mathbf{B}((nL - kA)|_{kZ}) = \mathbf{B}_+(L|_{kZ}) = \mathbf{B}_+(L|_Z)$ by assumption. Taking any section s of $(mkL - kA)|_{kZ}$, we may restrict to a section on kD and then lift to s' a section of $k(mL' - A')$. By construction $s|_{Y_2}, s'$ glue along $Y_1 \cap Y_2 \subseteq D$ to give a corresponding section of $k(mL - A)$ and the result follows. We may perform this gluing by **??**. □

Lemma 6.3.4. *Let X be a projective scheme over an excellent Noetherian base with L a nef line bundle on X. Suppose that $\mathbf{B}_+(L) = \mathbb{E}(L)$ and that Z is closed subscheme of X with $\mathbb{E}(L) \subseteq Z$. Then $\mathbf{B}_+(L|_Z) = \mathbb{E}(L|_Z)$.*

Proof. Choose $m > 0$, and A ample on X with $\mathbf{B}_+(L) = \mathbf{B}(mL - A)$ and $\mathbf{B}_+(L|_Z) = \mathbf{B}((mL - A)|_Z)$. Then we have that $\mathbf{B}((mL - A)|_Z) \subseteq \mathbf{B}(mL - A) \cap Z$ by restriction.

On the other hand, since $\mathbb{E}(L) \subseteq Z$, we have that $\mathbb{E}(L|_Z) = \mathbb{E}(L)$. Hence we have that

$$\mathbf{B}_+(L|_Z) \subseteq \mathbf{B}((mL - A)|_Z) \subseteq \mathbf{B}(mL - A) \cap Z = \mathbb{E}(L) \cap Z = \mathbb{E}(L|_Z).$$

It is always the case that $\mathbb{E}(L|_Z) \subseteq \mathbf{B}_+(L|_Z)$ and hence equality holds.

□

Theorem 6.3.5. *Let X be a projective scheme over an excellent Noetherian base S with L a nef line bundle on X. Suppose that $\mathbf{B}_+(L|_{X_\mathbb{Q}}) = \mathbb{E}(L|_{X_\mathbb{Q}})$. Then in fact $\mathbf{B}_+(L) = \mathbb{E}(L) = \mathbf{B}_+(L|_{X_{red}})$.*

Proof. It is immediate that $\mathbb{E}(L) \subseteq \mathbf{B}_+(L)$. Since $\mathbb{E}(L) = \mathbb{E}(L|_{X_{red}})$ it suffices to show only that $\mathbf{B}_+(L) \subseteq \mathbb{E}(L)$. We may assume therefore that $\mathbb{E}(L) \neq X$ and L is big, or the result follows immediately.

The proof will be by Noetherian induction. So we assume that the result holds on every proper closed subscheme of X. The question is local on the base, so we may assume that S is a $\mathbb{Z}_{(p)}$ scheme for some $p > 0$. Note that by **??** we have that $\mathbb{E}(L|_{X_{red,\mathbb{Q}}}) = \mathbf{B}_+(L|_{X_{red,\mathbb{Q}}})$

Step 1: Find a non-vanishing section t of $mL - A$.

Take A ample and $m > 0$ with $\mathbf{SB}(mL - A) = \mathbf{B}_+(L)$ and $\mathbf{SB}((mL - A)|_{X_{red}}) = \mathbf{B}_+(L|_{X_{red}})$. Then we have $\mathbb{E}(L|_{X_{red,\mathbb{Q}}}) = \mathbf{SB}((mL - A)|_{X_{red,\mathbb{Q}}})$ also. Suppose first that $\mathbf{SB}((mL - A)|_{X_\mathbb{Q}}) \neq X_\mathbb{Q}$. Then there is some non-zero section t of $mL - A$ which does not vanish everywhere on X_{red}.

Otherwise we have $\mathbb{E}(L) = \mathbf{SB}((mL - A)|_{X_{red,\mathbb{Q}}}) = X_{red}$, that is

$$H^0(X_{red,\mathbb{Q}}, k(mL - A)|_{X_{red,\mathbb{Q}}}) = 0$$

for all k. Since $\mathbb{E}(L|_{X_{red}}) = \mathbb{E}(L) \neq X$, $L|_{X_{red}}$ is still big. Now by **??** there is a section $s \in H^0(X_{red}, (mL - A)|_{X_{red}})$ which does not vanish on any component on which $L|_{X_{red}}$ is big. In particular it does not vanish everywhere. Then since $H^0(X_{red,\mathbb{Q}}, (mL - A)|_{X_{red,\mathbb{Q}}}) = 0$ we may use [`?witaszek2020keel`, Proposition 3.5] to lift s to a section t

of $H^0(X, p^e(mL - A))$ for some $e > 0$ with $t|_{X_{red}} = s^{p^e}$. After replacing L and A with their $p^{e^{th}}$ powers, t is precisely the non-vanishing section we seek.

Step 2: Reduce to $Z = Z(t)$.

By construction we have $\mathbb{E}(L) \subseteq Z$, since $\mathbf{B}_+(L) \subseteq Z$. By **??**, then, we have that $\mathbf{B}_+(L|_{kZ_\mathbb{Q}}) = \mathbb{E}(L|_{kZ_\mathbb{Q}})$ for $k \geq 1$, so the hypotheses of the theorem are still satisfied by kZ. Hence by the induction hypotheses we may assume $\mathbf{B}_+(L|_{kZ}) = \mathbb{E}(L|_{kZ}) = \mathbf{B}_+(L|_{Z_{red}})$ for all $k \geq 1$. Therefore we can apply **??** to deduce the result. □

Remark 6.3.6. *It is not clear in what generality the assumptions of this theorem should hold. Certainly if $S_\mathbb{Q}$ is a field they hold by* [Bir17]*. Even when $S_\mathbb{Q}$ is of finite type over a field however it is not known whether the condition holds. The arguments of* [Bir17] *do not hold in this relative setting as they rely heavily on certain cohomology groups being vector spaces over a field. One possible remedy, when $S_\mathbb{Q}$ is of finite type over a field, is to find a suitable compactification and reduce to the case that $X_\mathbb{Q}$ is projective over a field.*

Lemma 6.3.7. *Let X be a projective scheme over an excellent base S. Suppose that L is a semiample line bundle, inducing $\pi : X \to Y$ with $\pi_*\mathcal{O}_X = \mathcal{O}_Y$. Then we have equalities*

$$\mathbb{E}(L) = \mathbf{B}_+(L) = \mathit{Exc}(\pi)$$

where $\mathit{Exc}(\pi)$ is the union of closed, integral subschemes $Z \subseteq X$ such that $Z \to \pi(Z)$ is not an isomorphism at the generic point.

Proof. The morphism π is proper and it's own Stein factorisation. So by Zariski's Main Theorem [Sta, Tag 03GW], $\mathrm{Exc}(\pi)$ is precisely the complement of the locus on which π is finite, or equally the locus on which it has finite fibres.

After replacing L with a multiple we have $L = \pi^*A$ for some ample A on Y.

Take any hyperplane H on X, let $\mathcal{I} = \pi_*\mathcal{O}_X(-H)$ be the ideal sheaf induced on Y, so that we have $\pi_*(\mathcal{O}_X(kL - H)) = \mathcal{O}_Y(kA) \otimes \mathcal{I}$.

Suppose that $x \in X \setminus \mathrm{Exc}(\pi)$, then we may assume H does not contain x and so the co-support of I does not contain $\pi(x)$. Choose $k >> 0$ such that $\mathcal{O}_Y(kA) \otimes I$ is globally generated. Hence there is a section $s \in H^0(Y, \mathcal{O}_Y(kA) \otimes \mathcal{I})$ not vanishing at $\pi(x)$.

However by adjunction we have natural isomorphisms

$$H^0(Y, \mathcal{O}_Y(kA) \otimes \mathcal{I}) \simeq H^0(Y, \pi_*(\mathcal{O}_X(kL - H))) \simeq H^0(X, kL - H).$$

The corresponding section $s' \in H^0(X, kL - H)$ does not vanish at x by construction.

Hence we have inclusions $\mathbb{E}(L) \subseteq \mathbf{B}_+(L) \subseteq \mathrm{Exc}(\pi)$ and it remains to show that $\mathrm{Exc}(\pi) \subseteq \mathbb{E}(L)$. More precisely it is enough to show that if V is any closed, integral subscheme of X such that $L|_V$ is big then $V \to \pi(V)$ is generically an isomorphism.

Suppose then that $L' = L|_V$ is big, so we have a section s of $kL' - A$ for $k >> 0$ and A ample on V. Since V is integral, by assumption, this induces an inclusion $\mathcal{O}_V(A) \hookrightarrow \mathcal{O}_V(kL')$. Now $\pi_V : V \to \pi(V)$ is generically an isomorphism if and only if it is generically finite, and hence if and only if it's Stein factorisation is so. Therefore we may freely replace π_V with its Stein factorisation and assume that π_V is induced by generating sections of kL'. Then the inclusion $\mathcal{O}_V(A) \hookrightarrow \mathcal{O}_V(kL')$ ensures that π_V is generically an isomorphism, completing the proof. □

Corollary 6.3.8. *Let X be a projective scheme over an excellent Noetherian base S with L a nef line bundle on X. Suppose that one of the following holds:*

1. *$S_{\mathbb{Q}}$ has dimension 0;*
2. *$L|_{X_{\mathbb{Q}}}$ is semiample;*

Then $\mathbf{B}_+(L) = \mathbb{E}(L)$.

Proof. By **??**, it is enough to know $\mathbf{B}_+(L|_{X_{\mathbb{Q}}}) = \mathbb{E}(L|_{X_{\mathbb{Q}}})$. In case (1) this follows from [Bir17, Theorem 1.3], since each connected component of $X_{\mathbb{Q}}$ is projective over a field. In case (2) this is the content of **??**. □

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