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On Gauge Invariant Regularization of Fermion Currents★

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Abstract. We compare Schwinger and complex powers methods to construct regularized fermion currents. We show that, although both of them are gauge invariant, they are not always yield the same result.

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A difficulty specific to quantum field theories is the occurrence of infinities and hence the necessity of regularizing and renormalizing the theory. Whenever a field theory possesses a classical symmetry—and hence a conserved current—it is desirable to have at hand regularization procedures preserving that symmetry★★.

The calculation of vacuum expectation values of vector currents involves the evaluation of the Green function for the particle fields at the diagonal, so a regularization is required. In a classic paper, Julian Schwinger introduced a point-splitting method to regularize fermion currents maintaining gauge symmetry on the quantum level [1].

More recently, the so-called ζ-function method, based on complex powers of pseudodifferential operators [2], has proved to be a very valuable gauge invariant regularizing tool (see, for example [3]). Some time ago we used it to obtain fermion currents in two and three dimensional models [4].

It is the aim of this Letter to compare the results obtained by the above-mentioned methods.

Let M be a n-dimensional spin closed manifold endowed with a Riemannian metric tensor $g_{\mu\nu}$. For any covector a_μ defined on M, we adopt the usual convention $\not{a} = \gamma^\mu a_\mu$, where the Dirac matrices γ satisfy $\gamma^\mu(x)\gamma^\nu(x) + \gamma^\nu(x)\gamma^\mu(x) = 2\, g^{\mu\nu}(x)$. Let $\not{D} = i\not{\nabla} + \not{A}$ be a Euclidean Dirac operator coupled to a gauge field A_μ, where the

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★★As it is well known, it is not always possible to preserve all the classical symmetries present simultaneously and *anomalies* can arise.

covariant derivative ∇ is given by $\nabla_\mu = \partial_\mu - \Gamma_\mu$, with Γ_μ the spin connection associated with Levi-Civita's. The operator $\not{D}$ is elliptic and, since its principal symbol has only real eigenvalues, it fulfills the Agmon cone condition [2]. Thus, the complex powers $\not{D}^s$ can be constructed following Seeley [2]. For $\mathrm{Re}\, s < 0$ we can write

$$\not{D}^s := \frac{i}{2\pi} \int_\Gamma \lambda^s (\not{D} - \lambda)^{-1} \mathrm{d}\lambda, \tag{1}$$

where Γ is a contour enclosing the spectrum of $\not{D}$, and we define $\not{D}^s$ for $\mathrm{Re}\, s \geqslant 0$ by using $\not{D}^{s+1} = \not{D}^s \circ \not{D}$.

For each $s \in \mathbb{C}$, $\not{D}^s$ turns out to be a pseudodifferential operator of order s and so, if $Re s < -n$, its Schwartz kernel $K_s(x, y)$ is a continuous function. The evaluation at the diagonal $x = y$ of this kernel, $K_s(x, x)$, admits a meromorphic extension to the whole complex s-plane $\mathbb{C}$, with at most simple poles at $s \in \mathbb{Z}^-$. This extension will be also denoted by $K_s(x, x)$.

Since $K_{-1}(x, y)$ coincides with the Green function for $x \neq y$, the finite part of $K_s(x, x)$ at $s = -1$ can be used to define gauge-invariant regularized fermion currents [4]:

$$J^\mu(x) := -\mathrm{tr}\left(\gamma^\mu(x) \underset{s=-1}{\mathrm{FP}} K_s(x, x) \right). \tag{2}$$

Notice that this definition makes sense. In fact, owing to the density character of $K_s(x, x)$ (see, for instance, [5]) and the vectorial nature of the γ matrices, the right-hand side in (2) is a vector density.

In order to compare this regularizing procedure with Schwinger's, it is convenient to consider the kernels $K_s(x, x)$ within the framework developed within [5]. Since we are interested in studying the behaviour of these kernels for $s \to -1$, we shall carry out our analysis just for $-1 \leqslant \mathrm{Re}\, s < 0$.

By considering the finite expansion (see, for instance, [6])

$$\sigma(\not{D}^s) = \sum_{\ell=0}^{N} c_{s-\ell}(x, \xi) + r_N(x, \xi, s), \tag{3}$$

with $N = n - 1$, of the symbol of the operator $\not{D}^s$, with $c_{s-\ell}(x, \xi)$ positively homogeneous of degree $s - \ell$ for $|\xi| \geqslant 1$, we can write, for $s \neq -1$ the Schwartz kernel of this operator as

$$K_s(x, y) = \sum_{\ell=0}^{N} H_{-n-s+\ell}\ (x, x - y) + R_N(x, x - y, s), \tag{4}$$

where $H_{-n-s+\ell}(x, u)$ is the Fourier transform in the variable ξ of $\widetilde{c}_{s-\ell}(x, \xi)$, the homogeneous extension of $c_{s-\ell}(x, \xi)$, evaluated at $u = x - y$ (i.e. $H_{-n-s+\ell}(x, u) = \frac{1}{(2\pi)^n} \int \widetilde{c}_{s-\ell}(x, \xi)\ \mathrm{e}^{i\xi.u}\ d\xi$), and consequently u-homogeneous of degree $-n - s + \ell$

and $R_N(x, u, s)$ is that of $r_N(x, \xi, s) - \sum_{\ell=0}^{N}(\widetilde{c}_{s-\ell} - c_{s-\ell})(x, \xi)$. Note that $(\widetilde{c}_{s-\ell} - c_{s-\ell})(x, \xi) \equiv 0$ for $|\xi| \geqslant 1$.

Now, for $u \neq 0$, simple poles can arise at $s = -1$ in H_{-n-s+N} and in $R_N(x, u, s)$ [5]. Since $K_s(x, x - u)$ is holomorphic in the variable s for $u \neq 0$, these poles cancel each other. In fact, they are due to the singularity of $\widetilde{c}_{s-N}(x, \xi)$ at $\xi = 0$ and then

$$\operatorname*{res}_{s=-1} R_N(x, u, s) = - \operatorname*{res}_{s=-1} H_{-n-s+N}\ (x, u). \tag{5}$$

Thus, for $u \neq 0$, we have for $G(x, y)$, the Green function of $\not{D}$,

$$G(x, y) = \lim_{s \to -1} K_s(x, y) = \sum_{\ell=0}^{N} G_{-n+1+\ell}\ (x, u) + R_G(x, u), \tag{6}$$

with

$$G_{-n+1+\ell}\ (x, u) = \lim_{s \to -1} H_{-n-s+\ell}\ (x, u) \quad \text{for} \quad \ell < N,$$

$$G_{-n+1+N}\ (x, u) = \operatorname*{FP}_{s=-1} H_{-n-s+N}\ (x, u) \quad \text{and} \quad R_G(x, u) = \operatorname*{FP}_{s=-1} R_N(x, u, s).$$

It is worth noticing that a logarithmic term can arise in $\mathrm{FP}_{s=-1} H_{-n-s+N}\ (x, u)$.

Then, taking into account that, for $s \neq -1$ [5],

$$K_s(x, x) = R_N(x, 0, s), \tag{7}$$

we have

$$\operatorname*{FP}_{s=-1} K_s(x, x) = R_G(x, 0). \tag{8}$$

As we shall see below, this last expression furnishes the link between the two regularization methods.

On the other hand, the fermionic currents regularized according to Schwinger's prescription are given by [1]

$$J^\mu(x) = - \operatorname*{Sch\text{-}lim}_{y \to x} \mathrm{tr} \left(\gamma^\mu G(x, y)\, \mathrm{e}^{i \int_x^y A.\mathrm{d}z} \right), \tag{9}$$

where

$$\int_x^y A.\mathrm{d}z = - \int_0^1 A_\mu(x - tu)\, u^\mu\, \mathrm{d}t. \tag{10}$$

The Schwinger limit, $\text{Sch-lim}_{y \to x}$, is defined for each term in the expansion in u-homogeneous functions (and logarithmic ones if they appear) of $\gamma^\mu G(x, y)\, \mathrm{e}^{i(\int_x^y A.\mathrm{d}z)}$ in the following way: the usual limit when the latter exists, vanishes for negative degrees and for logarithmic terms, and coincides with the mean value at $|u| = 1$ for terms of zero degree. The exponential factor was introduced by Schwinger [1] in order to maintain gauge invariance.

From (2), (8) and (9) we see that both methods yield the same result for J^μ if and only if

$$\operatorname*{Sch-lim}_{y\to x} \operatorname{tr}\left(\gamma^\mu \sum_{\ell=0}^{N} G_{-n+1+\ell}\ (x,u)\ \mathrm{e}^{i\int_x^y A.\mathrm{d}z}\right) = 0 \tag{11}$$

since, being $R_G(x,u)$ continuous at $x = y$,

$$\begin{aligned}&\operatorname*{Sch-lim}_{y\to x} \operatorname{tr}\left(\gamma^\mu R_G(x,u)\ e^{i\int_x^y A.dz}\right)\\ &\quad= \operatorname*{Sch-lim}_{y\to x} \operatorname{tr}\left(\gamma^\mu R_G(x,u)\right)\\ &\quad= \lim_{u\to 0} \operatorname{tr}\left(\gamma^\mu R_G(x,u)\right) = \operatorname{tr}\left(\gamma^\mu \operatorname*{FP}_{s=-1} K_s(x,x)\right).\end{aligned} \tag{12}$$

Now, we shall see how this works in $n = 2$, 3 and 4. By computing the $G_{-n+1+\ell}\ (x,u)$'s we shall be able to establish when (11) holds and so, when both methods yield the same regularized currents.

It will be enough for our purposes to consider a flat coordinate patch. In Cartesian coordinates

$$\not{D} = \gamma^\mu D_\mu = \gamma^\mu(i\partial_\mu + A_\mu), \tag{13}$$

where the algebra of the γ-matrices is

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\ \delta^{\mu\nu}. \tag{14}$$

Its symbol, $\sigma(\not{D}; x, \xi)$, is

$$\sigma(\not{D}; x, \xi) = -\ \not{\xi} - \not{A}(x). \tag{15}$$

The symbol of the resolvent, $\sigma((\not{D} - \lambda)^{-1}; x, \xi)$, has an asymptotic expansion $\sum_\ell \widetilde{C}_{-1-\ell}(x, \xi, \lambda)$, where $\widetilde{C}_{-1-\ell}(x, \xi, \lambda)$ is homogeneous in ξ and λ of degree $-1 - \ell$ [2]. Then

$$(\not{D} - \lambda)^{-1}\ \varphi(x) \sim \frac{1}{(2\pi)^{n/2}} \int \sum_\ell \widetilde{C}_{-1-\ell}(x, \xi, \lambda)\ \mathrm{e}^{i\xi.x}\ \widehat{\varphi}(\xi)\ \mathrm{d}\xi. \tag{16}$$

Applying $\not{D} - \lambda$ to Equation (3) we get recursive equations for determining the $\widetilde{C}_{-1-\ell}(x, \xi, \lambda)$'s:

$$\begin{aligned}&-(\not{\xi} + \lambda)\ \widetilde{C}_{-1}(x, \xi, \lambda) = 1,\\ &\not{D}_x \widetilde{C}_{-1-\ell}(x, \xi, \lambda)\ - (\not{\xi} + \lambda)\ \widetilde{C}_{-1-\ell-1}(x, \xi, \lambda) = 0.\end{aligned} \tag{17}$$

Owing to the particular features of the Dirac operator, the standard symbolic cal-

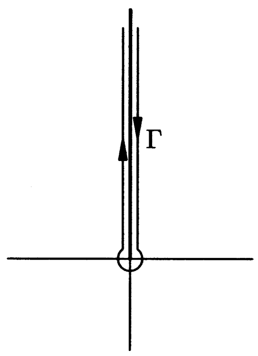

Figure 1. The Γ curve in the λ-plane.

culus [2] simplifies remarkably in our case. In fact, the solution of (17) can be written in a very concise form:

$$\widetilde{C}_{-1-\ell}(x,\xi,\lambda) = -\frac{(\not{\xi}-\lambda)}{\xi^2-\lambda^2}\left[\not{D}_x \frac{(\not{\xi}-\lambda)}{\xi^2-\lambda^2}\right]^{\ell}. \tag{18}$$

Now, from Equation (1),

$$\begin{aligned} &H_{-n-s+\ell}(x,u) \\ &= \frac{1}{(2\pi)^n}\int \widetilde{c}_{s-\ell}(x,\xi)\, \mathrm{e}^{i\xi.u}\, \mathrm{d}\lambda\, \mathrm{d}\xi \\ &= \frac{i}{(2\pi)^{n+1}}\int\int_{\Gamma} \widetilde{C}_{-1-\ell}(x,\xi,\lambda)\, \lambda^s\, \mathrm{e}^{i\xi.u}\, \mathrm{d}\lambda\, \mathrm{d}\xi, \end{aligned} \tag{19}$$

where the contour Γ can be chosen as shown in Figure 1. Therefore,

$$\begin{aligned} &H_{-n-s+\ell}(x,u) \\ &= \frac{-i}{(2\pi)^{n+1}}\int\int_{\Gamma} \frac{(\not{\xi}-\lambda)}{(\xi^2-\lambda^2)^{\ell+1}}\, [\not{D}_x\, (\not{\xi}-\lambda)]^{\ell}\, \lambda^s\, \mathrm{e}^{i\xi.u}\, \mathrm{d}\lambda\, \mathrm{d}\xi \\ &= \frac{-i}{(2\pi)^{n+1}}\int\int_{\Gamma} \frac{(-i\,\not{\partial}_u-\lambda)}{(\xi^2-\lambda^2)^{\ell+1}}\, [\not{D}_x\, (-i\,\not{\partial}_u-\lambda)]^{\ell}\, \lambda^s\, \mathrm{e}^{i\xi.u}\, \mathrm{d}\lambda\, \mathrm{d}\xi. \end{aligned} \tag{20}$$

Taking into account that, for any polynomial $P(\lambda)$,

$$\begin{aligned} &\frac{i}{2\pi}\int_{\Gamma} \frac{\lambda^s\, P(\lambda)}{(\xi^2-\lambda^2)^{\ell+1}}\, \mathrm{d}\lambda \\ &= \frac{i}{2\pi}\left\{\int_{\infty}^{0} \frac{(z\, e^{i\frac{\pi}{2}})^s\, P(iz)}{(\xi^2+z^2)^{\ell+1}}\, i\, \mathrm{d}z + \int_{0}^{\infty} \frac{(z\, \mathrm{e}^{-i\frac{3\pi}{2}})^s\, P(iz)}{(\xi^2+z^2)^{\ell+1}}\, i\, \mathrm{d}z\right\} \\ &= \frac{i}{\pi}\, \mathrm{e}^{-i\frac{\pi}{2}s}\, \sin(\pi s)\, P(-\partial_a)\left[\int_0^{\infty} \frac{z^s\, \mathrm{e}^{-iaz}}{(\xi^2+z^2)^{\ell+1}}\, \mathrm{d}z\right]_{a=0}, \end{aligned} \tag{21}$$

we can write

$$
\begin{aligned}
&H_{-n-s+\ell}(x,u)\\
&\quad=\frac{-i}{\pi}\ \mathrm{e}^{-i\frac{\pi}{2}s}\ \sin(\pi s)(-i\ \partial\!\!\!/_u+\partial_a)\ [\ D\!\!\!\!/_x\ (-i\ \partial\!\!\!/_u+\partial_a)]^{\ell}\times\\
&\quad\quad\times\sum_{k=0}^{\ell+1}\frac{(-ia)^k}{k!}\int_0^{\infty}z^{s+k}\frac{1}{(2\pi)^n}\int\frac{1}{(\xi^2+z^2)^{\ell+1}}\ \ \mathrm{e}^{i\xi.u}\ \mathrm{d}\xi\ \mathrm{d}z\Bigg|_{a=0}.
\end{aligned}
\tag{22}
$$

Now, the integrals in (22) can be performed using the known identities

$$
\frac{1}{(2\pi)^n}\int(\xi^2+z^2)^s\ \mathrm{e}^{i\xi.u}\ \mathrm{d}\xi=\frac{2^{1+s}}{(2\pi)^{\frac{n}{2}}}\ \frac{1}{\Gamma(-s)}\ \left(\frac{z}{u}\right)^{\frac{n}{2}+s}\ \mathbf{K}_{\frac{n}{2}+s}(zu),
\tag{23}
$$

where $\mathbf{K}_\mu$ is a Bessel function (see, for instance, [8]) and

$$
\int_0^{\infty}z^{\mu}\ \mathbf{K}_\nu(zu)\ \mathrm{d}z=2^{\mu-1}\ u^{-\mu-1}\ \Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right)
\tag{24}
$$

(see, for example, [7]).

Finally, we get the following expression for $H_{-n-s+\ell}(x,u)$:

$$
\begin{aligned}
&H_{-n-s+\ell}(x,u)\\
&\quad=\frac{-i\ 2^{s-2\ell-2}}{\pi^{\frac{n}{2}+1}\ \ell!}\mathrm{e}^{-i\frac{\pi}{2}s}\ \sin(\pi s)\times\\
&\quad\quad\times(-i\ \partial\!\!\!/_u+\partial_a)\ [\ D\!\!\!\!/_x\ (-i\ \partial\!\!\!/_u+\partial_a)]^{\ell}\sum_{k=0}^{\ell+1}\frac{(-ia)^k}{k!}\times\\
&\quad\quad\times\Gamma\left(\frac{1+s+k}{2}\right)\Gamma\left(\frac{s+k+n-1-2\ell}{2}\right)\ u^{-s-n+2\ell+1-k}\ \Bigg|_{a=0}.
\end{aligned}
\tag{25}
$$

The first four $H_{-n-s+\ell}(x,u)$ terms, obtained from (25) after a straightforward but tedious computation involving γ-matrices's algebra and derivatives, are shown in Table I. There, as usual, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu=-i(D_\mu A_\nu-D_\nu A_\mu)$. It is worth noticing that the first terms of the exponential

$$
\mathrm{e}^{-i\int_x^y A.\mathrm{d}z}=1+i(u.A)-\frac{(u.D)(u.A)}{2!}-i\frac{(u.D)(u.D)(u.A)}{3!}+\ldots
\tag{26}
$$

start to appear as an overall factor in the sum of the expansion (4) for $K_s(x,y)$.

Now, we shall compute the sum in expression (11) in order to see whether both methods coincide or not. Taking into account that

$$
G_{-n+1+\ell}\ (x,u)=\lim_{s\to-1}\ H_{-n-s+\ell}\ (x,u)\quad\text{for }\ell<N
$$

Table I. The first four $H_{-n-s+\ell}(x,u)$.

$$H_{-n-s}(x,u) = \frac{2^{s-1}}{\pi^{n2+1}}\ \mathrm{e}^{-i\frac{\pi}{2}s}\ \sin(\pi s)\times$$
$$\times\left[\ \Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s+1}{2}\right)u^{-n-s-1}\ \not{u}\ -\ \Gamma\left(\frac{2+s}{2}\right)\Gamma\left(\frac{n+s}{2}\right)u^{-n-s}\ \right]$$

$$H_{-n-s+1}(x,u) = \frac{2^{s-1}}{\pi^{\frac{n}{2}+1}}\ \mathrm{e}^{-i\frac{\pi}{2}s}\ \sin(\pi s)\times$$
$$\times\left[\ \Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s+1}{2}\right)u^{-n-s-1}\ \not{u}-\ \Gamma\left(\frac{2+s}{2}\right)\Gamma\left(\frac{n+s}{2}\right)u^{-n-s}\ \right]i(u.A)$$

$$H_{-n-s+2}(x,u) = \frac{2^{s-1}}{\pi^{\frac{n}{2}+1}}\ \mathrm{e}^{-i\frac{\pi}{2}s}\ \sin(\pi s)\ \times$$
$$\times\left\{\left[\ \Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s+1}{2}\right)u^{-n-s-1}\ \not{u}-\ \Gamma\left(\frac{2+s}{2}\right)\Gamma\left(\frac{n+s}{2}\right)u^{-n-s}\ \right]\left(-\frac{(u.D)(u.A)}{2!}\right)+\right.$$
$$\left.+\frac{i}{8}\left[\ \Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s-1}{2}\right)u^{-n-s+1}\ u_\rho\gamma^\mu\gamma^\rho\gamma^\nu+\Gamma\left(\frac{2+s}{2}\right)\Gamma\left(\frac{n+s-2}{2}\right)u^{-n-s+2}\ \gamma^\mu\gamma^\nu\right]F_{\mu\nu}\right\}$$

$$H_{-n-s+3}(x,u) = \frac{2^{s-1}}{\pi^{\frac{n}{2}+1}}\ \mathrm{e}^{-i\frac{\pi}{2}s}\ \sin(\pi s)\ \times$$
$$\times\left\{\left[\ \Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s+1}{2}\right)u^{-n-s-1}\ \not{u}-\ \Gamma\left(\frac{2+s}{2}\right)\Gamma\left(\frac{n+s}{2}\right)u^{-n-s}\ \right]\left(-i\frac{(u.D)(u.D)(u.A)}{3!}\right)+\right.$$
$$+\frac{i}{8}\left[\ \Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s-1}{2}\right)u^{-n-s+1}\ u_\rho\gamma^\mu\gamma^\rho\gamma^\nu+\Gamma\left(\frac{2+s}{2}\right)\Gamma\left(\frac{n+s-2}{2}\right)u^{-n-s+2}\ \gamma^\mu\gamma^\nu\right]F_{\mu\nu}\ i(u.A)+$$
$$+\frac{1}{24}\left[\ \Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s-1}{2}\right)u^{-n-s+1}\left(-\tfrac{3}{2}u_\rho u^\sigma\gamma^\mu\gamma^\rho\gamma^\nu\partial_\sigma F_{\mu\nu}-u^\mu u_\rho\gamma^\rho\partial^\nu F_{\mu\nu}+u^\mu u^\nu\gamma^\rho\partial_\nu F_{\mu\rho}\right)+\right.$$
$$+\Gamma\left(\frac{2+s}{2}\right)\Gamma\left(\frac{n+s-2}{2}\right)u^{-n-s+2}\left(-\tfrac{3}{2}u^\mu\gamma^\nu\gamma^\rho\partial_\mu F_{\nu\rho}+u^\mu\partial^\nu F_{\mu\nu}\right)+$$
$$\left.\left.+\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{n+s-3}{2}\right)u^{-n-s+3}\ \gamma^\nu\partial^\mu F_{\mu\nu}\ \right]\right\}$$

and

$$G_{-n+1+N}\ (x,u) = \underset{s=-1}{\mathrm{FP}}\ H_{-n-s+N}\ (x,u),$$

from Table I, we get the following relations.

For $n=2$, we have

$$\sum_{\ell=0}^{1} G_{-2+1+\ell}\ (x,u)\ \mathrm{e}^{i\int_x^y A.\mathrm{d}z} = -\frac{i}{2\pi}\frac{\not{u}}{u^2}(1+\mathrm{o}(u^2)), \tag{27}$$

so it is clear that (11) holds in this case.

For $n = 3$, we get

$$\sum_{\ell=0}^{2} G_{-3+1+\ell}\ (x, u)\ \mathrm{e}^{i\int_x^y A.\mathrm{d}z} = -\frac{i}{4\pi}\frac{\not{u}}{u^3}(1 + \mathrm{o}(u^3)) + \frac{1}{16\pi}\Big[\frac{u_\rho}{u}\gamma^\mu\gamma^\rho\gamma^\nu + \gamma^\mu\gamma^\nu\Big]F_{\mu\nu}, \tag{28}$$

and so

$$\underset{y\to x}{\text{Sch-lim}}\ \mathrm{tr}\left(\gamma^\mu \sum_{\ell=0}^{2} G_{-3+1+\ell}\ (x, u)\ \mathrm{e}^{i\int_x^y A.\mathrm{d}z}\right) = \frac{1}{16\pi}\mathrm{tr}[\gamma^\mu\gamma^\rho\gamma^\nu]F_{\rho\nu}, \tag{29}$$

which does or does not vanish depending on the γ's representation (it does not vanish if the 2×2 Pauli matrices are chosen).

Finally, we consider $n = 4$. In this case, a pole is present in$H_{-4-s+3}(x, u)$ at $s = -1$. After computing the finite part in order to get $G_{-4+1+3}(x, u)$ we have

$$\begin{aligned}\sum_{\ell=0}^{3} G_{-4+1+\ell}\ (x, u)\ \mathrm{e}^{i\int_x^y A.\mathrm{d}z} &= -\frac{i}{2\pi^2}\frac{\not{u}}{u^4}(1 + \mathrm{o}(u^4)) + \frac{1}{16\pi^2}\frac{u_\rho}{u^2}\gamma^\mu\gamma^\rho\gamma^\nu F_{\mu\nu}(1 + \mathrm{o}(u^2)) - \\ &- \frac{i}{48\pi^2}\frac{u_\rho u^\sigma}{u^2}(-\tfrac{3}{2}\gamma^\mu\gamma^\rho\gamma^\nu\partial_\sigma F_{\mu\nu} - \gamma^\rho\partial^\mu F_{\sigma\mu} + \gamma^\mu\partial^\rho F_{\sigma\mu}) - \\ &- \frac{i}{24\pi^2}(\ln 2 - \ln u - \frac{i\pi}{2} + \Gamma'(1))\gamma^\nu\partial^\mu F_{\mu\nu},\end{aligned} \tag{30}$$

which, in general, clearly yields a nonzero result for expression (11).

So, we see that although Schwinger and complex powers methods are both gauge-invariant, they only coincide for the two-dimensional case. In 3 dimensions, the coincidence depends on the representation chosen for the γ-matrices, while for $n = 4$ they in general disagree.

Had we considered the general case, additional terms depending on the Riemannian curvature would have appeared. Nevertheless, those terms coud not counterbalance the computed $F_{\mu\nu}$-depending terms which produced the difference between both methods.

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