Finite Fields and Their Applications 83 (2022) 102093

Contents lists available at ScienceDirect

Finite Fields and Their Applications

www.elsevier.com/locate/ffa

On the constructions of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear generalized Hadamard codes ☆,☆☆

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A R T I C L E I N F O

Article history:
Received 23 May 2022
Accepted 11 July 2022
Available online 21 July 2022
Communicated by James W.P. Hirschfeld

MSC:
94B25
94B60

Keywords:
Generalized Hadamard code
Gray map
$\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear code

A B S T R A C T

The $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive codes are subgroups of $\mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$, and can be seen as linear codes over $\mathbb{Z}_p$ when $\alpha_2 = 0$, $\mathbb{Z}_{p^2}$-additive codes when $\alpha_1 = 0$, or $\mathbb{Z}_2\mathbb{Z}_4$-additive codes when $p = 2$. A $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear generalized Hadamard (GH) code is a GH code over $\mathbb{Z}_p$ which is the Gray map image of a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code. In this paper, we generalize some known results for $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $p = 2$ to any $p \geq 3$ prime when $\alpha_1 \neq 0$. First, we give a recursive construction of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH codes of type $(\alpha_1, \alpha_2; t_1, t_2)$ with $t_1, t_2 \geq 1$. We also present many different recursive constructions of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH codes having the same type, and show that we obtain permutation equivalent codes after applying the Gray map. Finally, according to some computational results, we see that, unlike $\mathbb{Z}_4$-linear GH codes, when $p \geq 3$ prime, the $\mathbb{Z}_{p^2}$-linear GH codes are not included in the family of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$. Indeed, we observe that the

☆ This work has been partially supported by the Spanish Ministerio de Ciencia e Innovación under Grant PID2019-104664GB-I00 (AEI / 10.13039/501100011033) and by the Catalan AGAUR scholarship 2020 FI SDUR 00475.

☆☆ The material in this paper was presented in part at the 12th International Workshop on Coding and Cryptography (WCC 2022), 7-11 March 2022 - Virtual [1].

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https://doi.org/10.1016/j.ffa.2022.102093

constructed codes are not equivalent to the $\mathbb{Z}_{p^s}$-linear GH codes for any $s \geq 2$.

1. Introduction

Let $\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$ be the ring of integers modulo p and p^2, respectively, where p is a prime. Let $\mathbb{Z}_p^n$ and $\mathbb{Z}_{p^2}^n$ denote the set of all n-tuples over $\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$, respectively. In this paper, the elements of $\mathbb{Z}_p^n$ and $\mathbb{Z}_{p^2}^n$ will be called vectors of length n.

A code over $\mathbb{Z}_p$ of length n is a nonempty subset of $\mathbb{Z}_p^n$, and it is linear if it is a subspace of $\mathbb{Z}_p^n$. Similarly, a nonempty subset of $\mathbb{Z}_{p^2}^n$ is a $\mathbb{Z}_{p^2}$-additive if it is a subgroup of $\mathbb{Z}_{p^2}^n$. A $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code is a subgroup of $\mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$. Note that a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code is a linear code over $\mathbb{Z}_p$ when $\alpha_2 = 0$, a $\mathbb{Z}_{p^2}$-additive code when $\alpha_1 = 0$, or a $\mathbb{Z}_2\mathbb{Z}_4$-additive code when $p = 2$. The order of a vector $\mathbf{u} \in \mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$, denoted by $o(\mathbf{u})$, is the smallest positive integer m such that $m\mathbf{u} = (0, \ldots, 0 \mid 0, \ldots, 0)$.

The Hamming weight of a vector $u \in \mathbb{Z}_p^n$, denoted by $\mathrm{wt}_H(u)$, is the number of nonzero coordinates of u. The Hamming distance of two vectors $u, v \in \mathbb{Z}_p^n$, denoted by $d_H(u, v)$, is the number of coordinates in which they differ. Note that $d_H(u, v) = \mathrm{wt}_H(u - v)$. The minimum distance of a code C over $\mathbb{Z}_p$ is $d(C) = \min\{d_H(u, v) : u, v \in C, u \neq v\}$.

In [2], a Gray map from $\mathbb{Z}_4$ to $\mathbb{Z}_2^2$ is defined as $\phi(0) = (0, 0)$, $\phi(1) = (0, 1)$, $\phi(2) = (1, 1)$ and $\phi(3) = (1, 0)$. There exist different generalizations of this Gray map, which go from $\mathbb{Z}_{2^s}$ to $\mathbb{Z}_2^{2^{s-1}}$ [3–7]. The one given in [6] can be defined in terms of the elements of a Hadamard code [7], and Carlet's Gray map [3] is a particular case of the one given in [7] satisfying $\sum \lambda_i \phi(2^i) = \phi(\sum \lambda_i 2^i)$ [8]. In this paper, we focus on a generalization of Carlet's Gray map from $\mathbb{Z}_{p^s}$ to $\mathbb{Z}_p^{p^{s-1}}$, also denoted by ϕ, which is a particular case of the one given in [9]. Let $\Phi : \mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2} \to \mathbb{Z}_p^n$, where $n = \alpha_1 + p\alpha_2$, be an extension of the Gray map ϕ given by

$$\Phi((y \mid y')) = (y, \phi(y'_1), \ldots, \phi(y'_{\alpha_2})),$$

for any $y \in \mathbb{Z}_p^{\alpha_1}$ and $y' = (y'_1, \ldots, y'_{\alpha_2}) \in \mathbb{Z}_{p^2}^{\alpha_2}$.

Let $\mathcal{C} \subseteq \mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$ be a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code. We say that its Gray map image $C = \Phi(\mathcal{C})$ is a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear code of length $\alpha_1 + p\alpha_2$. Since $\mathcal{C}$ can be seen as a subgroup of $\mathbb{Z}_{p^2}^{\alpha_1+\alpha_2}$, it is isomorphic to an abelian structure $\mathbb{Z}_{p^2}^{t_1} \times \mathbb{Z}_p^{t_2}$, and we say that $\mathcal{C}$, or equivalently $C = \Phi(\mathcal{C})$, is of type $(\alpha_1, \alpha_2; t_1, t_2)$. Note that $|\mathcal{C}| = p^{2t_1+t_2}$. Unlike linear codes over finite fields, linear codes over rings do not have a basis, but there exist generator matrices for these codes having minimum number of rows, that is, $t_1 + t_2$ rows.

A generalized Hadamard (GH) matrix $H(p, \lambda) = (h_{ij})$ of order $N = p\lambda$ over $\mathbb{Z}_p$ is a $p\lambda \times p\lambda$ matrix with entries from $\mathbb{Z}_p$ with the property that for every i, j, $1 \leq i < j \leq p\lambda$, each of the multisets $\{h_{is} - h_{js} : 1 \leq s \leq p\lambda\}$ contains every element of $\mathbb{Z}_p$ exactly λ times [10]. An ordinary Hadamard matrix of order 4μ corresponds to a GH matrix $H(2, \lambda)$ over

$\mathbb{Z}_2$, where $\lambda = 2\mu$ [11]. Two GH matrices H_1 and H_2 of order N are said to be equivalent if one can be obtained from the other by a permutation of the rows and columns and adding the same element of $\mathbb{Z}_p$ to all the coordinates in a row or in a column.

We can always change the first row and column of a GH matrix into zeros and we obtain an equivalent GH matrix which is called normalized. From a normalized GH matrix H, we denote by F_H the code consisting of the rows of H, and $C_H = \cup_{\alpha\in\mathbb{Z}_p}(F_H + \alpha\mathbf{1})$, where $F_H + \alpha\mathbf{1} = \{h + \alpha\mathbf{1} : h \in F_H\}$ and $\mathbf{1}$ denotes the all-one vector. The code C_H over $\mathbb{Z}_p$ is called generalized Hadamard (GH) code [12]. Note that C_H is generally a non-linear code over $\mathbb{Z}_p$. Moreover, if it is of length N, it has pN codewords and minimum distance $N(p-1)/p$.

The $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive codes such that after applying the Gray map Φ give GH codes are called $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH codes and the corresponding images are called $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes. The classification of $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes of length 2^t with $\alpha_1 = 0$ and $\alpha_1 \neq 0$ is given in [13,14], showing that there are $\lfloor(t-1)/2\rfloor$ and $\lfloor t/2\rfloor$ such non-equivalent codes, respectively. Moreover, in [15], it is shown that each $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 = 0$ is equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 \neq 0$, so indeed there are only $\lfloor t/2\rfloor$ non-equivalent $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes of length 2^t. Later, in [8,16–18], an iterative construction for $\mathbb{Z}_{p^s}$-linear GH codes is described, the linearity is established, and a partial classification is obtained, giving the exact amount of non-equivalent non-linear such codes for some parameters.

This paper is focused on $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes of length p^t with $\alpha_1 \neq 0$ and $p \geq 3$ prime, generalizing some results given for $p = 2$ in [14,19] related to the construction of such codes. For $p = 3$ and $2 \leq t \leq 8$, these codes are compared with the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes of length p^t with $\alpha_1 = 0$ studied in [16]. This paper is organized as follows. In Section 2, we recall some properties of the Gray map considered in this paper. In Section 3, we describe a recursive construction of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes of type $(\alpha_1, \alpha_2; t_1, t_2)$ with $t_1, t_2 \geq 1$, $\alpha_1 \neq 0$, and p prime. In Section 4, we present many different recursive constructions of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes having the same type, and show that they give permutation equivalent codes. Finally, in Section 5, we show some computational results for $p = 3$, which point out that, unlike $\mathbb{Z}_4$-linear and $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes, when $p \geq 3$ prime, the $\mathbb{Z}_{p^2}$-linear GH codes are not included in the family of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$. Moreover, we observe that, for $p = 3$, the codes constructed in this paper are not equivalent to the $\mathbb{Z}_{p^s}$-linear GH codes with $s \geq 2$ considered in [16] using the same Gray map.

2. Preliminary results

In this section, we give the definition of the Gray map considered in this paper for elements of $\mathbb{Z}_{p^2}$. We also include some of its properties used in the paper.

We consider the following Gray map ϕ, given in [3,20], for $s = 2$:

$$\begin{aligned} \phi : \mathbb{Z}_{p^2} &\longrightarrow \mathbb{Z}_p^p \\ u &\mapsto (u_0, u_1)M, \end{aligned} \tag{1}$$

where $u \in \mathbb{Z}_{p^2}$; $[u_0, u_1]_p$ is the p-ary expansion of u, that is, $u = u_0 + u_1 p$ with $u_0, u_1 \in \mathbb{Z}_p$; and M is the following matrix of size $2 \times p$:

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & p-1 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Let $u', v' \in \mathbb{Z}_{p^2}$ and $[u'_0, u'_1]_p$, $[v'_0, v'_1]_p$ be the p-ary expansions of u' and v', respectively, i.e. $u' = u'_0 + u'_1 p$ and $v' = v'_0 + v'_1 p$. We define the operation "$\odot_p$" between elements u' and v' in $\mathbb{Z}_{p^2}$ as $u' \odot_p v' = \xi_0 + \xi_1 p$, where

$$\xi_i = \begin{cases} 1 & \text{if} \quad u'_i + v'_i \geq p, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the p-ary expansion of $u' \odot_p v'$ is $[\xi_0, \xi_1]_p$, where $\xi_0, \xi_1 \in \{0, 1\}$. For $u, v \in \mathbb{Z}_p$, we define $u \odot_p v = 1$ if $u + v \geq p$ and 0 otherwise. We denote in the same way, "$\odot_p$", the component-wise operation. For $\mathbf{u} = (u \mid u'), \mathbf{v} = (v \mid v') \in \mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$, we denote $\mathbf{u} \odot_p \mathbf{v} = (u \odot_p v \mid u' \odot_p v')$. Note that $p(\mathbf{u} \odot_p \mathbf{v}) = (\mathbf{0} \mid p(u' \odot_p v'))$.

From [16], we have the following results:

Lemma 2.1. [16] *Let $u \in \mathbb{Z}_{p^2}$ and $\lambda \in \mathbb{Z}_p$. Then, $\phi(u + \lambda p) = \phi(u) + (\lambda, \lambda, \ldots, \lambda)$.*

Corollary 2.1. [16] *Let $\lambda, \mu \in \mathbb{Z}_p$. Then, $\phi(\lambda\mu p) = \lambda\phi(\mu p) = \lambda\mu\phi(p)$.*

Corollary 2.2. [16] *Let $u, v \in \mathbb{Z}_{p^2}$. Then, $\phi(u) + \phi(v) = \phi(u + v - p(u \odot_p v))$.*

Corollary 2.3. [16] *Let $u, v \in \mathbb{Z}_{p^2}$. Then, $\phi(pu + v) = \phi(pu) + \phi(v)$.*

Corollary 2.4. [16] *For $u, v \in \mathbb{Z}_{p^2}$, $\phi(u+v) = \phi(u) + \phi(v) + (\xi_0, \xi_0, \ldots, \xi_0)$, where $\xi_0 = 1$ if $u_0 + v_0 \geq p$ and 0 otherwise.*

Proposition 2.1. [16] *Let $u, v \in \mathbb{Z}_{p^2}$ be two distinct elements. Then, $\phi(u) - \phi(v) = \phi(u - v) = (\lambda, \ldots, \lambda)$ if $u - v = \lambda p \in p\mathbb{Z}_{p^2}$, and $\phi(u) - \phi(v)$ contains every element of $\mathbb{Z}_p$ exactly once if $u - v \in \mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}$.*

Proposition 2.2. [16] *Let $u, v \in \mathbb{Z}_{p^2}$. Then, $d_H(\phi(u), \phi(v)) = \mathrm{wt}_H(\phi(u - v))$.*

From [21], the homogeneous weight of an element $u \in \mathbb{Z}_{p^2}$ is defined by

$$\mathrm{wt}^*(u) = \begin{cases} 0 & \text{if} \quad u = 0, \\ p & \text{if} \quad u \in p\mathbb{Z}_{p^2} \setminus \{0\}, \\ p-1 & \text{otherwise,} \end{cases} \tag{2}$$

and the homogeneous weight of a vector $u = (u_1, \dots, u_n) \in \mathbb{Z}_{p^2}^n$ is $\mathrm{wt}^*(u) = \sum_{i=1}^n \mathrm{wt}^*(u_i)$. The corresponding homogeneous distance of $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n) \in \mathbb{Z}_{p^2}^n$ is defined as follows:

$$d^*(u,v) = \sum_{i=1}^n \mathrm{wt}^*(u_i - v_i). \tag{3}$$

The Gray map Φ over $\mathbb{Z}_{p^2}^n$ is an isometry which transforms homogeneous distances defined in $\mathbb{Z}_{p^2}^n$ to Hamming distances defined in $\mathbb{Z}_p^{np}$ [20].

Then, we define the homogeneous weight of $\mathbf{u} = (u \mid u') \in \mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$ as $\mathrm{wt}^*(\mathbf{u}) = \mathrm{wt}_H(u) + \mathrm{wt}^*(u')$. From (3), the corresponding homogeneous distance of $\mathbf{u} = (u \mid u')$ and $\mathbf{v} = (v \mid v') \in \mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$ is defined as follows:

$$d^*(\mathbf{u},\mathbf{v}) = \mathrm{wt}_H(u-v) + \mathrm{wt}^*(u'-v'). \tag{4}$$

Note that the extension of the Gray map Φ over $\mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$ is also an isometry by using this homogeneous metric, that is, $d^*(\mathbf{u},\mathbf{v}) = d_H(\Phi(\mathbf{u}),\Phi(\mathbf{v}))$ for all $\mathbf{u},\mathbf{v} \in \mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$. Moreover, the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear codes obtained from this Gray map Φ are distance invariant by Proposition 2.2.

3. Construction of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes

The description of a generator matrix having minimum number of rows for $\mathbb{Z}_2\mathbb{Z}_4$-additive GH codes with $\alpha_1 \neq 0$, as long as an iterative construction of these matrices, is given in [14,19]. In this section, we generalize these results for $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH codes with $\alpha_1 \neq 0$ and any $p \geq 3$ prime. Specifically, we define an iterative construction for the generator matrices and establish that they generate $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH codes. The proof that the codes are GH is completely different from the binary case.

Let $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{p^2-1}$ be the vectors having the elements $0, 1, 2, \dots, p^2-1$ repeated in each coordinate, respectively. Let

$$A_p^{1,1} = \left(\begin{array}{cccc|cccc} 1 & 1 & \cdots & 1 & p & p & \cdots & p \\ 0 & 1 & \cdots & p-1 & 1 & 2 & \cdots & p-1 \end{array}\right). \tag{5}$$

Any matrix $A_p^{t_1,t_2}$ with $t_1 \geq 1, t_2 \geq 2$ or $t_1 \geq 2, t_2 \geq 1$ can be obtained by applying the following iterative construction. First, if A is a generator matrix of a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code, that is, a subgroup of $\mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$, then we denote by A_1 the submatrix of A with the first α_1 columns over $\mathbb{Z}_p$, and A_2 the submatrix with the last α_2 columns over $\mathbb{Z}_{p^2}$. We start with $A_p^{1,1}$. Then, if we have a matrix $A = A_p^{t_1,t_2}$, we may construct the matrices

$$A_p^{t_1,t_2+1} = \left(\begin{array}{cccc|cccc} A_1 & A_1 & \cdots & A_1 & A_2 & A_2 & \cdots & A_2 \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{p-1} & p\cdot\mathbf{0} & p\cdot\mathbf{1} & \cdots & p\cdot(\mathbf{p-1}) \end{array}\right) \tag{6}$$

and

$$A_p^{t_1+1,t_2} = \left(\begin{array}{cccc|ccccccc} A_1 & A_1 & \cdots & A_1 & pA_1 & \cdots & pA_1 & A_2 & A_2 & \cdots & A_2 \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{p-1} & \mathbf{1} & \cdots & \mathbf{p-1} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{p^2-1} \end{array}\right). \tag{7}$$

Example 3.1. Let

$$A_3^{1,1} = \left(\begin{array}{ccc|cc} 1 & 1 & 1 & 3 & 3 \\ 0 & 1 & 2 & 1 & 2 \end{array}\right)$$

be the matrix described in (5) for $p = 3$. By using the constructions described in (6) and (7), we obtain $A_3^{1,2}$ and $A_3^{2,1}$, respectively, as follows:

$$A_3^{1,2} = \left(\begin{array}{ccccccccc|cccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & 3 & 3 & 6 & 6 \end{array}\right),$$

$$A_3^{2,1} = \left(\begin{array}{ccccccccc|ccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & \cdots & 3 & 3 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 3 & 6 & 0 & 3 & 6 & 1 & 2 & \cdots & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0 & \cdots & 8 & 8 \end{array}\right).$$

In this paper, we consider that the matrices $A_p^{t_1,t_2}$ are constructed recursively starting from $A_p^{1,1}$ in the following way. First, we add $t_1 - 1$ rows of order p^2, up to obtain $A_p^{t_1,1}$; and then we add $t_2 - 1$ rows of order p up to achieve $A_p^{t_1,t_2}$. Note that in the first row there is always the row $(\mathbf{1} \mid \mathbf{p})$.

The $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code generated by $A_p^{t_1,t_2}$ is denoted by $\mathcal{H}_p^{t_1,t_2}$, and the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear code $\Phi(\mathcal{H}_p^{t_1,t_2})$ by $H_p^{t_1,t_2}$. We also write A^{t_1,t_2}, $\mathcal{H}^{t_1,t_2}$, and H^{t_1,t_2} instead of $A_p^{t_1,t_2}$, $\mathcal{H}_p^{t_1,t_2}$, and $H_p^{t_1,t_2}$, respectively, when the value of p is clear by the context.

Proposition 3.1. *Let $t_1, t_2 \geq 1$ and p prime. Then, $\mathcal{H}_p^{t_1,t_2}$ is a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code of type*

$$(p^{t_1+t_2-1}, (p-1)\sum_{i=1}^{t_1} p^{t_1+t_2+i-3}; t_1, t_2). \tag{8}$$

Proof. First, we prove this proposition for the code $\mathcal{H}_p^{t_1,1}$ by induction on $t_1 \geq 1$. Note that, if $t_1 = 1$, the code is of type $(p, p-1; 1, 1)$, which coincides with (8) since $\sum_{i=1}^{t_1} p^{t_1+t_2+i-3} = 1$ when $t_1 = t_2 = 1$. Assume that the type of $\mathcal{H}_p^{t_1,1}$ is $(\alpha_1', \alpha_2'; t_1, 1)$, where $\alpha_1' = p^{t_1}$ and $\alpha_2' = (p-1)\sum_{i=1}^{t_1} p^{t_1+i-2}$. By using construction (7), $\mathcal{H}_p^{t_1+1,1}$ is of type $(\alpha_1, \alpha_2; t_1+1, 1)$, where $\alpha_1 = p \cdot \alpha_1' = p \cdot p^{t_1} = p^{t_1+1}$ and $\alpha_2 = (p-1)\alpha_1' + p^2 \cdot \alpha_2' = (p-1)p^{t_1} + p^2(p-1)\sum_{i=1}^{t_1} p^{t_1+i-2} = (p-1)\sum_{i=0}^{t_1} p^{t_1+i} = (p-1)\sum_{i=1}^{t_1+1} p^{(t_1+1)+i-2}$. Therefore, for $t_1 \geq 1$, the type of the code $\mathcal{H}_p^{t_1,1}$ is $(p^{t_1}, (p-1)\sum_{i=1}^{t_1} p^{t_1+i-2}; t_1, 1)$.

Next, from the type of the code $\mathcal{H}_p^{t_1,1}$, we prove the proposition for the code $\mathcal{H}_p^{t_1,t_2}$ by induction on $t_2 \geq 1$. Assume that the type of $\mathcal{H}_p^{t_1,t_2}$ is $(\alpha_1', \alpha_2'; t_1, t_2)$, where $\alpha_1' = p^{t_1+t_2-1}$ and $\alpha_2' = (p-1)\sum_{i=1}^{t_1} p^{t_1+t_2+i-3}$. By using construction (6), we have that

$\mathcal{H}_p^{t_1,t_2+1}$ is of type $(\alpha_1, \alpha_2; t_1, t_2+1)$, where $\alpha_1 = p \cdot \alpha_1' = p \cdot p^{t_1+t_2-1} = p^{t_1+(t_2+1)-1}$ and $\alpha_2 = p \cdot \alpha_2' = p \cdot (p-1)\sum_{i=1}^{t_1} p^{t_1+t_2+i-3} = (p-1)\sum_{i=1}^{t_1} p^{t_1+(t_2+1)+i-3}$. This completes the proof. $\square$

When we include all the elements of $\mathbb{Z}_p$ (resp. $\mathbb{Z}_{p^2}$) as coordinates of a vector, we place them in increasing order. We denote by N_p the set $\{0, 1, \dots, p-1\} \subset \mathbb{Z}_{p^2}$ and $N_p^- = N_p \setminus \{0\}$. As before, when including all the elements in those sets as coordinates of a vector, we place them in increasing order. For a set $S \subseteq \mathbb{Z}_{p^2}$ and $\lambda \in \mathbb{Z}_{p^2}$, we define $\lambda S = \{\lambda j : j \in S\}$. For example, $N_3 = \{0, 1, 2\} \subset \mathbb{Z}_9$, $N_3^- = \{1, 2\} \subset \mathbb{Z}_9$, $2N_3^- = \{2, 4\}$, $3\mathbb{Z}_9 = \{0, 3, 6\}$, $(\mathbb{Z}_3, \mathbb{Z}_3) = (0, 1, 2, 0, 1, 2) \in \mathbb{Z}_3^6$ and $(\mathbb{Z}_3 \mid \mathbb{Z}_9, 2N_3^-) = (0, 1, 2 \mid 0, 1, 2, 3, 4, 5, 6, 7, 8, 2, 4) \in \mathbb{Z}_3^3 \times \mathbb{Z}_9^{11}$.

Lemma 3.1. *Let $\lambda \in N_p^-$ and $\mu \in \mathbb{Z}_{p^2}$. Then,*

1. $\lambda \mathbb{Z}_{p^2} + \mu = \mathbb{Z}_{p^2}$,
2. $\lambda p \mathbb{Z}_{p^2} + \mu = \begin{cases} p\mathbb{Z}_{p^2} & \text{if } \mu \in p\mathbb{Z}_{p^2}, \\ p\mathbb{Z}_{p^2} + \mu & \text{if } \mu \in \mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}, \end{cases}$
3. $\lambda(p\mathbb{Z}_{p^2}, \overset{p-1}{\dots}, p\mathbb{Z}_{p^2}) + \mu(\mathbf{1}, \dots, \mathbf{p-1})$ *is a permutation of* $(\mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2})$ *if* $\mu \in N_p^-$.
4. $(\mu N_p^-, \overset{p^2}{\dots}, \mu N_p^-) + \lambda(\mathbf{0}, \dots, \mathbf{p^2-1})$ *is a permutation of* $(\mathbb{Z}_{p^2}, \overset{p-1}{\dots}, \mathbb{Z}_{p^2})$.

Proof. Items 1 and 2 follow from the fact that $\mathbb{Z}_{p^2}$ is a ring and $p\mathbb{Z}_{p^2}$ is a proper ideal of $\mathbb{Z}_{p^2}$.

For Item 3, first, we have that $\lambda p \mathbb{Z}_{p^2} = p\mathbb{Z}_{p^2}$ since $\lambda \in N_p^-$. Then, note that all coordinates of $v = (p\mathbb{Z}_{p^2}, \overset{p-1}{\dots}, p\mathbb{Z}_{p^2}) + \mu(\mathbf{1}, \dots, \mathbf{p-1})$ are distinct elements of $\mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}$. Since v is a vector of length $p^2 - p$ and $|\mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}| = p^2 - p$, we have that v is a permutation of the vector $(\mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2})$.

For Item 4, let $\mu N_p^- = \{a_1, a_2, \dots, a_{p-1}\} \subset \mathbb{Z}_{p^2}$. By Item 1, we have that $\lambda \mathbb{Z}_{p^2} = \mathbb{Z}_{p^2}$. Then, $(\mu N_p^-, \overset{p^2}{\dots}, \mu N_p^-) + \lambda(\mathbf{0}, \dots, \mathbf{p^2-1})$ is a permutation of

$$(\mathbb{Z}_{p^2}, \overset{p-1}{\dots}, \mathbb{Z}_{p^2}) + (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{p-1}), \tag{9}$$

where $\mathbf{a}_i = (a_i, \overset{p^2}{\dots}, a_i)$ for all $i \in \{1, \dots, p-1\}$. Again, since $\mathbb{Z}_{p^2} + a_i = \mathbb{Z}_{p^2}$, (9) is a permutation of $(\mathbb{Z}_{p^2}, \overset{p-1}{\dots}, \mathbb{Z}_{p^2})$. $\square$

Lemma 3.2. *Let $\mathcal{H}_p^{t_1,1}$ be the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code generated by the matrix $A_p^{t_1,1}$ with $t_1 \geq 2$ and p prime. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{t_1}$ be the row vectors of $A_p^{t_1,1}$ of order p^2. Let $\mathbf{v} = (v \mid v') \in \mathcal{H}_p^{t_1,1}$ such that $\mathbf{v} = \sum_{i=1}^{t_1} \lambda_i \mathbf{v}_i$, where $\lambda_i \in N_p$ and at least one $\lambda_i \neq 0$. Then, v' contains every element of $p\mathbb{Z}_{p^2}$ the same number of times and one of the following conditions is satisfied:*

1. *There exist $\lambda \in N_p^-$ such that v' contains every element of λN_p^- the same number of times and every element of $\mathbb{Z}_{p^2} \setminus (p\mathbb{Z}_{p^2} \cup \lambda N_p^-)$ zero times.*

2. *There exist $\lambda \in N_p^-$ such that v' contains every element of λN_p^- the same number of times and every element of $\mathbb{Z}_{p^2} \setminus (p\mathbb{Z}_{p^2} \cup \lambda N_p^-)$ the same number of times.*
3. *Every element of $\mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}$ appears in v' the same number of times.*

Proof. We prove this lemma by induction on t_1. Let $\mathbf{v}_1 = (v_1 \mid v_1')$ and $\mathbf{v}_2 = (v_2 \mid v_2')$ be the row vectors of $A_p^{2,1}$ of order p^2. We have that

$$\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \left(\begin{array}{cccc|cccccc} \mathbb{Z}_p & \mathbb{Z}_p & \cdots & \mathbb{Z}_p & p\mathbb{Z}_{p^2} & \cdots & p\mathbb{Z}_{p^2} & N_p^- & \cdots & N_p^- \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{p-1} & \mathbf{1} & \cdots & \mathbf{p-1} & \mathbf{0} & \cdots & \mathbf{p^2-1} \end{array}\right). \tag{10}$$

Let $\mathbf{v} = (v \mid v') \in \mathcal{H}_p^{2,1}$ such that $\mathbf{v} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2$, where $\lambda_1, \lambda_2 \in N_p$ and $(\lambda_1, \lambda_2) \neq (0,0)$. Thus, $v' = \lambda_1 v_1' + \lambda_2 v_2'$. If $\lambda_2 = 0$, then v' satisfies the first condition since $\lambda_1 p\mathbb{Z}_{p^2} = p\mathbb{Z}_{p^2}$ by Lemma 3.1. If $\lambda_1 = 0$, then v' satisfies the second condition since $\lambda_2\mathbb{Z}_{p^2} = \mathbb{Z}_{p^2}$ by Lemma 3.1. If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, then after applying a suitable permutation of coordinates, we can write v' as

$$\lambda_1(p\mathbb{Z}_{p^2}, \overset{p-1}{\ldots}, p\mathbb{Z}_{p^2}, \mathbf{1}, \ldots, \mathbf{p-1}) + \lambda_2(\mathbf{1}, \ldots, \mathbf{p-1}, \mathbb{Z}_{p^2}, \overset{p-1}{\ldots}, \mathbb{Z}_{p^2}). \tag{11}$$

By Lemma 3.1, we have that $\lambda_1(p\mathbb{Z}_{p^2}, \overset{p-1}{\ldots}, p\mathbb{Z}_{p^2}) + \lambda_2(\mathbf{1}, \ldots, \mathbf{p-1})$ is a permutation of the vector $(\mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2})$ and $\lambda_1(\mathbf{1}, \ldots, \mathbf{p-1}) + \lambda_2(\mathbb{Z}_{p^2}, \overset{p-1}{\ldots}, \mathbb{Z}_{p^2})$ is a permutation of the vector $(\mathbb{Z}_{p^2}, \overset{p-1}{\ldots}, \mathbb{Z}_{p^2})$. Therefore, v' is a permutation of the vector $(\mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}, \mathbb{Z}_{p^2}, \overset{p-1}{\ldots}, \mathbb{Z}_{p^2})$, so it satisfies the third condition. Hence, the statement is true for $t_1 = 2$.

Assume that the lemma holds for the code $\mathcal{H}_p^{t_1,1}$ with $t_1 \geq 2$. Now, we have to show that it is also true for $\mathcal{H}_p^{t_1+1,1}$. Let $\mathbf{w}_1, \ldots, \mathbf{w}_{t_1+1}$ be the row vectors of $A_p^{t_1+1,1}$ of order p^2 such that $\mathbf{w}_{t_1+1} = (\mathbf{0}, \ldots, \mathbf{p-1} \mid \mathbf{1}, \ldots, \mathbf{p-1}, \mathbf{0}, \ldots, \mathbf{p^2-1})$. Let $\mathbf{w} = (w \mid w') = \sum_{i=1}^{t_1+1} \lambda_i \mathbf{w}_i$, where $\lambda_i \in N_p$ and at least one $\lambda_i \neq 0$. We have to show that w' contains every element of $p\mathbb{Z}_{p^2}$ the same number of times and satisfies one of the three conditions.

Let $\{\mathbf{v}_i = (v_i \mid v_i')\}_{1\leq i\leq t_1}$ be the set of all row vectors of $A_p^{t_1,1}$ of order p^2. Let $\mathbf{v} = (v \mid v') = \sum_{i=1}^{t_1} \lambda_i \mathbf{v}_i$. By construction, note that $v = \mathbf{0}$ or v contains every element of $\mathbb{Z}_p$ the same number of times. Indeed, v is a codeword of a simplex code of length p^{t_1} over $\mathbb{Z}_p$. Therefore, $v = \mathbf{0}$ or $v = (\mathbb{Z}_p, \overset{p^{t_1-1}}{\ldots}, \mathbb{Z}_p)$ up to a permutation of coordinates. On the one hand, if $\lambda_i = 0$ for all $i \in \{1, \ldots, t_1\}$, then $\lambda_{t_1+1} \neq 0$. In this case, $w' = \lambda_{t_1+1}(\mathbf{1}, \ldots, \mathbf{p-1}, \mathbf{0}, \ldots, \mathbf{p^2-1})$ and it satisfies the second condition since $\lambda_{t_1+1}\mathbb{Z}_{p^2} = \mathbb{Z}_{p^2}$ by Lemma 3.1.

On the other hand, if there exists $i \in \{1, \ldots, t_1\}$ such that $\lambda_i \neq 0$, then $v = (\mathbb{Z}_p, \overset{p^{t_1-1}}{\ldots}, \mathbb{Z}_p)$. If we consider v over $\mathbb{Z}_{p^2}$, we have that $v = (N_p, \overset{p^{t_1-1}}{\ldots}, N_p)$. In this case, up to a permutation of coordinates, we can write w' as

$$(V, \overset{p-1}{\ldots}, V, v', \overset{p^2}{\ldots}, v') + \lambda_{t_1+1}(\mathbf{1}, \ldots, \mathbf{p-1}, \mathbf{0}, \ldots, \mathbf{p^2-1}), \tag{12}$$

where $V = pv = (p\mathbb{Z}_{p^2}, \overset{p^{t_1-1}}{\ldots}, p\mathbb{Z}_{p^2})$ since $pN_p = p\mathbb{Z}_{p^2}$. If $\lambda_{t_1+1} = 0$, then w' satisfies the same condition as v'. Finally, we consider the general case, when $\lambda_{t_1+1} \neq 0$. Since v'

satisfies one of the three conditions, there is a permutation of coordinates π and $\lambda \in N_p^-$ such that

$$\pi(v') = (p\mathbb{Z}_{p^2}, \overset{m}{\ldots}, p\mathbb{Z}_{p^2}, \lambda N_p^-, \overset{n}{\ldots}, \lambda N_p^-, I_\lambda, \overset{r}{\ldots}, I_\lambda),$$

where $I_\lambda = (\mathbb{Z}_{p^2} \setminus (p\mathbb{Z}_{p^2} \cup \lambda N_p^-))$, for some integers $m, n, r \geq 0$. Note that $r = 0$ if v' satisfies the first condition, and $n = r$ if v' satisfies the third condition. Thus, we can write (12) as

$$(V, \overset{p-1}{\ldots}, V, \pi(v'), \overset{p^2}{\ldots}, \pi(v')) + \lambda_{t_1+1}(\mathbf{1}, \ldots, \mathbf{p-1}, \mathbf{0}, \ldots, \mathbf{p^2-1}). \tag{13}$$

First, by Lemma 3.1, we have that $(p\mathbb{Z}_{p^2}, \overset{p-1}{\ldots}, p\mathbb{Z}_{p^2}) + \lambda_{t_1+1}(\mathbf{1}, \ldots, \mathbf{p-1})$ is a permutation of $(\mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2})$, so $(V, \overset{p-1}{\ldots}, V) + \lambda_{t_1+1}(\mathbf{1}, \ldots, \mathbf{p-1})$ is a permutation of $(\mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}, \overset{p^{t_1-1}}{\ldots}, \mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2})$. Next, by Lemma 3.1, we have that $(p\mathbb{Z}_{p^2}, \overset{p^2}{\ldots}, p\mathbb{Z}_{p^2}) + \lambda_{t_1+1}(\mathbf{0}, \ldots, \mathbf{p^2-1})$ is a permutation of

$$(p\mathbb{Z}_{p^2}, \overset{p}{\ldots}, p\mathbb{Z}_{p^2}, \mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}, \overset{p}{\ldots}, \mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}), \tag{14}$$

and $(\lambda N_p^-, \overset{p^2}{\ldots}, \lambda N_p^-) + \lambda_{t_1+1}(\mathbf{0}, \ldots, \mathbf{p^2-1})$ is a permutation of $(\mathbb{Z}_{p^2}, \overset{p-1}{\ldots}, \mathbb{Z}_{p^2})$. Again by Lemma 3.1, $(I_\lambda, \overset{p^2}{\ldots}, I_\lambda) + \lambda_{t_1+1}(\mathbf{0}, \ldots, \mathbf{p^2-1})$ is a permutation of $(\mathbb{Z}_{p^2}, \overset{p^2-2p+1}{\ldots}, \mathbb{Z}_{p^2})$ since $|I_\lambda| = p^2 - 2p + 1$. Therefore, (13) is a permutation of

$$(\mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}, \overset{k_1}{\ldots}, \mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}, p\mathbb{Z}_{p^2}, \overset{k_2}{\ldots}, p\mathbb{Z}_{p^2}, \mathbb{Z}_{p^2}, \overset{k_3}{\ldots}, \mathbb{Z}_{p^2})$$

for $k_1 = p^{t_1} - 1 + mp$, $k_2 = mp$, and $k_3 = n(p-1) + r(p^2 - 2p + 1)$, so it satisfies the third condition. Note that the elements in $p\mathbb{Z}_{p^2}$ appear $k_2 + k_3$ times and the elements in $\mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}$ appear $k_1 + k_3$ times. $\square$

Example 3.2. Let $\mathbf{v}_1 = (v_1 \mid v_1')$ and $\mathbf{v}_2 = (v_2 \mid v_2')$ be the row vectors of $A_3^{2,1}$ of order 9. We have that

$$\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \left(\begin{array}{ccc|ccccc} \mathbb{Z}_3 & \mathbb{Z}_3 & \mathbb{Z}_3 & 3\mathbb{Z}_9 & 3\mathbb{Z}_9 & N_3^- & \cdots & N_3^- \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{2} & \mathbf{0} & \ldots & \mathbf{8} \end{array}\right).$$

Let $\mathbf{v} = (v \mid v') \in \mathcal{H}_3^{2,1}$ such that $\mathbf{v} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2$, where $\lambda_1, \lambda_2 \in N_3 = \{0, 1, 2\}$ and $(\lambda_1, \lambda_2) \neq (0, 0)$. Thus, $v' = \lambda_1 v_1' + \lambda_2 v_2'$. Now,

$$\begin{aligned} v' &\in \bigcup_{\substack{\lambda_1,\lambda_2 \in N_3 \\ (\lambda_1,\lambda_2)\neq(0,0)}} \{\lambda_1 v_1' + \lambda_2 v_2'\} \\ &= \{v_1', 2v_1', v_2', 2v_2', v_1' + v_2', 2v_1' + v_2', v_1' + 2v_2', 2v_1' + 2v_2'\} \\ &= \{(0, 3, 6, 0, 3, 6, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2), \end{aligned}$$

$$(0,6,3,0,6,3,2,4,2,4,2,4,2,4,2,4,2,4,2,4,2,4,2,4),$$
$$(1,1,1,2,2,2,0,0,1,1,2,2,3,3,4,4,5,5,6,6,7,7,8,8),$$
$$(2,2,2,4,4,4,0,0,2,2,4,4,6,6,8,8,1,1,3,3,5,5,7,7),$$
$$(1,4,7,2,5,8,1,2,2,3,3,4,4,5,5,6,6,7,7,8,8,0,0,1),$$
$$(1,7,4,2,8,5,2,4,3,5,4,6,5,7,6,8,7,0,8,1,0,2,1,3),$$
$$(2,5,8,4,7,1,1,2,3,4,5,6,7,8,0,1,2,3,4,5,6,7,8,0),$$
$$(2,8,5,4,1,7,2,4,4,6,6,8,8,1,1,3,3,5,5,7,7,0,0,2)\}.$$

Note that v'_1 and $2v'_1$ satisfy the first condition of Lemma 3.2, v'_2 and $2v'_2$ satisfy the second condition, and the remaining vectors the third condition. Therefore, v' satisfies Lemma 3.2.

Corollary 3.1. *Let $\mathcal{H}_p^{t_1,t_2}$ be the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code generated by the matrix $A_p^{t_1,t_2}$ with $t_1 \geq 2$, $t_2 \geq 1$, and p prime. Let $\mathbf{u} = (u \mid u') \in \mathcal{H}_p^{t_1,t_2}$ such that $o(\mathbf{u}) = p^2$. Then, u' contains every element of $p\mathbb{Z}_{p^2}$ the same number of times and the remaining coordinates are from $\mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}$.*

Proof. Let $\{\mathbf{v}_i\}_{1\leq i\leq t_1}$ be the set of row vectors of $A_p^{t_1,t_2}$ of order p^2. Since $o(\mathbf{u}) = p^2$, $\mathbf{u}$ can be expressed as $\mathbf{u} = \sum_{i=1}^{t_1} \lambda_i \mathbf{v}_i + \mathbf{w}$, where $\lambda_i \in N_p$, at least one $\lambda_i \neq 0$, and $\mathbf{w}$ is a codeword of order at most p. If $\mathbf{w} = \mathbf{0}$, then from Lemma 3.2 and construction (6), u' holds the property. If $\mathbf{w} \neq \mathbf{0}$, then from Lemma 3.2, construction (6), and Item 2 of Lemma 3.1, u' also holds the property. $\square$

Lemma 3.3. *Let $\mathcal{H}_p^{t_1,1}$ be the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code of type $(\alpha_1, \alpha_2; t_1, 1)$ generated by the matrix $A_p^{t_1,1}$ with $t_1 \geq 2$ and p prime. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{t_1}$ be the row vectors of $A_p^{t_1,1}$ of order p^2. Let $\mathbf{v} = (v \mid v') \in \mathcal{H}_p^{t_1,1}$ such that $\mathbf{v} = \sum_{i=1}^{t_1} \lambda_i p\mathbf{v}_i$, where $\lambda_i \in N_p$ and at least one $\lambda_i \neq 0$. Then, $v = (0, \overset{\alpha_1}{\ldots}, 0)$, and v' contains every element of $p\mathbb{Z}_{p^2} \setminus \{0\}$ exactly $\alpha_1/p + (p\alpha_2 - (p-1)\alpha_1)/p^2$ times and $(p\alpha_2 - (p-1)\alpha_1)/p^2$ times the element 0.*

Proof. We prove this lemma by induction on t_1. Let $\mathbf{v}_1 = (v_1 \mid v'_1)$ and $\mathbf{v}_2 = (v_2 \mid v'_2)$ be the row vectors of $A_p^{2,1}$ of order p^2. Recall that $A_p^{2,1}$ is the matrix given in (10). Let $\mathbf{v} = (v \mid v') \in \mathcal{H}_p^{2,1}$ such that $\mathbf{v} = \lambda_1 p\mathbf{v}_1 + \lambda_2 p\mathbf{v}_2$, where $\lambda_1, \lambda_2 \in N_p$ and $(\lambda_1, \lambda_2) \neq (0, 0)$. Thus, $v = (0, \overset{p^2}{\ldots}, 0)$, and $v' = \lambda_1 p v'_1 + \lambda_2 p v'_2$. If $\lambda_2 = 0$, then the first $p(p-1)$ coordinates of v' are 0 and the remaining $p^2(p-1)$ coordinates contain every element of $p\mathbb{Z}_{p^2} \setminus \{0\}$ exactly p^2 times. If $\lambda_1 = 0$, then the first $p(p-1)$ coordinates of v' contain every element of $p\mathbb{Z}_{p^2} \setminus \{0\}$ exactly p times and the remaining $p^2(p-1)$ coordinates contain every element of $p\mathbb{Z}_{p^2}$ exactly $p(p-1)$ times. Therefore, every element $a \in p\mathbb{Z}_{p^2} \setminus \{0\}$ appears $p + p(p-1) = p^2$ times in v', and the element 0 appears $p(p-1)$ times. If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, then after applying a suitable permutation of coordinates of v', we can write v' as

$$\lambda_1(\mathbf{0}, V, \overset{p}{\ldots}, V) + \lambda_2(V, W, \overset{p}{\ldots}, W),$$

where $\mathbf{0}$ is the all-zero vector of length $(p-1)p$, $V = p(\mathbf{1}, \ldots, \mathbf{p-1})$ of length $(p-1)p$, and $W = (p\mathbb{Z}_{p^2}, \overset{p-1}{\ldots}, p\mathbb{Z}_{p^2})$. Therefore, by Lemma 3.1, we have that every $a \in p\mathbb{Z}_{p^2} \setminus \{0\}$ appears $p + (p-1)p = p^2$ times in v', and the element 0 appears $(p-1)p$ times. Since $\alpha_1 = p^2$ and $\alpha_2 = (p-1)(p+p^2)$ by Proposition 3.1, the statement is true for $t_1 = 2$.

Assume that the lemma holds for the code $\mathcal{H}_p^{t_1,1}$ of type $(\alpha_1, \alpha_2; t_1, 1)$ with $t_1 \geq 2$. Now, we have to show that it is also true for $\mathcal{H}_p^{t_1+1,1}$. Note that the type of $\mathcal{H}_p^{t_1+1,1}$ is $(\alpha_1', \alpha_2'; t_1+1, 1)$, where $\alpha_1' = p\alpha_1$ and $\alpha_2' = (p-1)\alpha_1 + p^2\alpha_2$. Let $\mathbf{w}_1, \ldots, \mathbf{w}_{t_1+1}$ be the row vectors of $A_p^{t_1+1,1}$ of order p^2. Let $\mathbf{w} = (w \mid w') = \sum_{i=1}^{t_1+1} \lambda_i p\mathbf{w}_i$, where $\lambda_i \in N_p$ and at least one $\lambda_i \neq 0$. Let $\{\mathbf{v}_i = (v_i \mid v_i')\}_{1 \leq i \leq t_1}$ be the set of all row vectors of $A_p^{t_1,1}$ of order p^2. Let $\mathbf{v} = (v \mid v') = \sum_{i=1}^{t_1} \lambda_i \mathbf{v}_i$. We have that

$$w' = (\mathbf{0}, \overset{p-1}{\ldots}, \mathbf{0}, pv', \overset{p^2}{\ldots}, pv') + \lambda_{t_1+1} p(\mathbf{1}, \ldots, \mathbf{p-1}, \mathbf{0}, \ldots, \mathbf{p^2-1}).$$

If $\lambda_{t_1+1} = 0$, there is at least one $i \in \{1, \ldots, t_1\}$ such that $\lambda_i \neq 0$, so $v' \neq 0$. By induction hypothesis, every $a \in p\mathbb{Z}_{p^2} \setminus \{0\}$ appears

$$p^2\left(\frac{\alpha_1}{p} + \frac{p\alpha_2 - (p-1)\alpha_1}{p^2}\right) = \alpha_1 + p\alpha_2 = \frac{\alpha_1'}{p} + \frac{p\alpha_2' - (p-1)\alpha_1'}{p^2}$$

times in w', and the element 0 appears

$$\alpha_1(p-1) + p^2\left(\frac{p\alpha_2 - (p-1)\alpha_1}{p^2}\right) = p\alpha_2 = \frac{p\alpha_2' - (p-1)\alpha_1'}{p^2}$$

times. Now, assume that $\lambda_{t_1+1} \neq 0$. Note that $p(0, 1, \ldots, p^2-1) = (p\mathbb{Z}_{p^2}, \overset{p}{\ldots}, p\mathbb{Z}_{p^2})$. Then, by Lemma 3.1, we have that $(pv', \overset{p^2}{\ldots}, pv') + \lambda_{t_1+1} p(\mathbf{0}, \ldots, \mathbf{p^2-1})$ contains every element of $p\mathbb{Z}_{p^2}$ exactly $p^2\alpha_2/p = p\alpha_2$ times. Therefore, since every $a \in p\mathbb{Z}_{p^2} \setminus \{0\}$ appears $\alpha_1 + p\alpha_2$ times in w', and the element 0 appears $p\alpha_2$ times, the result follows. □

Example 3.3. Let $\mathbf{v}_1 = (v_1 \mid v_1')$ and $\mathbf{v}_2 = (v_2 \mid v_2')$ be the row vectors of $A_3^{2,1}$ of order 9, which are shown in Example 3.2. Let $\mathbf{v} = (v \mid v') \in \mathcal{H}_3^{2,1}$ such that $\mathbf{v} = \lambda_1 3\mathbf{v}_1 + \lambda_2 3\mathbf{v}_2$, where $\lambda_1, \lambda_2 \in N_3 = \{0, 1, 2\}$ and $(\lambda_1, \lambda_2) \neq (0, 0)$. Thus, $v' = \lambda_1 3v_1' + \lambda_2 3v_2'$. Now,

$$\begin{aligned} v' &\in \bigcup_{\substack{\lambda_1,\lambda_2 \in N_3 \\ (\lambda_1,\lambda_2) \neq (0,0)}} \{\lambda_1 3v_1' + \lambda_2 3v_2'\} \\ &= \{3v_1', 6v_1', 3v_2', 6v_2', 3v_1' + 3v_2', 6v_1' + 3v_2', 3v_1' + 6v_2', 6v_1' + 6v_2'\} \\ &= \{(0,0,0,0,0,0,3,6,3,6,3,6,3,6,3,6,3,6,3,6,3,6,3,6), \\ &\quad (0,0,0,0,0,0,6,3,6,3,6,3,6,3,6,3,6,3,6,3,6,3,6,3), \\ &\quad (3,3,3,6,6,6,0,0,3,3,6,6,0,0,3,3,6,6,0,0,3,3,6,6), \end{aligned}$$

$$(6,6,6,3,3,3,0,0,6,6,3,3,0,0,6,6,3,3,0,0,6,6,3,3),$$
$$(3,3,3,6,6,6,3,6,6,0,0,3,3,6,6,0,0,3,3,6,6,0,0,3),$$
$$(3,3,3,6,6,6,6,3,0,6,3,0,6,3,0,6,3,0,6,3,0,6,3,0),$$
$$(6,6,6,3,3,3,3,6,0,3,6,0,3,6,0,3,6,0,3,6,0,3,6,0),$$
$$(6,6,6,3,3,3,6,3,3,0,0,6,6,3,3,0,0,6,6,3,3,0,0,6)\}.$$

Note that, in any case, every element $a \in 3\mathbb{Z}_9 \setminus \{0\} = \{3,6\}$ appears 9 times in v', and the element 0 appears 6 times.

Theorem 3.1. *The $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code $\mathcal{H}_p^{1,1}$ generated by the matrix*

$$A_p^{1,1} = \left(\begin{array}{cccc|cccc} 1 & 1 & \cdots & 1 & p & p & \cdots & p \\ 0 & 1 & \cdots & p-1 & 1 & 2 & \cdots & p-1 \end{array}\right)$$

is a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(p, p-1; 1, 1)$.

Proof. Note that $\mathcal{H}_p^{1,1}$ can be partitioned into p disjoint sets $A_0, A_1, \ldots, A_{p-1}$, where $A_0 = \{\lambda(0,1,\ldots,p-1 \mid 1,2,\ldots,p-1) : \lambda \in \mathbb{Z}_{p^2}\}$ and $A_i = A_0 + i(\mathbf{1} \mid \mathbf{p})$, $i \in \{1,\ldots,p-1\}$. Therefore, we can also partition $\Phi(\mathcal{H}_p^{1,1})$ into p disjoint sets $\Phi(A_0), \Phi(A_1), \ldots, \Phi(A_{p-1})$, where $\Phi(A_i) = \Phi(A_0) + i \cdot \mathbf{1}$, $i \in \{1,\ldots,p-1\}$, by Lemma 2.1. Thus, it is enough to show that $\Phi(A_0)$ is a GH matrix $H(p,p)$. We take two distinct elements $\mathbf{u}, \mathbf{v} \in A_0$. We have to show that $\Phi(\mathbf{u}) - \Phi(\mathbf{v})$ contains every element of $\mathbb{Z}_p$ exactly p times.

Let $\mathbf{u} = \lambda_1(0,1,\ldots,p-1 \mid 1,2,\ldots,p-1)$ and $\mathbf{v} = \lambda_2(0,1,\ldots,p-1 \mid 1,2,\ldots,p-1)$, where $\lambda_1 \neq \lambda_2 \in \mathbb{Z}_{p^2}$. Then, $\mathbf{u} - \mathbf{v} = (\lambda_1 - \lambda_2)(0,1,\ldots,p-1 \mid 1,2,\ldots,p-1) = (x_0,\ldots,x_{p-1} \mid y_1,\ldots,y_{p-1})$. Now. we consider two cases. On the one hand, if $\lambda_1 - \lambda_2 \in p\mathbb{Z}_{p^2} \setminus \{0\}$, then the first p coordinates of $\mathbf{u} - \mathbf{v}$ are 0, and the last $p-1$ coordinates contain exactly all the elements of order p from $\mathbb{Z}_{p^2}$, that is, the elements μp for $\mu \in \{1,\ldots,p-1\}$. Since $\phi(\mu p) = (\mu,\mu,\ldots,\mu)$, $\Phi(\mathbf{u} - \mathbf{v})$ contains every element of $\mathbb{Z}_p$ exactly p times. Therefore, $\Phi(\mathbf{u}) - \Phi(\mathbf{v})$ contains every element of $\mathbb{Z}_p$ exactly p times, by Proposition 2.1. On the other hand, if $\lambda_1 - \lambda_2 \in \mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}$, then $y_i \in \mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}$, $i \in \{1,\ldots,p-1\}$. In the first p coordinates, $\Phi(\mathbf{u}) - \Phi(\mathbf{v})$ coincides with $(\lambda_1 - \lambda_2)(0,1,\ldots,p-1)$, so in these coordinates, it contains every element of $\mathbb{Z}_p$ once. Since $y_i \in \mathbb{Z}_{p^2} \setminus p\mathbb{Z}_{p^2}$, $\Phi(\mathbf{u}) - \Phi(\mathbf{v})$ contains every element of $\mathbb{Z}_p$ exactly $1 + 1 \cdot (p-1) = p$ times, by Proposition 2.1. Therefore, in any case, $\Phi(A_0)$ is a GH matrix $H(p,p)$. $\square$

Example 3.4. The $\mathbb{Z}_3\mathbb{Z}_9$-additive code $\mathcal{H}_3^{1,1}$ generated by the matrix $A_3^{1,1}$, given in Example 3.1, is a $\mathbb{Z}_3\mathbb{Z}_9$-additive GH code of type $(3,2;1,1)$. Indeed, we have that $H_3^{1,1} = \Phi(\mathcal{H}_3^{1,1}) = \cup_{\lambda \in \mathbb{Z}_3}(\Phi(A_0) + \lambda\mathbf{1})$, where $A_0 = \{\lambda(0,1,2 \mid 1,2) : \lambda \in \mathbb{Z}_9\}$, and then $\Phi(A_0)$ consists of all the rows of the GH matrix

$$H(3,3)=\begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&1&2&0&1&2&0&2&1\\ 0&2&1&0&2&1&1&2&0\\ 0&0&0&1&1&1&2&2&2\\ 0&1&2&1&2&0&2&1&0\\ 0&2&1&1&0&2&0&1&2\\ 0&0&0&2&2&2&1&1&1\\ 0&1&2&2&0&1&1&0&2\\ 0&2&1&2&1&0&2&0&1 \end{pmatrix}. \tag{15}$$

The $\mathbb{Z}_3\mathbb{Z}_9$-linear code $H_3^{1,1}$ has length $N=9$, $pN=3\cdot 9=27$ codewords and minimum distance $N(p-1)/p=9(3-1)/3=6$.

Proposition 3.2. *The $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code $\mathcal{H}_p^{t_1,1}$ generated by the matrix $A_p^{t_1,1}$, with $t_1\geq 2$ and p prime, is a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code.*

Proof. Let $\mathcal{H}_p^{t_1,1}$ be the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code of type $(\alpha_1,\alpha_2;t_1,1)$ generated by $A_p^{t_1,1}$. We can write that $\mathcal{H}_p^{t_1,1}=\cup_{\lambda\in\mathbb{Z}_p}(A_0+\lambda\cdot(\mathbf{1}\mid\mathbf{p}))$, where A_0 is the set of all codewords of the code generated by $A_p^{t_1,1}$ after removing the row $(\mathbf{1}\mid\mathbf{p})$. Let $H_p^{t_1,1}=\Phi(\mathcal{H}_p^{t_1,1})$. By Lemma 2.1, $H_p^{t_1,1}=\cup_{\lambda\in\mathbb{Z}_p}(\Phi(A_0)+\lambda\cdot\mathbf{1})$. If we prove that $\Phi(A_0)$ corresponds to the rows of a GH matrix $H(p,\alpha_2+\alpha_1/p)$, then the result follows. We take two distinct elements $\mathbf{u},\mathbf{v}\in A_0$. Now, we have to show that $\Phi(\mathbf{u})-\Phi(\mathbf{v})$ contains every element of $\mathbb{Z}_p$ exactly $\alpha_2+\alpha_1/p$ times.

We consider two cases depending on the order of $\mathbf{u}-\mathbf{v}$. First, let $\mathrm{o}(\mathbf{u}-\mathbf{v})=p$. Then, by Lemma 3.3, the number of 0 in $\Phi(\mathbf{u}-\mathbf{v})$ is

$$\alpha_1+p\cdot\frac{p\alpha_2-(p-1)\alpha_1}{p^2}=\alpha_2+\frac{\alpha_1}{p},$$

and the number of times an element $a\in\mathbb{Z}_p\setminus\{0\}$ appears in $\Phi(\mathbf{u}-\mathbf{v})$ is

$$p\cdot(\frac{\alpha_1}{p}+\frac{p\alpha_2-(p-1)\alpha_1}{p^2})=\alpha_2+\frac{\alpha_1}{p}.$$

Thus, in this case, $\Phi(\mathbf{u}-\mathbf{v})=\Phi(\mathbf{u})-\Phi(\mathbf{v})$ contains every element of $\mathbb{Z}_p$ exactly $\alpha_2+\alpha_1/p$ times, by Proposition 2.1.

Second, let $\mathrm{o}(\mathbf{u}-\mathbf{v})=p^2$. Let $\mathbf{u}-\mathbf{v}=(z\mid z')$. We have that z contains every element of $\mathbb{Z}_p$ exactly α_1/p times. Moreover, by Corollary 3.1, z' contains every element of $p\mathbb{Z}_{p^2}$ exactly α times, $\alpha>0$, and the remaining $\alpha_2-p\alpha$ coordinates of z' are from $\mathbb{Z}_{p^2}\setminus p\mathbb{Z}_{p^2}$. Therefore, by Proposition 2.1, $\Phi(\mathbf{u})-\Phi(\mathbf{v})$ contains every element of $\mathbb{Z}_p$ exactly $\alpha_1/p+p\alpha+(\alpha_2-p\alpha)\cdot 1=\alpha_2+\alpha_1/p$ times. $\square$

Proposition 3.3. *Let $\mathcal{H}_p^{t_1,t_2}$ be a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(\alpha_1, \alpha_2; t_1, t_2)$ with $t_1, t_2 \geq 1$ and p prime. Then, with the above construction (6), $\mathcal{H}_p^{t_1,t_2+1}$ is a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(p\alpha_1, p\alpha_2; t_1, t_2 + 1)$.*

Proof. We can write that $\mathcal{H}_p^{t_1,t_2} = \cup_{\lambda\in\mathbb{Z}_p}(A_0 + \lambda \cdot (\mathbf{1} \mid \mathbf{p}))$, where A_0 is the set of all codewords of the code generated by A^{t_1,t_2} after removing the row $(\mathbf{1} \mid \mathbf{p})$. Let $H_p^{t_1,t_2} = \Phi(\mathcal{H}_p^{t_1,t_2})$. By Lemma 2.1, $H_p^{t_1,t_2} = \cup_{\lambda\in\mathbb{Z}_p}(\Phi(A_0) + \lambda \cdot \mathbf{1})$. Since $\mathcal{H}_p^{t_1,t_2}$ is a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(\alpha_1, \alpha_2; t_1, t_2)$, then $\Phi(A_0)$ corresponds to the rows of a GH matrix $H(p, \alpha_2 + \alpha_1/p)$.

Let $H_p^{t_1,t_2+1} = \Phi(\mathcal{H}_p^{t_1,t_2+1})$. By applying a suitable permutation on the coordinates of the elements of $H_p^{t_1,t_2+1}$, we can get an equivalent code $H_*^{t_1,t_2+1}$ of length $p\alpha_1 + p^2\alpha_2$ such that $H_*^{t_1,t_2+1} = \cup_{\mu\in\mathbb{Z}_p}(\Phi(B_0)+\mu\cdot\mathbf{1})$, where $\Phi(B_0) = \{(\Phi(A_0), \Phi(A_0), \dots, \Phi(A_0))+ \lambda \cdot (\mathbf{0}, \mathbf{1}, \dots, \mathbf{p-1}) : \lambda \in \mathbb{Z}_p\}$. Since $\Phi(A_0)$ corresponds to the rows of a GH matrix $H(p, \alpha_2 + \alpha_1/p)$, then $\Phi(B_0)$ corresponds to the rows of a GH matrix $H(p, \alpha_1 + p\alpha_2)$. Therefore, $\mathcal{H}_p^{t_1,t_2+1}$ is a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(p\alpha_1, p\alpha_2; t_1, t_2 + 1)$. □

Theorem 3.2. *The $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive code $\mathcal{H}_p^{t_1,t_2}$ generated by the matrix $A_p^{t_1,t_2}$, with $t_1, t_2 \geq 1$ and p prime, is a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code.*

Proof. It follows from Theorem 3.1, and Propositions 3.2 and 3.3. □

Example 3.5. Let $\mathcal{H}_3^{1,2}$ be the $\mathbb{Z}_3\mathbb{Z}_9$-additive code generated by the matrix $A_3^{1,2}$ given in Example 3.1. By Theorem 3.2, $H_3^{1,2} = \Phi(\mathcal{H}_3^{1,2})$ is a $\mathbb{Z}_3\mathbb{Z}_9$-linear GH code of type $(9, 6; 1, 2)$. Actually, we can write $H_3^{1,2} = \cup_{\lambda\in\mathbb{Z}_3}(F_H + \lambda\mathbf{1})$, where F_H consists of all the rows of a GH matrix $H(3, 9)$. Also, note that $H_3^{1,2}$ has length $N = 27$, $pN = 3 \cdot 27 = 81$ codewords and minimum distance $N(p - 1)/p = 27(3 - 1)/3 = 18$.

Example 3.6. Let $\mathcal{H}_3^{2,1}$ be the $\mathbb{Z}_3\mathbb{Z}_9$-additive code generated by the matrix $A_3^{2,1}$ given in Example 3.1. By Theorem 3.2, $H_3^{2,1} = \Phi(\mathcal{H}_3^{2,1})$ is a $\mathbb{Z}_3\mathbb{Z}_9$-linear GH code of type $(9, 24; 2, 1)$, which has length $N = 81$, $pN = 3 \cdot 81 = 243$ codewords and minimum distance $N(p - 1)/p = 81(3 - 1)/3 = 54$.

Proposition 3.4. *Let $\mathcal{H}_p^{t_1,t_2}$ be a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(\alpha_1, \alpha_2; t_1, t_2)$ with $t_1, t_2 \geq 1$ and p prime. Let $H_p^{t_1,t_2}$ be the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code of length p^t, with $t \geq 2$. Then, $\alpha_1 = p^{t-t_1}$, $\alpha_2 = p^{t-1} - p^{t-t_1-1}$ and $t = 2t_1 + t_2 - 1$.*

Proof. Since $H_p^{t_1,t_2}$ is a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code of length $\alpha_1 + p\alpha_2 = p^t$, then $|H_p^{t_1,t_2}| = p \cdot p^t = p^{t+1}$. Note that $|H_p^{t_1,t_2}| = |\mathcal{H}_p^{t_1,t_2}| = p^{2t_1+t_2}$, and hence $t = 2t_1 + t_2 - 1$. By Proposition 3.1, $\alpha_1 = p^{t_1+t_2-1} = p^{t-t_1}$. Then, since $\alpha_1 + p\alpha_2 = p^t$, $\alpha_2 = p^{t-1} - p^{t-t_1-1}$. □

Remark 3.1. Let $\mathcal{H} = \mathcal{H}_p^{t_1,t_2}$ be a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(\alpha_1, \alpha_2; t_1, t_2)$ with $t_1, t_2 \geq 1$ and p prime. Let $H = \Phi(\mathcal{H}_p^{t_1,t_2})$ be the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code of length $\alpha_1 + p\alpha_2$. Then, since H is a GH code, its minimum distance is

$$\frac{(p-1)(\alpha_1 + p\alpha_2)}{p}.$$

Let $\mathcal{H}_1$ be the punctured code of $\mathcal{H}$ by deleting the last α_2 coordinates over $\mathbb{Z}_{p^2}$. Note that, by construction, $\mathcal{H}_1$ is a GH code over $\mathbb{Z}_p$ of length α_1 and minimum distance $(p-1)\alpha_1/p$.

Remark 3.2. Since the length of the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH code $\Phi(\mathcal{H}_p^{1,1})$ is p^2, its minimum distance is $(p-1)p^2/p = p(p-1)$ by Remark 3.1.

Remark 3.3. The above constructions (6) and (7) give always $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_2 \neq 0$ since the starting matrix $A_p^{1,1}$ has $\alpha_2 \neq 0$. If $\alpha_2 = 0$, the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes coincide with the codes obtained from a Sylvester GH matrix, so they are always linear and of type $(p^{t_2-1}, 0; 0, t_2)$ [12]. Therefore, we only focus on the ones with $\alpha_2 \neq 0$ to study whether they are linear or not.

4. Same type equivalent $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes

In this section, we see that if we consider other starting matrices, instead of the matrix $A_p^{1,1}$ given in (5), and apply the same recursive constructions (6) and (7), or (6) and a new construction more general than (7), we also obtain $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH codes with $\alpha_1 \neq 0$. Indeed, the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes, after applying the Gray map Φ, are permutation equivalent to the codes $\Phi(\mathcal{H}_p^{t_1,t_2})$ of the same type constructed in Section 3.

Let $\mathcal{S}_n$ be the symmetric group of permutations on the set $\{1, \dots, n\}$. A permutation $\pi \in \mathcal{S}_n$ acts linearly on vectors $(c_1, \dots, c_n) \in \mathbb{Z}_p^n$ by permuting their coordinates as follows: $\pi(c_1, \dots, c_n) = (c_{\pi^{-1}(1)}, \dots, c_{\pi^{-1}(n)})$. Given two permutations $\pi_1 \in \mathcal{S}_n$ and $\pi_2 \in \mathcal{S}_m$, we define $(\pi_1|\pi_2) \in \mathcal{S}_{n+m}$, where π_1 acts on the coordinates $\{1, \dots, n\}$ and π_2 on $\{n+1, \dots, n+m\}$.

Two codes C_1 and C_2 over $\mathbb{Z}_p$ of length n are said to be monomially equivalent (or just equivalent) provided there is a monomial matrix M such that $C_2 = \{\mathbf{c}M : \mathbf{c} \in C_1\}$. Recall that a monomial matrix is a square matrix with exactly one nonzero entry in each row and column. They are said to be permutation equivalent if there is a permutation matrix P such that $C_2 = \{\mathbf{c}P : \mathbf{c} \in C_1\}$. Recall that a permutation matrix is a square matrix with exactly one 1 in each row and column and 0s elsewhere. A permutation matrix represents a permutation of coordinates, so we can also say that they are permutation equivalent if there is a permutation of coordinates $\pi \in \mathcal{S}_n$ such that $C_2 = \{\pi(\mathbf{c}) : \mathbf{c} \in C_1\}$.

Remark 4.1. Let $\tau = (\tau_1 \mid \tau_2) \in \mathcal{S}_{\alpha_1+\alpha_2}$. Then, there is a permutation $\tau' = (\tau_1 \mid \tau_2') \in \mathcal{S}_{\alpha_1+p\alpha_2}$ such that, for all $\mathbf{u} \in \mathbb{Z}_p^{\alpha_1} \times \mathbb{Z}_{p^2}^{\alpha_2}$,

$$\Phi(\tau(\mathbf{u})) = \tau'(\Phi(\mathbf{u})).$$

Example 4.1. Let $p = 3$ and $\alpha_1 = \alpha_2 = 2$. Let $\tau = (1,2)(3,4) \in \mathcal{S}_4$. Then, $\tau' = (1,2)(3,6)(4,7)(5,8) \in \mathcal{S}_8$. For $\mathbf{u} = (u_1, u_2 \mid u_1', u_2') \in \mathbb{Z}_3^2 \times \mathbb{Z}_9^2$, $\tau(\mathbf{u}) = (u_2, u_1 \mid u_2', u_1')$ and we have $\Phi(\tau(\mathbf{u})) = (u_2, u_1, u_{2,1}', u_{2,2}', u_{2,3}', u_{1,1}', u_{1,2}', u_{1,3}')$, where $\phi(u_i') = (u_{i,1}', u_{i,2}', u_{i,3}')$ for $i \in \{1,2\}$. It is easy to see that $\Phi(\tau(\mathbf{u})) = \tau'(\Phi(\mathbf{u}))$.

Remark 4.2. Let $a_i = i + \mu_i p \in \mathbb{Z}_{p^2}$, where $i \in N_p^-$ and $\mu_i \in N_p$. Since $\phi(a_i)$ and $\phi(i)$ contain every element of $\mathbb{Z}_p$ exactly once by the definition of ϕ, there is a permutation $\pi_i \in \mathcal{S}_p$ such that $\pi_i(\phi(a_i)) = \phi(i)$ for all $i \in \{1, 2, \dots, p-1\}$. Moreover, from this permutation π_i, we can define a permutation $\sigma_i \in \mathcal{S}_{p^2}$ on the set of the elements of $\mathbb{Z}_{p^2}$ such that $\sigma_i(c) = b$ if and only if $\pi_i^{-1}(\phi(c)) = \phi(b)$. Then, $\pi_i(\phi(b)) = \phi(c)$ and $\pi_i(\phi(\sigma_i(c))) = \phi(c)$, or equivalently,

$$(\pi_i \mid \cdots \mid \pi_i)(\Phi(\sigma_i(w))) = \Phi(w), \tag{16}$$

where $w = (0, 1, \dots, p^2 - 1)$.

Example 4.2. Let $p = 3$. Recall that the Gray map from $\mathbb{Z}_9$ to $\mathbb{Z}_3^3$ is defined as

$$\begin{aligned}
&\phi(0) = (0,0,0),\ \phi(1) = (0,1,2),\ \phi(2) = (0,2,1),\\
&\phi(3) = (1,1,1),\ \phi(4) = (1,2,0),\ \phi(5) = (1,0,2),\\
&\phi(6) = (2,2,2),\ \phi(7) = (2,0,1),\ \phi(8) = (2,1,0).
\end{aligned}$$

For $a_1 = 4 \in \mathbb{Z}_9$, since $\phi(4) = (1,2,0)$ and $\phi(1) = (0,1,2)$, $\pi_1 = (1,2,3) \in \mathcal{S}_3$. For $a_2 = 5 \in \mathbb{Z}_9$, since $\phi(5) = (1,0,2)$ and $\phi(2) = (0,2,1)$, $\pi_2 = (1,3,2) \in \mathcal{S}_3$. The permutation $\sigma_1 \in \mathcal{S}_9$ satisfies (16), that is,

$$\begin{aligned}
&\Phi(\sigma_1(0,1,2,3,4,5,6,7,8)) = (\pi_1^{-1} \mid \cdots \mid \pi_1^{-1})(\Phi(0,1,2,3,4,5,6,7,8)) =\\
&\quad(0,0,0,\ 1,2,0,\ 2,1,0,\ 1,1,1,\ 2,0,1,\ 0,2,1,\ 2,2,2,\ 0,1,2,\ 1,0,2) =\\
&\quad\Phi(0,4,8,3,7,2,6,1,5).
\end{aligned}$$

Therefore, $\sigma_1(0,1,2,3,4,5,6,7,8) = (0,4,8,3,7,2,6,1,5)$, that is, $\sigma_1 = (2,8,5)(3,6,9) \in \mathcal{S}_9$. Similarly, $\sigma_2(0,1,2,3,4,5,6,7,8) = (0,7,5,3,1,8,6,4,2)$ and $\sigma_2 = (2,5,8)(3,9,6) \in \mathcal{S}_9$. Then, for $i \in \{1,2\}$, we have that

$$(\pi_i \mid \cdots \mid \pi_i)(\Phi(\sigma_i(0,1,2,3,4,5,6,7,8))) = \Phi(0,1,2,3,4,5,6,7,8).$$

Lemma 4.1. *Let $a, a', b \in \mathbb{Z}_{p^2}$, such that $a \equiv a' \mod p$. Then, $p(a \odot_p b) = p(a' \odot_p b)$.*

Proof. Let $[a_0, a_1]_p$, $[a'_0, a'_1]_p$, and $[b_0, b_1]_p$ be the p-ary expansions of a, a', and b, respectively. By definition, we have that $a \odot_p b = \xi_0 + \xi_1 p$ and $a' \odot_p b = \xi'_0 + \xi'_1 p$, where

$$\xi_i = \begin{cases} 1 & \text{if} \quad a_i + b_i \geq p, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \xi'_i = \begin{cases} 1 & \text{if} \quad a'_i + b_i \geq p, \\ 0 & \text{otherwise.} \end{cases}$$

Since $a \equiv a' \mod p$, we have that $a_0 = a'_0$ and hence $\xi_0 = \xi'_0$. Therefore, $p(a \odot_p b) = \xi_0 p = \xi'_0 p = p(a' \odot_p b)$. □

Corollary 4.1. *Let $a, b, \bar{a}, \bar{b} \in \mathbb{Z}_{p^2}$, such that $\bar{a} \equiv a \mod p$ and $\bar{b} \equiv b \mod p$. Then,*

$$p(a \odot_p b) = p(\bar{a} \odot_p \bar{b}).$$

Lemma 4.2. *Let $a_i = i + \mu_i p \in \mathbb{Z}_{p^2}$, where $i \in N_p^-$ and $\mu_i \in N_p$. Let $\pi_i \in \mathcal{S}_p$ such that $\pi_i(\phi(a_i)) = \phi(i)$. Let $b, c \in \mathbb{Z}_{p^2}$ satisfying $\pi_i(\phi(b)) = \phi(c)$. Then, $b \equiv c \mod p$ and $\pi_i(\phi(\lambda b)) = \phi(\lambda c)$ for all $\lambda \in N_p$.*

Proof. First, we show that $b \equiv c \mod p$. Let $y = (0, 1, \ldots, p-1) \in \mathbb{Z}_p^p$. By the definition of ϕ, $\phi(a_i) = iy + \mu_i \cdot \mathbf{1}$. Thus, $\pi_i(\phi(a_i)) = i(\pi_i(y)) + \mu_i \cdot \mathbf{1}$. Since $\pi_i(\phi(a_i)) = \phi(i)$, we have that

$$i(\pi_i(y)) + \mu_i \cdot \mathbf{1} = \phi(i). \tag{17}$$

Let $b = j + \mu_j p$ and $c = k + \mu_k p$, where $j, k, \mu_j, \mu_k \in N_p$. By the definition of ϕ, $\phi(b) = jy + \mu_j \cdot \mathbf{1}$ and $\phi(c) = ky + \mu_k \cdot \mathbf{1}$. Thus, $\pi_i(\phi(b)) = j\pi_i(y) + \mu_j \cdot \mathbf{1}$. Since $\pi_i(\phi(b)) = \phi(c)$, we have that $j\pi_i(y) + \mu_j \cdot \mathbf{1} = ky + \mu_k \cdot \mathbf{1}$. Then, $ij\pi_i(y) + i\mu_j \cdot \mathbf{1} = iky + i\mu_k \cdot \mathbf{1}$. By (17), we have that $ji(\pi_i(y)) + j\mu_i \cdot \mathbf{1} = j\phi(i)$ and then

$$j\phi(i) - j\mu_i \cdot \mathbf{1} + i\mu_j \cdot \mathbf{1} = iky + i\mu_k \cdot \mathbf{1}. \tag{18}$$

By the definition of ϕ, $\phi(i) = iy$. Thus, from (18), we have that $(j-k)iy + (i\mu_j - j\mu_i - i\mu_k) \cdot \mathbf{1} = \mathbf{0}$, and therefore, $j = k$, that is, $b \equiv c \mod p$.

Now, we prove that $\pi_i(\phi(\lambda b)) = \phi(\lambda c)$ for all $\lambda \in N_p$, by induction on λ. For $\lambda = 0$, it is true trivially, and for $\lambda = 1$, it is true by the condition given in the statement. Assume that the lemma is true for $\lambda \in N_p^-$. By Corollaries 2.1, 2.2, and 2.3, $\phi((\lambda+1)b) = \phi(\lambda b) + \phi(b) + \phi(p(\lambda b \odot_p b))$. Then,

$$\begin{aligned} \pi_i(\phi((\lambda+1)b)) &= \pi_i(\phi(\lambda b)) + \pi_i(\phi(b)) + \pi_i(\phi(p(\lambda b \odot_p b))) \\ &= \phi(\lambda c) + \phi(c) + \pi_i(\phi(p(\lambda b \odot_p b))), \end{aligned} \tag{19}$$

by induction hypothesis. By the definition of ϕ, $\pi_i(\phi(p(\lambda b \odot_p b))) = \phi(p(\lambda b \odot_p b))$. Since $b \equiv c \mod p$, we have that $p(\lambda b \odot_p b) = p(\lambda c \odot_p c)$ by Corollary 4.1. Therefore, from (19), $\pi_i(\phi((\lambda+1)b)) = \phi(\lambda c) + \phi(c) + \phi(p(\lambda c \odot_p c))) = \phi((\lambda+1)c)$. This completes the proof. □

Corollary 4.2. *Let $a_i = i + \mu_i p \in \mathbb{Z}_{p^2}$, where $i \in N_p^-$ and $\mu_i \in N_p$. Let $\pi_i \in \mathcal{S}_p$ such that $\pi_i(\phi(a_i)) = \phi(i)$. Then, $\pi_i(\phi(\lambda a_i)) = \phi(\lambda i)$ for all $\lambda \in N_p$.*

Proposition 4.1. *Let $a = (a_1, \dots, a_{p-1}) \in \mathbb{Z}_{p^2}^{p-1}$ such that $\{pa_1, \dots, pa_{p-1}\} = p\mathbb{Z}_{p^2} \setminus \{0\}$. Consider the matrix*

$$A_{p,a}^{1,1} = \left(\begin{array}{cccc|cccc} 1 & 1 & \cdots & 1 & p & p & \cdots & p \\ 0 & 1 & \cdots & p-1 & a_1 & a_2 & \cdots & a_{p-1} \end{array}\right).$$

The code generated by $A_{p,a}^{1,1}$, denoted by $\mathcal{H}_{p,a}^{1,1}$, is a $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code of type $(p, p-1; 1, 1)$. Moreover, the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear code is permutation equivalent to $\Phi(\mathcal{H}_p^{1,1})$.

Proof. First, note that there is a permutation $\bar{\sigma} \in \mathcal{S}_{p-1}$ such that $\bar{\sigma}(a_1, \dots, a_{p-1}) = (a'_1, \dots, a'_{p-1})$, where $(pa'_1, \dots, pa'_{p-1}) = (p, 2p, \dots, (p-1)p)$. Thus, there is a permutation $\sigma = (\mathrm{Id} \mid \bar{\sigma}) \in \mathcal{S}_{2p-1}$ such that $\sigma(\mathcal{H}_{p,a}^{1,1}) = \mathcal{H}_{p,a'}^{1,1}$, where $a' = (a'_1, \dots, a'_{p-1})$. By Remark 4.1, σ induces a permutation in the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear code. Therefore, we can assume that the coordinates of $a = (a_1, \dots, a_2)$ are in such an order that $(pa_1, \dots, pa_{p-1}) = (p, 2p, \dots, (p-1)p)$, that is, $a_i = i + \mu_i p$, where $i \in N_p^-$ and $\mu_i \in N_p$.

For any $i \in N_p^-$, by Remark 4.2, there is a permutation $\pi_i \in \mathcal{S}_p$ such that $\pi_i(\phi(a_i)) = \phi(i)$. Let $\pi_{1,1} = (\mathrm{Id} \mid \pi_1 \mid \pi_2 \mid \cdots \mid \pi_{p-1}) \in \mathcal{S}_{p^2}$, where $\mathrm{Id} \in \mathcal{S}_p$ is the identity permutation. Next, we show that $\pi_{1,1}(\Phi(\mathcal{H}_{p,a}^{1,1})) = \Phi(\mathcal{H}_p^{1,1})$. Consider the matrices

$$A_{p,a}^{1,1} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \left(\begin{array}{c|c} \mathbf{1} & \mathbf{p} \\ \mathbb{Z}_p & a \end{array}\right) \text{ and } A_p^{1,1} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \left(\begin{array}{c|c} \mathbf{1} & \mathbf{p} \\ \mathbb{Z}_p & N_p^- \end{array}\right).$$

Note that $\mathbf{v}_1 = \mathbf{u}_1$ and $p\mathbf{v}_2 = p\mathbf{u}_2$. If $\mathbf{x} \in \mathcal{H}_{p,a}^{1,1}$, then $\mathbf{x}$ can be expressed as $\mathbf{x} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2$, where $\alpha \in N_p \subset \mathbb{Z}_{p^2}$ and $\beta \in \mathbb{Z}_{p^2}$. By Corollary 2.3, we have that $\Phi(\mathbf{x}) = \Phi(\alpha\mathbf{v}_1) + \Phi(\beta\mathbf{v}_2)$ and also $\Phi(\beta\mathbf{v}_2) = \Phi(\lambda\mathbf{v}_2) + \Phi(\mu p\mathbf{v}_2)$, where $\beta = \lambda + \mu p$ and $\lambda, \mu \in N_p$. By Corollary 2.1, $\Phi(\mu p\mathbf{v}_2) = \mu\Phi(p\mathbf{v}_2)$ and $\Phi(\alpha\mathbf{v}_1) = \alpha\Phi(\mathbf{v}_1)$. Therefore, $\Phi(\mathbf{x}) = \Phi(\lambda\mathbf{v}_2) + \mu\Phi(p\mathbf{v}_2) + \alpha\Phi(\mathbf{v}_1)$. Note that $\pi_{1,1}(\Phi(p\mathbf{v}_2)) = \Phi(p\mathbf{v}_2)$ and $\pi_{1,1}(\Phi(\mathbf{v}_1)) = \Phi(\mathbf{v}_1)$. Thus, in order to show that $\pi_{1,1}(\Phi(\mathcal{H}_{p,a}^{1,1})) = \Phi(\mathcal{H}_p^{1,1})$, we only need to check that $\pi_{1,1}(\Phi(\lambda\mathbf{v}_2)) = \Phi(\lambda\mathbf{u}_2)$. Since $\pi_{1,1}$ is the identity in the first p coordinates, we just need to prove that $\pi_i(\phi(\lambda a_i)) = \phi(\lambda i)$ for all $i \in N_p^-$ and $\lambda \in N_p$, which is true by Corollary 4.2. $\square$

Example 4.3. Let $p = 3$. Let $S = \{(a_1, a_2) : (3a_1, 3a_2) = (3, 6) = 3\mathbb{Z}_9 \setminus \{0\}\}$. Note that $S = \{(1, 2), (1, 5), (1, 8), (4, 2), (4, 5), (4, 8), (7, 2), (7, 5), (7, 8)\}$ and it can also be written as $\{(1 + 3x, 2 + 3y) \in \mathbb{Z}_9^2 : x, y \in N_3\}$. Therefore, in this case, we can consider 9 different starting matrices

$$A_{3,a}^{1,1} = \left(\begin{array}{ccc|cc} 1 & 1 & 1 & 3 & 3 \\ 0 & 1 & 2 & a_1 & a_2 \end{array}\right),$$

where $a = (a_1, a_2) \in S$. By Proposition 4.1, these matrices generate 9 different $\mathbb{Z}_3\mathbb{Z}_9$-additive codes $\mathcal{H}^{1,1}_{3,a}$ of type $(3, 2; 1, 1)$ whose corresponding $\mathbb{Z}_3\mathbb{Z}_9$-linear codes, $\Phi(\mathcal{H}^{1,1}_{3,a})$, are permutation equivalent to each other. Note that if $a = (1, 2)$, we obtain the matrix $A^{1,1}_3$ given in Example 3.1. Moreover, it is clear that if we permute the coordinates of a, we also obtain $\mathbb{Z}_3\mathbb{Z}_9$-linear codes which are permutation equivalent to the previous ones.

Example 4.4. Let $p = 2$. Consider $a = (a_1) \in \mathbb{Z}_4$. We have that the possible values for a_1 such that $\{2a_1\} = \{2\}$ are 1 and 3. If $a_1 = 1$, then the matrix $A^{1,1}_{2,a} = A^{1,1}_2$. In the case that $a_1 = 3$, then $A^{1,1}_{2,a}$ is the matrix $A^{1,1}_2$ after multiplying by 3 the last row, so $\mathcal{H}^{1,1}_{2,a} = \mathcal{H}^{1,1}_2$.

Theorem 4.1. *Let $a = (a_1, \dots, a_{p-1}) \in \mathbb{Z}^{p-1}_{p^2}$ such that $\{pa_1, \dots, pa_{p-1}\} = p\mathbb{Z}_{p^2} \setminus \{0\}$. Let $A^{t_1,t_2}_{p,a}$ be the matrix obtained by using constructions (6) and (7), starting with the following matrix*

$$A^{1,1}_{p,a} = \left(\begin{array}{cccc|cccc} 1 & 1 & \cdots & 1 & p & p & \cdots & p \\ 0 & 1 & \cdots & p-1 & a_1 & a_2 & \cdots & a_{p-1} \end{array}\right), \tag{20}$$

instead of $A^{1,1}_p$. Then, the codes generated by $A^{t_1,t_2}_{p,a}$, denoted by $\mathcal{H}^{t_1,t_2}_{p,a}$, are $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH codes of type $(\alpha_1, \alpha_2; t_1, t_2)$ with $\alpha_1 \neq 0$. Moreover, the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear codes are permutation equivalent to $\Phi(\mathcal{H}^{t_1,t_2}_p)$.

Proof. As in the proof of Proposition 4.1, we can assume that the coordinates of $a = (a_1, \dots, a_{p-1})$ are in such an order that $(pa_1, \dots, pa_{p-1}) = (p, 2p, \dots, (p-1)p)$. Therefore, $a_i = i + \mu_i p$, where $i \in N^-_p$ and $\mu_i \in N_p$. For each $i \in N^-_p$, by Remark 4.2, there is a permutation $\pi_i \in \mathcal{S}_p$ such that $\pi_i(\phi(a_i)) = \phi(i)$ and $\sigma_i \in \mathcal{S}_{p^2}$ on the set of the elements of $\mathbb{Z}_{p^2}$ such that $\sigma_i(c) = b$ if and only if $\pi^{-1}_i(\phi(c)) = \phi(b)$. Again, by Remark 4.2, we also have (16), that is, $(\pi_i \mid \cdots \mid \pi_i)(\Phi(\sigma_i(w))) = \Phi(w)$, where $w = (0, 1, \dots, p^2 - 1)$.

Let $A^{t_1,t_2}_{p,a} = (A_1 \mid A'_2)$ be the matrix $A^{1,1}_{p,a}$ if $t_1 = t_2 = 1$, or the matrix obtained by applying (6) and (7) recursively from $A^{1,1}_{p,a}$ if $t_1 > 1$ or $t_2 > 1$. Let $A^{t_1,t_2}_p = (A_1 \mid A_2)$ be the matrix defined in Section 3. By construction, the second row of $A^{t_1,t_2}_{p,a}$ contains the block of coordinates $a_1, \dots, a_{p-1}$ repeated $p^{t_2-1}(p^2)^{t_1-1} = p^{2t_1+t_2-3}$ times. For the ℓ-th block of coordinates $a_1, \dots, a_{p-1}$, where $\ell \in \{1, \dots p^{2t_1+t_2-3}\}$, define I_ℓ as the coordinate positions corresponding to $\Phi(a_1, \dots, a_{p-1})$ within the Gray map image of this second row. Let $\pi_{t_1,t_2} \in \mathcal{S}_{\alpha_1+p\alpha_2}$ be the permutation such that it coincides with $(\pi_1 \mid \cdots \mid \pi_{p-1})$ when it is restricted to the coordinates I_ℓ, for all $\ell \in \{1, \dots, p^{2t_1+t_2-3}\}$, and it is the identity elsewhere. Note that $\pi_{1,1} = (\mathrm{Id} \mid \pi_1 \mid \cdots \mid \pi_{p-1}) \in \mathcal{S}_{p^2}$, where $\mathrm{Id} \in \mathcal{S}_p$.

Now, we consider the matrix $\bar{A}^{t_1,t_2}_{p,a} = (A_1 \mid \bar{A}_2)$ constructed with the following recursive construction. We start with the matrix $\bar{A}^{1,1}_{p,a} = A^{1,1}_{p,a}$ and if $\bar{A}^{t_1,t_2}_{p,a} = (A_1 \mid \bar{A}_2)$, then

$$\bar{A}^{t_1,t_2+1}_{p,a} = \left(\begin{array}{cccc|cccc} A_1 & A_1 & \cdots & A_1 & \bar{A}_2 & \bar{A}_2 & \cdots & \bar{A}_2 \\ \mathbf{0} & \mathbf{1} & \cdots & \mathbf{p-1} & p\cdot\mathbf{0} & p\cdot\mathbf{1} & \cdots & p\cdot(\mathbf{p-1}) \end{array}\right) \tag{21}$$

and

$$\bar{A}_{p,a}^{t_1+1,t_2} = \sigma_{t_1+1,t_2}\left(\begin{array}{ccc|cccccc} A_1 & \cdots & A_1 & pA_1 & \cdots & pA_1 & \bar{A}_2 & \cdots & \bar{A}_2 \\ \mathbf{0} & \cdots & \mathbf{p-1} & \mathbf{1} & \cdots & \mathbf{p-1} & \mathbf{0} & \cdots & \mathbf{p^2-1} \end{array}\right), \quad (22)$$

where σ_{t_1+1,t_2} permutes the columns as follows. Let B be the matrix $\bar{A}_{p,a}^{t_1+1,t_2}$ before applying σ_{t_1+1,t_2}. For each block of coordinates $a_1, \dots, a_{p-1}$ in the second row of B and the corresponding coordinates of the last row of B, we consider the following submatrix of B:

$$\begin{pmatrix} a_1 & \cdots & a_{p-1} & a_1 & \cdots & a_{p-1} & \cdots & a_1 & \cdots & a_{p-1} \\ 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & p^2-1 & \cdots & p^2-1 \end{pmatrix}. \quad (23)$$

Note that there are $p^{t_2-1}(p^2)^{t_1-1} = p^{2t_1+t_2-3}$ such submatrices, named $S_1, \dots, S_{p^{2t_1+t_2-3}}$. Then, σ_{t_1+1,t_2} is the permutation such that it coincides with σ_i when it is restricted to the p^2 columns corresponding to the columns having a_i in the first row of the submatrix S_ℓ for all $i \in N_p^-$ and $\ell \in \{1, \dots, p^{2t_1+t_2-3}\}$, and it is the identity elsewhere. By (16) and the definition of π_{t_1+1,t_2}, we have that

$$\pi_{t_1+1,t_2}(\Phi(\sigma_{t_1+1,t_2}(\mathbf{0}, \mathbf{1}, \dots, \mathbf{p-1} \mid \mathbf{1}, \dots, \mathbf{p-1}, \mathbf{0}, \mathbf{1}, \dots, \mathbf{p^2-1})) = (\mathbf{0}, \mathbf{1}, \dots, \mathbf{p-1} \mid \mathbf{1}, \cdots, \mathbf{p-1}, \mathbf{0}, \mathbf{1}, \dots, \mathbf{p^2-1}). \quad (24)$$

Let $\bar{\mathcal{H}}_{p,a}^{t_1,t_2}$ be the code generated by $\bar{A}_{p,a}^{t_1,t_2}$. Since $\bar{\mathcal{H}}_{p,a}^{t_1,t_2}$ is permutation equivalent to $\mathcal{H}_{p,a}^{t_1,t_2}$, we also have that $\Phi(\bar{\mathcal{H}}_{p,a}^{t_1,t_2})$ is permutation equivalent to $\Phi(\mathcal{H}_{p,a}^{t_1,t_2})$ by Remark 4.1. Therefore, to prove the statement, we just need to show that $\Phi(\bar{\mathcal{H}}_{p,a}^{t_1,t_2})$ is permutation equivalent to $\Phi(\mathcal{H}_p^{t_1,t_2})$. We proceed by induction. Let $\mathbf{v}_j = (v_j \mid v_j')$ be the j-th row of $\bar{A}_{p,a}^{t_1,t_2} = (A_1 \mid \bar{A}_2)$ and $\mathbf{u}_j = (u_j \mid u_j')$ be the j-th row of $A_p^{t_1,t_2}$, $j \in \{1, \dots, t_1+t_2\}$.

First, assume $t_1 = t_2 = 1$. Note that v_2' contains the coordinates $a_1, \dots, a_{p-1}$ exactly once and, we recall that $\pi_{1,1} = (\mathrm{Id} \mid \bar{\pi}_{1,1}) \in \mathcal{S}_{p^2}$, where $\mathrm{Id} \in \mathcal{S}_p$ and $\bar{\pi}_{1,1} = (\pi_1 \mid \cdots \mid \pi_{p-1})$. By the proof of Proposition 4.1, $\pi_{1,1}(\Phi(\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2)) = \Phi(\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2)$ for all $\lambda_1, \lambda_2 \in \mathbb{Z}_{p^2}$, so $\pi_{1,1}(\Phi(\mathcal{H}_{p,a}^{1,1})) = \pi_{1,1}(\Phi(\bar{\mathcal{H}}_{p,a}^{1,1})) = \Phi(\mathcal{H}_p^{1,1})$, and $\bar{\pi}_{1,1}(\Phi(\lambda_1 v_1' + \lambda_2 v_2')) = \Phi(\lambda_1 u_1' + \lambda_2 u_2')$.

Now, assume that the permutation $\pi_{t_1,t_2} = (\mathrm{Id} \mid \bar{\pi}_{t_1,t_2})$, where $\mathrm{Id} \in \mathcal{S}_{\alpha_1}$, satisfies

$$\pi_{t_1,t_2}(\Phi(\sum_{j=1}^{t_1+t_2} \lambda_j \mathbf{v}_j)) = \Phi(\sum_{j=1}^{t_1+t_2} \lambda_j \mathbf{u}_j)$$

for all $\lambda_j \in \mathbb{Z}_{p^2}$, so $\pi_{t_1,t_2}(\Phi(\bar{\mathcal{H}}_{p,a}^{t_1,t_2})) = \Phi(\mathcal{H}_p^{t_1,t_2})$ and in particular

$$\bar{\pi}_{t_1,t_2}(\Phi(\sum_{j=1}^{t_1+t_2} \lambda_j v'_j)) = \Phi(\sum_{j=1}^{t_1+t_2} \lambda_j u'_j). \tag{25}$$

Note that π_{t_1,t_2} is defined from the second row of $A^{t_1,t_2}_{p,a}$, which coincides with the second row of $\bar{A}^{t_1,t_2}_{p,a}$ by construction.

First, consider the code $\bar{\mathcal{H}}^{t_1,t_2+1}_{p,a}$ generated by $\bar{A}^{t_1,t_2+1}_{p,a}$ given in (21) and $\mathcal{H}^{t_1,t_2+1}_{p}$ generated by (6). Note that, for $j \in \{1,\dots,t_1+t_2\}$, the j-th row of $\bar{A}^{t_1,t_2+1}_{p,a}$ and A^{t_1,t_2+1}_{p} are $(v_j,\dots,v_j \mid v'_j,\dots,v'_j)$ and $(u_j,\dots,u_j \mid u'_j,\dots,u'_j)$, respectively, where $v_j = u_j$. By the construction of $\bar{A}^{t_1,t_2+1}_{p,a}$, we have that $\pi_{t_1,t_2+1} = (\mathrm{Id} \mid \bar{\pi}_{t_1,t_2+1})$, where $\mathrm{Id} \in \mathcal{S}_{p\alpha_1}$ and $\bar{\pi}_{t_1,t_2+1} = (\bar{\pi}_{t_1,t_2} \mid \cdots \mid \bar{\pi}_{t_1,t_2}) \in \mathcal{S}_{p^2\alpha_2}$. Moreover, since the first $p\alpha_1$ coordinates are exactly the same in both codes $\Phi(\bar{\mathcal{H}}^{t_1,t_2+1}_{p,a})$ and $\Phi(\mathcal{H}^{t_1,t_2+1}_{p})$, and π_{t_1,t_2+1} is the identity in these coordinates, we can focus on the last ones. Let $\mathbf{v} = (v \mid v') \in \bar{\mathcal{H}}^{t_1,t_2+1}_{p,a}$. We have that $v' = \sum_{j=1}^{t_1+t_2} \lambda_j(v'_j,\dots,v'_j) + \lambda w$, where $w = (p\cdot\mathbf{0}, p\cdot\mathbf{1},\dots,p\cdot(\mathbf{p-1}))$. Then, $\Phi(v') = \Phi(\sum_{j=1}^{t_1+t_2} \lambda_j(v'_j,\dots,v'_j)) + \Phi(\lambda w)$ by Corollary 2.3. Applying (25) and since $\bar{\pi}_{t_1,t_2+1}(\Phi(\lambda w)) = \Phi(\lambda w)$, we have that

$$\begin{aligned}\bar{\pi}_{t_1,t_2+1}(\Phi(v')) =&\bar{\pi}_{t_1,t_2+1}(\Phi(\sum_{j=1}^{t_1+t_2} \lambda_j(v'_j,\cdots,v'_j))) + \bar{\pi}_{t_1,t_2+1}(\Phi(\lambda w))\\ =&\Phi(\sum_{j=1}^{t_1+t_2} \lambda_j(u'_j,\dots,u'_j)) + \Phi(\lambda w).\end{aligned}$$

Applying again Corollary 2.3, we have that $\Phi(\bar{\mathcal{H}}^{t_1,t_2+1}_{p,a})$ and $\Phi(\mathcal{H}^{t_1,t_2+1}_{p})$ are permutation equivalent by using the permutation π_{t_1,t_2+1}.

Second, consider the code $\bar{\mathcal{H}}^{t_1+1,t_2}_{p,a}$ generated by $\bar{A}^{t_1+1,t_2}_{p,a}$ given in (22), and $\mathcal{H}^{t_1+1,t_2}_{p}$ generated by (7). We have that, for $j \in \{1,\dots,t_1+t_2\}$, the j-th row of $\bar{A}^{t_1+1,t_2}_{p,a}$ and A^{t_1+1,t_2}_{p} are $(v_j,\dots,v_j \mid pv_j,\dots,pv_j,v'_j,\dots,v'_j)$ and $(u_j,\dots,u_j \mid pu_j,\dots,pu_j,u'_j,\dots,u'_j)$, respectively, where $v_j = u_j$. Moreover, the (t_1+1+t_2)-th row of $\bar{A}^{t_1+1,t_2}_{p,a}$ and A^{t_1+1,t_2}_{p} are $\sigma_{t_1+1,t_2}(\mathbf{0},\mathbf{1},\dots,\mathbf{p-1} \mid \mathbf{1},\dots,\mathbf{p-1},\ \mathbf{0},\mathbf{1},\dots,\mathbf{p^2-1})$ and $(\mathbf{0},\mathbf{1},\dots,\mathbf{p-1} \mid \mathbf{1},\dots,\mathbf{p-1},\mathbf{0},\mathbf{1},\dots,\mathbf{p^2-1})$, respectively. The first $p\alpha_1 + (p-1)\alpha_1 = (2p-1)\alpha_1$ coordinates are exactly the same in both codes $\bar{\mathcal{H}}^{t_1+1,t_2}_{p,a}$ and $\mathcal{H}^{t_1+1,t_2}_{p}$, and hence the first $p\alpha_1 + p(p-1)\alpha_1 = p^2\alpha_1$ coordinates are also the same in both codes $\Phi(\bar{\mathcal{H}}^{t_1+1,t_2}_{p,a})$ and $\Phi(\mathcal{H}^{t_1+1,t_2}_{p})$. By the definition of $\bar{\pi}_{t_1,t_2}$ and σ_{t_1,t_2}, we have that $\pi_{t_1+1,t_2} = (\mathrm{Id} \mid \bar{\pi}_{t_1+1,t_2})$, where $\mathrm{Id} \in \mathcal{S}_{p^2\alpha_1}$ and $\bar{\pi}_{t_1+1,t_2} = (\bar{\pi}_{t_1,t_2} \mid \cdots \mid \bar{\pi}_{t_1,t_2})$, and $\sigma_{t_1+1,t_2} = (\mathrm{Id} \mid \bar{\sigma})$, where $\mathrm{Id} \in \mathcal{S}_{(2p-1)\alpha_1}$. Therefore, we can focus on the last coordinates, that is, we consider

$$\sum_{j=1}^{t_1+t_2} \lambda_j(v'_j,\dots,v'_j) + \lambda\bar{\sigma}(w), \tag{26}$$

where $w = (\mathbf{0},\mathbf{1},\dots,\mathbf{p^2-1})$. From (26), by Corollaries 2.2 and 2.3, we have that

$$\begin{aligned}&\Phi(\sum_{j=1}^{t_1+t_2}\lambda_j(v'_j,\dots,v'_j)+\lambda\bar{\sigma}(w))\\&=\Phi(\sum_{j=1}^{t_1+t_2}\lambda_j(v'_j,\dots,v'_j))+\Phi(\lambda\bar{\sigma}(w))+\Phi(p((\sum_{j=1}^{t_1+t_2}\lambda_j(v'_j,\dots,v'_j))\odot_p\lambda\bar{\sigma}(w))).\end{aligned}\tag{27}$$

Applying (25), $\bar{\pi}_{t_1+1,t_2}(\Phi(\sum_{j=1}^{t_1+t_2}\lambda_j(v'_j,\dots,v'_j)))=\Phi(\sum_{j=1}^{t_1+t_2}\lambda_j(u'_j,\dots,u'_j))$ and, from (24) and Lemma 4.2,

$$\bar{\pi}_{t_1+1,t_2}(\Phi(\lambda\bar{\sigma}(w)))=\Phi(\lambda w).\tag{28}$$

By Lemma 4.2, note that $\bar{\sigma}(w)\equiv w \mod p$. Moreover, $v'_j\equiv u'_j \bmod p$ for all $j\in\{1,\dots,t_1+t_2\}$ by construction. Thus, we also have that

$$\begin{aligned}&\bar{\pi}_{t_1+1,t_2}(\Phi(p((\sum_{j=1}^{t_1+t_2}\lambda_j(v'_j,\dots,v'_j))\odot_p\lambda\bar{\sigma}(w)))\\&=\Phi(p((\sum_{j=1}^{t_1+t_2}\lambda_j(u'_j,\dots,u'_j))\odot_p\lambda w)),\end{aligned}\tag{29}$$

by Corollary 4.1 and because $\bar{\pi}_{t_1+1,t_2}$ fixes the Gray map image of elements of order p by construction. Therefore, from the previous equations, we have that

$$\begin{aligned}&\bar{\pi}_{t_1+1,t_2}(\Phi(\sum_{j=1}^{t_1+t_2}\lambda_j(v'_j,\dots,v'_j)+\lambda\bar{\sigma}(w)))\\&=\Phi(\sum_{j=1}^{t_1+t_2}\lambda_j(u'_j,\dots,u'_j))+\Phi(\lambda w)+\Phi(p((\sum_{j=1}^{t_1+t_2}\lambda_j(u'_j,\dots,u'_j))\odot_p\lambda w))\\&=\Phi(\sum_{j=1}^{t_1+t_2}\lambda_j(u'_j,\dots,u'_j)+\lambda w),\end{aligned}$$

by Corollary 2.3. Therefore, the codes $\Phi(\bar{\mathcal{H}}_{p,a}^{t_1+1,t_2})$ and $\Phi(\mathcal{H}_p^{t_1+1,t_2})$ are permutation equivalent by using the permutation π_{t_1+1,t_2}, and the result follows. $\square$

Theorem 4.2. *Let $a=(a_1,\dots,a_{p-1}),b=(b_1,\dots,b_{p-1})\in\mathbb{Z}_{p^2}^{p-1}$ such that $\{pa_1,\dots,pa_{p-1}\}=\{pb_1,\dots,pb_{p-1}\}=p\mathbb{Z}_{p^2}\setminus\{0\}$. Let $A_{p,a}^{t_1,t_2}$ be the matrix obtained by using constructions (6) and (7), starting with the matrix $A_{p,a}^{1,1}$ given in (20). From $A_{p,a}^{t_1,t_2}=(A_1\mid A_2)$, we apply the following construction*

$$A_{p,a,b}^{t_1+1,t_2}=\left(\begin{array}{ccc|cccccc}A_1&\cdots&A_1&pA_1&\cdots&pA_1&A_2&\cdots&A_2\\\mathbf{0}&\cdots&\mathbf{p-1}&\mathbf{b}_1&\cdots&\mathbf{b}_{p-1}&\mathbf{0}&\cdots&\mathbf{p^2-1}\end{array}\right).\tag{30}$$

Then, the codes generated by $A_{p,a,b}^{t_1+1,t_2}$, denoted by $\mathcal{H}_{p,a,b}^{t_1+1,t_2}$, are $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH codes of type $(\alpha_1, \alpha_2; t_1+1, t_2)$ with $\alpha_1 \neq 0$. Moreover, the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear codes $\Phi(\mathcal{H}_{p,a,b}^{t_1+1,t_2})$ are permutation equivalent to $\Phi(\mathcal{H}_p^{t_1+1,t_2})$.

Proof. Let $A_{p,a}^{t_1+1,t_2}$ be the matrix obtained from $A_{p,a}^{t_1,t_2}$ by using construction (7). Let $\mathcal{H}_{p,a}^{t_1+1,t_2}$ be the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH code generated by $A_{p,a}^{t_1+1,t_2}$. By Theorem 4.1, we just need to show that $\Phi(\mathcal{H}_{p,a,b}^{t_1+1,t_2})$ is permutation equivalent to $\Phi(\mathcal{H}_{p,a}^{t_1+1,t_2})$.

As in the proof of Proposition 4.1 and Theorem 4.1, we can assume that the coordinates of $b = (b_1, \dots, b_{p-1})$ are in such an order that $(pb_1, \dots, pb_{p-1}) = (p, 2p, \dots, (p-1)p)$. Therefore, $b_i = i + \mu_i p$, where $i \in N_p^-$ and $\mu_i \in N_p$. Then, by using the same argument as in the proof of Theorem 4.1, we have that the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear code obtained from $A_{p,a,b}^{t_1+1,t_2}$ is permutation equivalent to $\Phi(\mathcal{H}_{p,a}^{t_1,+1,t_2})$. $\square$

Example 4.5. Let $p = 2$. As in Example 4.4, we have that the possible values for b_1 such that $\{2b_1\} = \{2\}$ are 1 and 3. If $b_1 = 1$, then $A_{2,a,b}^{t_1+1,t_2} = A_{2,a}^{t_1+1,t_2}$. Otherwise, if $b_1 = 3$, then $A_{2,a,b}^{t_1+1,t_2}$ is the matrix $A_{2,a}^{t_1+1,t_2}$ after changing the sign of the columns where b_1 appears in the last row of $A_{2,a,b}^{t_1+1,t_2}$. In this last case, the $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{H}_{2,a,b}^{t_1+1,t_2}$ and $\mathcal{H}_{2,a}^{t_1+1,t_2}$ are monomially equivalent [22], hence their corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear codes are permutation equivalent.

By following the same arguments as in the proofs of Theorems 4.1 and 4.2, we can construct $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear codes that are permutation equivalent to $\Phi(\mathcal{H}_p^{t_1,t_2})$, by starting from $A_{p,a}^{1,1}$, $a = (a_1, \dots, a_{p-1}) \in \mathbb{Z}_{p^2}^{p-1}$ satisfying $\{pa_1, \dots, pa_{p-1}\} = p\mathbb{Z}_{p^2} \setminus \{0\}$, and by using constructions (6) and (30) recursively. Moreover, each time the construction (30) is applied, the vector b can be different as long as it satisfies that $\{pb_1, \dots, pb_{p-1}\} = p\mathbb{Z}_{p^2} \setminus \{0\}$ as shown in the following corollary.

Corollary 4.3. *Let $a = (a_1, \dots, a_{p-1}) \in \mathbb{Z}_{p^2}^{p-1}$ such that $\{pa_1, \dots, pa_{p-1}\} = p\mathbb{Z}_{p^2} \setminus \{0\}$. Let $L = [b_1, \dots, b_{t_1-1}]$ such that $b_i = (b_{i,1}, \dots, b_{i,p-1}) \in \mathbb{Z}_{p^2}^{p-1}$ and $\{pb_{i,1}, \dots, pb_{i,p-1}\} = p\mathbb{Z}_{p^2} \setminus \{0\}$ for all $i \in \{1, \dots, t_1-1\}$. Let $A_{p,a,L}^{t_1,t_2}$ be the matrix obtained by using (6) and (30), starting with $A_{p,a}^{1,1}$ given in (20) and using b_i in (30) to construct $A_{p,a,L}^{i+1,j}$ from $A_{p,a,L}^{i,j}$ for $i \in \{1, \dots, t_1-1\}, j \in \{1, \dots, t_2\}$. Then, the codes generated by $A_{p,a,L}^{t_1,t_2}$, denoted by $\mathcal{H}_{p,a,L}^{t_1,t_2}$, are $\mathbb{Z}_p\mathbb{Z}_{p^2}$-additive GH codes of type $(\alpha_1, \alpha_2; t_1, t_2)$ with $\alpha_1 \neq 0$. Moreover, the corresponding $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear codes $\Phi(\mathcal{H}_{p,a,L}^{t_1,t_2})$ are permutation equivalent to $\Phi(\mathcal{H}_p^{t_1,t_2})$.*

Note that if $a = (1, 2, \dots, p-1)$, then $\mathcal{H}_{p,a}^{t_1,t_2} = \mathcal{H}_p^{t_1,t_2}$ for all $t_1, t_2 \geq 1$. If $b = (1, 2, \dots, p-1)$, since matrix (30) coincides with matrix (7), then $\mathcal{H}_{p,a,b}^{t_1,t_2} = \mathcal{H}_{p,a}^{t_1,t_2}$. Similarly, if $L = [b, \dots, b]$, where $b = (1, 2, \dots, p-1)$, then $\mathcal{H}_{p,a,L}^{t_1,t_2} = \mathcal{H}_{p,a}^{t_1,t_2}$.

Example 4.6. Let S be the set given in Example 4.3. By Theorem 4.2, from each one of the possible starting matrices $A_{3,a}^{1,1}$, described in Example 4.3, we can construct different $\mathbb{Z}_3\mathbb{Z}_9$-linear GH codes of type $(\alpha_1, \alpha_2; 2, 1)$, which are all permutation equivalent to each other. Specifically, for any $a = (a_1, a_2) \in S$ and $b_1 = (b_{1,1}, b_{1,2}) \in S$, the $\mathbb{Z}_3\mathbb{Z}_9$-linear codes corresponding to the $\mathbb{Z}_3\mathbb{Z}_9$-additive codes generated by

$$A^{2,1}_{3,a,b_1} = \left(\begin{array}{ccc|ccccc} 111 & 111 & 111 & 333 & 333 & 3\ 3 & \cdots & 3\ 3 \\ 012 & 012 & 012 & 036 & 036 & a_1a_2 & \cdots & a_1a_2 \\ 000 & 111 & 222 & \mathbf{b}_{1,1} & \mathbf{b}_{1,2} & 0\ 0 & \cdots & 8\ 8 \end{array}\right), \tag{31}$$

where $\mathbf{b}_{1,1} = (b_{1,1}, b_{1,1}, b_{1,1})$ and $\mathbf{b}_{1,2} = (b_{1,2}, b_{1,2}, b_{1,2})$, are all permutation equivalent to each other. Similarly, by Corollary 4.3, for any $a, b_1, b_2 \in S$, $b_2 = (b_{2,1}, b_{2,2})$, $L = [b_1, b_2]$ and $A^{2,1}_{3,a,b_1} = (A_1 \mid A_2)$ as in (31), the $\mathbb{Z}_3\mathbb{Z}_9$-linear GH codes obtained from the following matrix

$$A^{3,1}_{3,a,L} = \left(\begin{array}{ccc|cccccc} A_1 & A_1 & A_1 & 3A_1 & 3A_1 & A_2 & A_2 & \cdots & A_2 \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{b}_{2,1} & \mathbf{b}_{2,2} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{8} \end{array}\right),$$

where $\mathbf{b}_{2,1} = (b_{2,1}, \dots, b_{2,1}), \mathbf{b}_{2,2} = (b_{2,2}, \dots, b_{2,2}) \in \mathbb{Z}_9^9$, are all permutation equivalent to each other.

Remark 4.3. Let $t_1, t_2 \geq 1$, $a \in \mathbb{Z}_{p^2}^{p-1}$ and $L = [b_1, \dots, b_{t_1-1}]$, where $b_i \in \mathbb{Z}_{p^2}^{p-1}$ for all $i \in \{1, \dots, t_1 - 1\}$. From Theorem 4.1 and Corollary 4.3, we have that if a and $b_i \in \mathbb{Z}_{p^2}^{p-1}$ satisfy the condition given in these results for all $i \in \{1, \dots, t_1 - 1\}$, then the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear codes obtained from matrices $A^{t_1,t_2}_{p,a}$ and $A^{t_1,t_2}_{p,a,L}$ are GH codes, and they are permutation equivalent to the codes obtained from A^{t_1,t_2}_{p}. It is easy to prove that if a or b_i, for some $i \in \{1, \dots, t_1 - 1\}$, do not satisfy that property, then the $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear codes obtained from $A^{t_1,t_2}_{p,a}$ and $A^{t_1,t_2}_{p,a,L}$ are not GH codes. Indeed, $\Phi(\mathbf{u}) - \Phi(\mathbf{0})$ would not contain every element of $\mathbb{Z}_p$ the same number of times, if $\mathbf{u}$ was a multiple of the row containing a or b_i.

5. Conclusions and further research

In this paper, we have seen that there are many different ways to construct $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes of type $(\alpha_1, \alpha_2; t_1, t_2)$ with $t_1, t_2 \geq 1$, $\alpha_1 \neq 0$, and p prime. However, we have proved that for all of them we obtain permutation equivalent codes. Thus, to study these codes, we can always focus on the construction given in Section 3. This construction generalizes the known recursive construction of $\mathbb{Z}_2\mathbb{Z}_4$-additive Hadamard codes with $\alpha_1 \neq 0$ given in [19].

Two structural properties of codes over $\mathbb{Z}_p$ are the rank and dimension of the kernel. The rank of a code C over $\mathbb{Z}_p$ is simply the dimension of the linear span, $\langle C \rangle$, of C. The kernel of a code C over $\mathbb{Z}_p$ is defined as $\mathrm{K}(C) = \{x \in \mathbb{Z}_p^n : x + C = C\}$ [23,24]. If the all-zero vector belongs to C, then $\mathrm{K}(C)$ is a linear subcode of C. Note also that if C is linear, then $K(C) = C = \langle C \rangle$. We denote the rank of C as r and the dimension of the kernel as k. The parameters (r, k) can be used to distinguish between non-equivalent codes, since equivalent ones have the same value of rank and dimension of the kernel.

Table 1
Type and parameters (r, k) of $\mathbb{Z}_9$-linear and $\mathbb{Z}_3\mathbb{Z}_9$-linear GH codes.

p	t	$\mathbb{Z}_9$-linear GH codes		$\mathbb{Z}_3\mathbb{Z}_9$-linear GH codes	
		$(0, \alpha_2; t_1, t_2)$	(r, k)	$(\alpha_1, \alpha_2; t_1, t_2)$	(r, k)
3	2	$(0, 3; 1, 1)$	$(3, 3)$	$(9, 0; 0, 3)$	$(3, 3)$
				$(3, 2; 1, 1)$	$(4, 2)$
	3	$(0, 9; 1, 2)$	$(4, 4)$	$(27, 0; 0, 4)$	$(4, 4)$
		$(0, 9; 2, 0)$	$(5, 2)$	$(9, 6; 1, 2)$	$(5, 3)$
	4	$(0, 27; 1, 3)$	$(5, 5)$	$(81, 0; 0, 5)$	$(5, 5)$
		$(0, 27; 2, 1)$	$(6, 3)$	$(27, 18; 1, 3)$	$(6, 4)$
				$(9, 24; 2, 1)$	$(10, 3)$
	5	$(0, 81; 1, 4)$	$(6, 6)$	$(243, 0; 0, 6)$	$(6, 6)$
		$(0, 81; 2, 2)$	$(7, 4)$	$(81, 54; 1, 4)$	$(7, 5)$
		$(0, 81; 3, 0)$	$(11, 3)$	$(27, 72; 2, 2)$	$(11, 4)$
	6	$(0, 243; 1, 5)$	$(7, 7)$	$(729, 0; 0, 7)$	$(7, 7)$
		$(0, 243; 2, 3)$	$(8, 5)$	$(243, 162; 1, 5)$	$(8, 6)$
		$(0, 243; 3, 1)$	$(12, 4)$	$(81, 216; 2, 3)$	$(12, 5)$
				$(27, 234; 3, 1)$	$(20, 4)$
	7	$(0, 729; 1, 6)$	$(8, 8)$	$(2187, 0; 0, 8)$	$(8, 8)$
		$(0, 729; 2, 4)$	$(9, 6)$	$(729, 486; 1, 6)$	$(9, 7)$
		$(0, 729; 3, 2)$	$(13, 5)$	$(243, 648; 2, 4)$	$(13, 6)$
		$(0, 729; 4, 0)$	$(21, 4)$	$(81, 702; 3, 2)$	$(21, 5)$
	8	$(0, 2187; 1, 7)$	$(9, 9)$	$(6561, 0; 0, 9)$	$(9, 9)$
		$(0, 2187; 2, 5)$	$(10, 7)$	$(2187, 1458; 1, 7)$	$(10, 8)$
		$(0, 2187; 3, 3)$	$(14, 6)$	$(729, 1944; 2, 5)$	$(14, 7)$
		$(0, 2187; 4, 1)$	$(22, 5)$	$(243, 2106; 3, 3)$	$(22, 6)$
				$(81, 2160; 4, 1)$	$(35, 5)$

In [13,14], the rank and dimension of the kernel are used to classify $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 = 0$ and $\alpha_1 \neq 0$, respectively. Moreover, it is also known that the family of $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 \neq 0$ includes the family of $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard codes with $\alpha_1 = 0$ [15], since each $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 = 0$ is equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard code with $\alpha_1 \neq 0$. The rank and dimension of the kernel have also been used to classify $\mathbb{Z}_{p^s}$-linear GH codes of length p^t, with $s \geq 2$ and p prime [8,16–18].

Table 1 shows the type $(\alpha_1, \alpha_2; t_1, t_2)$ and parameters (r, k) for all $\mathbb{Z}_3\mathbb{Z}_9$-linear GH codes of length 3^t, with $\alpha_1 \neq 0$ and $2 \leq t \leq 8$, considered in this paper. It also includes the type $(0, \alpha_2; t_1, t_2)$ and parameters (r, k) for all $\mathbb{Z}_9$-linear GH codes of the same length considered in [16].

By looking at Table 1, we have that all $\mathbb{Z}_3\mathbb{Z}_9$-linear GH codes of length 3^t, with $\alpha_1 \neq 0$ and $2 \leq t \leq 8$, are pairwise non-equivalent since all of them have a different value of the kernel. This means that there are at least $\lfloor t/2 \rfloor + 1$ such codes for $2 \leq t \leq 8$. Moreover, we can see that these non-linear codes are also non-equivalent to any $\mathbb{Z}_9$-linear GH code of the same length. Similar results can be obtained computationally for $p = 5$ and $p = 7$. This means that, unlike for $p = 2$, in general, for $p \geq 3$ prime, the $\mathbb{Z}_{p^2}$-linear GH codes are not included in the family of $\mathbb{Z}_p\mathbb{Z}_{p^2}$-linear GH codes with $\alpha_1 \neq 0$. More generally, we can also observe that the $\mathbb{Z}_3\mathbb{Z}_9$-linear GH codes of length 3^t, with $\alpha_1 \neq 0$ and $2 \leq t \leq 8$, are not equivalent to any $\mathbb{Z}_{3^s}$-linear GH code of the same length with $s \geq 2$, by comparing the values of (r, k) given in Table 1 with the ones given in [16, Tables 4 and 5]. Further research on this topic would be to prove these results for any $t \geq 2$ and $p \geq 3$ prime.

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