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UNIQUENESS OF THE LIMIT CYCLES FOR COMPLEX DIFFERENTIAL EQUATIONS WITH TWO MONOMIALS

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ABSTRACT. We prove that any complex differential equation with two monomials of the form $\dot{z} = az^k\bar{z}^l + bz^m\bar{z}^n$, with k, l, m, n non-negative integers and $a, b \in \mathbb{C}$, has one limit cycle at most. Moreover, we characterise when such a limit exists and prove that then it is hyperbolic. For an arbitrary equation of the above form, we also solve the centre-focus problem and examine the number, position, and type of its critical points. In particular, we prove a Berlinskiĭ-type result regarding the geometrical distribution of the critical points stabilities.

1. INTRODUCTION

In this work, we will prove that any complex polynomial differential equation inside the family

$$\dot{z} = az^k\bar{z}^l + bz^m\bar{z}^n, \; z \in \mathbb{C}, \tag{1.1}$$

with $k, l, m, n \in \mathbb{Z}^+ \cup \{0\}, k + l < m + n$, and $a, b \in \mathbb{C}\backslash\{0\}$ has one limit cycle at most. Recall that a limit cycle γ is a periodic orbit such that, in at least one of the connected components of $\mathbb{R}^2 \setminus \gamma$, has initial conditions (as close to γ as desired) that do not belong to a periodic orbit.

Note that easier cases $k + l = m + n$ or $ab = 0$ need not be considered because they give rise to particular planar homogeneous vector fields and the global phase portraits of the general homogeneous polynomial vector fields are well known (see, e.g., [2]). Particularly, they do not have limit cycles, and the centre-focus problem is completely solved: the vanishing of a given single integral distinguishes between both possibilities.

Hence, complex differential equations with a single monomial have no limit cycles. However, it was proved in [10], that there is no upper bound for the number of limit cycles of the differential equations defined by three monomials

$$\dot{z} = az^k\bar{z}^l + bz^m\bar{z}^n + cz^p\bar{z}^q.$$

Therefore, our results fill the gap between the one and the three monomial cases, where as we will see, their dynamic complexity renders this question nontrivial.

Before presenting our results in more detail, we briefly recall some concepts that will appear in this study. A simple critical point of a vector field is a critical point for which the determinant of its associated Jacobian matrix is nonzero. When the sign of the determinant is negative, the critical point

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is a saddle (index -1), whereas it is an anti-saddle (index $+1$) when it is positive. For analytic vector fields, anti-saddles are the foci, nodes, or centres. Moreover, in this analytical setting, the limit cycles can be defined as isolated periodic solutions in the set of all periodic orbits of the equation. A weak focus is an anti-saddle of a centre or focus type, at which the divergence vanishes. A centre is considered reversible (with respect to a straight line) if, after translating it to the origin and performing a suitable rotation, it is invariant by the change in the variable and time $(z,t) \to (\bar{z},-t)$. Finally, the limit cycle is named hyperbolic if its associated Poincaré return map has a simple fixed point.

As we prove in this work, family (1.1) exhibits a large variety of behaviours despite its apparent simplicity. For instance, when $q := l-k+m-n \neq 0$, the equation has $|q|$ nonzero critical points, all of which are located on a circle $\mathbb{S}^1$ centred at the origin. When $q>0$ (resp. $q<0$) all are anti-saddles (resp. saddles). Moreover, we show that, when one of these critical points is a weak focus, it is indeed a centre. However, this is not the case of the origin: it can be a weak focus of order one and not being a centre. We also solve the centre-focus problem for all critical points, proving that all centres are reversible. We investigate the number of nonzero centres that the equation can have. In particular, as a consequence of Lagrange's theorem on the cardinality of the subgroups of finite groups, we prove that this number is a divisor of $q>0$ and it is not bounded for the full family.

The following is the main theorem:

Theorem A. *Any differential equation from the family* (1.1) *has at most one limit cycle, and such a limit cycle exists if and only if* $k-l=m-n=1$, $\operatorname{Re}(a)\operatorname{Re}(b)<0$ *and* $a/b \notin \mathbb{R}^-$. *Moreover, it is the circle* $|z|^2 = (-\operatorname{Re}(a)/\operatorname{Re}(b))^{n-l}$, *which is hyperbolic, and its stability depends on the sign of* $-\operatorname{Re}(a)$.

If we consider $\operatorname{Re}(a)$ as a bifurcation parameter, this limit cycle appears by an Andronov-Hopf-type bifurcation occurring at the origin when $\operatorname{Re}(a)=0$ and $\operatorname{Re}(b)\neq 0$.

We stress that family (1.1) is one of the few nontrivial families for which the sometimes called *Coppel's problem*, [5], has some hope of being resolved. Recall that, although he proposed it for quadratic systems, it can be naturally extended to other polynomial systems. The problem in his own words was: "Ideally one might hope to characterize the phase portraits of quadratic systems by means of algebraic inequalities on the coefficients." In general, for quadratic systems, such a solution is impossible (see [7]). Typically, one of the main difficulties for this solution is the question of existence and number of limit cycles. The fact that, for our differential equation, both the centre-focus problem and existence of limit cycles can be solved, provides some hope for this case. We also address the case of nonzero critical points of index $+1$ by studying their stabilities distribution. The next main issue in Coppel's problem for our family is characterizing the appearance of homoclinic or heteroclinic solutions. However, we did not consider this question in this study.

The remainder of this paper is organised as follows. The study of the critical points is discussed in Section 2. We also prove a Berlinskiĭ-type result regarding the relative position and the stability of critical points (see Subsection 2.3). We prove our main theorem in Section 3. Section 3 also contains some results regarding more general differential equations than family (1.1) (see, e.g., Propositions 3.3 or 3.5) useful for proving Theorem A. Particularly, the former proposition is a natural extension of the classical result for quadratic systems $\dot{z} = X_1(z,\bar{z}) + X_2(z,\bar{z})$, which indicates that they do not have periodic orbits surrounding a node. Proposition 3.3 is based on the theory of rotated vector fields, [6, 18, 19].

Finally, to illustrate the dynamical richness of family (1.1), we end the paper with a very short section exhibiting some of the phase portraits that this equation has.

Through a notation, along the study, we use $a = r_a e^{i\alpha}, b = r_b e^{i\beta}$, $r_a, r_b \geq 0$, and $\alpha, \beta \in [0, 2\pi)$. We also will write $q = l - k + m - n$, $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$ and sgn for the sign function.

2. Results on critical points

In this section, we examine the number and type of critical points, the centre-focus problem, and some Berlinskiĭ-type results for the differential Equation (1.1). We start with a preliminary computational result, borrowed from [10].

Lemma 2.1. *Consider the differential equation $\dot{z} = F(z,\bar{z})$, and denote its associated vector field as $X(x,y) = \big(\mathrm{Re}(F(z,\bar{z})), \mathrm{Im}(F(z,\bar{z}))\big)$, where $z = x + iy$. Then,*

(i) Its expression in polar coordinates $z = re^{i\theta}$ is

$$\dot{r} = \frac{1}{r}\,\mathrm{Re}\left(\bar{z}F(z,\bar{z})\big|_{z=re^{i\theta}}\right), \quad \dot{\theta} = \frac{1}{r^2}\,\mathrm{Im}\left(\bar{z}F(z,\bar{z})\big|_{z=re^{i\theta}}\right).$$

(ii) Its divergence is written as $\mathrm{div}(X) = 2\,\mathrm{Re}\left(\frac{\partial}{\partial z}F\right)$.

(iii) The determinant of its differential dX is $\det(dX) = \left|\frac{\partial}{\partial z}F\right|^2 - \left|\frac{\partial}{\partial \bar{z}}F\right|^2$.

2.1. Number and type of critical points.

We begin with a preliminary result that simplifies the computations for nonzero critical points and is also useful in studying the centre-focus problem and stability of the simple critical points of Equation (1.1).

Lemma 2.2. *If a differential equation of the form (1.1) has a nonzero critical point, $Re^{i\psi}, R \neq 0$, then, after a linear change of coordinates and positive constant rescaling of time, this equation can be written as*

$$\dot{z} = c(z^k\bar{z}^l - z^m\bar{z}^n), \quad \text{where} \quad c = e^{i(\alpha+(k-l-1)\psi)}. \tag{2.1}$$

Moreover, when $q = l - k + m - n = 0$, Equation (2.1) has $|z| = 1$ full of critical points. When $q \neq 0$, it has exactly $|q|$ nonzero critical points; that is, $z = z_j = \omega^j$, $j = 0, 1, \ldots|q| - 1$, where $\omega = e^{2\pi i/|q|}$, and they are located at the $|q|$th roots of unity. Finally, if X is a vector field associated to (2.1), then for all $j = 0, 1, \ldots|q| - 1$, it holds that

$$\begin{aligned} &\det(dX(z_j)) = (m-k)^2 - (n-l)^2, &(2.2)\\ &\mathrm{div}(X)(z_j) = 2(k-m)\,\mathrm{Re}(cz_j^{k-l-1}) = 2(k-m)\,\mathrm{Re}(cz_j^{m-n-1}). \end{aligned}$$

Proof. Set $Z = Re^{\mathrm{i}\psi}$. By taking the new variable w such that $z = wZ$ and a new time s such that $\mathrm{d}s/\mathrm{d}t = |a||Z|^{k+l-1}$, we obtain that Equation (1.1) is transformed into

$$w' = c(w^k\bar{w}^l - w^m\bar{w}^n) \quad \text{where} \quad c = \frac{aZ^{k-1}\bar{Z}^l}{|a||Z|^{k+l-1}} = \mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}(k-l-1)\psi},$$

where the prime symbol denotes the derivative with respect to s. By renaming the new variable as the old one, we obtain Equation (2.1). Clearly, nonzero critical points must satisfy $z^k\bar{z}^l - z^m\bar{z}^n = 0$, or equivalently, if $z = re^{\mathrm{i}\theta}$, $r = 1$ and $\big(\mathrm{e}^{\mathrm{i}\theta}\big)^q = 1$, thus achieving the stated result.

Hence, considering that $|c| = 1$, at the critical points, $z^{k-1}\bar{z}^l = z^{m-1}\bar{z}^n$, and using Lemma 2.1, we obtain

$$\begin{aligned}\det(dX(z_j)) &= \left(\left|\frac{\partial F}{\partial z}\right|^2 - \left|\frac{\partial F}{\partial \bar{z}}\right|^2\right)\Big|_{z=z_j} \\ &= \big|c(k-m)z_j^{k-1}\bar{z}_j^l\big|^2 - \big|c(l-n)z_j^{k-1}\bar{z}_j^l\big|^2 = (m-k)^2 - (n-l)^2. \end{aligned} \tag{2.3}$$

Similarly, following Lemma 2.1, we obtain

$$\operatorname{div}(X)|_{z=z_j} = 2\operatorname{Re}\left(\frac{\partial F}{\partial z}\right)\Big|_{z=z_j} = 2(k-m)\operatorname{Re}(cz_j^{k-1}\bar{z}_j^l).$$

□

The following proposition determines the critical point type of the differential Equation (1.1) as follows.

Proposition 2.3. *Let us consider a differential equation of the form* (1.1) *and set $q = l - k + m - n$. Then,*

(i) The origin is a critical point if and only if $k + l > 0$; in this case its index is $k - l$. Moreover, when $k - l > 1$ it has $2(k-l) - 2$ elliptic sectors; when $k - l = 1$ it is a node, focus, or centre; and when $k - l \leq 0$ it has $2|k-l| + 2$ hyperbolic sectors.

(ii) If $q \neq 0$, it has $|q|$ nonzero critical points, all of them are simple and located on a circle centred at the origin. Moreover, when $q > 0$ (resp. $q < 0$) all of them are anti-saddles (resp. saddles).

(iii) If $q = 0$ and $a/b \in \mathbb{R}^-$, it has a circle centred at the origin filled with critical points.

(iv) If $q = 0$ and $a/b \notin \mathbb{R}^-$, it does not have nonzero critical points.

Proof. Denote the right-hand side of Equation (1.1) by $\dot{z} = F(z, \bar{z})$. Taking the polar coordinates $z = re^{\mathrm{i}\theta}$, the critical points satisfy

$$r_a\, r^{k+l}\mathrm{e}^{\mathrm{i}(\alpha+(k-l)\theta)} + r_b\, r^{m+n}\mathrm{e}^{\mathrm{i}(\beta+(m-n)\theta)} = 0. \tag{2.4}$$

(i) It is clear that the origin is a critical point if and only if $k + l > 0$. Let us now assume that $k + l > 0$ and examine the index of this critical point. Note that

$$\frac{F(z,\bar{z})|_{z=re^{\mathrm{i}\theta},\bar{z}=re^{-\mathrm{i}\theta}}}{r^{k+l}} = r_a\mathrm{e}^{\mathrm{i}[(k-l)\theta+\alpha)]} + r_b r^{m+n-k-l}\mathrm{e}^{\mathrm{i}[(m-n)\theta+\beta)]}.$$

Hence, for small enough r, the right-hand side function on the circle $|z| = r$, provides $k - l$ turns in the clockwise (resp. counter-clockwise) sense when $k - l > 0$ (resp. $k - l < 0$), which is precisely the definition of having index

$k-l$. In fact, when $k-l \neq 0$ the critical point is formed by $2(k-l)-2$ elliptic sectors when $k-l>1$, or $2|k-l|+2$ hyperbolic sectors when $k-l<0$. The behaviour in the case $k-l \notin \{0,1\}$ can be proved, for example, using polar coordinates and following the approach used in [2] (we omit the details). Let us prove that, if $k-l=0$, it is formed by two hyperbolic sectors. In this case, equation (1.1) can be written as

$$\dot{z} = a|z|^{2k} + bz^m\bar{z}^n.$$

By time rescaling $\mathrm{d}s/\mathrm{d}t = |z|^{2k}$, we determine that the origin is not a critical point, or equivalently recovering the original differential equation, that the critical point has exactly two hyperbolic sectors. When $k-l=1$, with a similar rescaling, we find that it behaves as a nondegenerated critical point of index $+1$; that is, the origin is either a focus, centre, or node.

$(ii)-(iv)$ Regarding the critical points that are different from the origin by solving Equation (2.4), we obtain

$$r^{k+l-m-n}\mathrm{e}^{\mathrm{i}(\alpha-\beta+\pi-q\theta)} = r_b/r_a.$$

Thus, when $q=0$ and $\alpha-\beta \notin \{\pi,-\pi\}$, that is, $a/b \notin \mathbb{R}^-$, Equation (1.1) does not have nonzero critical points, and item (iv) follows. Otherwise, the differential equation has some nonzero critical points, and we can apply Lemma 2.2. Thus, all the results stated in the proposition follow, except the one related to the characterisation of the type of critical point when $q \neq 0$. This characterization is a simple consequence of equality (2.2) because

$$(m-k)^2-(n-l)^2 = \operatorname{sgn}(q)\big|(m-k)^2-(n-l)^2\big| \neq 0,$$

and, hence, the sign of the determinant at each nonzero critical point is given by the sign of q and is independent of this point. To prove the last equality, recall that $m+n-k-l>0$. Assume, for example, that $q<0$. Then, $-q=-l+k-m+n>0$. By joining both inequalities and their sum, we obtain

$$n-l>k-m, \quad n-l>m-k \quad \text{and} \quad n-l>0.$$

Hence,

$$\begin{aligned} |n-l| = n-l > |m-k| \quad &\Longrightarrow \quad (m-k)^2-(n-l)^2<0 \\ &\Longrightarrow (m-k)^2-(n-l)^2 = \operatorname{sgn}(q)|(m-k)^2-(n-l)^2|, \end{aligned}$$

as intended. Case $q>0$ follows similarly. □

Subsequently, we present a completely different proof of the fact that, when $q \neq 0$, all nonzero critical points have the same index. We include this new proof because it is used in the proof of Theorem A and, moreover, because it is more qualitative. This proof uses the following lemma.

Lemma 2.4. *Let $\widetilde{X}$ be the compactification by adding a point (to be called infinity) of the vector field associated to* (1.1). *Then, infinity is a critical point of $\widetilde{X}$ on $\mathbb{S}^2$, and its index is $2+n-m$.*

Proof. The compactification described in the statement is achieved by executing the change of variable $w=z^{-1}$ and by introducing a new time s

satisfying $\mathrm{d}t/\mathrm{d}s = |w|^{2(m+n)}$. Hence, we arrive at

$$w' = -bw^{2+n}\bar{w}^m - aw^{2+m+n-k}\bar{w}^{m+n-l}, \tag{2.5}$$

where the prime symbol denotes the derivative with respect to s. Note that $2+m+n-k+m+n-l > 2+n+m$, because $k+l < m+n$. Because the infinity of the Equation (1.1) is the origin of this last equation, from Proposition 2.3, infinity has index $2+n-m$. $\square$

Alternative proof of item (ii) of Proposition 2.3. We prove that, if all nonzero critical points are simple (i.e., their indices are $+1$ or -1), they in fact have the same index. We compactify the differential equation to $\mathbb{S}^2$ by adding a critical point at infinity, as in the proof of Lemma 2.4, we obtain the new vector field $\widetilde{X}$.

Recall that, if a vector field Y on the sphere has finitely many critical points, for example, p_j, $j = 1, 2, \ldots, N$, Poincaré–Hopf theorem, [11, 14, 16], asserts that

$$\sum_{j=1}^{N} \mathrm{ind}_Y(p_j) = 2. \tag{2.6}$$

where $\mathrm{ind}_Y(p_j)$ denotes the index of p_j.

Under hypothesis $q \neq 0$, the compactified vector field, $\widetilde{X}$ associated to Equation (1.1) has $|q|$ nonzero finite critical points, infinity, by Lemma 2.4 with index $2+n-m$, and the origin (unless $k = l = 0$) with index $k-l$, by item (i) of Proposition 2.3. Hence, through the abuse of language and to apply Equation (2.6), we consider that it has $|q|+2$ critical points because when the origin is not a critical point, the same formula works because its index is $k-l = 0$. Hence, if we call $p_1, p_2, \ldots, p_{|q|}$ the $|q|$ nonzero finite critical points, we obtain

$$2 = \sum_{j=1}^{|q|+2} \mathrm{ind}_{\widetilde{X}}(p_j) = \sum_{j=1}^{|q|} \mathrm{ind}_{\widetilde{X}}(p_j) + k - l + 2 + n - m,$$

or equivalently, $\sum_{j=1}^{|q|} \mathrm{ind}_{\widetilde{X}}(p_j) = q$. As all p_j are simple critical points, $\mathrm{ind}_{\widetilde{X}}(p_j) \in \{-1, +1\}$. Then, it holds that $\mathrm{ind}_{\widetilde{X}}(p_j) = \mathrm{sgn}(q)$, for all $j = 1, 2, \ldots, |q|$, as we intended to prove. $\square$

2.2. **Centre-focus problem.** We will use the well-known Poincaré reversibility criterion several times. The following is a suitable version for our interest: if the origin of a smooth planar differential equation is a monodromic critical point and the equation is invariant by the change of variable and time $(z, -t) \longrightarrow (\bar{z}, t)$, then the origin is a centre.

The following theorem solves the centre-focus problem for any equation from family (1.1).

Theorem 2.5. *Consider a differential equation of form* (1.1)*. The following holds:*

- *(i) When* $m - n = 1$ *the origin is a centre if and only if* $k - l = 1$, $\mathrm{Im}(a) \neq 0$ *and* $\mathrm{Re}(a) = \mathrm{Re}(b) = 0$.
- *(ii) When* $m - n \neq 1$ *the origin is a centre if and only if* $k - l = 1$, $\mathrm{Im}(a) \neq 0$ *and* $\mathrm{Re}(a) = 0$.

(iii) *It has a nonzero centre at* $z = R\mathrm{e}^{\mathrm{i}\psi}$, $R \neq 0$, *if and only if the point has index* $+1$ *and the divergence vanishes at this point. Specifically, if and only if* $q > 0$ *and* $\mathrm{Re}(a\mathrm{e}^{\mathrm{i}(k-l-1)\psi}) = 0$, *where the later condition is equivalent to* $\mathrm{Re}(b\mathrm{e}^{\mathrm{i}(m-n-1)\psi}) = 0$.

Moreover, all centres are reversible.

Proof. (*i*)-(*ii*) By Proposition 2.3, the index of the origin is $k - l$. To have a centre at the origin, this index must be $+1$. Hence, we can write the differential equation as

$$\dot{z} = a|z|^{2l}z + bz^m\bar{z}^n. \tag{2.7}$$

Our proof of the characterisation of centres for Equation (2.7) extends the results of [9, Lem. 3.2] which covers the case $l = 0$. By Lemma 2.1, the expression of Equation (2.7) in polar coordinates is:

$$\dot{r} = \frac{1}{r}\,\mathrm{Re}\left(G(r,\theta)\right), \quad \dot{\theta} = \frac{1}{r^2}\,\mathrm{Im}\left(G(r,\theta)\right), \tag{2.8}$$

where $G(r,\theta) = \bar{z}F(z,\bar{z})\big|_{z=r\mathrm{e}^{i\theta}} = ar^{2l+2} + br^{m+n+1}\mathrm{e}^{\mathrm{i}(m-n-1)\theta}$. Clearly, the origin of Equation (2.7) corresponds to the solution $r = 0$ of Equation (2.8). Note that $m + n > 2l + 1$,

$$\dot{\theta} = r^{2l}\Big(\mathrm{Im}(a) + \mathrm{Im}\left(\mathrm{e}^{\mathrm{i}(m-n-1)\theta}\right)r^{m+n-2l-1}\Big),$$

and hence $\mathrm{Im}(a) \neq 0$ is a necessary condition for having a monodromic critical point at the origin. Moreover, when $\mathrm{Im}(a) \neq 0$, in a neighbourhood of $r = 0$, system (2.8) can be studied by using the non-autonomous differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{Re}(a)r + \mathrm{Re}\left(b\mathrm{e}^{\mathrm{i}(m-n-1)\theta}\right)r^{m+n-2l}}{\mathrm{Im}(a) + \mathrm{Im}\left(b\mathrm{e}^{\mathrm{i}(m-n-1)\theta}\right)r^{m+n-2l-1}} = H(r,\theta). \tag{2.9}$$

The stability of $r = 0$ is determined by the sign of

$$\sigma = \int_0^{2\pi} \frac{\partial}{\partial r}H(r,\theta)\Big|_{r=0}\,\mathrm{d}\theta,$$

see [17]. Simple computations give that $\sigma = 2\pi\,\mathrm{Re}(a)/\,\mathrm{Im}(a)$. Hence $\mathrm{Re}(a) = 0$ is a necessary condition to have a centre at the origin.

If $m - n = 1$, differential equation (2.9), with $\mathrm{Re}(a) = 0$, writes as

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{Re}(b)r^{2n-2l+1}}{\mathrm{Im}(a) + \mathrm{Im}(b)r^{2n-2l}}.$$

From it, it is clear that in this case Equation (2.7) has a centre at the origin if and only if $\mathrm{Im}(a) \neq 0$, $\mathrm{Re}(a) = 0$ and $\mathrm{Re}(b) = 0$, and item (*i*) follows. It is easy to see that this centre is reversible.

If $m - n \neq 1$, we prove that when $\mathrm{Re}(a) = 0$ the origin is a reversible centre with respect to a straight line passing through the origin. Consider a new variable $z = \mathrm{e}^{\mathrm{i}\eta}w$. Then, Equation (2.7) is rewritten as

$$w' = a|w|^{2l}w + b\mathrm{e}^{\mathrm{i}(m-n-1)\eta}w^m\bar{w}^n.$$

As $m - n - 1 \neq 0$, we can choose η such that $\mathrm{Re}(b\mathrm{e}^{\mathrm{i}(m-n-1)\eta}) = 0$; that is, we have reduced the general case to the situation

$$z' = a|z|^{2l}z + bz^m\bar{z}^n, \quad \text{with} \quad \mathrm{Re}(a) = 0,\ \mathrm{Re}(b) = 0,$$

where we renamed the new variables and parameters as the old ones. The origin of this differential equation is a monodromic critical point and the differential equation is invariant by the change of variables $(z,t) \longrightarrow (\bar{z},-t)$. Hence, it satisfies the hypotheses of the Poincaré reversibility criterion, and the origin is a reversible centre. Thus item (ii) is proved.

(iii) Let $z = R\mathrm{e}^{\mathrm{i}\psi}, R \neq 0$, be a nonzero critical point of centre-type of Equation (1.1). As we have proved in Proposition 2.3, to have anti-saddles inequality $q > 0$ must be satisfied. Another necessary condition to have a centre at this point is that the divergence of its associated vector field X at this point is zero. By Lemmas 2.1 and 2.2, we can transform Equation (1.1) into Equation (2.1), where this critical point moves to $z = 1$. Then, we obtain

$$\operatorname{div}(X)\Big|_{z=1} = 2(k-m)\operatorname{Re}(a\mathrm{e}^{\mathrm{i}((k-l-1)\psi)}).$$

Note that $k - m \neq 0$ because $q|_{k=m} = l - n < 0$, as $k + l < n + m$. Hence, $\operatorname{Re}(a\mathrm{e}^{\mathrm{i}(k-l-1)\psi}) = 0$ is a necessary condition for obtaining a centre at this point. To observe that this condition is equivalent to $\operatorname{Re}(b\mathrm{e}^{\mathrm{i}(m-n-1)\psi}) = 0$, simply note that, on the nonzero critical points, $az^k\bar{z}^l = -bz^m\bar{z}^n$.

To end the proof, we need to show that under the conditions $q > 0$ and $\operatorname{Re}(a\mathrm{e}^{\mathrm{i}(k-l-1)\psi}) = 0$, the point $R\mathrm{e}^{\mathrm{i}\psi}$ is a centre. Evidently, this is a weak focus and therefore monodromic. To prove that it is a centre, we apply Poincaré reversibility criterion.

To this aim, using Lemma 2.2, we move the nonzero critical point to $z = 1$, obtaining a new differential equation

$$\dot{z} = c(z^k\bar{z}^l - z^m\bar{z}^n), \quad \text{where} \quad c = \pm i.$$

because $c = \mathrm{e}^{\mathrm{i}(\alpha+(k-l-1)\psi)}$ and $\operatorname{Re}(c) = 0$. Subsequently, we perform a translation to place this real critical point at the origin, leading to the new differential equation

$$\dot{z} = \pm i\big((z+1)^k(\bar{z}+1)^l - (z+1)^m(\bar{z}+1)^n\big).$$

It can be easily proven that, for any choice of sign, this equation is invariant under a change of the variable and time $(z,t) \longrightarrow (\bar{z},-t)$. Consequently, the origin is a reversible centre. □

One can wonder whether the number of centres for an equation of type (1.1) is limited. The following proposition answers this question.

Proposition 2.6. *Consider a differential equation inside family* (1.1)*, with* $q = l - k + m - n > 0$ *nonzero anti-saddles. If it has* $p > 0$ *nonzero centres, then* p *divides* q*. Moreover, for each* $s \in \mathbb{N}$*, there is a differential equation of the form* (1.1) *with* s *nonzero centres.*

Proof. Assume that Equation (1.1) has at least one nonzero centre. By using Lemma 2.2 and Theorem 2.5, the differential equation can be written as

$$\dot{z} = \mathrm{i}(z^k\bar{z}^l - z^m\bar{z}^n),$$

where possibly we have changed t by $-t$. Moreover, all nonzero critical points are located at $z = z_j = \omega^j$, $j = 0, 1, \ldots, q-1$. Here, $\omega = \mathrm{e}^{2\pi\mathrm{i}/q}$ is a primitive qth root of unity, and the equation has a centre at z_j if and only if $\operatorname{Re}\big(i(\omega^j)^{k-l-1}\big) = \operatorname{Re}(i\varpi^j) = \operatorname{Im}(\varpi^j) = 0$, where $\varpi = \omega^{k-l-1}$ is also a qth root of unity, that is, $\varpi^q = 1$, but not necessarily primitive. In short, the number of centres coincide with the cardinal, $\mathrm{card}(\mathcal{G})$ of the set

$$\mathcal{G} = \{j \in \mathbb{Z}_q \,:\, \varpi^j \in \mathbb{R}\},$$

where $\mathbb{Z}_q$ is the group of integers modulo q. Clearly, $\mathcal{G}$ is a subgroup of $\mathbb{Z}_q$, and by the well-known Lagrange's theorem, $p = \mathrm{card}(\mathcal{G})$ divides $\mathrm{card}(\mathbb{Z}_q) = q$, as we intended to prove.

To obtain an example with exactly s nonzero centres, it suffices to consider

$$\dot{z} = \mathrm{i}(z - z^m \bar{z}^{m-s-1}), \quad \text{with} \quad m \geq s+1.$$

For this equation, $q = l - k + m - n = s$ and $k - l - 1 = 0$. Hence, $\varpi = 1$, $\mathcal{G} = \mathbb{Z}_p$, and the differential equation has $p = q = s$ nonzero centres. Moreover, by item (i) of Theorem 2.5, the origin is a centre.

Another simpler example is the holomorphic differential equation

$$\dot{z} = \mathrm{i}(z - z^{s+1})$$

(see, e.g., [1, 12, 13]). Here, again $q = s$ and $k - l - 1 = 0$, and the same reasoning can also be applied. In this case, the origin is again another centre. In fact, new differential equations obtained by multiplying the right-hand side of the differential equations by $(z\bar{z})^l, l \in \mathbb{N}$, also have s centres. This is because they have the same phase portraits as the corresponding older ones. □

The exact number of centres for an equation of type (1.1) is studied in more detail in the next subsection (see Proposition 2.8). From these forthcoming results, it follows again that the number of nonzero centres is a divisor of q. However, we have decided to include the proof above because it is simpler and uses the nice Lagrange's theorem.

2.3. **Berlinskiĭ type results.** Recall that Berlinskiĭ's theorem is a result for quadratic systems relating the types of critical points (saddles and anti-saddles) with their geometrical positions. Specifically, if a quadratic system has four critical points, and their convex hull is a quadrilateral, along its boundary, their indices alternate. If the convex hull is a triangle, then the three points at the vertices have the same index, whereas the interior point has the opposite one (see [3, 5]). Recently, it has been extended to other classes of vector fields, [1, 4, 15].

In our context, we are interested in the case where $q > 0$ and consequently all nonzero critical points have index $+1$. We already know that all of them lie on a circle centred at the origin and are anti-saddles (see Proposition 2.3). Hence, these q points are ordered as points in $\mathbb{S}^1$, and only three types of critical points exist: attractors $(-)$, repellers $(+)$, and centres (0). From Theorem 2.5, for a given critical point z, its symbol coincides with the sign of divergence of the vector field associated to Equation (1.1), which is called the stability index and is denoted as $s(z)$.

This subsection aims to investigate which chains of q-ordered symbols $+, -, 0$ (in a circular order) are possible. This is interesting because, in general, different chains correspond to non-conjugated phase portraits. Our results for these chains are called Berlinskiĭ-type results.

From Lemma 2.2 and Theorem 2.5, we can reduce the problem to a simple and appealing geometrical question. Although we will not examine this in detail, we now describe this reduction and some simple consequences.

Recall that, by Lemma 2.2, it is not restrictive to study this question using a simpler differential equation

$$\dot{z} = -\mathrm{e}^{\mathrm{i}\delta}(z^k\bar{z}^l - z^m\bar{z}^n), \quad \text{where} \quad \delta \in [0, 2\pi). \tag{2.10}$$

The stability indices are given in the following result, where "sgn" denotes the sign function and $\operatorname{sgn}(0) = 0$. Note that we have added a minus sign in front of the differential equation to simplify the expressions.

Proposition 2.7. *Consider the differential Equation* (2.10) *with $q = l-k+m-n > 0$. Their nonzero critical points are $z = z_j = \omega^j$, $j = 0, 1, \ldots, q-1$, where $\omega = \mathrm{e}^{2\pi\mathrm{i}/q}$ is a primitive qth root of unity, and their stability indices are*

$$s(z_j) = \operatorname{sgn}\big(\operatorname{Re}(\mathrm{e}^{\delta\mathrm{i}} z_j^{k-l-1})\big) = \operatorname{sgn}\big(\operatorname{Re}\big(\mathrm{e}^{\delta\mathrm{i}}\varpi^j\big)\big),$$

where $\varpi = \mathrm{e}^{(2\pi(k-l-1)\mathrm{i})/q}$, is another qth root of unity, which is not necessarily primitive.

Proof. Note that, adding $q = l-k+m-n > 0$ with $m+n-k-l > 0$, we obtain that $2(m-k) > 0$. Hence, from Lemma 2.2 and Theorem 2.5, the stability index of z_j is

$$s(z_j) = \operatorname{sgn}\big((k-m)\operatorname{Re}(-\mathrm{e}^{\delta\mathrm{i}} z_j^{k-l-1})\big) = \operatorname{sgn}\big(\operatorname{Re}(\mathrm{e}^{\delta\mathrm{i}} z_j^{k-l-1})\big),$$

and the result follows. □

From these results, a procedure to determine which Berlinskiĭ-type configurations are possible for Equation (2.10) is presented. Note that they coincide with those of Equation (1.1). Let $\mathcal{L}$ be the line through the origin, with a slope $\tan(\delta)$. The procedure is as follows: First, compute the qth root of unity ϖ. Then, for each $j = 0, 1, \ldots, q-1$, according the region where ϖ^j lies between the two connected components of $\mathbb{C}^2 \setminus \mathcal{L}$ and $\mathcal{L}$, we obtain the values $s(z_j)$. Finally, the configuration is $[s(z_0), s(z_1), \ldots, s(z_{q-1})]$. Note that, when a nonzero critical point of centre type exists, it is not restrictive to take $\mathcal{L} = \{z : \operatorname{Re}(z) = 0\}$; and then, these configurations always start with $s(z_0) = 0$.

We present our initial findings regarding possible configurations.

Proposition 2.8. *For $q = l-k+m-n > 0$, set $D = \gcd(|k-l-1|, q) = \gcd(|k-l-1|, |m-n-1|)$. Then, there exist P and Q positive integers such that $(k-l-1)/q = P/Q$, where $q = DQ$, $k-l-1 = DP \in \mathbb{Z}$, and $\gcd(P, Q) = 1$.*

Subsequently, each configuration for Equation (2.1) *with $q > 0$ is formed by the repetition of D identical basis blocks of Q symbols. Moreover,*

- *(i) if Equation* (1.1) *has some nonzero centres and Q is odd the basis block has only one* 0, *$(Q-1)/2$ symbols $+$ and $(Q-1)/2$ symbols $-$,*
- *(ii) if Equation* (1.1) *has some nonzero centres and Q is even the basis block has two* 0, *$(Q-2)/2$ symbols $+$ and $(Q-2)/2$ symbols $-$,*
- *(iii) if Equation* (1.1) *has no nonzero centre and Q is odd the basis block has $(Q+1)/2$ symbols $+$ and $(Q-1)/2$ symbols $-$, or vice versa,*

(iii) if Equation (1.1) *has no nonzero centre and* Q *is even the basis block has* $Q/2$ *symbols* $+$ *and* $Q/2$ *symbols* $-$.

Proof. The equality of both greatest common divisors is simple because $q+k-l-1 = m-n-1$. We can reduce our proof by studying the normal form (2.1). Note that if $(k-l-1)/q = P/Q$, then the qth root of unity ϖ in the statement of Proposition 2.7 is indeed a primitive Qth root of the unity. Hence, a geometrical interpretation of how to obtain the stability indices of the points z_j can be given. Recall that z_j are the roots of the unity and, therefore, the corners of a regular $q$gon, R_q. Let us describe this interpretation as follows:

- Mark the points of the regular Qgon, R_q that correspond to the points $\varpi^0, \varpi^1, \ldots, \varpi^{Q-1}$.
- Turn R_q by an angle δ. The marked points are then $\mathrm{e}^{\delta i}\varpi^j, j = 0, 1, \ldots, Q-1$ and form a turned regular Qgon.
- The sign $s(z_j)$ only depends on which of the three sets, $\{z : \mathrm{Re}(z) = 0\}$, $\{z : \pm \mathrm{Re}(z) > 0\}$, contains the marked point $\mathrm{e}^{\delta i}\varpi^j$. Clearly, its position only depends on $j \mod Q$.

Consequently, each configuration is formed by the repetition of D identical basis blocks of Q symbols.

A centre appears when the angle δ is such that one of the vertices of the turned Qgon touches the imaginary axis $\{z : \mathrm{Re}(z) = 0\}$. Then, only one configuration exists when Q is odd, or two when Q is even for such points. Based on these results, one can determine the centre for each basis block. The results regarding the nonzero signs simply follow from the symmetry of each regular Qgon. $\square$

First, we start with some simple scenarios that appear for all q :

(i) When $k-l-1 = 0$ then $\varpi = 1$ and all $s(z_j) = \mathrm{Re}(\mathrm{e}^{\mathrm{i}\delta})$. Therefore, in this case, all symbols are equal, and they can be either all 0, or all $+$, or all $-$.

(ii) If $2|k-l-1| = q$, then $\varpi = -1$ and $s(z_j) = \mathrm{Re}(\mathrm{e}^{\mathrm{i}\delta}(-1)^j)$. If moreover $\mathrm{e}^{\delta \mathrm{i}} \notin \{1, -1\}$, then the symbols $+$ and $-$ are alternating.

(iii) If $k-l = 2$, then $\varpi = \omega$ is a primitive q-root of the unity. If $\mathrm{e}^{\delta\mathrm{i}} \notin \{1, -1\}$, then the configurations are formed by $q/2$ nonzero and equal consecutive symbols and $q/2$ opposite consecutive symbols when q is even, and something similar when q is odd. However, one of the chains has one more symbol. Similarly, one or two 0 symbols appear, rather than the other ones, when $\mathrm{e}^{\delta\mathrm{i}} \in \{1, -1\}$.

To determine the relative positions of the $0, +$ and $-$ signs for each basis block, it is also relevant the value Q given by Proposition 2.8. In fact, it is sufficient to investigate only one of these blocks, which were repeated D times. To illustrate different possibilities, we fixed $q = 6$ and $k-l-1 \in \{0, 1, 2, 3\}$. The three remaining cases $(k-l-1 \in \{4, 5, 6\})$ are the consequences of the four previous ones. These are the possible configurations, taking suitable values of δ.

(i) Case $0/6 = 0/1 : (0,0,0,0,0,0)$, $(+,+,+,+,+,+)$, and $(-,-,-,-,-,-)$.

(ii) Case $1/6 : (+,+,+,-,-,-)$, $(0,+,+,0,-,-)$.
(iii) Case $2/6 = 1/3 : (+,+,-,+,+,-)$, $(-,-,+,-,-,+)$, $(0,+,-,0,+,-)$ and $(0,-,+,0,-,+)$.
(iv) Case $3/6 = 1/2 : (+,-,+,-,+,-)$ and $(0,0,0,0,0,0)$.

3. Study of limit cycles. Proof of Theorem A

This section aims to prove Theorem A. We begin by providing two results regarding the nonexistence of limit cycles. The first deals with the case in which an equation of the form (1.1) has infinitely many critical points. Note that, in particular, the next result shows that there exist differential equations of the form (1.1), with a centre at the origin and infinity, simultaneously.

Lemma 3.1. *If an equation of the form* (1.1) *has infinitely many critical points, then it does not have limit cycles. Moreover, it has periodic orbits if and only if $k-l=m-n=1$ (then $q=0$),* $\mathrm{Re}(a)=\mathrm{Re}(b)=0$ *and* $\mathrm{Im}(a)\,\mathrm{Im}(b)<0$.

Proof. As proved in Proposition 2.3, the condition for having infinitely many critical points is $q=0$ (then $k-l=m-n=j$ for some $j\in\mathbb{Z}$) and $a=cb$ for some $c\in\mathbb{R}^-$. If this is the case, family (1.1) writes as

$$\dot{z} = bz^j(z\bar{z})^l\big(c+(z\bar{z})^{n-l}\big), \quad \text{with} \quad n>l.$$

Because $(z\bar{z})^l(c+(z\bar{z})^{n-l})$ is real, we can do a time rescaling to eliminate the circle of critical points, $c+(z\bar{z})^{n-l}=0$, and also the factor $(z\bar{z})^l$ arriving at $\dot{z}=bz^j$. This last differential equation has no limit cycles and has periodic orbits if and only if $j=1$, $\mathrm{Re}(b)=0$ and $\mathrm{Im}(b)\neq 0$. Thus, the lemma follows. Note that, in the last case, the differential equation has a centre at the origin and at infinity simultaneously. □

As a consequence of the above lemma we prove the following result.

Corollary 3.2. *Any differential equation of the form* (1.1) *with $q=0$ has no limit cycles, unless $k-l=m-n=1$. In this case a limit cycle exists if and only if* $\mathrm{Re}(a)\,\mathrm{Re}(b)<0$ *and* $a/b\notin\mathbb{R}^-$.

Proof. By Proposition 2.3, two cases should be considered, either the differential equation has infinitely many critical points or the origin is the unique critical point. In the first situation, from Lemma 3.1 no limit cycle exists. In the second one, because by Proposition 2.3 the index of the origin is $k-l$, a periodic orbit exists only when $k-l=1$. As $q=0$, then $m-n=1$.

Finally, in this case, the expression for Equation (1.1) in polar coordinates is very simple and can be written as

$$\begin{cases} \dot{r} = \mathrm{Re}(a)r^{2l+1}+\mathrm{Re}(b)r^{2n+1}, \\ \dot{\theta} = \mathrm{Im}(a)r^{2l}+\mathrm{Im}(b)r^{2n}. \end{cases}$$

Hence, a differential equation of the form (1.1) has a limit cycle (the circle $r^{2(n-l)}=-\mathrm{Re}(a)/\mathrm{Re}(b)$, as $n-l\neq 0$) if and only if $\mathrm{Re}(a)\,\mathrm{Re}(b)<0$ and $a/b\notin\mathbb{R}^-$. Note that this second condition avoids the fact that this circle is full of critical points. □

We continue with the result that restricts the existence of limit cycles surrounding the origin for a more general family of differential equations.

Proposition 3.3. *Consider the differential equation*

$$\dot{z} = X_N(z,\bar{z}) + X_M(z,\bar{z}),\ 0 \le N < M, \tag{3.1}$$

where X_j is a homogeneous vector field of degree j for variables z and $\bar{z}$. If one of the following conditions holds,

(i) the differential equation $\dot{z} = X_N(z,\bar{z})$ has an invariant straight line through the origin and M is even,

(ii) the differential equation $\dot{z} = X_M(z,\bar{z})$ has an invariant straight line through the origin and N is even,

then it has no periodic orbits surrounding the origin.

We make the following observations prior to proving this proposition. Most homogeneous differential equations $\dot{z} = X_r(z,\bar{z})$ have invariant straight lines through the origin. For example, it suffices that the origin has index different from 0 and 1, or that it has some elliptic or hyperbolic sector (see [2]). Furthermore, if the index is 1 and the point is of nodal type, or the point has index 0 and is formed by two hyperbolic sectors, an invariant straight line also exists. As mentioned previously, the above result is a natural extension of the classical result for quadratic systems, $\dot{z} = X_1(z,\bar{z}) + X_2(z,\bar{z})$. It asserts that these systems do not have periodic orbit surrounding a node. Moreover, the fact that some quadratic systems with a focus at the origin do have limit cycles surrounding it implies that the condition of item (i) that states that $\dot{z} = X_N(z,\bar{z})$ has an invariant straight line through the origin cannot be removed when $N = 1$. For $N > 1$ it suffices to consider differential equations of the form $\dot{z} = a(z\bar{z})^s z + (z\bar{z})^s X_2(z,\bar{z})$, $s \in \mathbb{N}$, where the differential equation $\dot{z} = az + X_2(z,\bar{z})$ has a limit cycle surrounding the origin.

Proof of Proposition 3.3. We prove item (i). Specifically, we observe that the invariant line for $\dot{z} = X_N(z,\bar{z})$ is a line without contact for the equation (3.1). Item (ii) follows similarly.

First, we consider Equation (3.1) with $N > 0$. In this case, the differential equation can be written in polar coordinates as

$$\left\{ \begin{array}{rcl} \dot{r} & = & u_N(\theta) r^N + u_M(\theta) r^M, \\ \dot{\theta} & = & v_N(\theta) r^{N-1} + v_M(\theta) r^{M-1}, \end{array} \right. \tag{3.2}$$

where $v_j(\theta)$ is a homogeneous trigonometric polynomial of degree $j+1$.

Let $\theta = \theta^*$ be the half line that corresponds to the invariant straight line of $\dot{z} = X_N(z,\bar{z})$. Note that $v_N(\theta^*) = 0$ and

$$\dot{\theta}|_{\theta=\theta^*} = v_M(\theta^*) r^{M-1}.$$

If we now consider the half-line that differs π radians from θ^*, we obtain $v_N(\theta^* + \pi) = 0$ and

$$\dot{\theta}|_{\theta=\theta^*+\pi} = v_M(\theta^* + \pi) r^{M-1} = -v_M(\theta^*) r^{M-1},$$

because v_M is a homogeneous trigonometric polynomial of degree $M+1$, which is odd. If $v_M(\theta^*) = 0$ the half-lines $\theta = \theta^*$ and $\theta = \theta^* + \pi$ are invariant and form an invariant line through the origin. Hence, Equation (3.1) has no periodic orbit surrounding the origin. Now, we assume that $v_M(\theta^*) \neq 0$. Thus, the signs of $\dot\theta|_{\theta=\theta^*}$ and $\dot\theta|_{\theta=\theta^*+\pi}$ are different. Consequently, $\dot\theta$ increases when $\theta = \theta^*$ and decreases when $\theta = \theta^* + \pi$ (or vice versa). Hence, no periodic orbit can cross the entire straight line, which is without contact (except at the origin). Consequently, no periodic orbit can surround the origin of Equation (3.1).

We now consider $N = 0$. In this case, the vector field $\dot z = X_N(z, \bar z) = a \in \mathbb{C}$ does not have critical points. Hence, the complete Equation (3.1) can be transformed into the same polar system (3.2) (but $r = 0$ is not a solution of the system), and the arguments presented in the case $N \neq 0$ work in the same manner. □

Application of the previous proposition to Equation (1.1) achieves the following result.

Corollary 3.4. *If $k + l$ is even and $m - n \neq 1$, then Equation* (1.1) *has no periodic orbits surrounding the origin (and, possibly, other critical points). The same occurs if $m + n$ is even and $k - l \neq 1$.*

Proof. Let us prove the second assertion that covers the case $m+n$ even and $k - l \neq 1$. The first follows by using the same concepts. We use item (i) in Proposition 3.3. Note that $M = m + n$ is even, and here $X_N(z, \bar z) = az^k \bar z^l$. Writing $\dot z = az^k \bar z^l$ in polar coordinates, clearly, unless $k - l = 1$, it always has an invariant straight line through the origin. In fact, when $k - l \neq 1$ and $k + l \neq 0$ these lines are the separatrices between consecutive elliptic or hyperbolic sectors of $\dot z = az^k \bar z^l$, whose origin has index $k - l$, see item (i) of Proposition 2.3. When $k + l = 0$ (that is $k = l = 0$), the differential equation is simply $\dot z = a$ and does not have critical points but an invariant line passing through the origin exists. Hence, the result follows. □

The following result, based on the properties of so-called families of rotated vector fields, will be useful in proving the nonexistence of limit cycles in several situations (see [6, 18, 19], for more details about this theory). Recall that the period annulus of a centre is its largest open neighbourhood filled of periodic orbits.

Proposition 3.5. *Let the origin be a centre for a smooth differential equation $\dot z = \mathrm{i}F(z, \bar z)$ and let $\mathcal{U}$ be its period annulus. Then, for $\delta \notin \{\pi/2, -\pi/2\}$ the differential equation $\dot z = \mathrm{e}^{\mathrm{i}\delta} F(z, \bar z)$ does not exhibit periodic orbits intersecting the set $\mathcal{U}$. Moreover, if F is analytic, then it does not have periodic orbits surrounding only the origin.*

Proof. The first part is well known and is a consequence of the classical theory of rotated vector fields.

Let us prove the second part concerning the case F analytic. Assume that, to arrive at a contradiction, for $\delta = \delta^* \notin \{\pi/2, -\pi/2\}$ the differential equation has a periodic orbit γ surrounding only its origin. Then, γ becomes a curve without contact for the differential equation when $\delta = \pi/2$. Because for this value of δ the origin is a centre, we would have a positive or negative invariant region (the region surrounded by γ) containing a continuum of periodic orbits. This situation is impossible for analytical differential equations. □

Proof of Theorem A. For the proof, we distinguish the following three cases:

(i) $m-n=1, k-l=1$,
(ii) $m-n\neq 1$,
(iii) $m-n=1, k-l\neq 1$,

and we prove that a limit cycle can exist only under the hypotheses of case (i), and when it exists, it is hyperbolic.

Suppose that case (i) occurs, $m-n=1, k-l=1$. In Corollary 3.2, we have already characterised the existence and uniqueness of the limit cycle in this case. It exists if and only if $\operatorname{Re}(a)\operatorname{Re}(b)<0$ and $a/b\notin\mathbb{R}^-$. Let us prove its hyperbolicity. Recall that, under these conditions, the differential equation is written in polar coordinates as the integrable system

$$\begin{cases} \dot r = \operatorname{Re}(a)r^{2l+1}+\operatorname{Re}(b)r^{2n+1}, \\ \dot\theta = \operatorname{Im}(a)r^{2l}+\operatorname{Im}(b)r^{2n}, \end{cases}$$

with $l<n$. Because the limit cycle γ is explicit, $\operatorname{Re}(a)+\operatorname{Re}(b)r^{2(n-l)}=0$, its hyperbolicity and stability are given by the sign of

$$\int_0^T \operatorname{div}(X)(z(t),\bar z(t))\,\mathrm{d}t,$$

where X is the vector field associated to the differential equation, $z=z(t)$ is its time parameterization and T is its period, see [8]. Using Lemma 2.1 we compute this divergence as

$$\begin{aligned}\operatorname{div}(X) =& 2\operatorname{Re}\big(\frac{\partial}{\partial z}F\big) = 2\operatorname{Re}\big((l+1)a(z\bar z)^l + b(n+1)(z\bar z)^n\big)\\ =&2(l+1)\operatorname{Re}(a)(z\bar z)^l+2(n+1)\operatorname{Re}(b)(z\bar z)^n.\end{aligned}$$

Moreover, on the limit cycle, $\operatorname{Re}(b)(z\bar z)^n|_\gamma = -\operatorname{Re}(a)(z\bar z)^l|_\gamma$. Hence,

$$\operatorname{div}(X)|_\gamma = 2(l-n)\operatorname{Re}(a)(z\bar z)^l|_\gamma = -2|l-n|\operatorname{Re}(a)\Big(\frac{-\operatorname{Re}(a)}{\operatorname{Re}(b)}\Big)^{l/(n-l)}.$$

Consequently,

$$\int_0^T \operatorname{div}(X)(z(t),\bar z(t))\,\mathrm{d}t = -2|l-n|\operatorname{Re}(a)\Big(\frac{-\operatorname{Re}(a)}{\operatorname{Re}(b)}\Big)^{l/(n-l)}T.$$

This proves that γ is hyperbolic, and an attractor (resp. repeller) if $\operatorname{Re}(a)>0$ (resp. $\operatorname{Re}(a)<0$). Its stability is the opposite to that of the origin, which is given by the sign of $\operatorname{Re}(a)$. Thus, the proof for the first case ends.

Suppose that case (ii) occurs; then, $m-n\neq 1$. By Corollary 3.2, differential Equation (1.1) has no limit cycle when $q=0$. Then, we can assume that $q\neq 0$; hence, this differential equation has nonzero critical points. Recall that, if a periodic orbit γ exists, it must surround a set of critical points whose indices sum is one. Recall also that all nonzero critical points have the same index, which coincides with the sign of q (see Proposition 2.3). Hence, there are only two possibilities:

(I) The periodic orbit γ does not surround the origin. In this case, the periodic orbit only surrounds a single nonzero critical point of index $+1$ and $q>0$.
(II) The periodic orbit γ surrounds the origin and possibly other critical points.

Let us prove that possibility (I) does not hold. As $q \neq 0$, by applying Lemma 2.2, the differential Equation (1.1) can be written as

$$\dot{z} = \mathrm{e}^{\delta \mathrm{i}}(z^k \bar{z}^l - z^m \bar{z}^n), \tag{3.3}$$

for a certain $\delta \in [0, 2\pi)$. Now, it suffices to prove that Equation (3.3) does not have limit cycles, surrounding only the point $z = 1$. First, we translate this critical point to the origin and obtain

$$\dot{z} = \mathrm{e}^{\delta \mathrm{i}}\big((z+1)^k(\bar{z}+1)^l - (z+1)^m(\bar{z}+1)^n\big).$$

Moreover, by the proof of item *(iii)* of Theorem 2.5, we find that this translated differential equation has a centre at the origin if and only if $\delta \in \{\pi/2, -\pi/2\}$. For other values of δ, by Proposition 3.5, we find that it does not have periodic orbits surrounding the origin that correspond to the nonzero anti-saddle in the original equation, as we intended to prove.

Let us now prove that possibility (II) cannot occur. Assume that Equation (1.1) has a limit cycle γ surrounding the origin and possibly some nonzero critical points. Because $m - n \neq 1$, to obtain this limit cycle by applying Corollary 3.4, $k - l$ must be odd. When $k - l \neq 1$, by applying Corollary 3.4, to have such a limit cycle, $m - n$ must also be odd. Hence, we can assume that $m - n$ and $k - l$ are both odd; consequently, $m + n$ and $k + l$ are also odd. Then, Equation (1.1) has a symmetry: it is invariant with respect to the change of variables $(z, t) \longrightarrow (-z, -t)$. Thus, $-\gamma$ is a periodic orbit too. If γ surrounds the origin and other critical points, but not all of them, $-\gamma \cap \gamma \neq \emptyset$, which is in contradiction with the uniqueness of solution for the differential equation. Hence, $\gamma = -\gamma$, and furthermore γ surrounds either only the origin or all critical points.

Observe that, if a periodic orbit exists surrounding all the critical points, then the infinity index must be $+1$. From Lemma 2.4, this index is $2+n-m$. Because of the hypothesis $m - n \neq 1$, then $2 + n - m \neq 1$; this possibility is excluded.

If a limit cycle exists surrounding only the origin, it has index $+1$. According to Proposition 2.3, the origin index is $k - l$. Hence, as we are assuming $k - l \neq 1$, this possibility is also excluded and no limit cycle exist.

Finally, consider the case $k - l = 1$ and let us prove again that no limit cycle can exist surrounding the origin and, possibly, other critical points. To do so, note that, under our assumptions $q \neq 0$ and then the differential equation has nonzero critical points. Recall that the index of the origin is $k - l = 1$, and if a limit cycle surrounds it, as the nonzero critical points are all simple and of the same index, the limit cycle must surround only the origin. Hence, applying Lemma 2.2 to Equation (1.1), it can be expressed as Equation (3.3) for a certain $\delta \in [0, 2\pi)$. From Theorem 2.5(ii), the origin is a centre if and only if $\delta \in \{\pi/2, -\pi/2\}$. For the other values of δ, from Proposition 3.5, Equation (1.1) does not have periodic orbits surrounding the origin. Hence, the proof of this case is complete.

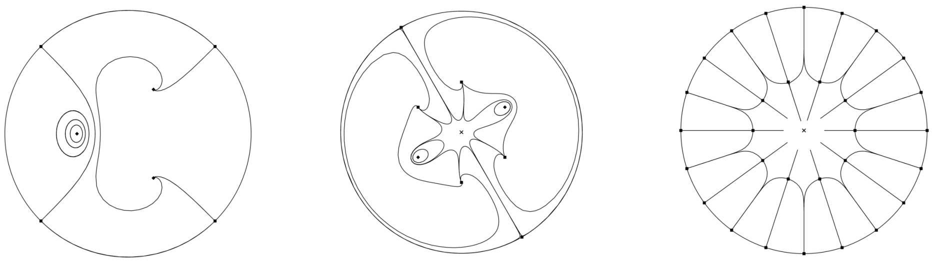
To end the proof, assume that, in order to arrive at a contradiction, case (iii) occurs, $m - n = 1, k - l \neq 1$, and that Equation (1.1) has a limit cycle γ. Then, if we do the change of variable and time used in the proof of Lemma 2.4, $w = z^{-1}$ and $\mathrm{d}t/\mathrm{d}s = |w|^{2(m+n)}$, we obtain expression (2.5); that is, $w' = -bw^{2+n}\bar{w}^m - aw^{2+m+n-k}\bar{w}^{m+n-l}$, where $2+m+n-k+m+n-l >$

$2+n+m$, which is also a differential equation of the form (1.1). Note that the previous Equation (2.5) has $\gamma^* = \{w \in \mathbb{C} : w^{-1} \in \gamma\}$ as a limit cycle, because Equations (2.5) and (1.1) are topologically equivalent, but note that the corresponding value of $m-n$ is now $(2+m+n-k)-(m+n-l) = l-k+2 \neq 1$. Consequently, we would have a limit cycle under the hypotheses of case (ii), which is a contradiction. □

4. Phase portraits

We show some phase portraits in the Poincaré disk of Equation (1.1) using the free software "Polynomial Planar Phase Portraits" typically abbreviated as $P4$, and which is introduced in [8, Chap. 9]. In Figure 1 we want to illustrate the dynamic richness of this family despite its apparent simplicity. The parameters set in this figure are (k,l,m,n,a,b): (a) $(1,1,4,1,i,i)$, (b) $(0,4,4,2,-(1+\sqrt{3})/2,-(1+\sqrt{3})/2)$, and (c) $(3,2,0,9,1,-1)$.

We remark that these configurations are straightforward applications of Proposition 2.8 and Theorem 2.5. From these results, we conclude that, in Figure 1(a), the origin is a critical point with two hyperbolic sectors as $k-l=0$; in Figure 1(b), the origin is a critical point with 10 hyperbolic sectors as $k-l=-4$; and in Figure 1(c), the origin is a critical point of index $k-l=+1$.



(a) (b) (c)

Figure 1. Phase portraits in the Poincaré disk of some examples of Equation (1.1) with nonzero finite critical points: (a) one centre and two focus, (b) two centres and four nodes, and (c) ten saddles.

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