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ANALYTICAL AND EXPERIMENTAL STUDY OF BEAM BENDING VIBRATION

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Abstract: This study presents free bending vibration analysis of beams of glass epoxy material. The analytical and the modal model (experimental) methods were used to conduct the investigation. The experiment represents basic concepts of the modal model analysis method, which allows us to find the beam's natural frequencies and vibrational mode shapes. The analytical solution was found using Maple and compared to the frequencies obtained experimentally.

Keywords: *Transverse vibration, Modal analysis, Analytical solution, Free-Free beam, Cantilever Beam.*

1. INTRODUCTION

Vibrational problems are critical in most engineering applications. Hence, understanding the vibration effects in an application is essential. In the last decades, the Modal analysis has become one of the most used methods for determining and optimizing the dynamic characteristics of engineering applications and different fields such as mechanical, aeronautical, civil, biomechanical, acoustic engineering and music instruments, transportation, and many other areas. Modal analysis determines the dynamic properties of a given system in the frequency domain.
Free vibration happens when no external vibration force is applied on the system. The system, in this case, the beams, starts vibrating due to initial displacement or by using a tool like a shaker. The first part of this investigation consists of the exact solution for the two boundary conditions (constraints on the beam due to its supports). They are the Free-Free and Fixed-Free beams. Maple was used to find the exact solution and plot each case's mode shapes.
In recent decades, modal analysis has been used in many specialized engineering fields related to dynamic structural analysis [1]. In the experiments, the system

vibrates due to a shaker then the displacement is measured in time using a sensor. The sensor is attached to the system, either transducer (accelerometer) or noncontact like lasers or cameras. When the shaker starts vibrating the beam, the accelerometer will collect data as displacement (amplitude) in time for the specific point of the beam. This data is sent and displayed on a computer.
This study aims to understand the vibration of a simple model (geometry) for wind turbine blades: the fixed-free beam, also known as the cantilever beam. After that, in future studies, the system will be improved into a more complicated beam by adding more conditions or changing the beam, such as a change in the cross-section area, pre-bending, moving mass on the beam, or attaching a spiral spring to the beam.

2. ANALYTICAL STUDY

In this section, the equation of motion and boundary conditions are given, which are used to obtain the characteristic equation and eigenvalues. Then the mode shapes are plotted.

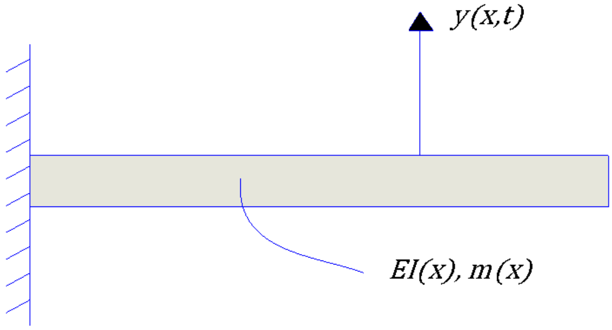

Figure 1. *Cantilever Beam*

Figure 1 shows the schematic setup of the problem (cantilever beam). By applying the Euler–Bernoulli theory on the system to derive the equation of motion, the following equation is obtained [2]:

$$EI\,\frac{\partial^4}{\partial x^4}y(x,t)+\rho A\,\frac{\partial^2}{\partial x^2}y(x,t)=0. \tag{1}$$

The constants of this equation are E (Young's modulus of elasticity), I (the moment of inertia), ρ (the density of the beam), and A (the cross-section area), while

$y(x,t)$ is the deflection function. By using the method of separation of variables and rearranging some terms, the following equation is obtained:

$$\frac{d^4}{\mathrm{d}x^4}y(x) - \beta^4 y(x) = 0. \tag{2}$$

The wave number β in this equation can be expressed as follows

$$\beta^4 = \omega^2 \frac{m}{EI}, \tag{3}$$

where ω is the angular frequency, and m(x) shown in Figure 1 is the mass per unit length that is constant for our case, i.e., m(x)=m.
The spatial part of the solution can be presented as follow:

$$y(x) = A\sin(\beta x) + B\cos(\beta x) + C\sinh(\beta\, x) + D\cosh(\,\beta\, x). \tag{4}$$

The boundary conditions, the characteristic equation, and the eigenvalues are presented in Table 1. The characteristic equation for each case is obtained by using the solution in equation (4) to solve the equation of motion (1) using the boundary conditions in Table 1. Further details can be found in [3].

Table 1
Boundary conditions, Characteristics equation, and Eigenvalues for Free-Free and Fixed-Free beams

Free-Free		**Fixed-Free**	
Right end	**Left end**	**Right end**	**Left end**
$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x) = 0$	$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x) = 0$	$y(x) = 0$	$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x) = 0$
$\frac{\mathrm{d}^3}{\mathrm{d}x^3}y(x) = 0$	$\frac{\mathrm{d}^3}{\mathrm{d}x^3}y(x) = 0$	$\frac{d}{dx}y(x) = 0$	$\frac{\mathrm{d}^3}{\mathrm{d}x^3}y(x) = 0$
Characteristics equation	$\cos(\beta_n L)\cosh(\beta_n L) - 1 = 0$ $n = [0,1,2,\ldots]$	$\cos(\beta_n L)\cosh(\beta_n L) + 1 = 0$ $n = [1,2,\ldots]$	
Eigenvalues(β_nL)	[0., 4.730, 7.853, 10.99, 14.137,]	[1.875, 4.694, 7.854, 10.995, 14.137]	

Natural frequencies are calculated from the eigenvalues listed in Table 1 and they are compared with the frequencies obtained experimentally as shown in Table 3 and Table 4, whereas the mode shapes are obtained from the deflection function. The free-vibration solution for the beam with free-free ends is obtained as follows

$$y(x) = -\frac{(\cos(\beta l) - \cosh(\beta l))\sin(\beta x)}{sin(\beta l) - \sinh\,(\beta l)} + cos(\beta x) - \frac{(\cos(\beta l) - \cosh(\beta l))\,\sinh\,(\beta x)}{sin(\beta l) - \sinh\,(\beta l)} + \cosh(\beta x). \tag{5}$$

The first five mode shapes for the free-free ends are plotted in Figure 2.

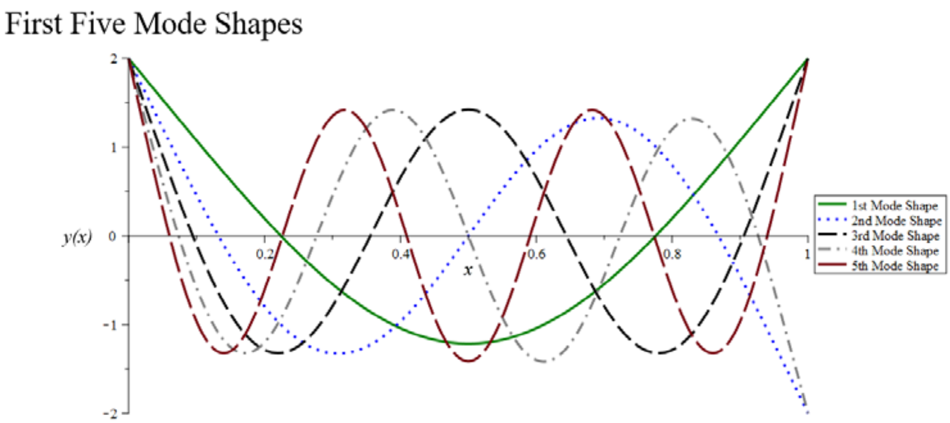

Figure 2. *First Five mode shapes for Free-Free beam. The x-axis presents the dimensionless length, while the y-axis is the deflection*

The free vibration solution for the cantilever beam is the following:

$$y(x) = \frac{(\cosh\,(\beta x) - \cos\,(\beta x))\sinh\,(\beta l)}{\sin\,(\beta l) + sinh\,(\beta l)} + (\cosh(\beta x) - \cos\,(\beta x))\,\sin(\beta l) - \left(\cos(\beta l) + \cosh(\beta l)\right)(\sinh(\beta x) + \sin(\beta x). \tag{6}$$

The first five mode shapes for the cantilever beam are plotted in Figure 3. Figure 2 and Figure 3 illustrate the beam's deflection y(x) along the x-axis for the first five mode shapes for the Free-Free and Fixed-Free cases, respectively. For instance, when the beam gets excited at the first natural frequency, the beam will be deflected into the shape of the first mode shape in the figure.

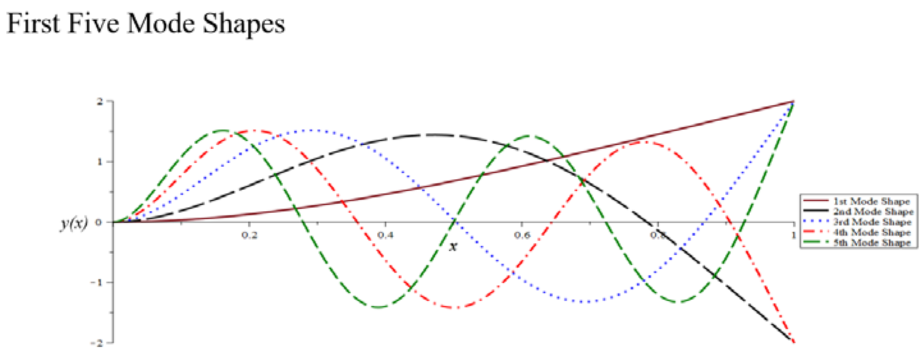

Figure 3. *First Five mode shapes for Cantilever beam. The x-axis presents the dimensionless length, while the y-axis is the deflection*

3. EXPERIMENTAL METHOD (MODAL MODEL ANALYSIS)

Modal analysis studies the dynamic properties of systems in the frequency domain. Such analysis is carried out to study the free vibration of machines and structures such as a car, a turbine blade, or a wind turbine. The experiments need the following items to be carried out:

- Sensors like transducers (accelerometers) or noncontact like lasers or cameras to measure the system answer caused by an excitation.
- Force transducer to measure the excitation force.
- An analog-to-digital converter front end (DAQ) and a host PC are to view the data in which Fourier series is used to analyse the data where the resulting transfer function demonstrates the natural frequencies.

The experiment for this study consists of the following items shown in Figure a:

1. A shaker.
2. Data acquisition board (DAQ).
3. Impedance head (accelerometer and force transducer in one housing).
4. A computer.

The measurements are carried out with the following software setup:

- Frequency range: 0 – 18750 Hz
- Frequency resolution: 0.732 Hz
- Excitation signal: white noise
- Number of averages: 30 (linear average)
- Hanning window
- Measuring transfer functions (FRF): a/F; v/F, and for each FRF also the coherence function

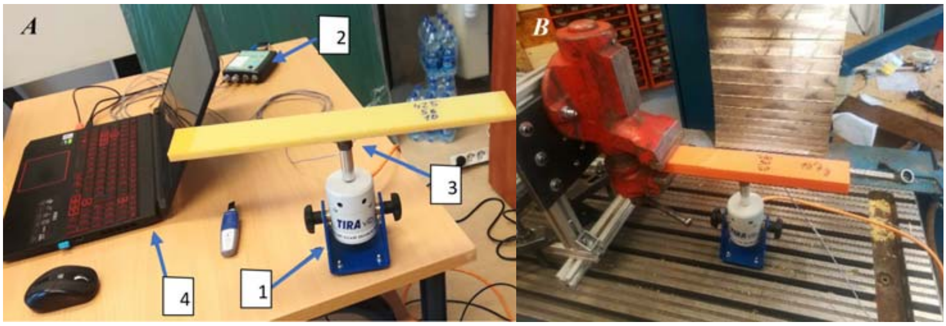

Figure 4. *Experimental setup for (A) Free-Free beam and (B) Cantilever beam*

In order to determine the modal parameters (natural frequencies, damping, and mode shapes) of a specimen, the so-called frequency response functions (FRF) have to be constructed from the measurements. The FRFs are functions that are constructed, in this case, from the excitation force and from the response function of the system to the excitation. The response functions are in this measurement the acceleration (in the excitation point measured with impedance head) and the velocity (at the end of the beam, measured with laser vibrometer). The FRFs are in general, recorded in several points of the specimen geometry, so the mode shapes can also be calculated. In our case, we only measured 2 points, so the experimental mode shapes could not be calculated.
Three specimens are selected for the modal model analysis with the data in Table 2. The data was imported into MATLAB to plot figures for different cases. These figures are followed by tables showing the experimental and analytical (theoretical) values of the natural frequencies and the error between them.

Table 2
Parameters for the three specimens

Specimen	Material	Dimensions (m)			Mass (kg)	Young's Modulus (GPa)
		Length	Width	Height		
1	Glass epoxy	0.390	0.06	0.015	0.655	10.5
2	Glass epoxy	0.425	0.058	0.01	0.462	12
3	Printed	0.3	0.05	0.02	0.197	14

Glass epoxy has a range of different values for Young's modulus in the transverse direction. In papers [4], [5], and [6] a range for its possible values can be found. For our purposes, we selected the values in table (2), which minimizes the error among a large number of frequencies.

Two different boundary conditions were applied for every specimen (Free-Free, and fixed-free), and the experimental results were compared to the exact solution for each case.
Figure 5 illustrates the behaviour of the first specimen (15 mm) in the Frequency-Acc/Force domain, where the peaks in the figures represent the resonance frequencies. The left part is for the free-free case, while the right side is for the fixed-free beam.

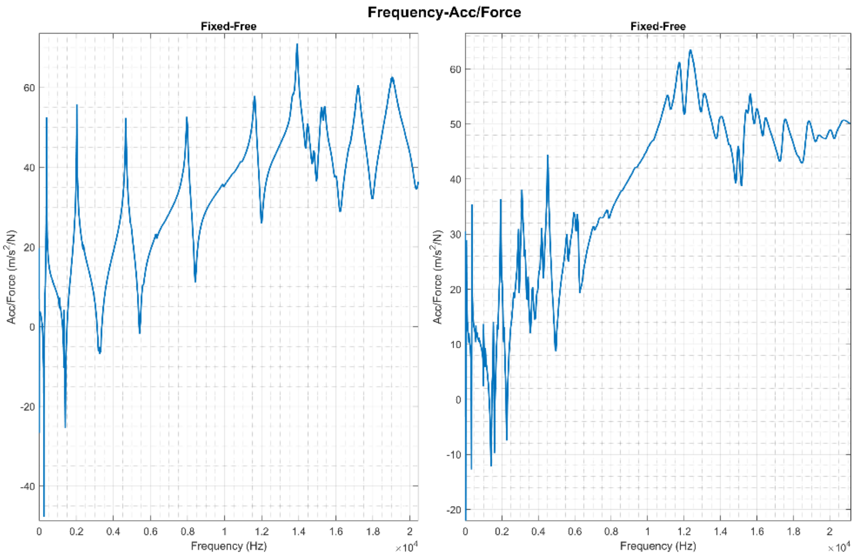

Figure 5. *Frequency-Acc/Force domain for the first specimen (15 mm)*

Table 3
Experimental and analytical results for different frequencies and the error between them for the first specimen (15 mm)

15 mm	Experiment (rad/sec)	Analytical (rad/sec)	Error %
	Fixed-Free		
ω_1	361.2	380.2	4.988
ω_2	2245.7	2382.7	5.747
ω_4	12135.3	13073.7	7.177
	Free-Free		
ω_1	2475.8	2419.3	2.335
ω_3	12733.5	13073.9	2.603
ω_5	29346.5	32284.3	9.099

Table 3 represents the values for different natural frequencies analytically and experimentally with the error for the first specimen (15 mm). The unit of the frequency in the experiment is Hz, so it was converted into rad/sec to compare it with the analytical value.
Figure 6 illustrates the first beam's behaviour in the Frequency-Acc/Force domain. The left part is for the Free-Free case, while the right side is for the Fixed-Free beam.
Table 4 represents the values for different natural frequencies analytically and experimentally with the error for the second specimen.

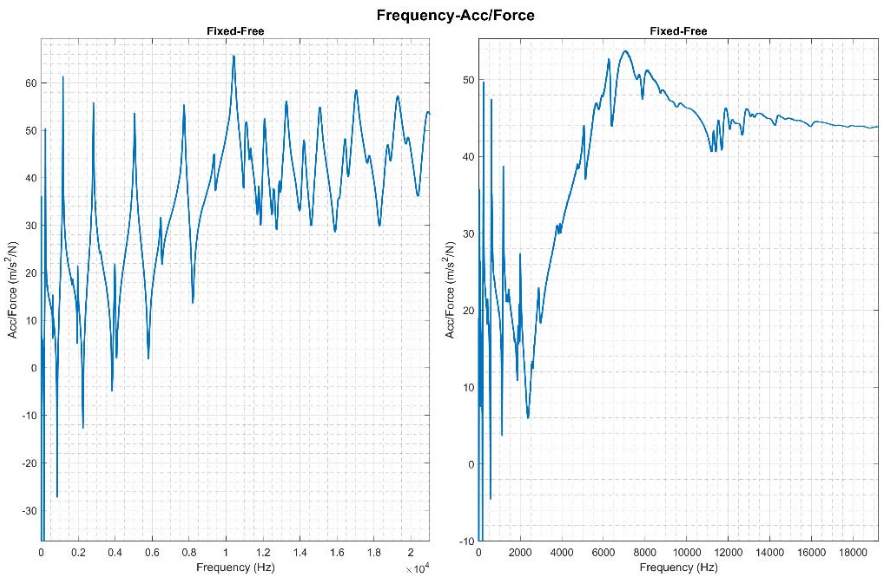

Figure 6. *Frequency-Acc/Force domain for the second specimen (10 mm)*

The error increases with the increase in frequency due to two main reasons. The first one is the inhomogeneous elasticity of the material, and the second one is an assumption applied in the mathematical model considered for the system. The shaker is attached to the beam at a specific point. At that point, the shaker acts as a pin-support, yet the shaker's force on the beam is assumed to be zero in the mathematical model. At higher frequency (displacement), the force increases, which increases its effect on the system and the error's value as well.

Table 4

Experimental and analytical results for different frequencies and the error between them for the second specimen (10 mm)

10 mm	Experiment (rad/sec)	Analytical (rad/sec)	Error %
	Fixed-Free		
ω_1	227.6	218.1	4.360
ω_2	1417.3	1366.8	3.698
ω_4	3815.0	3827.1	0.318
	Free-Free		
ω_1	1389.7	1387.8	0.138
ω_3	7367.7	7499.8	1.761
ω_5	17708.2	18519.9	4.382

In paper [7], the researchers compared the natural frequencies obtained by the FEM and the experimental method. The error between the frequencies from both methods kept increasing with the increase in the number of the natural frequencies. This might be because the FEM uses a complete mathematical model. Nevertheless, the increase in error might be due to the used material, human error, or numerical errors.

4. RESONANCE FREQUENCIES FROM THE MODAL METHOD

The modal model method can be used to construct the mode shapes of the beam experimentally [8] by placing sensors (for example, accelerometers) at different points along the beam and comparing the results.

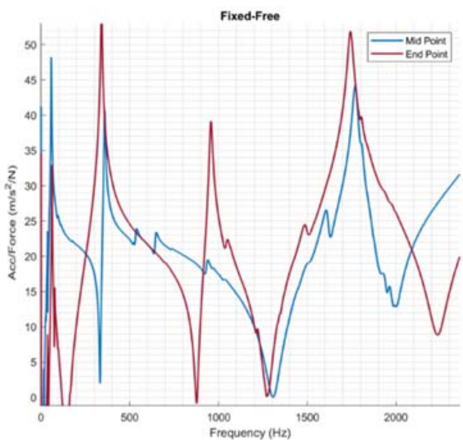

Figure 7. *The resonance frequencies from the experimental data*

In Figure 7, we can see the data obtained in the same experiment for two different points. One point is at the beam's free end, while the other one is in the middle. The beam has zero displacements at the middle point for the third natural frequency. Hence, the sensor could not detect any significant movement (deflection). The mode shapes can be built by tracking such pieces of information from the experimental data and from the geometrical location of the sensors.

5. SUMMARY AND CONCLUSION

The free vibration of a Free-Free Beam and a Cantilever beam has been investigated analytically and experimentally with glass epoxy material. The relative error between the exact theoretical solution and experimental values is at its minimum (less than 1%) at small frequencies and increases at higher frequencies to more than 10%. This is due to the inhomogeneity of the material and the mathematical model, which does not accommodate the force between the beam and the shaker. The paper also showed the possibility of building the vibrational mode shapes from the experimental data.

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