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ENERGY CONCENTRATION OF THE P-LANDAU-LIFSCHITZ FUNCTIONAL WITH RADIAL STRUCTURE

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Abstract. This paper is concerned with the asymptotic behavior of a p-Landau-Lifschitz type functional with radial structure as parameter goes to zero. We study the concentration compactness and give several global properties in the case of $p > 2$.

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1. INTRODUCTION

Let $B = \{x \in \mathbb{R}^2; x_1^2 + x_2^2 < 1\}$. Denote $\mathbb{S}^1 = \{x = (x_1 + ix_2, x_3) \in \mathbb{C} \times \mathbb{R}; x_1^2 + x_2^2 = 1, x_3 = 0\}$ and $\mathbb{S}^2 = \{x \in \mathbb{C} \times \mathbb{R}; x_1^2 + x_2^2 + x_3^2 = 1\}$. Let $g(x) = (e^{id\theta}, 0)$ where $x = (\cos\theta, \sin\theta)$ on ∂B, $d \in N$. We are concerned with the minimizer of the energy functional of p-Landau-Lifschitz type

$$E_\varepsilon(u,B) = \frac{1}{p}\int_B |\nabla u|^p dx + \frac{1}{2\varepsilon^p}\int_B u_3^2 dx \quad (p > 2)$$

in the function class

$$W = \{u(x) = (\sin f(r)e^{id\theta}, \cos f(r)) \in W^{1,p}(B, \mathbb{S}^2); u|_{\partial B} = g\},$$

which is named the radial minimizer of $E_\varepsilon(u,B)$.

When $p = 2$, the functional $E_\varepsilon(u,B)$ was introduced in the study of some simplified model of high-energy physics, which controls the statics of planar ferromagnets and antiferromagnets (cf. [8] and [15]). In addition, it is helpful to understand the dynamics of singularities appearing in the liquid crystals (cf. [2, 7, 12, 14] and [6]). In particular, the authors of [7] discussed the asymptotic behaviour of the radial minimizer of $E_\varepsilon(u,B)$ in §5. When the penalization term $\frac{1}{2\varepsilon^2}\int_B u_3^2 dx$ is replaced by $\frac{1}{4\varepsilon^2}\int_B (1-|u|^2)^2 dx$ and $\mathbb{S}^2$ is replaced by $\mathbb{C}$, the functional becomes the Ginzburg-Landau energy introduced in the theory of superconductors (cf. [3] and the references therein). Nineteen problems were proposed in [3]. Comte and Mironescu studied Problem 7 in [4, 5, 13]. Problem 7 and Theorems VII.2 and VII.3 in [3] describe the global properties of the Ginzburg-Landau functional. For the Landau-Lifschitz functional, Theorem 4.2 in [7] shows analogous results of Theorems VII.2 and VII.3 in [3].

When $p > 2$, Lei studied the behaviour of minimizers of $E_\varepsilon(u,B)$ as $\varepsilon \to 0$ (cf. [10]). In addition, he also proved the $W^{1,p}_{loc}$ convergence of the radial minimizers, and obtained some estimates of the convergent rate of the radial minimizer (cf. [9]). For the p-Ginzburg-Landau functional, the behaviour of radial minimizers was studied in [1] and [11]. In particular, the analogous global properties are shown in [11].

In polar coordinates, for $u(x) = (\sin f(r)e^{id\theta}, \cos f(r))$, we have

$$|\nabla u| = (f_r^2 + d^2 r^{-2}\sin^2 f)^{1/2}.$$

Sometimes we denote $\sin f(r)e^{id\theta}$ by u'. If we denote

$$V = \{f \in W^{1,p}_{loc}(0,1]; r^{1/p} f_r, r^{(1-p)/p}\sin f \in L^p(0,1), f(r) \geq 0, f(1) = \frac{\pi}{2}\},$$

then $V = \{f(r); u(x) = (\sin f(r)e^{id\theta}, \cos f(r)) \in W\}$.

Substituting $u(x) = (\sin f(r)e^{id\theta}, \cos f(r)) \in W$ into $E_\varepsilon(u,B)$ we obtain

$$E_\varepsilon(u,B) = 2\pi E_\varepsilon(f,(0,1)),$$

where

$$E_\varepsilon(f,(0,1)) = \int_0^1 [\frac{1}{p}(f_r^2 + d^2 r^{-2}\sin^2 f)^{p/2} + \frac{1}{2\varepsilon^p}\cos^2 f]rdr.$$

This shows that $u = (\sin f(r)e^{id\theta}, \cos f(r)) \in W$ is the minimizer of $E_\varepsilon(u,B)$ if and only if $f(r) \in V$ is the minimizer of $E_\varepsilon(f,(0,1))$. Applying the direct method in the calculus of variations we can see that the functional $E_\varepsilon(u,B)$ achieves its minimum on W by a function $u_\varepsilon(x) = (\sin f_\varepsilon(r)e^{id\theta}, \cos f_\varepsilon(r))$, hence $f_\varepsilon(r)$ is the minimizer of $E_\varepsilon(f,(0,1))$.

Recall some results in [9]. Let $u_\varepsilon = (\sin f_\varepsilon(r)e^{id\theta}, \cos f_\varepsilon(r))$ be a radial minimizer of $E_\varepsilon(u,B)$ on W. Then Theorem 1.1 in [9] shows that for any $\gamma \in (0,1)$, there exists a constant $h = h(\gamma)$ which is independent of $\varepsilon \in (0,1)$ such that

$$Z_\varepsilon = \{x \in B; |u_{\varepsilon 3}| > \gamma\} \subset B(0,h\varepsilon). \tag{1.1}$$

This implies that all the points where $u_{\varepsilon 3}^2 = 1$ are contained in $B(0,h\varepsilon)$. Hence as $\varepsilon \to 0$, these points converge to 0. Furthermore, Proposition 3.2 and Theorem 1.3 in [9] show that for any compact subset $K \subset \overline{B} \setminus \{0\}$, there exists a positive constant C (independent of ε), such that

$$E_\varepsilon(u_\varepsilon, K) \leq C \tag{1.2}$$

$$\sup_{x \in K} |u_{\varepsilon 3}(x)| \leq C\varepsilon^{\frac{p-2}{2}}. \tag{1.3}$$

Here $K=\overline{B}\setminus B(0,\eta)$. In addition, Proposition 2.1 in [9] shows

$$E_\varepsilon(u_\varepsilon,B)\le C\varepsilon^{2-p}. \tag{1.4}$$

In this paper, we will study the global properties of the p-Landau-Lifschitz model, which are described by the concentration properties.

Theorem 1. *Let $u_\varepsilon=(\sin f_\varepsilon(r)e^{id\theta},\cos f_\varepsilon(r))$ be a radial minimizer of $E_\varepsilon(u,B)$ on W. Then as $\varepsilon\to 0$, there exists a subsequence ε_k such that*

$$\frac{1}{2\varepsilon_k^2}|u_{\varepsilon_k 3}|^2\to L_1\delta_o,\quad \text{weakly star in } C(\overline{B}), \tag{1.5}$$

$$\varepsilon_k^{p-2}|\nabla u_{\varepsilon_k}|^p\to \frac{2p}{p-2}L_1\delta_o,\quad \text{weakly star in } C(\overline{B}). \tag{1.6}$$

Here δ_o is the Dirac mass at the origin, and the positive constant L_1 satisfies

$$\frac{\pi d^p}{p}\sup_{\gamma\in(0,1)}(1-\gamma^2)^{p/2}h^{2-p}(\gamma)\le L_1\le(1-\frac{2}{p})\min_W E_1(u,B)+\frac{2\pi d^p}{p^2}, \tag{1.7}$$

where $h(\gamma)$ is a positive constant in (1.1).

Theorem 2. *Let $u_\varepsilon=(\sin f_\varepsilon(r)e^{id\theta},\cos f_\varepsilon(r))$ be a radial minimizer of $E_\varepsilon(u,B)$ on W. Then for any $\alpha\ge 2-4/p$, we can find a subsequence ε_k of ε, and constants $L_3>0$ and $L_4\ge 0$ which are independent of ε, such that as $k\to\infty$,*

$$|u_{\varepsilon_k 3}|^\alpha|\nabla u_{\varepsilon_k}|^2\to L_3\delta_o,\quad \text{weakly star in } C(\overline{B}), \tag{1.8}$$

$$\varepsilon_k^{p-2}|\det(\nabla u'_{\varepsilon_k})|^{p/2}\to L_4\delta_o,\quad \text{weakly star in } C(\overline{B}). \tag{1.9}$$

The related results in higher dimension to Theorems 2 and 1 can be found in [16] and [17].

2. Proof of Theorem 1

2.1. *Proofs of (1.5) and (1.6)*

In view of (1.4), there exist two Radon measures ω_1 and ω_2, such that as $\varepsilon\to 0$,

$$\varepsilon_k^{p-2}|\nabla u_{\varepsilon_k}|^p\to\omega_1,\quad \text{weakly star in}\quad C(\overline{B}), \tag{2.1}$$

$$\frac{1}{2\varepsilon_k^2}u_{\varepsilon_k 3}^2\to\omega_2,\quad \text{weakly star in}\quad C(\overline{B}), \tag{2.2}$$

for some subsequence ε_k of ε. Sometimes we also denote u_{ε_k} by u_ε for convenience. Furthermore, (1.2) implies that as $\varepsilon\to 0$,

$$\varepsilon^{p-2}\int_K|\nabla u_\varepsilon|^p dx\to 0,$$

$$\frac{1}{2\varepsilon^2}\int_K u_{\varepsilon 3}^2 dx\to 0,$$

where K is an arbitrary compact subset of $B\setminus\{0\}$. These results lead to $\mathrm{supp}(\omega_i)\subset\{0\}$ for $i=1,2$. Then we can find constants L_1 and L_2 such that

$$\omega_1 = L_2\delta_o, \quad \omega_2 = L_1\delta_o. \tag{2.3}$$

Next, we shall point out the relation between L_1 and L_2. It is not difficult to see that the radial minimizer u_ε solves the system

$$-\operatorname{div}[|\nabla u|^{p-2}\nabla u] = u|\nabla u|^p + \frac{1}{\varepsilon^p}(uu_3^2 - u_3 e_3) \ \text{ in } \ B. \tag{2.4}$$

Multiplying (2.4) by $x\cdot\nabla u$ and integrating by parts, we can obtain the Pohozaev type identity

$$\begin{aligned}
&-\int_{\partial B_R(0)} |x||\nabla u|^{p-2}|\partial_\nu u|^2 ds + \int_{B_R(0)} |\nabla u|^p dx - \frac{2}{p}\int_{B_R(0)} |\nabla u|^p dx \\
&\quad + \frac{1}{p}\int_{\partial B_R(0)} |x||\nabla u|^p ds = -\frac{1}{2\varepsilon^p}\int_{\partial B_R(0)} |x|u_3^2 ds + \frac{1}{\varepsilon^p}\int_{B_R(0)} u_3^2 dx
\end{aligned} \tag{2.5}$$

for any $R\in(0,1]$. Hereafter, we denote f_ε by f. By (1.2) and the mean value theorem, there exists $\sigma\in(1/4,1/2)$ such that

$$r[(f_r)^2 + d^2 r^{-2}(n-1)\sin^2 f]^{p/2}|_{r=\sigma} + \frac{r}{\varepsilon^p}\cos^2 f|_{r=\sigma} \le C. \tag{2.6}$$

Then, we take $R=\sigma$ in (2.5) and multiply it by ε^{p-2} to obtain

$$\begin{aligned}
&-\varepsilon^{p-2}\sigma^2[(f_r)^2 + \frac{d^2}{r^2}\sin^2 f]^{p/2}|_{r=\sigma} + (1-\frac{2}{p})\varepsilon^{p-2}\int_0^\sigma [(f_r)^2 + \frac{d^2}{r^2}\sin^2 f]^{p/2} r dr \\
&\quad + \frac{\sigma^2}{p}\varepsilon^{p-2}[(f_r)^2 + \frac{d^2}{r^2}\sin^2 f]^{p/2}|_{r=\sigma} = -\frac{\sigma^2}{2\varepsilon^2}\cos^2 f|_{r=\sigma} + \frac{1}{\varepsilon^2}\int_0^\sigma \cos^2 f r dr.
\end{aligned}$$

Using (2.6), we get

$$(1-\frac{2}{p})\varepsilon^{p-2}\int_{B_\sigma(0)} |\nabla u_\varepsilon|^p dx - \frac{1}{\varepsilon^2}\int_{B_\sigma(0)} u_{\varepsilon 3}^2 dx \to 0 \tag{2.7}$$

as $\varepsilon\to 0$. Combining this result with (2.1)-(2.3), we obtain

$$L_2 = \frac{2p}{p-2}L_1.$$

Thus, (1.5) and (1.6) are proved.

2.2. *Proof of (1.7)*

Step 1. Upper bound
Similar to the proof of Proposition 2.1 in [9], it is easy to derive

$$\varepsilon^{p-2}E_\varepsilon(u_\varepsilon, B) \le \frac{2\pi d^p}{p(p-2)} + \min_W E_1(u,B) + C\varepsilon^{p-2}. \tag{2.8}$$

Here $C>0$ is independent of ε. On the other hand, (1.5) and (1.6) lead to

$$\lim_{\varepsilon\to 0}[\frac{\varepsilon^{p-2}}{p}|\nabla u_\varepsilon|^p+\frac{1}{2\varepsilon^2}u_{\varepsilon 3}^2]=\frac{p}{p-2}L_1\delta_o,\quad \text{weakly star in}\quad C(\bar{B}). \tag{2.9}$$

This result, together with (2.8), implies the upper bound of L_1 in (1.7).

Step 2. Lower bound

From (1.1), we can deduce that, for any $\sigma>0$, there exists $C=C(\sigma)>0$ independent of ε, such that

$$\begin{aligned}\int_{h\varepsilon}^{\sigma}[(f_r)^2+d^2r^{-2}\sin^2 f]^{p/2}rdr &\geq d^p\int_{h\varepsilon}^{\sigma}r^{1-p}\sin^p f dr\\ &\geq \frac{d^p}{p-2}(1-\gamma^2)^{p/2}h^{2-p}(\gamma)\varepsilon^{2-p}-C(\sigma).\end{aligned} \tag{2.10}$$

Applying (2.7), we obtain that

$$\lim_{\varepsilon\to 0}\varepsilon^{p-2}E_\varepsilon(u_\varepsilon,B_\sigma(0))=\pi\lim_{\varepsilon\to 0}\varepsilon^{p-2}\int_0^{\sigma}[(f_r)^2+d^2r^{-2}\sin^2 f]^{p/2}rdr.$$

Inserting (2.10) into this result, we deduce that for any $\eta\in(0,1)$,

$$\lim_{\varepsilon\to 0}\varepsilon^{p-2}E_\varepsilon(u_\varepsilon,B_\sigma(0))\geq\frac{\pi d^p}{p-2}(1-\gamma^2)^{p/2}h^{2-p}(\gamma).$$

Taking the supremum and writing

$$H:=\sup_{\gamma\in(0,1)}(1-\gamma^2)^{p/2}h^{2-p}(\gamma),$$

we have

$$\lim_{\varepsilon\to 0}\varepsilon^{p-2}E_\varepsilon(u_\varepsilon,B_\sigma(0))\geq\frac{\pi d^p}{p-2}H.$$

Combining this with (2.9), we can get $\frac{p}{p-2}L_1\geq\frac{\pi d^p}{p-2}H$. This means $L_1\geq\frac{\pi d^p}{p}H$, thus we obtain the lower bound of L_1 in (1.7).

3. Proof of Theorem 2

3.1. *Proof of (1.8)*

According to Proposition 2.2 in [18], there exists a constant $C=C(h)>0$ which is independent of ε, such that

$$\|\nabla u_\varepsilon\|_{L^\infty(B(0,h\varepsilon))}\leq C\varepsilon^{-1}. \tag{3.1}$$

Therefore,

$$\int_{B(0,h\varepsilon)}|\nabla u_\varepsilon|^2|u_{\varepsilon 3}|^\alpha dx\leq\frac{C}{\varepsilon^2}\pi(h\varepsilon)^2\leq C. \tag{3.2}$$

Next, using Hölder's inequality and (1.3) and (1.2), we see that as $\varepsilon\to 0$,

$$\int_{B\setminus B(0,\sigma)}|\nabla u_\varepsilon|^2|u_{\varepsilon 3}|^\alpha dx\leq[\int_{B\setminus B(0,\sigma)}|\nabla u_\varepsilon|^p dx]^{\frac{2}{p}}[\int_{B\setminus B(0,\sigma)}|u_{\varepsilon 3}|^{\frac{p\alpha}{p-2}}dx]^{\frac{p-2}{p}}\to 0. \tag{3.3}$$

By the same derivation of (13) in [13], we also get from $\|u_{\varepsilon 3}\|_{L^2(B)} \leq C\varepsilon$ (which is deduced by (1.4)) that

$$\int_{h\varepsilon}^{\sigma} \frac{d^2}{r^2}(\sin f)^2(\cos f)^{\alpha} r dr \leq C \tag{3.4}$$

by using Hölder's inequality. In addition, noting $\alpha \geq 2-\frac{4}{p}$, using Hölder's inequality and (1.4), we also deduce that

$$\int_{h\varepsilon}^{\sigma}(f_r)^2(\cos f)^{\alpha} r dr \leq C\int_{h\varepsilon}^{\sigma}(f_r)^2(\cos f)^{2-\frac{4}{p}} r dr$$
$$\leq C(\int_{h\varepsilon}^{\sigma}(\cos f)^2 r dr)^{1-\frac{2}{p}}(\int_{h\varepsilon}^{\sigma}(f_r)^p r dr)^{\frac{2}{p}} \leq C\varepsilon^{2(1-\frac{2}{p})+\frac{2}{p}(2-p)} \leq C. \tag{3.5}$$

Combining this result with (3.2)-(3.4), and noting $|\nabla u|^2 = (f_r)^2 + \frac{d^2}{r^2}(\sin f)^2$, we obtain that $|\nabla u_\varepsilon|^2|u_{\varepsilon 3}|^{\alpha}$ is bounded in $L^1(B)$. Thus, there exists a Radon measure ω_3 such that

$$\lim_{\varepsilon\to 0}|\nabla u_\varepsilon|^2|u_{\varepsilon 3}|^{\alpha} = \omega_3, \quad \text{weakly star in} \quad C(\overline{B}).$$

By virtue of (3.3), $supp(\omega_3) \subset \{0\}$. Hence we can find $L_3 \geq 0$ such that $\omega_3 = L_3\delta_o$.

We claim $L_3 > 0$. Since $f(r) \in C[0,1]$ and $f(0) = 0$ (see Remark in p.68 of [9]), $f(h\varepsilon) \geq 1/2$ (which can be deduced by (1.1) with $\gamma = \cos(1/2)$), there must exist $r_\varepsilon \in (0, h\varepsilon)$ such that $f(r_\varepsilon) = 1/4$. Using (3.1), we can find a sufficiently small positive constant δ which is independent of ε, such that

$$\frac{1}{8} \leq f(x) \leq \frac{3}{8}, \quad r \in (r_\varepsilon(1-\delta), r_\varepsilon(1+\delta)).$$

Therefore,

$$\int_{B(0,r_\varepsilon(1+\delta))\setminus B(0,r_\varepsilon(1-\delta))}(\cos f)^{\alpha}|\nabla u_\varepsilon|^2 dx \geq 2\pi d^2(\sin\frac{1}{8})^2(\cos\frac{3}{8})^{\alpha}\int_{r_\varepsilon(1-\delta)}^{r_\varepsilon(1+\delta)}\frac{dr}{r} > 0.$$

This implies $L_3 > 0$. Equation (1.8) is proved.

3.2. *Proof of (1.9)*

By a direct calculation, it follows

$$\det(\nabla u'_\varepsilon) = \frac{d}{r^2}(\sin f\cos f)(x\cdot\nabla f). \tag{3.6}$$

Using Hölder's inequality and (1.2), we get

$$\int_{B\setminus B(0,\sigma)}|\det(\nabla u'_\varepsilon)|^{p/2}dx \leq C.$$

This means that when $\varepsilon \to 0$,

$$\varepsilon^{p-2}\int_{B\setminus B(0,\sigma)}|\det(\nabla u'_\varepsilon)|^{p/2}dx \to 0. \tag{3.7}$$

In addition, in view of $|\det(\nabla u'_\varepsilon)| \le \frac{1}{2}|\nabla u'_\varepsilon|^2$, we can deduce from (1.4) that

$$\varepsilon^{p-2}\int_{B(0,\sigma)} |\det(\nabla u'_\varepsilon)|^{p/2}dx \le C\varepsilon^{p-2}\int_{B(0,\sigma)} |\nabla u'_\varepsilon|^p dx \le C.$$

Combining this with (3.7) yields the upper bound of $\varepsilon^{p-2}|\det(\nabla u'_\varepsilon)|^{p/2}$ in $L^1(B)$. Then, we can find a Radon measure ω_4 such that

$$\lim_{\varepsilon\to 0}\varepsilon^{p-2}|\det(\nabla u'_\varepsilon)|^{p/2} = \omega_4, \ \text{ weakly star in } \ C(\overline{B}).$$

In view of (3.7), $supp(\omega_4) \subset \{0\}$. There exists a constant $L_4 \ge 0$ such that $\omega_4 = L_4\delta_o$. The proof of Theorem 2 is completed.

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