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BLOW-UP AND DECAY OF SOLUTIONS FOR A DELAYED TIMOSHENKO EQUATION WITH VARIABLE-EXPONENTS

HAZAL YÜKSEKKAYA AND ERHAN PIŞKIN

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Abstract. This work deals with a Timoshenko equation with delay term and variable exponents. Firstly, we obtain the blow up of solutions for negative initial energy in a finite time. Later, we establish the decay results by using an integral inequality due to Komornik. These, improve and extend the previous studies in the literature.

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1. INTRODUCTION

This paper is interested to study the following problem:

$$\begin{cases} u_{tt}+\Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u - \Delta u_t & \\ +\mu_1 u_t\left(x,t\right)\left|u_t\right|^{m(x)-2}\left(x,t\right) & \text{in } \Omega\times R^+, \\ +\mu_2 u_t\left(x,t-\tau\right)\left|u_t\right|^{m(x)-2}\left(x,t-\tau\right) & \\ = bu\left|u\right|^{p(x)-2} & \\ u\left(x,t\right)=\frac{\partial u(x,t)}{\partial \upsilon}=0 & \text{on } x\in\partial\Omega,\ t\in[0,\infty), \\ u\left(x,0\right)=u_0\left(x\right),\ u_t\left(x,0\right)=u_1\left(x\right) & \text{in } \Omega, \\ u_t\left(x,t-\tau\right)=f_0\left(x,t-\tau\right) & \text{in } \Omega\times(0,\tau), \end{cases} \tag{1.1}$$

which is contained a Timoshenko equation over $\Omega\times R^+$ with the Dirichlet-Neumann conditions on $\partial\Omega$ and the initial conditions on Ω and finally a initial condition related to the presence of the delay time $\tau>0$, given in $\Omega\times(0,\tau)$ where Ω is a bounded domain in R^n with sufficiently smooth boundary. μ_1 is a positive constant, μ_2 is a real number, $b\geq 0$ is a constant and υ is the unit outward normal vector on $\partial\Omega$. $M(s)$ is a positive C^1-function given as $M(s)=1+s^\gamma$ for $s\geq 0$, $\gamma>0$. The exponents $m(\cdot)$

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and $p(\cdot)$ are given continuous functions on $\overline{\Omega}$ and satisfy

$$\begin{cases} 2 \le m^- \le m(x) \le m^+ \le m^* \\ 2 \le p^- \le p(x) \le p^+ \le p^*, \end{cases} \tag{1.2}$$

where

$$m^- = \operatorname*{ess\,inf}_{x\in\Omega} m(x), \qquad m^+ = \operatorname*{ess\,inf}_{x\in\Omega} m(x)$$
$$p^- = \operatorname*{ess\,inf}_{x\in\Omega} p(x), \qquad p^+ = \operatorname*{ess\,inf}_{x\in\Omega} p(x),$$

and

$$\begin{cases} 2 < p^*, m^* < \infty & \text{if } n \le 4, \\ 2 < p^*, m^* < \frac{2n}{n-4} & \text{if } n > 4. \end{cases}$$

The Timoshenko equation is among the famous wave equation's model which describe extensible beam theory. It has been introduced in 1921 by Timoshenko [31]. For detailed information on derivation the equation, see [9]. The problems with variable exponents arises in many branches in sciences such as nonlinear elasticity theory, electrorheological fluids and image processing [6,8,29]. Time delay appears in many practical problems such as thermal, biological, chemical, physical and economic phenomena [15].

Datko et al. [7], indicated that a small delay in a boundary control is a source of instability. In [20], Nicaise and Pignotti studied the equation as follows

$$u_{tt} - \Delta u + a_0 u_t(x,t) + a u_t(x, t-\tau) = 0,$$

where a_0, a are positive real parameters. They obtained that, under the condition $0 \le a \le a_0$, the system is exponentially stable. In the case $a \ge a_0$, they obtained a sequence of delays that shows the solution is instable. In [32], Xu et al. obtained the same result similar to the [20] for the one space dimension by adopting the spectral analysis approach. In [19], Nicaise et al. studied the wave equation in one space dimension in the case of time-varying delay. In that work, they showed that an exponential stability result under the condition

$$a \le \sqrt{1-d}\,a_0,$$

where d is a constant such that

$$\tau'(t) \le d < 1, \forall t > 0.$$

In [11], Feng studied the following equation

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u - \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0.$$

He obtained well-posedness of solutions with $|\mu_2| \le \mu_1$, and proved decay results under the assumption $|\mu_2| < \mu_1$.

Park [21], looked into the following equation

$$u_{tt}+\Delta^2 u-M\left(\|\nabla u\|^2\right)\Delta u+\sigma(t)\int_0^t g(t-s)\Delta u(s)\,ds \\ +a_0u_t+a_1u_t(t-\tau(t))=0.$$

He established decay results under the assumption $|a_1|<\sqrt{1-d}a_0$.

Antontsev et al. [2] concerned with the following equation with variable exponents

$$u_{tt}+\Delta^2 u-\Delta u_t+|u_t|^{p(x)-2}u_t=|u|^{q(x)-2}u. \tag{1.3}$$

They proved the local weak solutions and obtained the blow up results for the (1.3) under suitable conditions.

Antontsev et al. [5] examined the following equation with variable exponents

$$u_{tt}+\Delta^2 u-M\left(\|\nabla u\|^2\right)\Delta u+|u_t|^{p(x)-2}u_t=|u|^{q(x)-2}u. \tag{1.4}$$

They established the local existence and proved the nonexistence of solutions with negative initial energy for the equation (1.4).

When $M(s)\equiv 1$, and in the absence of $(+\Delta^2 u)$ term and without strong damping term $(-\Delta u_t)$, the equation (1.1) becomes the following equation

$$u_{tt}-\Delta u+\mu_1 u_t(x,t)|u_t|^{m(x)-2}(x,t) \\ +\mu_2 u_t(x,t-\tau)|u_t|^{m(x)-2}(x,t-\tau)=bu|u|^{p(x)-2}. \tag{1.5}$$

Messaoudi and Kafini [14] established the decay estimates and proved the global nonexistence of the equation (1.5).

In recent years, some other authors investigated related studies (see [1, 2, 12, 13, 16, 22–28, 30, 33–38]).

In the present paper, we consider the blow up and the decay results for the Timoshenko equation (1.1) with delay term and variable exponents. Our aim in this work is to study the Timoshenko equation with the strong damping term $(-\Delta u_t)$, delay term $(\mu_2 u_t(x,t-\tau))$ and variable exponents.

The plan of this paper is as follows: In Section 2, the definitions of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and Sobolev spaces $W^{1,p(\cdot)}(\Omega)$, as well as some of their properties, are stated. In Section 3, we prove the blow up of solutions for negative initial energy. In Section 4, we establish the decay results by using an integral inequality due to Komornik.

2. Preliminaries

Let us start by presents our functional spaces and some related results taken from [3, 4, 8, 10, 18].

Let $p\colon \Omega\to[1,\infty)$ be a measurable function. The variable exponent Lebesgue space with a variable exponent $p(\cdot)$ defined as

$$L^{p(\cdot)}(\Omega)=\left\{u\colon \Omega\to R \text{ measurable in } \Omega\ :\int_\Omega |u|^{p(\cdot)}dx<\infty\right\},$$

and inner with a Luxemburg-type norm

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

is a Banach space (see [8]).

We define the variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as follows

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega)\right\}.$$

Variable exponent Sobolev space with respect to the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)},$$

is a Banach space. The space $W_0^{1,p(\cdot)}(\Omega)$ is defined to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. The dual space of $W_0^{1,p(\cdot)}(\Omega)$ is $W_0^{-1,p'(\cdot)}(\Omega)$, defined in the same way as in the classical Sobolev spaces, where

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1.$$

Assume that

$$|p(x) - p(y)| \le -\frac{A}{\log|x-y|} \quad \text{and} \quad |m(x) - m(y)| \le -\frac{B}{\log|x-y|} \tag{2.1}$$

for all $x, y \in \Omega$, $A, B > 0$ and $0 < \delta < 1$ with $|x-y| < \delta$. (log-Hölder condition).

If $p \ge 1$ is a measurable function on Ω, then

$$\min\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\} \le \rho_{p(\cdot)}(u) \le \max\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\}$$

for a.e. $x \in \Omega$ and for any $u \in L^{p(\cdot)}(\Omega)$.

Let $p, q, s \ge 1$ be measurable functions defined on Ω such that

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)} \quad \text{for a.e. } y \in \Omega.$$

If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, then $fg \in L^{s(\cdot)}(\Omega)$ and

$$\|fg\|_{s(\cdot)} \le 2\|f\|_{p(\cdot)}\|g\|_{q(\cdot)}.$$

(Hölder's inequality).

Lemma 1 (pp. 506 in [4], Poincarė's inequality). *Suppose that $p(\cdot)$ satisfies (2.1) and let Ω be a bounded domain of R^n. Then,*

$$\|u\|_{p(\cdot)} \le c\|\nabla u\|_{p(\cdot)} \text{ for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

where $c = c(p^-, p^+, |\Omega|) > 0$.

Lemma 2 ([8]). *Let $m(\cdot) \in C\left(\overline{\Omega}\right)$ and $p\colon \Omega \to [1,\infty)$ be a measurable function, such that*

$$\underset{x\in\overline{\Omega}}{\operatorname{ess\ inf}} \left(m^*(x) - p(x)\right) > 0.$$

Then, the Sobolev embedding $W_0^{1,m(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is continuous and compact, where

$$\begin{cases} \frac{nm^-}{n-m^-} & \text{if } m^- < n, \\ \text{any number in } [1,\infty) & \text{if } m^- \geq n. \end{cases}$$

If in addition $m(\cdot)$ satisfies log-Hölder condition, then

$$\begin{cases} \frac{nm(x)}{n-m(x)} & \text{if } m(x) < n, \\ \text{any number in } [1,\infty) & \text{if } m(x) \geq n. \end{cases}$$

Remark 1. Let c be various positive constants which may be different from line to line. Then, we use the embedding

$$H_0^2(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^p(\Omega)$$

which satisfies

$$\|u\|_p \leq c\|\nabla u\| \leq c\|\Delta u\|,$$

where $2 \leq p < \infty$ $(n = 1,2)$, $2 \leq p \leq \frac{2n}{n-2}$ $(n \geq 3)$. Moreover,

$$\|u\|_p \leq c\|\Delta u\|, \qquad p = \begin{cases} \infty & \text{if } n < 4, \\ \text{any number in } [1,\infty) & \text{if } n = 4, \\ \frac{2n}{n-4} & \text{if } n > 4. \end{cases}$$

3. Blow up results

In this part, we give the blow up result of solutions under two conditions, the first one if the initial energy is negative and the second if the weight of the external force $b > 0$. Firstly, as in [20], we introduce the new function

$$z(x,\rho,t) = u_t(x,t-\tau\rho),\ x \in \Omega,\ \rho \in (0,1),\ t > 0 \tag{3.1}$$

which gives

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0,\ x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Then, the problem (1.1) takes the form

$$\begin{cases} u_{tt}+\Delta^2 u-M\left(\|\nabla u\|^2\right)\Delta u & \\ -\Delta u_t+\mu_1 u_t(x,t)|u_t(x,t)|^{m(x)-2} & \text{in } \Omega\times(0,\infty), \\ +\mu_2 z(x,1,t)|z(x,1,t)|^{m(x)-2} & \\ =bu|u|^{p(x)-2} & \\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0 & \text{in } \Omega\times(0,1)\times(0,\infty), \\ z(x,\rho,0)=f_0(x,-\rho\tau) & \text{in } \Omega\times(0,1), \\ u(x,t)=\frac{\partial u(x,t)}{\partial \upsilon}=0 & \text{on } x\in\partial\Omega,\ t\in[0,\infty), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) & \text{in } \Omega. \end{cases} \tag{3.2}$$

Similar to the work of Kafini and Messaoudi [14], we can write the following definition:

Definition 1. Fix $T>0$. We call (u,z) a strong solution of (3.2) if

$$\begin{aligned} u&\in W^{2,\infty}\left([0,T);L^2(\Omega)\right)\cap W^{1,\infty}\left([0,T);H_0^2(\Omega)\right)\\ &\qquad\cap L^\infty\left([0,T);H^2(\Omega)\cap H_0^2(\Omega)\right),\\ u_t&\in L^{m(\cdot)}(\Omega\times(0,T)),\\ z&\in W^{1,\infty}\left([0,1]\times[0,T);L^2(\Omega)\right)\cap L^\infty\left([0,1];L^{m(\cdot)}(\Omega)\cap[0,T)\right) \end{aligned}$$

and (u,z) satisfies the initial data and (3.2) in the following sense:

$$\begin{aligned} &\int_\Omega u_{tt}(\cdot,t)\,vdx+\int_\Omega \Delta^2 u(\cdot,t)\,vdx-\int_\Omega M\left(\|\nabla u(\cdot,t)\|^2\right)\Delta u(\cdot,t)\,vdx\\ &-\int_\Omega \Delta u_t(\cdot,t)\,vdx+\mu_1\int_\Omega |u_t(\cdot,t)|^{m(\cdot)-2}u_t(\cdot,t)\,vdx\\ &+\mu_2\int_\Omega |z(\cdot,1,t)|^{m(\cdot)-2}z(\cdot,1,t)\,vdx\\ &=b\int_\Omega |u(\cdot,t)|^{p(\cdot)-2}u(\cdot,t)\,vdx \end{aligned}$$

and

$$\tau\int_\Omega z_t(\cdot,\rho,t)\,wdx+\int_\Omega z_\rho(\cdot,\rho,t)\,wdx=0,$$

for a.e. $t\in[0,T)$ and for $(v,w)\in H_0^2(\Omega)\cap L^2(\Omega)$.

The energy functional related to (3.2) is given by

$$\begin{aligned} E(t)=&\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2+\frac{1}{2}\|\nabla u\|^2+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}\\ &+\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho-b\int_\Omega \frac{|u|^{p(x)}}{p(x)}dx, \end{aligned}$$

for $t \geq 0$, where ξ is a continuous function satisfying

$$\tau |\mu_2| (m(x) - 1) < \xi(x) < \tau (\mu_1 m(x) - |\mu_2|), \, x \in \overline{\Omega}. \tag{3.3}$$

The following lemma shows that the related energy of the problem is nonincreasing under the condition $\mu_1 > |\mu_2|$.

Lemma 3. *Let (u, z) be a solution of (3.2), such that*

$$E'(t) \leq -C_0 \int_\Omega \left(|u_t|^{m(x)} + |z(x,1,t)|^{m(x)} \right) dx \leq 0,$$

for some $C_0 > 0$.

Proof. Multiplying the first equation in (3.2) by u_t, integrating over Ω, then multiplying the second equation in (3.2) by $\frac{1}{\tau}\xi(x)|z|^{m(x)-2} z$ and integrating over $\Omega \times (0,1)$, then summing, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} \right. \\
&\quad \left. + \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho - b\int_\Omega \frac{|u|^{p(x)}}{p(x)} dx\right] \\
&= -\mu_1 \int_\Omega |u_t|^{m(x)} dx - \int_\Omega |\nabla u_t|^2 dx \\
&\quad -\frac{1}{\tau}\int_\Omega \int_0^1 \xi(x)|z(x,\rho,t)|^{m(x)-2} z z_\rho(x,\rho,t)\, d\rho dx \\
&\quad -\mu_2 \int_\Omega u_t z(x,1,t)|z(x,1,t)|^{m(x)-2} dx.
\end{aligned} \tag{3.4}$$

Now, we estimate the last two terms of the right hand side of (3.4) as follows,

$$\begin{aligned}
&-\frac{1}{\tau}\int_\Omega \int_0^1 \xi(x)|z(x,\rho,t)|^{m(x)-2} z z_\rho(x,\rho,t)\, d\rho dx \\
&= -\frac{1}{\tau}\int_\Omega \int_0^1 \frac{\partial}{\partial\rho}\left(\frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}\right) d\rho dx \\
&= \frac{1}{\tau}\int_\Omega \frac{\xi(x)}{m(x)}\left(|z(x,0,t)|^{m(x)} - |z(x,1,t)|^{m(x)}\right) dx \\
&= \int_\Omega \frac{\xi(x)}{\tau m(x)}|u_t|^{m(x)} dx - \int_\Omega \frac{\xi(x)}{\tau m(x)}|z(x,1,t)|^{m(x)}.
\end{aligned}$$

By using Young's inequality, $q = \frac{m(x)}{m(x)-1}$ and $q' = m(x)$ for the last term, we get

$$|u_t||z(x,1,t)|^{m(x)-1} \leq \frac{1}{m(x)}|u_t|^{m(x)} + \frac{m(x)-1}{m(x)}|z(x,1,t)|^{m(x)}.$$

Consequently, we conclude that

$$
\begin{aligned}
&-\mu_2 \int_\Omega u_t z \left|z(x,1,t)\right|^{m(x)-2} dx \\
&\leq |\mu_2| \left(\int_\Omega \frac{1}{m(x)} \left|u_t(t)\right|^{m(x)} dx + \int_\Omega \frac{m(x)-1}{m(x)} \left|z(x,1,t)\right|^{m(x)} dx \right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{dE(t)}{dt} \leq & -\int_\Omega \left(\mu_1 - \left(\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)} \right) \right) \left|u_t(t)\right|^{m(x)} dx \\
& -\int_\Omega \left(\frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)} \right) \left|z(x,1,t)\right|^{m(x)} dx.
\end{aligned}
$$

As a result, for all $x \in \overline{\Omega}$, the relation (3.3) yields

$$
f_1(x) = \mu_1 - \left(\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)} \right) > 0, \quad f_2(x) = \frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)} > 0.
$$

Since $m(x)$, and hence $\xi(x)$, is bounded, we infer that $f_1(x)$ and $f_2(x)$ are bounded. Hence, if we define

$$
C_0(x) = \min\{f_1(x),\ f_2(x)\} > 0 \text{ for any } x \in \overline{\Omega},
$$

and take $C_0 = \inf_{\overline{\Omega}} C_0(x)$, then $C_0(x) \geq C_0 > 0$. Therefore,

$$
E'(t) \leq -C_0 \left[\int_\Omega \left|u_t(t)\right|^{m(x)} dx + \int_\Omega \left|z(x,1,t)\right|^{m(x)} dx \right] \leq 0.
$$

□

In order to obtain the blow up result, we suppose in addition to (1.2) that $E(0) < 0$. We set

$$
H(t) = -E(t),
$$

therefore,

$$
H'(t) = -E'(t) \geq 0,
$$

$$
0 < H(0) \leq H(t) \leq b \int_\Omega \frac{|u|^{p(x)}}{p(x)} dx \leq \frac{b}{p^-} \rho(u),
$$

where

$$
\rho(u) = \rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)} dx.
$$

Lemma 4 (Lemma 3.2, Lemma 3.6 and Lemma 3.7 in [14]). *Assume that the exponents $m(\cdot)$ and $p(\cdot)$ satisfy*

$$
2 \leq m^- \leq m(x) \leq m^+ < p^- \leq p(x) \leq p^+ \leq 2 + \frac{4}{n-4} \text{ if } n > 4.
$$

Then, depending on Ω only, there exists a positive $C > 1$, such that

$$\rho^{s/p^-}(u) \leq C\left(\|\Delta u\|^2 + \rho(u)\right).$$

Then, for any $u \in H_0^2(\Omega)$ and $2 \leq s \leq p^-$, we have the following inequalities:

(i)

$$\|u\|_{p^-}^s \leq C\left(\|\Delta u\|^2 + \|u\|_{p^-}^{p^-}\right),$$

(ii)

$$\rho^{s/p^-}(u) \leq C\left(|H(t)| + \|u_t\|^2 + \rho(u) + \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho\right),$$

(iii)

$$\|u\|_{p^-}^s \leq C\left(|H(t)| + \|u_t\|^2 + \|u\|_{p^-}^{p^-} + \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho\right), \tag{3.5}$$

(iv)

$$\rho(u) \geq C\|u\|_{p^-}^{p^-}, \tag{3.6}$$

(v)

$$\int_\Omega |u|^{m(x)} dx \leq C\left(\rho^{m^-/p^-}(u) + \rho^{m^+/p^-}(u)\right). \tag{3.7}$$

To obtain the main result we have the theorem as follows:

Theorem 1. *Let the condition (2.1) holds and Lemma 4 be provided. Suppose further $E(0) < 0$, and the exponents $m(\cdot)$ and $p(\cdot)$ satisfy*

$$2 \leq m^- \leq m(x) \leq m^+ < p^- \leq p(x) \leq p^+ \leq 2 + \frac{4}{n-4} \qquad \text{if } n > 4.$$

Then, the solution of (3.2) blows up in finite time.

Proof. We define

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t dx + \frac{\varepsilon}{2}\|\nabla u\|^2, \tag{3.8}$$

for small ε to be chosen later and

$$0 \leq \alpha \leq \min\left\{\frac{p^- - 2}{2p^-}, \frac{p^- - m^+}{p^-(m^+ - 1)}\right\}. \tag{3.9}$$

A direct differentiation of (3.8) using the first equation in (3.2) gives

$$\begin{aligned} L'(t) = {} & (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega u_t^2 dx - \varepsilon\int_\Omega |\Delta u|^2 dx - \varepsilon\int_\Omega |\nabla u|^2 dx \\ & - \varepsilon\int_\Omega |\nabla u|^{2(\gamma+1)} dx + \varepsilon b\int_\Omega |u|^{p(x)} dx - \varepsilon\mu_1\int_\Omega u u_t(x,t)|u_t(x,t)|^{m(x)-2} dx \end{aligned}$$

$$-\varepsilon\mu_2\int_\Omega uz(x,1,t)\left|z(x,1,t)\right|^{m(x)-2}dx.$$

Recalling the definition of $H(t)$ and for $0<a<1$, we get

$$\begin{aligned}
L'(t) &\geq C_0(1-\alpha)H^{-\alpha}(t)\left[\int_\Omega |u_t(t)|^{m(x)}dx+\int_\Omega \left|z(x,1,t)\right|^{m(x)}dx\right]\\
&+\varepsilon\left((1-a)p^-H(t)+\frac{(1-a)p^-}{2}\|u_t\|^2+\frac{(1-a)p^-}{2}\|\Delta u\|^2\right)\\
&+\varepsilon\left(\frac{(1-a)p^-}{2}\|\nabla u\|^2+\frac{(1-a)p^-}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}\right)\\
&+\varepsilon(1-a)p^-\int_0^1\int_\Omega\frac{\xi(x)\left|z(x,\rho,t)\right|^{m(x)}}{m(x)}dxd\rho\\
&+\varepsilon\int_\Omega\left[u_t^2-|\Delta u|^2-|\nabla u|^2-|\nabla u|^{2(\gamma+1)}\right]dx\\
&+\varepsilon ab\int_\Omega |u|^{p(x)}dx-\varepsilon\mu_1\int_\Omega uu_t(x,t)\left|u_t(x,t)\right|^{m(x)-2}dx\\
&-\varepsilon\mu_2\int_\Omega uz(x,1,t)\left|z(x,1,t)\right|^{m(x)-2}dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
L'(t) &\geq C_0(1-\alpha)H^{-\alpha}(t)\left[\int_\Omega |u_t(t)|^{m(x)}dx+\int_\Omega \left|z(x,1,t)\right|^{m(x)}dx\right]\\
&+\varepsilon(1-a)p^-H(t)+\varepsilon\frac{(1-a)p^-+2}{2}\|u_t\|^2\\
&+\varepsilon\frac{(1-a)p^--2}{2}\|\Delta u\|^2+\varepsilon\frac{(1-a)p^--2}{2}\|\nabla u\|^2\\
&+\varepsilon\frac{(1-a)p^--2(\gamma+1)}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}\\
&+\varepsilon(1-a)p^-\int_0^1\int_\Omega\frac{\xi(x)\left|z(x,\rho,t)\right|^{m(x)}}{m(x)}dxd\rho+\varepsilon ab\rho(u)\\
&-\varepsilon\mu_1\int_\Omega uu_t(x,t)\left|u_t(x,t)\right|^{m(x)-2}dx\\
&-\varepsilon\mu_2\int_\Omega uz(x,1,t)\left|z(x,1,t)\right|^{m(x)-2}dx.
\end{aligned}$$

Utilizing Young's inequality, we obtain

$$\int_\Omega |u_t|^{m(x)-1}|u|\,dx\leq\frac{1}{m^-}\int_\Omega\delta^{m(x)}|u|^{m(x)}dx+\frac{m^+-1}{m^+}\int_\Omega\delta^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}dx \tag{3.10}$$

and

$$\int_{\Omega}|z(x,1,t)|^{m(x)-1}|u|\,dx \leq \frac{1}{m^{+}}\int_{\Omega}\delta^{m(x)}|u|^{m(x)}\,dx + \frac{m^{+}-1}{m^{+}}\int_{\Omega}\delta^{-\frac{m(x)}{m(x)-1}}|z(x,1,t)|^{m(x)}\,dx. \tag{3.11}$$

The estimates (3.10) and (3.11) remain valid if δ is time-dependent. Hence, taking δ such that

$$\delta^{-\frac{m(x)}{m(x)-1}} = kH^{-\alpha}(t),$$

for large $k \geq 1$ to be specified later, we obtain

$$\int_{\Omega}\delta^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}\,dx = kH^{-\alpha}(t)\int_{\Omega}|u_t|^{m(x)}\,dx, \tag{3.12}$$

$$\int_{\Omega}\delta^{-\frac{m(x)}{m(x)-1}}|z(x,1,t)|^{m(x)}\,dx = kH^{-\alpha}(t)\,|z(x,1,t)|^{m(x)}\,dx \tag{3.13}$$

and

$$\int_{\Omega}\delta^{m(x)}|u|^{m(x)}\,dx = \int_{\Omega}k^{1-m(x)}H^{\alpha(m(x)-1)}(t)\,|u|^{m(x)}\,dx \tag{3.14}$$

$$\leq \int_{\Omega}k^{1-m^{-}}H^{\alpha(m^{+}-1)}(t)\int_{\Omega}|u|^{m(x)}\,dx. \tag{3.15}$$

By using (3.6) and (3.7), we obtain

$$H^{\alpha(m^{+}-1)}(t)\int_{\Omega}|u|^{m(x)}\,dx \leq C\left[(\rho(u))^{m^{-}/p^{-}+\alpha(m^{+}-1)} + (\rho(u))^{m^{+}/p^{-}+\alpha(m^{+}-1)}\right]. \tag{3.16}$$

From (3.9), we infer that

$$s = m^{-}+\alpha p^{-}\left(m^{+}-1\right) \leq p^{-} \text{ and } s = m^{+}+\alpha p^{-}\left(m^{+}-1\right) \leq p^{-}.$$

Thus, Lemma 4 yields

$$H^{\alpha(m^{+}-1)}(t)\int_{\Omega}|u|^{m(x)}\,dx \leq C\left(\|\Delta u\|^{2}+\rho(u)\right). \tag{3.17}$$

By combining (3.10)-(3.17), we conclude that

$$\begin{aligned} L'(t) \geq{}& (1-\alpha)H^{-\alpha}(t)\left[C_0-\varepsilon\left(\frac{m^{+}-1}{m^{+}}\right)ck\right]\int_{\Omega}|u_t(t)|^{m(x)}\,dx \\ &+(1-\alpha)H^{-\alpha}(t)\left[C_0-\varepsilon\left(\frac{m^{+}-1}{m^{+}}\right)ck\right]\int_{\Omega}|z(x,1,t)|^{m(x)}\,dx \\ &+\varepsilon\left(\frac{(p^{-}-2)-ap^{-}}{2}-\frac{C}{m^{-}k^{m^{-}-1}}\right)\|\Delta u\|^{2}+\varepsilon\frac{(1-a)p^{-}-2}{2}\|\nabla u\|^{2} \\ &+\varepsilon(1-a)p^{-}H(t)+\varepsilon\frac{(1-a)p^{-}+2}{2}\|u_t\|^{2} \end{aligned} \tag{3.18}$$

$$+\varepsilon\frac{(1-a)\,p^{-}-2(\gamma+1)}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}+\varepsilon\left(ab-\frac{C}{m^{-}k^{m^{-}-1}}\right)\rho(u)$$
$$+\varepsilon(1-a)\,p^{-}\int_0^1\int_\Omega\frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho.$$

By choosing a small enough, such that

$$\frac{(1-a)\,p^{-}-2}{2}>0\quad\text{and}\quad\frac{(1-a)\,p^{-}-2(\gamma+1)}{2(\gamma+1)}>0,$$

and k so large that

$$\frac{(p^{-}-2)-ap^{-}}{2}-\frac{C}{m^{-}k^{m^{-}-1}}>0\quad\text{and}\quad ab-\frac{C}{m^{-}k^{m^{-}-1}}>0.$$

Once k and a are fixed, we choose ε small enough that

$$C_0-\varepsilon\left(\frac{m^{+}-1}{m^{+}}\right)ck>0,\qquad\qquad C_0-\varepsilon\left(\frac{m^{+}-1}{m^{+}}\right)ck>0$$

and

$$L(0)=H^{1-\alpha}(0)+\varepsilon\int_\Omega u_0(x)\,u_1(x)\,dx+\frac{\varepsilon}{2}\|\nabla u_0\|^2>0.$$

Therefore, (3.18) becomes

$$L'(t)\geq\varepsilon\eta\Big[H(t)+\|u_t\|^2+\|\Delta u\|^2+\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}$$
$$+\rho(u)+\int_0^1\int_\Omega\frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\Big],\tag{3.19}$$

for a constant $\eta>0$. As a result,

$$L(t)\geq L(0)>0\quad\forall t\geq 0.$$

Next, for some constants σ, $\Gamma>0$, we show $L'(t)\geq\Gamma L^{\sigma}(t)$. For this reason, we estimate

$$\left|\int_\Omega uu_t(x,t)\,dx\right|\leq\|u\|_2\|u_t\|_2\leq C\|u\|_{p^{-}}\|u_t\|_2,$$

which implies

$$\left|\int_\Omega uu_t(x,t)\,dx\right|^{1/(1-\alpha)}\leq C\|u\|_{p^{-}}^{1/(1-\alpha)}\|u_t\|_2^{1/(1-\alpha)}$$

and utilizing Young's inequality yields

$$\left|\int_\Omega uu_t(x,t)\,dx\right|^{1/(1-\alpha)}\leq C\left[\|u\|_{p^{-}}^{mu/(1-\alpha)}+\|u_t\|_2^{\Theta/(1-\alpha)}\right],$$

where $1/\mu+1/\Theta=1$. From (3.9), the choice of $\Theta=2(1-\alpha)$ will make $\mu/(1-\alpha)=2/(1-2\alpha)\leq p^{-}$. Hence,

$$\left|\int_{\Omega}uu_t(x,t)\,dx\right|^{1/(1-\alpha)}\leq C\left[\|u\|_{p^-}^{s}+\|u_t\|^2\right],$$

where $s=\mu/(1-\alpha)$. By (3.5), we obtain

$$\begin{aligned}\left|\int_{\Omega}uu_t(x,t)\,dx\right|^{1/(1-\alpha)}\leq C\Big[&|H(t)|+\|u_t\|^2+\rho(u)\\&+\int_0^1\int_{\Omega}\frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\Big].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}L^{1/(1-\alpha)}(t)&=\left[H^{(1-\alpha)}(t)+\varepsilon\int_{\Omega}uu_tdx+\frac{\varepsilon}{2}\|\nabla u\|^2\right]^{1/(1-\alpha)}\\&\leq 2^{\alpha/(1-\alpha)}\left[H(t)+\varepsilon^{1/(1-\alpha)}\left|\int_{\Omega}uu_tdx\right|^{1/(1-\alpha)}\right]\\&\leq C\left[|H(t)|+\|u_t\|^2+\rho(u)+\int_0^1\int_{\Omega}\frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\right].\end{aligned}$$

Thus, for some $\Psi>0$, from (3.19) we arrive at $L'(t)\geq\Psi L^{1/(1-\alpha)}(t)$. A simple integration over $(0,t)$ yields

$$L^{\alpha/(1-\alpha)}(t)\geq\frac{1}{L^{-\alpha/(1-\alpha)}(0)-\Psi\alpha t/(1-\alpha)}$$

which implies that the solution blows up in a finite time T^{*}, with

$$T^{*}\leq\frac{1-\alpha}{\Psi\alpha\,[L(0)]^{\alpha/(1-\alpha)}}.$$

As a result, the proof is completed. □

4. Decay results

In this part, we obtain the decay results for the problem (4.1) without source term (i.e. $b=0$). Similar to the beginning of the blow up section, we introduce a same function z defined in (3.1), hence the problem (1.1) becomes equivalent to:

$$\begin{cases} u_{tt}+\Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u-\Delta u_t & \\ +\mu_1 u_t(x,t)\left|u_t(x,t)\right|^{m(x)-2} & \text{in } \Omega\times(0,\infty), \\ +\mu_2 z(x,1,t)\left|z(x,1,t)\right|^{m(x)-2}=0, & \\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0 & \text{in } \Omega\times(0,1)\times(0,\infty), \\ z(x,\rho,0)=f_0(x,-\rho\tau) & \text{in } \Omega\times(0,1), \\ u(x,t)=\frac{\partial u(x,t)}{\partial \upsilon}=0 & \text{on } x\in\partial\Omega,\ t\in[0,\infty), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) & \text{in } \Omega. \end{cases} \tag{4.1}$$

The energy functional associated to (4.1) is given by

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2+\frac{1}{2}\|\nabla u\|^2+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \int_0^1\int_\Omega \frac{\xi(x)\left|z(x,\rho,t)\right|^{m(x)}}{m(x)}dxd\rho, \tag{4.2}$$

where ξ is the continuous function introduced in (3.3) and $t\geq 0$.

Similar to Lemma 3, we easily establish, for $\mu_1>|\mu_2|$ and for some $C_0>0$, that

$$E'(t)\leq -C_0\int_\Omega\left(|u_t|^{m(x)}+|z(x,1,t)|^{m(x)}\right)dx\leq 0.$$

Lemma 5 (Komornik, pp. 103 and pp. 124 in [17]). *Let $E:R^+\to R^+$ be a nonincreasing function, such that*

$$\int_s^\infty E^{1+\sigma}(t)\,dt\leq\frac{1}{\Omega}E^\sigma(0)E(s)=cE(s)\quad \forall s>0,$$

where σ, $\omega>0$. Then, we have

$$\begin{cases} E(t)\leq cE(0)/(1+t)^{1/\sigma} & \text{if } \sigma>0, \\ E(t)\leq cE(0)e^{-\omega t} & \text{if } \sigma=0, \end{cases}$$

for all $t\geq 0$.

We need the following technical lemma, before we state the main theorem:

Lemma 6 (Lemma 4.2 in [14]). *The functional*

$$F(t)=\tau\int_0^1\int_\Omega e^{-\rho\tau}\xi(x)\left|z(x,\rho,t)\right|^{m(x)}dxd\rho,$$

satisfies, along the solution of (4.1),

$$F'(t)\leq\int_\Omega\xi(x)|u_t|^{m(x)}dx-\tau e^{-\tau}\int_0^1\int_\Omega\xi(x)\left|z(x,\rho,t)\right|^{m(x)}dxd\rho.$$

Theorem 2. *Suppose that the condition (2.1) is satisfied and the exponents $m(\cdot)$ and $p(\cdot)$ satisfy*

$$2 \leq m^{-} \leq m(x) \leq m^{+} < p^{-} \leq p(x) \leq p^{+} \leq 2 + \frac{4}{n-4} \quad \text{if } n > 4.$$

Then, there exist two constants c, $\alpha > 0$ independent of t, such that, any global solution of (4.1) satisfies,

$$\begin{cases} E(t) \leq ce^{-\alpha t} & \text{if } m(x) = 2, \\ E(t) \leq cE(0)/(1+t)^{2/(m^{+}-2)} & \text{if } m^{+} > 2. \end{cases}$$

Proof. Multiplying the first equation of (4.1) by $uE^{q}(t)$, for $q > 0$ to be specified later, and integrating over $\Omega \times (s,T)$, $s < T$, to get

$$\int_{s}^{T} E^{q}(t) \int_{\Omega} \Big(uu_{tt} + u\Delta^{2}u - u\Delta u - \|\nabla u\|^{2\gamma} u\Delta u - u\Delta u_{t}$$
$$+\mu_{1} uu_{t} |u_{t}|^{m(x)-2} + \mu_{2} uz(x,1,t) |z(x,1,t)|^{m(x)-2} \Big) dxdt = 0,$$

which gives

$$\begin{aligned} \int_{s}^{T} E^{q}(t) \int_{\Omega} \Big(& \frac{d}{dt}(uu_{t}) - u_{t}^{2} + |\Delta u|^{2} + |\nabla u|^{2} \\ & + \|\nabla u\|^{2\gamma} |\nabla u|^{2} + \nabla u \nabla u_{t} + \mu_{1} uu_{t}(x,t) |u_{t}(x,t)|^{m(x)-2} \\ & + \mu_{2} uz(x,1,t) |z(x,1,t)|^{m(x)-2} \Big) dxdt = 0. \end{aligned} \tag{4.3}$$

Recalling the definition of $E(t)$ given in (4.2), adding and subtracting some terms and using the relation

$$\frac{d}{dt}\left(E^{q}(t) \int_{\Omega} uu_{t} dx \right) = qE^{q-1}(t) E'(t) \int_{\Omega} uu_{t} dx + E^{q}(t) \frac{d}{dt} \int_{\Omega} uu_{t} dx,$$

the equation (4.3) becomes,

$$\begin{aligned} 2\int_{s}^{T} E^{q+1}(t)\, dt = & -\int_{s}^{T} \frac{d}{dt}\left(E^{q}(t) \int_{\Omega} uu_{t} dx \right) dt + q\int_{s}^{T} E^{q-1}(t) E'(t) \int_{\Omega} uu_{t} dxdt \\ & - \frac{\gamma}{\gamma+1} \int_{s}^{T} E^{q} \int_{\Omega} \|\nabla u\|^{2\gamma} \left|\nabla u^{2}\right| dxdt + 2\int_{s}^{T} E^{q}(t) \int_{\Omega} u_{t}^{2} dxdt \\ & - \frac{1}{2} \int_{s}^{T} \frac{d}{dt}\left(E^{q}(t) \int_{\Omega} |\nabla u|^{2} dx \right) dt \\ & + \frac{q}{2} \int_{s}^{T} E^{q-1}(t) E'(t) \int_{\Omega} |\nabla u|^{2} dxdt \\ & - \mu_{1} \int_{s}^{T} E^{q}(t) \int_{\Omega} uu_{t} |u_{t}|^{m(x)-2} dxdt \end{aligned} \tag{4.4}$$

$$-\mu_2\int_s^T E^q(t)\int_\Omega uz(x,1,t)\,|z(x,1,t)|^{m(x)-2}\,dxdt$$
$$+2\int_s^T E^q(t)\int_0^1\int_\Omega \frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho dt.$$

Now, we estimate the parts on the right side of (4.4), respectively.

The first term is estimated as follows:

$$\begin{aligned}
&\left|-\int_s^T \frac{d}{dt}\left(E^q(t)\int_\Omega uu_t dx\right)dt\right| \\
&=\left|E^q(s)\int_\Omega uu_t(x,s)\,dx-E^q(T)\int_\Omega uu_t(x,T)\,dx\right| \\
&\le \frac{1}{2}E^q(s)\left[\int_\Omega u^2(x,s)\,dx+\int_\Omega u_t^2(x,s)\,dx\right] \\
&\quad+\frac{1}{2}E^q(T)\left[\int_\Omega u^2(x,T)\,dx+\int_\Omega u_t^2(x,T)\,dx\right] \\
&\le \frac{1}{2}E^q(s)\left[C_p\,\|\Delta u(s)\|_2^2+2E(s)\right] \\
&\quad+\frac{1}{2}E^q(T)\left[C_p\,\|\Delta u(T)\|_2^2+2E(T)\right] \\
&\le E^q(s)\,[C_pE(s)+E(s)]+E^q(T)\,[C_pE(T)+E(T)],
\end{aligned}$$

where C_p is the Poincaré constant. Recalling that $E(t)$ is decreasing, we infer that

$$\left|-\int_s^T \frac{d}{dt}\left(E^q(t)\int_\Omega uu_t dx\right)dt\right|\le cE^{q+1}(s)\le cE^q(0)\,E(s)\le cE(s). \tag{4.5}$$

In a similar way, we handle the term

$$\begin{aligned}
\left|q\int_s^T E^{q-1}(t)\,E'(t)\int_\Omega uu_t dxdt\right| &\le -q\int_s^T E^{q-1}(t)\,E'(t)\,[C_pE(T)+E(T)]\,dt \\
&\le -c\int_s^T E^q(t)\,E'(t)\le cE^{q+1}(s)\le cE(s).
\end{aligned}$$

We estimate the next term as follows,

$$\begin{aligned}
&\left|-\frac{\gamma}{\gamma+1}\int_s^T E^q\int_\Omega \|\nabla u\|^{2\gamma}\left|\nabla u^2\right|dxdt\right| = \left|-2\gamma\int_s^T E^q\left(\frac{\|\nabla u\|^{2\gamma}}{2(\gamma+1)}\int_\Omega\left|\nabla u^2\right|dx\right)dt\right| \\
&=\left|-2\gamma\int_s^T E^q\left(\frac{\|\nabla u\|^{2(\gamma+1)}}{2(\gamma+1)}\right)dt\right|\le\left|-2\gamma\int_s^T E^q(E(t))\,dt\right| \qquad (4.6)\\
&\le C^*\int_s^T E^{q+1}(t)\,dt\le C^*E(s)
\end{aligned}$$

where C^* is a generic constant.

To treat the other term, we set

$$\Omega_+ = \{x \in \Omega,\ |u_t(x,t)| \geq 1\} \quad \text{and} \quad \Omega_- = \{x \in \Omega,\ |u_t(x,t)| < 1\},$$

and by using the Hölder's and Young's inequalities, to obtain

$$\begin{aligned}
&\left| \int_s^T E^q(t) \int_\Omega u_t^2 dx dt \right| \\
&= \left| \int_s^T E^q(t) \left[\int_{\Omega_+} u_t^2 dx + \int_{\Omega_-} u_t^2 dx \right] dt \right| \\
&\leq c \int_s^T E^q(t) \left[\left(\int_{\Omega_+} |u_t|^{m^-} dx \right)^{2/m^-} + \left(\int_{\Omega_-} |u_t|^{m^+} dx \right)^{2/m^+} \right] dt \\
&\leq c \int_s^T E^q(t) \left[\left(\int_\Omega |u_t|^{m(x)} dx \right)^{2/m^-} + \left(\int_\Omega |u_t|^{m(x)} dx \right)^{2/m^+} \right] dt \\
&\leq c \int_s^T E^q(t) \left[\left(-E'(t) \right)^{2/m^-} + \left(-E'(t) \right)^{2/m^+} \right] dt \\
&\leq c\varepsilon \int_s^T [E(t)]^{qm^-/(m^- - 2)} dt + c(\varepsilon) \int_s^T \left(-E'(t) \right) dt \\
&\quad + c\varepsilon \int_s^T E(t)^{q+1} dt + c(\varepsilon) \int_s^T \left(-E'(t) \right)^{2(q+1)/m^+} dt.
\end{aligned}$$

For $m^- > 2$ and the choice of $q = m^+/2 - 1$ will make $\frac{qm^-}{m^- - 2} = q + 1 + \frac{m^+ - m^-}{m^- - 2}$. Therefore,

$$\begin{aligned}
\left| \int_s^T E^q(t) \int_\Omega u_t^2 dx dt \right| &\leq c\varepsilon \int_s^T E(t)^{q+1} dt + c\varepsilon [E(0)]^{\frac{m^+ - m^-}{m^- - 2}} \int_s^T [E(t)]^{q+1} dt \\
&\quad + c(\varepsilon) E(s) \leq c\varepsilon \int_s^T E(t)^{q+1} dt + c(\varepsilon) E(s). \qquad (4.7)
\end{aligned}$$

For the case $m^- = 2$ and the choice of $q = m^+/2 - 1$ will give a similar result.

The other term is estimated as follows:

$$\begin{aligned}
&\left| -\frac{1}{2} \int_s^T \frac{d}{dt} \left(E^q(t) \int_\Omega |\nabla u|^2 dx \right) dt \right| \leq \\
&\qquad \frac{1}{2} E^q(s) \int_\Omega |\Delta u(s)|^2 dx + \frac{1}{2} E^q(s) \int_\Omega |\Delta u(T)|^2 dx \leq \\
&\qquad\qquad cE^{q+2/m^+}(s) \leq c \left(E^{q-1+2/m^+}(0) \right) E(s) \leq \lambda E(s) \qquad (4.8)
\end{aligned}$$

where c and λ are positive constants.

Similarly,

$$\int_s^T E^{q-1}(t) E'(t) \int_\Omega |\nabla u|^2 dx dt \leq$$

$$cE^{q+2/m^+}(s) \le c\left(E^{q-1+2/m^+}(0)\right)E(s) \le \lambda_1 E(s) \quad (4.9)$$

where c and λ_1 are positive constants.

For the other term, utilizing Young's inequality, we conclude that

$$\begin{aligned}
&\left|-\mu_1 \int_s^T E^q(t)\int_\Omega u\,|u_t|^{m(x)-1}\,dxdt\right| \le \\
&\varepsilon \int_s^T E^q(t)\int_\Omega |u(t)|^{m(x)}\,dxdt + c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dxdt \le \\
&\varepsilon \int_s^T E^q(t)\left[\int_{\Omega_+} |u(t)|^{m^-}\,dx + \int_{\Omega_-} |u(t)|^{m^+}\,dx\right]dt \\
&+c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dxdt,
\end{aligned}$$

where we have used Young's inequality with

$$p(x) = \frac{m(x)}{m(x)-1},\ p'(x) = m(x),$$

thus,

$$c_\varepsilon(x) = (m(x)-1)\,m(x)^{m(x)/(1-m(x))}\,\varepsilon^{1/(1-m(x))}.$$

Hence, by using the embeddings $H_0^2(\Omega) \hookrightarrow L^{m^-}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{m^+}(\Omega)$, we conclude that

$$\begin{aligned}
&\left|-\mu_1 \int_s^T E^q(t)\int_\Omega u\,|u_t|^{m(x)-1}\,dxdt\right| \\
&\le \varepsilon \int_s^T E^q(t)\left[c\,\|\Delta u(s)\|_2^{m^-} + c\,\|\Delta u(s)\|_2^{m^+}\right]dt \\
&\quad + c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dxdt \\
&\le \varepsilon \int_s^T E^q(t)\left[cE^{(m^--2)/2}(0)\,E(t) + cE^{(m^+-2)/2}(0)\,E(t)\right]dt \\
&\quad + c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dxdt \\
&\le c\varepsilon \int_s^T E^{q+1}(t)\,dt + \int_s^T E^q(t)\int_\Omega c_\varepsilon(x)\,|u_t(t)|^{m(x)}\,dxdt.
\end{aligned} \quad (4.10)$$

The next term of (4.4) can be estimated in a similar attitude

$$\begin{aligned}
&\left|-\mu_2 \int_s^T E^q(t)\int_\Omega u\,|z(x,1,t)|^{m(x)-1}\,dxdt\right| \\
&\le \varepsilon \int_s^T E^q(t)\left[c\,\|\Delta u(s)\|_2^{m^-} + c\,\|\Delta u(s)\|_2^{m^+}\right]dt
\end{aligned}$$

$$
\begin{aligned}
&+c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)\,|z(x,1,t)|^{m(x)}\,dxdt \\
\leq\ & c\varepsilon\int_s^T E^{q+1}(t)\,dt+\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)\,|z(x,1,t)|^{m(x)}\,dxdt.
\end{aligned}
\tag{4.11}
$$

As $\xi(x)$ is bounded, using Lemma 6 and (4.2), we obtain

$$
\begin{aligned}
&2\int_s^T E^q(t)\int_0^1\int_\Omega \frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho dt \\
&\leq \frac{2\tau e^{-\tau}}{m^-}E^q(s)\,E(s)+\frac{2c}{m^-}E^{q+1}(T) \\
&\leq \frac{2\tau e^{-\tau}}{m^-}E^q(0)\,E(s)+\frac{2c}{m^-}E^q(T)\,E(s)\leq cE(s)
\end{aligned}
\tag{4.12}
$$

for some $c>0$.

By combining (4.4)-(4.12), we conclude that

$$
\begin{aligned}
\int_s^T E^{q+1}(t)\,dt \leq \varepsilon\int_s^T E^{q+1}(t)\,dt+cE(s)& \\
+c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)\,|z(x,1,t)|^{m(x)}\,dxdt.&
\end{aligned}
$$

At this point, the choice of ε small enough, gives

$$
\int_s^T E^{q+1}(t)\,dt \leq cE(s)+c\int_s^T E^q(t)\int_\Omega c_\varepsilon(x)\,|z(x,1,t)|^{m(x)}\,dxdt.
$$

Once ε is fixed, $c_\varepsilon(x)$ becomes bounded (i.e. $c_\varepsilon(x)\leq M$), as $m(x)$ is bounded. Therefore, we infer that

$$
\begin{aligned}
\int_s^T E^{q+1}(t)\,dt &\leq cE(s)+cM\int_s^T E^q(t)\int_\Omega |z(x,1,t)|^{m(x)}\,dxdt \\
&\leq cE(s)-C_0M\int_s^T E^q(t)\,E'(t)\,dt \\
&\leq cE(s)+\frac{C_0M}{q+1}\left[E^{q+1}(s)-E^{q+1}(T)\right]\leq cE(s).
\end{aligned}
$$

As $T\to\infty$, we obtain

$$
\int_s^\infty E^{q+1}(t)\,dt \leq cE(s).
$$

Thus, Komornik's Lemma is satisfied with $\sigma=q=m^+/2-1$, which implies the desired result. $\square$

5. Conclusions

In recent years, many papers have been published about decay or blow up of solutions for different type of wave equations (Kirchhoff, Petrovsky, Bessel, ...etc.) with different state of delay time (constant delay, time-varying delay,...etc.). However, to the best of our knowledge, there were no blow up and decay results for the Timoshenko equation with delay term and variable exponents. Firstly, we have been proved the blow up of solutions. Later, we have been obtained the decay results by using an integral inequality due to Komornik.

References

[1] S. N. Antontsev, J. Ferreira, E. Piskin, H. Yüksekkaya, and M. Shahrouzi, "Blow up and asymptotic behavior of solutions for a $p(x)$-Laplacian equation with delay term and variable exponents," *Electron. J. Differ. Equ*, pp. 1–20, 2021, doi: 10.22541/au.160975791.14681913/v1.

[2] S. Antontsev, J. Ferreira, and E. Piskin, "Existence and blow up of solutions for a strongly damped Petrovsky equation with variable-exponent nonlinearities," *Electron J. Differ. Equ*, vol. 6, pp. 1–18, 2021. [Online]. Available: http://ejde.math.txstate.eduorhttp://ejde.math.unt.edu

[3] S. N. Antontsev, "Wave equation with $p(x,t)$-Laplacian and damping term: Blow-up of solutions," *Comptes Rendus Mecanique*, vol. 339, no. 12, pp. 751–755, 2011, doi: 10.1016/j.crme.2011.09.001.

[4] S. N. Antontsev, "Wave equation with $p(x,t)$-Laplacian and damping term: existence and blow-up," *Differ. Equ. Appl*, vol. 3, no. 4, pp. 503–525, 2011, doi: 10.7153/dea-03-32.

[5] S. N. Antontsev, J. Ferreira, E. Pişkin, and S. M. S. Cordeiro, "Existence and non-existence of solutions for Timoshenko-type equations with variable exponents," *Nonlinear Analysis: Real World Applications*, vol. 61, p. 103341, 2021, doi: 10.1016/j.nonrwa.2021.103341.

[6] Y. Chen, S. Levine, and M. Rao, "Variable exponent, linear growth functionals in image restoration," *SIAM journal on Applied Mathematics*, vol. 66, no. 4, pp. 1383–1406, 2006, doi: 10.1137/050624522.

[7] R. Datko, J. Lagnese, and M. Polis, "An example on the effect of time delays in boundary feedback stabilization of wave equations," *SIAM Journal on Control and Optimization*, vol. 24, no. 1, pp. 152–156, 1986, doi: 10.1137/0324007.

[8] L. Diening, P. Harjulehto, P. Hästö, and M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents*. Springer, 2011. doi: 10.1007/978-3-642-18363-8.

[9] C. Doshi, "On the Analysis of the Timoshenko Beam Theory with and without internal damping," Ph.D. dissertation, 1979. [Online]. Available: http://scholarworks.rit.edu/theses

[10] X. Fan, J. Shen, and D. Zhao, "Sobolev embedding theorems for spaces Wk, $p(x)(\Omega)$," *Journal of Mathematical Analysis and Applications*, vol. 262, no. 2, pp. 749–760, 2001, doi: 10.1006/jmaa.2001.7618.

[11] B. Feng, "Global well-posedness and stability for a viscoelastic plate equation with a time delay," *Mathematical Problems in Engineering*, vol. 2015, 2015, doi: 10.1155/2015/585021.

[12] E. Guariglia and S. Silvestrov, *Fractional-Wavelet Analysis of Positive definite Distributions and Wavelets on* $\mathbb{D}'(\mathrm{C})$. Springer, 2016. doi: 10.1007/978-3-319-42105-6_16.

[13] E. Guariglia and K. Tamilvanan, "On the stability of radical septic functional equations," *Mathematics*, vol. 8, no. 12, p. 2229, 2020.

[14] M. Kafini and S. Messaoudi, "On the decay and global nonexistence of solutions to a damped wave equation with variable-exponent nonlinearity and delay," in *Annales Polonici Mathematici*,

vol. 122, doi: 10.4064/ap180524-31-10. Instytut Matematyczny Polskiej Akademii Nauk, 2019, pp. 49–70.

[15] M. Kafini and S. A. Messaoudi, "A blow-up result in a nonlinear wave equation with delay," *Mediterranean Journal of Mathematics*, vol. 13, no. 1, pp. 237–247, 2016, doi: 10.1007/s00009-014-0500-4.

[16] J.-R. Kang, "Global nonexistence of solutions for von Karman equations with variable exponents," *Applied Mathematics Letters*, vol. 86, pp. 249–255, 2018, doi: 10.1016/j.aml.2018.07.008.

[17] V. Komornik and V. Gattulli, "Exact controllability and stabilization. The multiplier method," *SIAM Review*, vol. 39, no. 2, pp. 351–351, 1997.

[18] O. Kovacik and J. Rakosnik, "On spaces $L^{p(x)}$ and $W^{k,p(x)}$," *Czechoslovak mathematical journal*, vol. 41, no. 4, pp. 592–618, 1991, doi: 10.21136/CMJ.1991.102493.

[19] S. Nicaise, J. Valein, and E. Fridman, "Stabilization of the heat and the wave equations with boundary time-varying delays. DCDS-S," 2009," doi: 10.3934/dcdss.2009.2.559.

[20] S. Nicaise and C. Pignotti, "Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks," *SIAM Journal on Control and Optimization*, vol. 45, no. 5, pp. 1561–1585, 2006, doi: 10.1137/060648891.

[21] S. H. Park, "Decay rate estimates for a weak viscoelastic beam equation with time-varying delay," *Applied Mathematics Letters*, vol. 31, pp. 46–51, 2014, doi: 10.1016/j.aml.2014.02.005.

[22] S.-H. Park and J.-R. Kang, "Blow-up of solutions for a viscoelastic wave equation with variable exponents," *Mathematical Methods in the Applied Sciences*, vol. 42, no. 6, pp. 2083–2097, 2019, doi: 10.1002/mma.5501.

[23] E. Pişkin and H. Yüksekkaya, "Nonexistence of global solutions of a delayed wave equation with variable-exponents," *Miskolc Mathematical Notes*, vol. 22, no. 2, pp. 841–859, 2021, doi: 10.18514/MMN.2021.3487.

[24] E. Pişkin and H. Yüksekkaya, "Blow up of solutions for a Timoshenko equation with damping terms," *Middle East Journal of Science*, vol. 4, no. 2, pp. 70–80, doi: 10.23884/mejs.2018.4.2.03.

[25] E. Pişkin and H. Yüksekkaya, "Mathematical behavior of the solutions of a class of hyperbolic-type equation," *Balıkesir Üniversitesi Fen Bilimleri Enstitüsü Dergisi*, vol. 20, no. 3, pp. 117–128, 2018, doi: 10.25092/baunfbed.483072.

[26] E. Pişkin and H. Yüksekkaya, "Non-existence of solutions for a Timoshenko equations with weak dissipation," *Mathematica Moravica*, vol. 22, no. 2, pp. 1–9, 2018, doi: 10.5937/MatMor1802001P.

[27] M. A. Ragusa, "Elliptic boundary value problem in vanishing mean oscillation hypothesis," *Commentationes Mathematicae Universitatis Carolinae*, vol. 40, no. 4, pp. 651–663, 1999. [Online]. Available: http://eudml.org/doc/248421

[28] M. A. Ragusa, "Dirichlet problem in Morrey spaces for elliptic equations in non divergence form with VMO coefficients," in *Proceedings of the Eighth International Colloquium on Differential Equations, Plovdiv, Bulgaria, 18–23 August, 1997*. De Gruyter, 2020, pp. 385–390.

[29] M. Ruzicka, *Electrorheological fluids: Modeling and mathematical theory*. Springer Science & Business Media, 2000.

[30] M. Shahrouzi, "On behaviour of solutions for a nonlinear viscoelastic equation with variable-exponent nonlinearities," *Computers & Mathematics with Applications*, vol. 75, no. 11, pp. 3946–3956, 2018, doi: 10.1016/j.camwa.2018.03.005.

[31] S. P. Timoshenko, "LXVI. On the correction for shear of the differential equation for transverse vibrations of prismatic bars," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 41, no. 245, pp. 744–746, 1921, doi: 10.1080/14786442108636264.

[32] G. Q. Xu, S. P. Yung, and L. K. Li, "Stabilization of wave systems with input delay in the boundary control," *ESAIM: Control, optimisation and calculus of variations*, vol. 12, no. 4, pp. 770–785, 2006, doi: 10.1051/cocv:2006021.

[33] H. Yüksekkaya and E. Pişkin, "Blow-up results for a Viscoelastic plate equation with distributed delay," *Journal of Universal Mathematics*, vol. 4, no. 2, pp. 128–139, 2021, doi: 10.33773/jum.957748.
[34] H. Yüksekkaya and E. Pişkin, "Nonexistence of global solutions for a Kirchhoff-type viscoelastic equation with distributed delay," *Journal of Universal Mathematics*, vol. 4, no. 2, pp. 271–282, 2021, doi: 10.33773/jum.957741.
[35] H. Yüksekkaya, E. Pișkin, S. M. Boulaaras, B. B. Cherif, and S. A. Zubair, "Existence, nonexistence, and stability of solutions for a delayed plate equation with the logarithmic source," *Advances in Mathematical Physics*, vol. 2021, 2021, doi: 10.1155/2021/8561626.
[36] H. Yüksekkaya and E. Pişkin, "Blow-up of solutions of a logarithmic viscoelastic plate equation with delay term," *Journal of Mathematical Sciences & Computational Mathematics*, vol. 3, no. 2, pp. 167–182, 2022, doi: 10.15864/jmscm.3203.
[37] H. Yüksekkaya and E. Pişkin, "Growth of solutions for a delayed Kirchhoff-type viscoelastic equation," *Journal of Mathematical Sciences & Computational Mathematics*, vol. 3, no. 2, pp. 234–246, 2022, doi: 10.15864/jmscm.3209.
[38] H. Yüksekkaya, E. Pişkin, S. M. Boulaaras, and B. B. Cherif, "Existence, decay, and blow-up of solutions for a higher-order Kirchhoff-type equation with delay term," *Journal of Function Spaces*, vol. 2021, 2021, doi: 10.1155/2021/4414545.

Authors' addresses

Hazal Yüksekkaya
(**Corresponding author**) Dicle University, Department of Mathematics, Diyarbakir, Turkey
E-mail address: hazally.kaya@gmail.com

Erhan Pişkin
Dicle University, Department of Mathematics, Diyarbakir, Turkey
E-mail address: episkin@dicle.edu.tr