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EXISTENCE OF SOLUTIONS FOR THIRD ORDER MULTI POINT IMPULSIVE BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Abstract. In this paper, we obtain sufficient conditions for existence of solutions of a third order m-point impulsive boundary value problem on time scales. To the best of our knowledge, there is hardly any work dealing with third order multi point dynamic impulsive BVPs. The reason may be the complex arguments caused by both impulsive perturbations and calculations on time scales. As an application, we give an example demonstrating our results.

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1. INTRODUCTION

Time scales theory was initiated by Stefan Hilger [5] to combine the analysis over continuous and discrete sets (see also [6]). Later, this theory was rapidly developed by many mathematicians and scientists working in other disciplines such as mechanics, electronics, neural networks, population models, economics, etc. [2]. In recent years impulsive dynamic equations have also attracted huge interest because the non-impulsive equations are usually insufficient for modelling evolutionary processes. In this work, we focus on multi-point dynamic impulsive boundary value problems. They are more adequate for modelling many real-world phenomena as it is more likely for a dynamical system to have multi-points of freedom. We may refer the reader to the book [1] which offers an excellent review and several examples of applications modelled by boundary value problems, and to the famous book of Bainov and Simeonov [11] for extensive knowledge about impulsive differential equations. Many good papers in literature deal with third-order m-point impulsive boundary value problems, however most of them have focused on differential BVPs on the real line $\mathbb{R}$, see for example [4, 7, 12, 13]. Regarding to dynamic BVPs on arbitrary time scales there are very few studies some of which are mentioned below.

In [9], Li and Li studied the following boundary value problem for the nonlinear third order impulsive dynamic system on time scales

$$\begin{cases} -u^{\Delta^3}(t) = f(t,u(t),u^{\Delta}(t),u^{\Delta^2}(t)), \quad t \in [0,T]_{\mathbb{T}} \backslash \Omega, \\ \Delta u(t_k) = I_k, \quad \Delta u^{\Delta}(t_k) = J_k, \quad \Delta u^{\Delta^2}(t_k) = L_k \quad k = 1,2,\ldots,m, \\ u(0) = \lambda u(\sigma(T)), \quad u^{\Delta}(0) = \lambda u^{\Delta}(\sigma(T)), \quad u^{\Delta^2}(0) = \lambda u^{\Delta^2}(\sigma(T)). \end{cases}$$

They obtained some sufficient conditions for the existence of solutions by using Schauder's fixed point theorem.

In [10], Liang and Zhang studied the third order m-point impulsive boundary value problem

$$\begin{cases} (\varphi(-u''(t)))' + a(t)f(u(t)) = 0, \quad t \neq t_k, \quad 0 < t < 1, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1,2,\ldots,N, \\ u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(1) = 0, \quad u''(0) = 0, \end{cases}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing positive homomorphism with $\varphi(0) = 0$. By using the five-functionals fixed point theorem, they provided sufficient conditions for existence of three positive solutions.

Later, Karaca and Fen [8] studied the following nonlinear third order m-point impulsive boundary value problem on time scales

$$\begin{cases} (\phi(u^{\Delta\Delta}(t)))^{\Delta} + q(t)f(t,u(t),u^{\Delta}(t),u^{\Delta^2}(t)) = 0,\ t \in J = [0,1]_{\mathbb{T}},\ t \neq t_k,\ 0 < t < 1, \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1,2,\ldots,n, \\ \Delta u^{\Delta}(t_k) = -J_k(u(t_k),u^{\Delta}(t_k)), \quad k = 1,2,\ldots,n, \\ au(0) - bu^{\Delta}(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad cu(1) + du^{\Delta}(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \\ u^{\Delta^2}(0) = 0, \end{cases}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing positive homomorphism with $\phi(0) = 0$. They obtained sufficient conditions for existence of solutions by using four functionals fixed point theorem to reach existence of at least one positive solution.

Motivated by the above studies, in this work we deal with the third order multi-point boundary value problem for impulsive dynamic equations of the form

$$\begin{cases} x^{\Delta^3}(t) = f(t,x(t),x^{\Delta}(t),x^{\Delta^2}(t)), \quad t \in J_0, \\ x^{\Delta^2}(t_k^+) = x^{\Delta^2}(t_k) + I_k(x(t_k),x^{\Delta}(t_k),x^{\Delta^2}(t_k)), \quad k \in \{1,\ldots,m\}, \\ x^{\Delta^2}(0) = \lambda x^{\Delta^2}(T), \\ x^{\Delta}(0) = \sum\limits_{j=1}^{n} \alpha_j x^{\Delta}(\xi_j), \\ x(0) = 0, \end{cases} \tag{1.1}$$

where $z(t_k^+) = \lim_{t \to t_k^+} z(t)$, $z(t_k^-) = z(t_k)$, $k \in \{1,\ldots,m\}$, and t_k, $k \in \{1,\ldots,m\}$, are right dense.

We define $J_0 = [0,T]_{\mathbb{T}} \backslash \{t_k\}_{k=1}^m$ and assume the following:

(A1): $f \in \mathcal{C}([0,T]_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$,

(A2): $I_k \in \mathcal{C}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$, $k \in \{1,\ldots,m\}$,

(A3): $0 \le \xi_1 < \xi_2 < \ldots < \xi_n \le T$,

(A4): there exist nonnegative constants α and γ_k, $k \in \{1,\ldots,n\}$, so that

$$\alpha = \limsup_{|u|+|v|+|w| \to \infty} \left(\max_{t \in [0,T]_{\mathbb{T}}} \frac{f(t,u,v,w)}{|u|+|v|+|w|} \right),$$

$$\gamma_k = \limsup_{|u|+|v|+|w| \to \infty} \frac{I_k(u,v,w)}{|u|+|v|+|w|}, \quad k \in \{1,\ldots,n\},$$

(A5): λ, $\alpha_j \in \mathbb{R}$, $j \in \{1,\ldots,n\}$, $\lambda \neq 1$, $1 - \sum_{j=1}^{n} \alpha_j \neq 0$.

Using an integral representation of the solutions of the BVP (1.1), sufficient conditions for the existence of at least one positive solution is obtained.

The paper is organized as follows. In the next section, we give some preliminary results. In Section 3, we formulate and prove our main result and we give an example that illustrates our main result.

2. Preliminaries

In the sequel we give some basic definitions for time scales extracted from [2, 3] and discuss further the models introduced above.

A time scale $\mathbb{T}$ is any closed subset of the real numbers. Some very well known examples are $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$ and the Cantor set.

Definition 1. For $t \in \mathbb{T}$, the forward jump operator $\sigma\colon \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, and the backward jump operator $\rho\colon \mathbb{T} \to \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. We note that $\sigma(t) \ge t$ and $\rho(t) \le t$ for any $t \in \mathbb{T}$.

Definition 2. t is said to be right scattered if $\sigma(t) > t$; right dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$; left scattered if $\rho(t) < t$, and left dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$. The graininess function $\mu\colon \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) = \sigma(t) - t$ for any $t \in \mathbb{T}$.

Definition 3. Assume that $a \le b$. Then we define the interval $[a,b]_{\mathbb{T}}$ in $\mathbb{T}$ by

$$[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}.$$

Definition 4. If $\mathbb{T}$ has a left scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^\kappa = \mathbb{T}$. In other words,

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \mathbb{T}. \end{cases}$$

Definition 5. For a function $f\colon \mathbb{T} \to \mathbb{R}$ we define $f^\sigma\colon \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$.

Definition 6. If $f\colon \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, we define $f^\Delta(t)$ to be the number with the property that given any $\varepsilon > 0$ there is a neighbourhood U of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta[\sigma(t) - s]\right| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$. Here, $f^\Delta(t)$ is called the delta derivative of f at t, and f is called delta differentiable in $\mathbb{T}^\kappa$ provided that $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. We shall talk about the second derivative f^{Δ^2} provided f^Δ is differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $f^{\Delta^2} = \left(f^\Delta\right)^\Delta : \mathbb{T}^{\kappa^2} \to \mathbb{R}$. Similarly, we define higher order derivatives $f^{\Delta^n} : \mathbb{T}^{\kappa^n} \to \mathbb{R}$.

Definition 7. A function $f\colon \mathbb{T} \to \mathbb{R}$ is called regulated provided that its right-sided limits exist at all right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.

Definition 8. A function $F\colon \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f\colon \mathbb{T} \to \mathbb{R}$ if

$$F^\Delta(t) = f(t) \text{ for all } t \in \mathbb{T}^\kappa.$$

We define the indefinite integral of f by

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant. We define the Cauchy integral by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for} \quad \text{any} \quad s, r \in \mathbb{T}.$$

Definition 9. $f\colon \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided that f is continuous at each right-dense point of $\mathbb{T}$ and has a finite left-dense limit at each left-dense point of $\mathbb{T}$. The set of rd-continuous functions will be denoted by $C_{rd}(\mathbb{T})$ and the set of differentiable functions that posseses rd-continuous derivatives is denoted by $C^1_{rd}(\mathbb{T})$.

In the following theorem, we list some properties of differentiation and integration on time scales that will be utilized in the paper. Its proof can be found in [2].

Theorem 1. *Let $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$ and $f, g \in C_{rd}$. Then,*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t;$
(ii) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t;$
(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t;$
(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$
(v) $\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t;$
(vi) $\int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t;$
(vii) $\int_a^a f(t)\Delta(t) = 0;$
(viii) *if* $|f(t)| \leq g(t)$ *on* $[a, b)$, *then* $\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t;$
(ix) *if* $f(t) \geq 0$ *for all* $a \leq t < b$, *then* $\int_a^b f(t)\Delta t \geq 0;$
(x) *if* $t \in \mathbb{T}^\kappa$, *then* $\int_t^{\sigma(t)} f(t)\Delta(t) = \mu(t)f(t).$

3. Integral representation and existence of solutions

This section consists of our main results and their illustrative examples. First, an integral representation of solutions is obtained and then, by an application of the Schauder fixed point theorem, the existence of the solutions is proved.

3.1. *Integral representation*

To find an integral representation of the solutions for the BVP (1.1), we first introduce the following BVP for the sake of brevity. Consider

$$\begin{cases} x^{\Delta^3}(t) = h(t), \quad t \in J_0, \\ x^{\Delta^2}(t_k^+) = x^{\Delta^2}(t_k) + I_k(x(t_k), x^{\Delta}(t_k), x^{\Delta^2}(t_k)), \quad k \in \{1,\dots,m\}, \\ x^{\Delta^2}(0) = \lambda x^{\Delta^2}(T), \\ x^{\Delta}(0) = \sum_{j=1}^{n} \alpha_j x^{\Delta}(\xi_j), \\ x(0) = 0, \end{cases} \tag{3.1}$$

where it is assumed that

(A6): $h \in C([0,T]_{\mathbb{T}})$.

For convenience, we introduce the notation $\beta = 1 - \sum_{j=1}^{n} \alpha_j$.

Lemma 1. *Suppose that the hypotheses* (A2), (A3), (A5) *and* (A6) *hold. Then x is a solution to the BVP* (3.1) *if and only if it is a solution to the integral equation*

$$\begin{aligned} x(t) = {} & \frac{\lambda}{1-\lambda}\left(t\left(\frac{\sum_{l=1}^{n}\alpha_l\xi_l}{\beta}\right) + p(t)\right)\left(\int_0^T h(s)\Delta s + \sum_{j=1}^{m} I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j))\right) \\ & + \frac{t}{\beta}\sum_{l=1}^{n}\alpha_l\left[\int_0^{\xi_l}(\xi_l - \sigma(s))h(s)\Delta s + \sum_{0<t_j<\xi_l}(\xi_l - t_j)I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j))\right] \\ & + \int_0^t q(t,\sigma(s))h(s)\Delta s + \sum_{0<t_j<t} q(t,t_j)I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)), \end{aligned} \tag{3.2}$$

where $p(t) = \int_0^t s\Delta s$ *and* $q(t,s) = p(t) - p(s) - (t-s)s$.

Proof. (1) Let x be a solution to the BVP (3.1). We integrate step by step both sides of the first equation of (3.1) from 0 to t to find

$$\begin{aligned} x^{\Delta^2}(t) &= x^{\Delta^2}(0) + \int_0^t h(s)\Delta s + \sum_{0<t_j<t}(x^{\Delta^2}(t_j^+) - x^{\Delta^2}(t_j)) \\ &= x^{\Delta^2}(0) + \int_0^t h(s)\Delta s + \sum_{0<t_j<t} I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)),\ t \in [0,T]_{\mathbb{T}}, \end{aligned} \tag{3.3}$$

which implies that

$$x^{\Delta^2}(T)=x^{\Delta^2}(0)+\int_0^T h(s)\Delta s+\sum_{j=1}^m I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j)).$$

Using the boundary condition

$$x^{\Delta^2}(0)=\lambda x^{\Delta^2}(T)$$

in the equation above, it follows that

$$(1-\lambda)x^{\Delta^2}(0)=\lambda\int_0^T h(s)\Delta s+\lambda\sum_{j=1}^m I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j)).$$

Then, setting the last equation in (3.3), we find

$$\begin{aligned}x^{\Delta^2}(t)=&\frac{\lambda}{1-\lambda}\left(\int_0^T h(s)\Delta s+\sum_{j=1}^m I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j))\right)\\&+\int_0^t h(s)\Delta s+\sum_{0<t_j<t} I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j)),\quad t\in[0,T]_{\mathbb{T}}.\end{aligned}\tag{3.4}$$

Now, integrating the last equation from 0 to t leads to

$$\begin{aligned}x^{\Delta}(t)=&x^{\Delta}(0)+t\frac{\lambda}{1-\lambda}\left(\int_0^T h(s)\Delta s+\sum_{j=1}^m I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j))\right)\\&+\int_0^t (t-\sigma(s))h(s)\Delta s\\&+\sum_{0<t_j<t} I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j))(t-t_j),\quad t\in[0,T]_{\mathbb{T}}\end{aligned}\tag{3.5}$$

which implies that

$$\begin{aligned}\sum_{l=1}^n \alpha_l x^{\Delta}(\xi_l)=&x^{\Delta}(0)\sum_{l=1}^n\alpha_l\\&+\left(\sum_{l=1}^n\alpha_l\xi_l\right)\frac{\lambda}{1-\lambda}\left(\int_0^T h(s)\Delta s+\sum_{j=1}^m I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j))\right)\\&+\sum_{l=1}^n\alpha_l\Bigg[\int_0^{\xi_l}(\xi_l-\sigma(s))h(s)\Delta s\\&+\sum_{0<t_j<\xi_l}(\xi_l-t_j)I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j))\Bigg].\end{aligned}$$

Applying the boundary condition

$$x^{\Delta}(0)=\sum_{l=1}^n\alpha_l x^{\Delta}(\xi_l)$$

one has

$$
\begin{aligned}
x^{\Delta}(0) = {} & \frac{\lambda}{1-\lambda}\left(\frac{\sum_{l=1}^{n}\alpha_l\xi_l}{\beta}\right)\left(\int_0^T h(s)\Delta s + \sum_{j=1}^{m} I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j))\right) \\
& + \frac{1}{\beta}\sum_{l=1}^{n}\alpha_l\left[\int_0^{\xi_l}(\xi_l - \sigma(s))h(s)\Delta s \right. \\
& \left. + \sum_{0<t_j<\xi_l}(\xi_l - t_j)I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j))\right].
\end{aligned}
$$

Hence, using the above equation in (3.5), we obtain

$$
\begin{aligned}
x^{\Delta}(t) = {} & \frac{\lambda}{1-\lambda}\left(\frac{\sum_{l=1}^{n}\alpha_l\xi_l}{\beta} + t\right)\left(\int_0^T h(s)\Delta s + \sum_{j=1}^{m} I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j))\right) \\
& + \frac{1}{\beta}\sum_{l=1}^{n}\alpha_l\left[\int_0^{\xi_l}(\xi_l - \sigma(s))h(s)\Delta s \right. \\
& \left. + \sum_{0<t_j<\xi_l}(\xi_l - t_j)I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j))\right] \\
& + \int_0^t (t-\sigma(s))h(s)\Delta s + \sum_{0<t_j<t}(t-t_j)I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)), \quad t \in [0,T]_{\mathbb{T}}.
\end{aligned}
$$

Finally, integration of the last equation results in

$$
\begin{aligned}
x(t) = {} & \frac{\lambda}{1-\lambda}\left(t\left(\frac{\sum_{l=1}^{n}\alpha_l\xi_l}{\beta}\right) + p(t)\right)\left(\int_0^T h(s)\Delta s + \sum_{j=1}^{m} I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j))\right) \\
& + \frac{t}{\beta}\sum_{l=1}^{n}\alpha_l\left[\int_0^{\xi_l}(\xi_l - \sigma(s))h(s)\Delta s \right. \\
& \left. + \sum_{0<t_j<\xi_l}(\xi_l - t_j)I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j))\right] \\
& + \int_0^t q(t,\sigma(s))h(s)\Delta s + \sum_{0<t_j<t} I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j))\int_{t_j}^t (s-t_j)\Delta s,
\end{aligned}
$$

where $t \in [0,T]_{\mathbb{T}}$. In view of

$$
q(t,t_j) = p(t) - p(t_j) - (t-t_j)t_j = \int_{t_j}^{t} s\Delta s - \int_{t_j}^{t} t_j\Delta s = \int_{t_j}^{t}(s-t_j)\Delta s,
$$

we obtain the integral representation of $x(t)$ defined by (3.2).

(2) Let x be a solution to the integral equation (3.2). Then, from $p(0)=0$, we have

$$x(0)=0.$$

So, differentiation of (3.1) gives

$$\begin{aligned} x^{\Delta}(t) &= \frac{\lambda}{1-\lambda}\left(\frac{\sum_{l=1}^{n}\alpha_l\xi_l}{\beta}\right)\left(\int_0^T h(s)\Delta s+\sum_{j=1}^{m} I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j))\right)\\ &\quad+\frac{1}{\beta}\sum_{l=1}^{n}\alpha_l\left[\int_0^{\xi_l}(\xi_l-\sigma(s))h(s)\Delta s+\sum_{0<t_j<\xi_l} I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j))(\xi_l-t_j)\right]\\ &\quad+t\frac{\lambda}{1-\lambda}\left(\int_0^T h(s)\Delta s+\sum_{j=1}^{m} I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j))\right)\\ &\quad+\int_0^t (t-\sigma(s))h(s)\Delta s+\sum_{0<t_j<t}(t-t_j)I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j)). \end{aligned} \tag{3.6}$$

Then, setting $t=0$ and $t=\xi_l$, respectively, it is seen that

$$\begin{aligned} x^{\Delta}(0) &= \frac{\lambda}{1-\lambda}\left(\frac{\sum_{l=1}^{n}\alpha_l\xi_l}{\beta}\right)\left(\int_0^T h(s)\Delta s+\sum_{j=1}^{m} I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j))\right)\\ &\quad+\frac{1}{\beta}\sum_{l=1}^{n}\alpha_l\left[\int_0^{\xi_l}(\xi_l-\sigma(s))h(s)\Delta s\right.\\ &\quad\left.+\sum_{0<t_j<\xi_l} I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j))(\xi_l-t_j)\sum_{l=1}^{n}\alpha_l x^{\Delta}(x_l)\right]\\ &=\sum_{l=1}^{n}\alpha_l x^{\Delta}(\xi_l). \end{aligned}$$

Also, it is not difficult to see that differentiation of (3.6) results in (3.4). Thus, setting $t=t_k$ in equation (3.4), one has

$$\begin{aligned} x^{\Delta^2}(t_k^+) &= \frac{\lambda}{1-\lambda}\left(\int_0^T h(s)\Delta s+\sum_{j=1}^{m} I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j))\right)\\ &\quad+\int_0^{t_k} h(s)\Delta s+\sum_{0<t_j<t_k^+} I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j)), \end{aligned}$$

$k\in\{1,\dots,m\}$. Hence,

$$x^{\Delta^2}(t_k^+)-x^{\Delta^2}(t_k^-)=I_k(x(t_k),x^{\Delta}(t_k),x^{\Delta^2}(t_k)),\quad k\in\{1,\dots,m\}$$

holds, as well. Finally, note that

$$x^{\Delta^3}(t) = h(t), \quad t \in [0,T]_{\mathbb{T}}.$$

Therefore, x is a solution to the BVP (3.1). This completes the proof. □

3.2. *Existence of solutions*

Lemma 2. *Suppose* $(A1)$,$(A2)$ *and* $(A4)$ *hold. Then there exist positive constants* Q *and* R *such that*

$$\begin{aligned} |f(t,u,v,w)| &\le Q(|u|+|v|+|w|)+R, \\ |I_k(u,v,w)| &\le Q(|u|+|v|+|w|)+R, \quad k \in \{1,\dots,m\}, \end{aligned}$$

where $t \in [0,T]_{\mathbb{T}}$ *and* $u,v,w \in \mathbb{R}$.

Proof. By the first condition of $(A4)$, it follows that there exists a positive constant N_1 such that

$$|f(t,u,v,w)| \le \alpha(|u|+|v|+|w|), \quad t \in [0,1], \quad |u|+|v|+|w| > N_1.$$

On the other hand, since $f \in \mathcal{C}([0,T]_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$, there exists a positive constant Q_1 such that

$$|f(t,u,v,w)| \le Q_1, \quad t \in [0,1], \quad |u|+|v|+|w| \le N_1.$$

Therefore

$$|f(t,u,v,w)| \le \alpha(|u|+|v|+|w|)+Q_1, \quad t \in [0,T]_{\mathbb{T}}, \quad u,v,w \in \mathbb{R}.$$

Similarly, from the second condition of $(A4)$, there exist positive constants N_{2j}, $j \in \{1,\dots,m\}$, such that

$$|I_j(u,v,w)| \le \delta_j(|u|+|v|+|w|), \quad |u|+|v|+|w| > N_{2j}, \quad j \in \{1,\dots,m\},$$

and from $I_j \in \mathcal{C}(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$, $j \in \{1,\dots,m\}$, there exist positive constants Q_{2j}, $j \in \{1,\dots,m\}$, such that

$$|I_j(u,v,w)| \le Q_{2j}, \quad |u|+|v|+|w| \le N_{2j}, \quad j \in \{1,\dots,m\}.$$

Consequently

$$|I_j(u,v,w)| \le \delta_j(|u|+|v|+|w|)+Q_{2j}, \quad u,v,w \in \mathbb{R},$$

$j \in \{1,\dots,m\}$. If we define

$$\begin{aligned} Q &= \max\{\alpha, \delta_j, \quad j \in \{1,\dots,m\}\}, \\ R &= \max\{Q_1, Q_{2j}, \quad j \in \{1,\dots,m\}\}, \end{aligned}$$

we get the desired inequalities. This completes the proof. □

In order to prove the existence of solutions, we will utilize the Schauder fixed point theorem. For this purpose, we first introduce the function spaces

$$PC([0,T]_{\mathbb{T}}) = \left\{ x \in \mathcal{C}(J_0), \quad \exists \lim_{t \to t_j^-} x(t), \quad \lim_{t \to t_j^+} x(t), x(t_j^-) = x(t_j), \quad j \in \{1,\dots,m\} \right\},$$

$$\begin{aligned} PC^1([0,T]_{\mathbb{T}}) = \Big\{ & x \in PC([0,T]_{\mathbb{T}}) : x^{\Delta} \in \mathcal{C}(J_0), \quad \exists \lim_{t \to t_j^+} x^{\Delta}(t), \\ & \lim_{t \to t_j^-} x^{\Delta}(t), x^{\Delta}(t_j^-) = x^{\Delta}(t_j), \quad j \in \{1,\dots,m\} \Big\} \end{aligned}$$

and

$$\begin{aligned} PC^2([0,T]_{\mathbb{T}}) = \Big\{ & x \in PC^1([0,T]_{\mathbb{T}}) : x^{\Delta^2} \in \mathcal{C}(J_0), \quad \exists \lim_{t \to t_j^+} x^{\Delta^2}(t), \quad \lim_{t \to t_j^-} x^{\Delta^2}(t), \\ & x^{\Delta^2}(t_j^-) = x^{\Delta^2}(t_j), \quad j \in \{1,\dots,m\} \Big\}, \end{aligned}$$

endowed with the norms

$$\|x\| = \sup_{t \in [0,T]_{\mathbb{T}}} |x(t)|,$$

$$\|x\| = \max \left\{ \sup_{t \in [0,T]_{\mathbb{T}}} |x(t)|, \quad \sup_{t \in [0,T]_{\mathbb{T}}} |x^{\Delta}(t)| \right\},$$

and

$$\|x\| = \max \left\{ \sup_{t \in [0,T]_{\mathbb{T}}} |x(t)|, \quad \sup_{t \in [0,T]_{\mathbb{T}}} \left|x^{\Delta}(t)\right|, \quad \sup_{t \in [0,T]_{\mathbb{T}}} \left|x^{\Delta^2}(t)\right| \right\},$$

respectively. Let $x \in PC^2([0,T]_{\mathbb{T}})$ and define the operator

$$\begin{aligned} Tx(t) = {} & \frac{\lambda}{1-\lambda} \left(t \left(\frac{\sum_{l=1}^{n} \alpha_l \xi_l}{\beta} \right) + p(t) \right) \\ & \times \left(\int_0^T f(s, x(s), x^{\Delta}(s), x^{\Delta^2}(s)))\Delta s + \sum_{j=1}^{m} I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)) \right) \\ & + \frac{t}{\beta} \sum_{l=1}^{n} \alpha_l \Bigg[\int_0^{\xi_l} (\xi_l - \sigma(s)) f(s, x(s), x^{\Delta}(s), x^{\Delta^2}(s)))\Delta s \\ & + \sum_{0 < t_j < \xi_l} (\xi_l - t_j) I_j(x(t_j), x^{\Delta}(t_j), x^{\Delta^2}(t_j)) \Bigg] \\ & + \int_0^t q(t, \sigma(s)) f(s, x(s), x^{\Delta}(s), x^{\Delta^2}(s)))\Delta s \end{aligned}$$

$$+\sum_{0<t_j<t} q(t,t_j)I_j(x(t_j),x^{\Delta}(t_j),x^{\Delta^2}(t_j)), \quad t\in[0,T]_{\mathbb{T}}.$$

Take a sufficiently large $A>0$ such that

$$\begin{aligned}A \geq \max\Bigg\{ & \left|\frac{\lambda}{1-\lambda}\right|\left(\sigma(T)\frac{\sum\limits_{l=1}^{n}|\alpha_l|\xi_l}{|\beta|}+(\sigma(T))^2\right)\Big(m+\sigma(T)\Big)\\ & +\frac{\sigma(T)}{|\beta|}\sum_{l=1}^{n}|\alpha_l|\Big[\xi_l^2+\sum_{j=1}^{n}(\xi_l+t_j)\Big]+4(\sigma(T))^3+(\sigma(T))^2,\\ & \left|\frac{\lambda}{1-\lambda}\right|\frac{\sum\limits_{l=1}^{n}|\alpha_l|\xi_l}{|\beta|}\Big(m+\sigma(T)\Big)+\frac{1}{|\beta|}\sum_{l=1}^{n}|\alpha_l|\Big[\xi_l^2+\sum_{j=1}^{m}(\xi_l+t_j)\Big]\\ & +\left(\left|\frac{\lambda}{1-\lambda}\right|+1\right)\sigma(T)\Big(m+\sigma(T)\Big),\left(\left|\frac{\lambda}{1-\lambda}\right|+1\right)\Big(m+\sigma(T)\Big),1\Bigg\}.\end{aligned}$$

Lemma 3. *Suppose* $(A1)$-$(A5)$ *hold. Then,* $T:PC^2([0,T]_{\mathbb{T}})\to PC^2([0,T]_{\mathbb{T}})$ *is a completely continuous operator.*

Proof. Let $D\subset PC^2([0,T]_{\mathbb{T}})$ be a bounded set. Then, there exists a positive constant B such that

$$\|x\|\leq B, \quad x\in D.$$

Take $x\in D$ arbitrarily. Then

$$\begin{aligned}|Tx(t)| &\leq \Bigg\{\left|\frac{\lambda}{1-\lambda}\right|\left(\sigma(T)\frac{\sum\limits_{l=1}^{n}|\alpha_l|\xi_l}{|\beta|}+(\sigma(T))^2\right)\Big(\sigma(T)+m\Big)\\ &\quad +\frac{\sigma(T)}{|\beta|}\sum_{l=1}^{n}|\alpha_l|\left[\xi_l^2+\sum_{j=1}^{n}(\xi_l+t_j)\right]+4(\sigma(T))^3+(\sigma(T))^2\Bigg\}(3QB+R)\\ &\leq A(3QB+R), \quad t\in[0,T]_{\mathbb{T}},\end{aligned}$$

and

$$\begin{aligned}|(Tx)^{\Delta}(t)| &\leq \Bigg\{\left|\frac{\lambda}{1-\lambda}\right|\left(\frac{\sum\limits_{l=1}^{n}|\alpha_l|\xi_l}{|\beta|}+\sigma(T)\right)\Big(\sigma(T)+m\Big)\\ &\quad +\frac{1}{|\beta|}\sum_{l=1}^{n}|\alpha_l|\left[\xi_l^2+\sum_{j=1}^{m}(\xi_l+t_j)\right]+\sigma(T)\Big(\sigma(T)+m\Big)\Bigg\}(3QB+R)\\ &\leq A(3QB+R), \quad t\in[0,T]_{\mathbb{T}},\end{aligned}$$

and

$$|(Tx)^{\Delta^2}(t)| \leq \left(\left|\frac{\lambda}{1-\lambda}\right|+1\right)\left(\sigma(T)+m\right)(3QB+R) \leq A(3QB+r), \quad t \in [0,T]_{\mathbb{T}}.$$

Consequently

$$\|Tx\| \leq A(3QB+R). \tag{3.7}$$

Moreover,

$$|(Tx)^{\Delta^3}(t)| = |f(t,x(t),x^{\Delta}(t),x^{\Delta^2}(t))| \leq 3QB+R \leq A(3QB+R), \;\; t \in [0,T]_{\mathbb{T}}.$$

Hence, by the Arzela-Ascoli theorem, the operator $T : PC^2([0,T]_{\mathbb{T}}) \to PC^2([0,T]_{\mathbb{T}})$ is completely continuous. □

Theorem 2. *Suppose that* $(A1)$-$(A5)$ *hold, and the nonnegative constants* A, B, Q *and* R *satisfy*

$$A(3QB+R) \leq B.$$

Then, the BVP (1.1) *has at least one solution.*

Proof. Let

$$S = \{x \in PC^2([0,T]_{\mathbb{T}}) : \|x\| \leq B\}.$$

By Lemma 2, $T : S \to PC^2([0,T]_{\mathbb{T}})$ is a completely continuous operator, and from (3.7), it is seen that

$$\|Tx\| \leq A(3QB+R) \leq B, \quad x \in D,$$

which means $T : S \to S$. Thus, applying Schauder fixed point theorem, we conclude that the operator T has a fixed point in S. This completes the proof. □

3.3. *An example*

We provide the following example in order to show that our main result is applicable for impulsive dynamic boundary value problems of the form (1.1).

Example 1. Let

$$\mathbb{T} = \left[0,\frac{1}{32}\right] \cup \left[\frac{1}{16},\frac{1}{8}\right] \cup \left[\frac{1}{4},\frac{1}{3}\right] \cup \left[\frac{1}{2},1\right],$$

where $\left[0,\frac{1}{32}\right]$, $\left[\frac{1}{16},\frac{1}{8}\right]$, $\left[\frac{1}{4},\frac{1}{3}\right]$ and $\left[\frac{1}{2},1\right]$ are the real-valued intervals. Let also,

$$T = \frac{1}{3}, \quad m = 4, \quad n = 3,$$

$$t_0 = 0, \quad t_1 = \frac{1}{36}, \quad t_2 = \frac{1}{16}, \quad t_3 = \frac{1}{10}, \quad t_4 = \frac{1}{4}, \quad t_5 = T = \frac{1}{3},$$

$$J = \left[0,\frac{1}{3}\right], \quad J_0 = \left(0,\frac{1}{32}\right) \cup \left(\frac{1}{16},\frac{1}{8}\right) \cup \left(\frac{1}{4},\frac{1}{3}\right],$$

$$\xi_1 = \frac{1}{38}, \quad \xi_2 = \frac{3}{32}, \quad \xi_3 = \frac{7}{24}.$$

Consider the BVP

$$\begin{cases} x^{\Delta^3}(t) = \frac{x(t)+x^{\Delta}(t)+x^{\Delta^2}(t)}{10^{20}(1+(x(t))^2)(1+(x^{\Delta}(t))^2)\left(1+(x^{\Delta^2}(t))^2\right)}, & t \in J_0, \\ x^{\Delta^2}(t_k^+) = x^{\Delta^2}(t_k) + \frac{x(t_k)}{10^{40}\left(1+(x^{\Delta}(t_k))^4\right)\left(1+\left(x^{\Delta^2}(t_k)\right)^8\right)}, & k \in \{1,\dots,4\}, \\ x^{\Delta^2}(0) = \frac{1}{2}x^{\Delta^2}\left(\frac{1}{3}\right), \\ x^{\Delta}(0) = \frac{1}{2}x^{\Delta}(\xi_1) + \frac{1}{8}x^{\Delta}(\xi_2) + \frac{1}{16}x^{\Delta}(\xi_3), \\ x(0) = 0. \end{cases} \tag{3.8}$$

Clearly,

$$f(t,u,v,w) = \frac{u+v+w}{10^{20}(1+u^2)(1+v^2)(1+w^2)}, \quad t \in J, \quad u,v,w \in \mathbb{R},$$

$$I_k(u,v,w) = \frac{u}{10^{40}(1+v^4)(1+w^8)}, \quad u,v,w \in \mathbb{R},$$

$$\sigma\left(\frac{1}{3}\right) = \frac{1}{2}, \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{8}, \quad \alpha_3 = \frac{1}{16}, \quad \lambda = \frac{1}{2}.$$

Thus, we have

$$|f(t,u,v,w)| \leq \frac{1}{10^{20}}(|u|+|v|+|w|), \tag{3.9}$$

$$|I_k(u,v,w)| \leq \frac{1}{10^{40}}|u| \tag{3.10}$$

which implies that

$$Q = \frac{1}{10^{20}}, \quad R = 0.$$

If we take

$$A = 10^{10} \quad \text{and} \quad B = \frac{1}{2},$$

then we obtain

$$A(3QB+R) = 3AQB = 3 \cdot 10^{10} \cdot \frac{1}{10^{20}} \cdot \frac{1}{2} < \frac{1}{2} = B. \tag{3.11}$$

Therefore, by (3.9)-(3.11) it follows that all the hypotheses of Theorem 2 hold. Hence, the BVP (3.8) has at least one solution.

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