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EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS FOR THE SECOND-ORDER m-POINT EIGENVALUE IMPULSIVE BOUNDARY VALUE PROBLEM

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Abstract. This paper is devoted to the study of the existence and nonexistence of positive solutions for a second-order m-point eigenvalue impulsive boundary value problem. We use the fixed point theorems on the cones in order to achieve our results.

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1. INTRODUCTION

Impulsive differential equations have recently become important in applied mathematical models for real processes that arise from phenomena studied in physics, ecology, biological systems, biotechnology and industrial robotic. Moreover, impulsive differential equations are richer in applications compared with the corresponding theory of ordinary differential equations. Many of the mathematical problems encountered in the study of impulsive differential equations cannot be treated with the usual techniques within the standard framework of ordinary differential equations. For the introduction of the basic theory of impulsive equations, see [3, 5, 6, 15, 21], and the papers [2, 4, 8, 11–14, 24–27].

In the literature, there are many studies on second-order impulsive boundary value problems, for these, we refer to reader to [1, 7, 10, 16–19, 22, 29, 31, 32]. In recent years, several authors have been working on the existence of positive solutions for second order m-point impulsive boundary value problems [9, 20, 28, 30].

Liu and Zhao [22] studied the following impulsive boundary value problem:

$$\begin{cases} -u''(t)+u(t)+p(t)u'(t)=\lambda f(t,u)+\mu g(t,u), & t\in[0,T],\\ \triangle u'(t_i)=I_i(u(t_i)), & i=1,2,\ldots,n,\\ u(0)=u(T)=0 \end{cases}$$

where $\lambda > 0$, $\mu \geq 0$, f, $g \in C([0,T] \times \mathbb{R}, \mathbb{R})$, $p \in L^\infty([0,T])$ satisfies $0 = t_0 < \frac{T}{4} \leq t_1 < t_2 < \cdots < t_n \leq \frac{3T}{4} < t_{n+1} = T$, $I_i\colon \mathbb{R} \to \mathbb{R}$, $i = 1,2,\ldots,n$ are continuous. They obtained some new existence of solutions by using variational methods combining with a three critical points theorem.

Motivated by the aforementioned results, in this study, we consider the following second-order m-point eigenvalue impulsive boundary value problem:

$$\begin{cases} \vartheta''(t) + q_1(t)\vartheta'(t) + q_2(t)\vartheta(t) + \kappa h(t,\vartheta(t)) = 0, \quad \kappa > 0 \\ \triangle\vartheta|_{t=t_j} = \kappa I_j(\vartheta(t_j)), \\ \triangle\vartheta'|_{t=t_j} = -\kappa J_j(\vartheta(t_j)), \\ a\vartheta(0) - b\vartheta'(0) = \sum_{i=1}^{m-2} \alpha_i \vartheta(\xi_i), \\ c\vartheta(1) + d\vartheta'(1) = \sum_{i=1}^{m-2} \beta_i \vartheta(\xi_i) \end{cases} \tag{1.1}$$

where $J = [0,1]$, $t \neq t_j$, $j = 1,2,\ldots,n$ with $0 < t_1 < t_2 < \cdots < t_n < 1$. $\triangle\vartheta|_{t=t_j}$ and $\triangle\vartheta'|_{t=t_j}$ denote the jump of $\vartheta(t)$ and $\vartheta'(t)$ at $t = t_j$, i.e.,

$$\triangle\vartheta|_{t=t_j} = \vartheta(t_j^+) - \vartheta(t_j^-), \qquad \triangle\vartheta'|_{t=t_j} = \vartheta'(t_j^+) - \vartheta'(t_j^-),$$

where $\vartheta(t_j^+)$, $\vartheta'(t_j^+)$ and $\vartheta(t_j^-)$, $\vartheta'(t_j^-)$ represent the right-hand limit and left-hand limit of $\vartheta(t)$ and $\vartheta'(t)$ at $t = t_j$, $j = 1,2,\ldots,n$, respectively.

Throughout this paper we assume that following conditions hold:

- (C1) $a,b,c,d \in (0,\infty)$, $\alpha_i, \beta_i \in [0,\infty)$, $\xi_i \in (0,1)$ for $i \in \{1,\ldots,m-2\}$, κ is a positive parameter,
- (C2) $h \in C(J \times \mathbb{R}^+, \mathbb{R}^+)$, $q_1 \in C(J, \mathbb{R}^+)$, $q_2 \in C(J, (-\infty, 0))$ such that for $t \in J$: $q_1(t) + q_2(t) \geq 0$,
- (C3) $I_j \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $J_j \in C(\mathbb{R}^+, \mathbb{R}^+)$ are bounded functions for $j = 1,2,\ldots,n$.

By using Guo-Krasnosel'skii's fixed point theorem [15], some criteria of existence and nonexistence of positive solutions for the second-order m-point eigenvalue impulsive boundary value problem (1.1) are established in terms of different values of κ. The problem (1.1) is more general, includes two-point, multi-point and impulsive problems as special cases. Thus, we generalize and improve some of the known results in the literature to some degree, and therefore our results may contribute to the literature in this field.

The main structure of this paper is as follows. In section 2, we provide some definitions and preliminary lemmas which are key tools for our main results. In section 3, we give and prove our main results.

2. Preliminaries

In this section, we first introduce some background definitions in Banach spaces, and then present auxiliary lemmas which will be used later.

Let $J' = J \setminus \{t_1, t_2, \dots, t_n\}$. $C(J, \mathbb{R}^+)$ denote the Banach space of all continuous mapping $\vartheta\colon J \to \mathbb{R}^+$ with the norm $\|\vartheta\| = \sup_{t \in J} |\vartheta(t)|$,

$$PC(J,\mathbb{R}^+) = \Big\{ \vartheta\colon J \to \mathbb{R}^+ \mid \vartheta \in C(J'),\ \vartheta(t_j^+) \text{ and } \vartheta(t_j^-) \text{ exist} \\ \text{and } \vartheta(t_j^-) = \vartheta(t_j),\ j = 1, 2, \dots, n \Big\}$$

is also a Banach space with norm $\|\vartheta\|_{PC} = \sup_{t \in J} |\vartheta(t)|$;

$$PC^1(J,\mathbb{R}^+) = \Big\{ \vartheta \in PC(J,\mathbb{R}^+) \mid \vartheta' \in PC(J'),\ \vartheta'(t_j^+) \text{ and } \vartheta'(t_j^-) \text{ exist} \\ \text{and } \vartheta'(t_j^-) = \vartheta'(t_j),\ j = 1, 2, \dots, n \Big\}$$

is a real Banach space with norm $\|\vartheta\|_{PC^1} = \max\{\|\vartheta\|_{PC}, \|\vartheta'\|_{PC}\}$ where

$$\|\vartheta\|_{PC} = \sup_{t \in J} |\vartheta(t)|, \qquad \|\vartheta'\|_{PC} = \sup_{t \in J} |\vartheta'(t)|.$$

A function $\vartheta \in PC^1(J,\mathbb{R}^+) \bigcap C^2(J',\mathbb{R})$ is called a solution of problem (1.1) if it satisfies (1.1).

Lemma 1 (Lemma 2.1 in [23])**.** *Let (C1–C2) hold. Let* φ *and* ψ *be the unique solution of boundary value problem:*

$$\begin{cases} \varphi''(t) + q_1(t)\varphi'(t) + q_2(t)\varphi(t) = 0, & t \in J, \\ \varphi(0) = b, \quad \varphi'(0) = a \end{cases}$$

and

$$\begin{cases} \psi''(t) + q_1(t)\psi'(t) + q_2(t)\psi(t) = 0, & t \in J, \\ \psi(1) = d, \quad \psi'(1) = -c \end{cases}$$

respectively. Then

(i) φ *is strictly increasing on* $[0,1]$, *and* $\varphi > 0$ *on* $(0,1]$;

(ii) ψ *is strictly decreasing on* $[0,1]$, *and* $\psi > 0$ *on* $[0,1)$.

Set $\rho := q(t) \begin{vmatrix} \psi(t) & \varphi(t) \\ \psi'(t) & \varphi'(t) \end{vmatrix} > 0$ with $q(t) = \exp\left(\int_0^t q_1(s)ds \right)$, $\forall t \in J$ and

$$\triangle = \begin{vmatrix} -\sum_{i=1}^{m-2} \alpha_i \varphi(\xi_i) & \rho - \sum_{i=1}^{m-2} \alpha_i \psi(\xi_i) \\ \dfrac{\rho}{q(1)} - \sum_{i=1}^{m-2} \beta_i \varphi(\xi_i) & -\sum_{i=1}^{m-2} \beta_i \psi(\xi_i) \end{vmatrix}.$$

Lemma 2. *Let (C1–C3) hold. Assume*

(C4) $\Delta :\neq 0$.

If $\vartheta \in PC^1(J,\mathbb{R}^+)\cap C^2(J')$ *is a solution of the equation*

$$\begin{aligned}\vartheta(t) = \kappa\int_0^1 G(t,s)q(s)h(s,\vartheta(s))ds + \kappa\sum_{j=1}^{n} G_s(t,s)|_{s=t_j}q(t_j)I_j(\vartheta(t_j)) \\ + \kappa\sum_{j=1}^{n} G(t,t_j)q(t_j)J_j(\vartheta(t_j)) + \kappa A(h)\varphi(t) + \kappa B(h)\psi(t)\end{aligned} \tag{2.1}$$

where

$$G(t,s) = \frac{1}{\rho}\begin{cases}\varphi(s)\psi(t), & s\le t,\\ \varphi(t)\psi(s), & t\le s,\end{cases} \tag{2.2}$$

$$\begin{aligned}K_i = \int_0^1 G(\xi_i,s)q(s)h(s,\vartheta(s))ds + \sum_{j=1}^{n} G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}}q(t_j)I_j(\vartheta(t_j)) \\ + \sum_{j=1}^{n} G(\xi_i,t_j)q(t_j)J_j(\vartheta(t_j)),\end{aligned} \tag{2.3}$$

$$A(h) = \frac{1}{\triangle}\begin{vmatrix}\sum_{i=1}^{m-2}\alpha_i K_i & \rho - \sum_{i=1}^{m-2}\alpha_i\psi(\xi_i)\\ \sum_{i=1}^{m-2}\beta_i K_i & -\sum_{i=1}^{m-2}\beta_i\psi(\xi_i)\end{vmatrix} \tag{2.4}$$

and

$$B(h) = \frac{1}{\triangle}\begin{vmatrix}-\sum_{i=1}^{m-2}\alpha_i\varphi(\xi_i) & \sum_{i=1}^{m-2}\alpha_i K_i\\ \frac{\rho}{q(1)} - \sum_{i=1}^{m-2}\beta_i\varphi(\xi_i) & \sum_{i=1}^{m-2}\beta_i K_i\end{vmatrix}, \tag{2.5}$$

then ϑ *is a solution of the impulsive boundary value problem* (1.1).

Proof. Let ϑ satisfies the integral equation (2.1), then we have

$$\begin{aligned}\vartheta(t) = &\frac{\kappa}{\rho}\int_0^t \varphi(s)\psi(t)q(s)h(s,\vartheta(s))ds + \frac{\kappa}{\rho}\int_t^1 \varphi(t)\psi(s)q(s)h(s,\vartheta(s))ds\\ &+ \frac{\kappa}{\rho}\sum_{t_j<t}\varphi'(t_j)\psi(t)q(t_j)I_j(\vartheta(t_j)) + \frac{\kappa}{\rho}\sum_{t<t_j}\varphi(t)\psi'(t_j)q(t_j)I_j(\vartheta(t_j))\\ &+ \frac{\kappa}{\rho}\sum_{t_j<t}\varphi(t_j)\psi(t)q(t_j)J_j(\vartheta(t_j)) + \frac{\kappa}{\rho}\sum_{t<t_j}\varphi(t)\psi(t_j)q(t_j)J_j(\vartheta(t_j))\\ &+ \kappa A(h)\varphi(t) + \kappa B(h)\psi(t).\end{aligned}$$

Also, $\vartheta'(t)$ and $\vartheta''(t)$ can be easily written. Thus by Liouville's formula, we have

$$\begin{aligned}\vartheta''(t)+q_1(t)\vartheta'(t)+q_2(t)\vartheta(t) &= \frac{\kappa}{\rho}\left[\varphi(t)\psi'(t)-\varphi'(t)\psi(t)\right]q(t)h(t,\vartheta(t))\\ &= \frac{\kappa}{\rho}\left[-W\left(\varphi(t),\psi(t)\right)\right]q(t)h(t,\vartheta(t))\\ &= \frac{\kappa}{\rho}\left[-\rho e^{-\int_0^t q_1(s)ds}\right]q(t)h(t,\vartheta(t))\\ &= -\kappa h(t,\vartheta(t)),\end{aligned}$$

$$\begin{aligned}\triangle\vartheta|_{t=t_j} &= \vartheta(t_j^+)-\vartheta(t_j^-) = \frac{\kappa}{\rho}\left[\varphi'(t_j)\psi(t_j)-\varphi(t_j)\psi'(t_j)\right]q(t_j)I_j(\vartheta(t_j))\\ &= \frac{\kappa}{\rho}q(t_j)\begin{vmatrix}\psi(t_j) & \varphi(t_j)\\ \psi'(t_j) & \varphi'(t_j)\end{vmatrix}I_j(\vartheta(t_j))\\ &= \frac{\kappa}{\rho}\begin{vmatrix}\psi(0) & \varphi(0)\\ \psi'(0) & \varphi'(0)\end{vmatrix}e^{-\int_0^{t_j} q_1(s)ds}q(t_j)I_j(\vartheta(t_j))\\ &= \kappa I_j(\vartheta(t_j)),\end{aligned}$$

and

$$\begin{aligned}\triangle\vartheta'|_{t=t_j} &= \vartheta'(t_j^+)-\vartheta'(t_j^-) = \frac{\kappa}{\rho}\left[\varphi(t_j)\psi'(t_j)-\varphi'(t_j)\psi(t_j)\right]q(t_j)J_j(\vartheta(t_j))\\ &= -\frac{\kappa}{\rho}\begin{vmatrix}\psi(t_j) & \varphi(t_j)\\ \psi'(t_j) & \varphi'(t_j)\end{vmatrix}q(t_j)J_j(\vartheta(t_j))\\ &= -\frac{\kappa}{\rho}\begin{vmatrix}\psi(0) & \varphi(0)\\ \psi'(0) & \varphi'(0)\end{vmatrix}e^{-\int_0^{t_j} q_1(s)ds}q(t_j)J_j(\vartheta(t_j))\\ &= -\kappa J_j(\vartheta(t_j)).\end{aligned}$$

Since

$$\begin{aligned}\vartheta(0) &= \frac{\kappa}{\rho}\int_0^1 \varphi(0)\psi(s)q(s)h(s,\vartheta(s))ds+\frac{\kappa}{\rho}\sum_{j=1}^{n}\varphi(0)\psi'(t_j)q(t_j)I_j(\vartheta(t_j))\\ &\quad+\frac{\kappa}{\rho}\sum_{j=1}^{n}\varphi(0)\psi(t_j)q(t_j)J_j(\vartheta(t_j))+\kappa A(h)\varphi(0)+\kappa B(h)\psi(0),\end{aligned}$$

$$\vartheta'(0) = \frac{\kappa}{\rho}\int_0^1 \varphi'(t)\psi(s)q(s)h(s,\vartheta(s))ds+\frac{\kappa}{\rho}\sum_{j=1}^{n}\varphi'(0)\psi'(t_j)q(t_j)I_j(\vartheta(t_j))$$

$$+\frac{\kappa}{\rho}\sum_{j=1}^{n}\varphi'(0)\psi(t_j)q(t_j)J_j(\vartheta(t_j))+\kappa A(h)\varphi'(0)+\kappa B(h)\psi'(0),$$

we have that

$$a\vartheta(0)-b\vartheta'(0)=\kappa\rho B(h)=\sum_{i=1}^{m-2}\alpha_i\left(\kappa A(h)\varphi(\xi_i)+\kappa B(h)\psi(\xi_i)+\kappa K_i\right). \tag{2.6}$$

Since

$$\vartheta(1)=\frac{\kappa}{\rho}\int_0^1\varphi(s)\psi(1)q(s)h(s,\vartheta(s))ds+\frac{\kappa}{\rho}\sum_{j=1}^{n}\varphi'(t_j)\psi(1)q(t_j)I_j(\vartheta(t_j))$$
$$+\frac{\kappa}{\rho}\sum_{j=1}^{n}\varphi(t_j)\psi(1)q(t_j)J_j(\vartheta(t_j))+\kappa A(h)\varphi(1)+\kappa B(h)\psi(1),$$

$$\vartheta'(1)=\frac{\kappa}{\rho}\int_0^1\varphi(s)\psi'(t)q(s)h(s,\vartheta(s))ds+\frac{\kappa}{\rho}\sum_{j=1}^{n}\varphi'(t_j)\psi'(1)q(t_j)I_j(\vartheta(t_j))$$
$$+\frac{\kappa}{\rho}\sum_{j=1}^{n}\varphi(t_j)\psi'(1)q(t_j)J_j(\vartheta(t_j))+\kappa A(h)\varphi'(1)+\kappa B(h)\psi'(1),$$

we have that

$$c\vartheta(1)+d\vartheta'(1)=\kappa\frac{\rho}{q(1)}A(h)=\sum_{i=1}^{m-2}\beta_i\left(\kappa A(h)\varphi(\xi_i)+\kappa B(h)\psi(\xi_i)+\kappa K_i\right). \tag{2.7}$$

From (2.6) and (2.7), we get that

$$\begin{cases} A(h)\left[-\sum_{i=1}^{m-2}\alpha_i\varphi(\xi_i)\right]+B(h)\left[\rho-\sum_{i=1}^{m-2}\alpha_i\psi(\xi_i)\right]=\sum_{i=1}^{m-2}\alpha_iK_i \\ A(h)\left[\frac{\rho}{q(1)}-\sum_{i=1}^{m-2}\beta_i\varphi(\xi_i)\right]+B(h)\left[-\sum_{i=1}^{m-2}\beta_i\psi(\xi_i)\right]=\sum_{i=1}^{m-2}\beta_iK_i \end{cases}$$

which implies that $A(h)$ and $B(h)$ satisfy (2.4) and (2.5), respectively. □

Lemma 3. *Let (C1)-(C4) hold. Assume*

(C5) $\triangle<0,\ \rho-\sum_{i=1}^{m-2}\alpha_i\psi(\xi_i)>0,\ \frac{\rho}{q(1)}-\sum_{i=1}^{m-2}\beta_i\varphi(\xi_i)>0$ *and*

$$\psi(t_j)J_j(\vartheta(t_j))+\psi'(t_j)I_j(\vartheta(t_j))>0 \quad \text{for } j=1,2,\dots,n.$$

Then for $\vartheta\in PC^1(J,\mathbb{R}^+)\cap C^2(J')$, *the solution* ϑ *of the problem* (1.1) *satisfies* $\vartheta(t)\geq 0$ *for* $t\in J$.

Proof. It is an immediate subsequence of the facts that $G(t,s)\geq 0$ on $J\times J$ and $A(h)\geq 0, B(h), I_j, J_j\geq 0$. □

Lemma 4. *Suppose that (C1)-(C5) hold, then for any $t,s \in J$, we have*

$$G(t,s) \le \delta G(s,s) \tag{2.8}$$

and

$$G_s(t,s) \le \delta G_s(t,s)|_{t=s} \tag{2.9}$$

where

$$\delta = \frac{\max\{\varphi(1),\psi(0)\}}{\min\{\varphi(0),\psi(1)\}} \tag{2.10}$$

and

$$G_s(t,s) = \frac{1}{\rho}\begin{cases} \varphi'(s)\psi(t), & s \le t, \\ \varphi(t)\psi'(s), & t < s, \end{cases} \tag{2.11}$$

for $t,s \in J$.

Proof. By the monotonicity of φ, ψ and the definition of $G(t,s)$, we have

$$\frac{G(t,s)}{G(s,s)} = \frac{\varphi(t)}{\varphi(s)} \le \frac{\varphi(1)}{\varphi(0)} \le \frac{\max\{\varphi(1),\psi(0)\}}{\min\{\varphi(0),\psi(1)\}} = \delta \quad \text{for } t \le s$$

and

$$\frac{G(t,s)}{G(s,s)} = \frac{\psi(t)}{\psi(s)} \le \frac{\psi(0)}{\psi(1)} \le \frac{\max\{\varphi(1),\psi(0)\}}{\min\{\varphi(0),\psi(1)\}} = \delta \quad \text{for } s \le t.$$

Similarly, by the monotonicity of φ, ψ and the definition of $G_s(t,s)$, we have $G_s(t,s) \le \delta G_s(t,s)|_{t=s}$ for $t,s \in J$. □

Lemma 5. *Suppose that (C1)-(C5) hold. Let $\sigma \in (0,\frac{1}{2})$ and $t \in [\sigma, 1-\sigma]$. Then we have*

$$G(t,s) \ge \omega G(s,s) \tag{2.12}$$

where

$$\omega = \frac{\min\{\varphi(\sigma),\psi(1-\sigma)\}}{\max\{\varphi(1),\psi(0)\}}. \tag{2.13}$$

Proof. By the monotonicity of φ, ψ and the definition of $G(t,s)$, for $t \in [\sigma, 1-\sigma]$, we have

$$\frac{G(t,s)}{G(s,s)} = \frac{\varphi(t)}{\varphi(s)} \ge \frac{\varphi(\sigma)}{\varphi(1)} \ge \frac{\min\{\varphi(\sigma),\psi(1-\sigma)\}}{\max\{\varphi(1),\psi(0)\}} = \omega \quad \text{for } t \le s$$

and

$$\frac{G(t,s)}{G(s,s)} = \frac{\psi(t)}{\psi(s)} \ge \frac{\psi(1-\sigma)}{\psi(0)} \ge \frac{\min\{\varphi(\sigma),\psi(1-\sigma)\}}{\max\{\varphi(1),\psi(0)\}} = \omega \quad \text{for } s \le t.$$

□

Lemma 6. *Let (C1)-(C5) hold. Assume*

(C6) $\sum_{i=1}^{m-2}(a\beta_i - c\alpha_i) \geq 0.$

Then the solution $\vartheta \in PC^1(J,\mathbb{R}^+) \cap C^2(J',\mathbb{R})$ *of the problem* (1.1) *satisfies* $\vartheta'(t) \geq 0$ *for* $t \in J$.

Proof. Assume that the inequality $\vartheta'(t) < 0$ holds. Since $\vartheta(t)$ is nonincreasing on J, one can verify that $\vartheta(1) < \vartheta(0)$.
From the boundary conditions of the problem (1.1), we have

$$\frac{1}{c}\sum_{i=1}^{m-2}\beta_i\vartheta(\xi_i) - \frac{d}{c}\vartheta'(1) < \sum_{i=1}^{m-2}\alpha_i\vartheta(\xi_i) + \frac{b}{a}\vartheta'(0).$$

This yields that

$$\frac{1}{c}\sum_{i=1}^{m-2}\beta_i\vartheta(\xi_i) - \frac{1}{a}\sum_{i=1}^{m-2}\alpha_i\vartheta(\xi_i) < \frac{d}{c}\vartheta'(1) + \frac{b}{a}\vartheta'(0) < 0.$$

The last inequality yields

$$a\sum_{i=1}^{m-2}\beta_i\vartheta(\xi_i) - c\sum_{i=1}^{m-2}\alpha_i\vartheta(\xi_i) < 0.$$

Therefore, we obtain that

$$\sum_{i=1}^{m-2}(a\beta_i - c\alpha_i)\vartheta(\xi_i) < 0.$$

According to Lemma 3, we have $\vartheta(\xi_i) \geq 0$. So, $\sum_{i=1}^{m-2}(a\beta_i - c\alpha_i) < 0$. However, this contradicts to condition (C6). Consequently, $\vartheta'(t) \geq 0$ for $t \in J$.

□

Let $K = \{\vartheta \in PC(J,\mathbb{R}^+) \mid \vartheta(t)$ is nonnegative, nondecreasing and concave on $J\}$. So, K is a cone of $PC(J,\mathbb{R}^+)$.

Lemma 7. *Let (C1)-(C6) hold and* $\vartheta(t) \in K$, $\sigma \in \left(0, \frac{1}{2}\right)$. *Then*

$$\min_{t\in[\sigma,1-\sigma]}\vartheta(t) \geq \sigma\|\vartheta\|_{PC}$$

where $\|\vartheta\|_{PC} = \sup_{t\in J}|\vartheta(t)|$.

Proof. Since $\vartheta \in K$, we know that $\vartheta(t)$ is concave on J. Thus, $\min_{t\in[\sigma,1-\sigma]}\vartheta(t) = \vartheta(\sigma)$ and $\|\vartheta\|_{PC} = \sup_{t\in J}|\vartheta(t)| = \vartheta(1)$. Since the graph of ϑ is concave down on J, we obtain

$$\frac{\vartheta(1) - \vartheta(0)}{1-0} \leq \frac{\vartheta(\sigma) - \vartheta(0)}{\sigma - 0},$$

i.e. $\vartheta(\sigma) \geq \sigma\vartheta(1) + (1-\sigma)\vartheta(0)$. So $\vartheta(\sigma) \geq \sigma\vartheta(1)$. The proof is complete. □

For any $r > 0$, let $K_r = \{\vartheta \in K \mid \|\vartheta\|_{PC} < r\}$, $\partial K_r = \{\vartheta \in K \mid \|\vartheta\|_{PC} = r\}$, and $\overline{K}_r = \{\vartheta \in K \mid \|\vartheta\|_{PC} \leq r\}$. Defining operator $T_\kappa \colon K \to PC(J, \mathbb{R}^+)$ as follows:

$$T_\kappa\vartheta(t) = \kappa\int_0^1 G(t,s)q(s)h(s,\vartheta(s))ds + \kappa\sum_{j=1}^n G_s(t,s)|_{s=t_j}q(t_j)I_j(\vartheta(t_j)) + \kappa\sum_{j=1}^n G(t,t_j)q(t_j)J_j(\vartheta(t_j)) + \kappa A(h)\varphi(t) + \kappa B(h)\psi(t). \tag{2.14}$$

It is well known that $\vartheta \in K$ is a positive fixed point of operator T_κ if and only if ϑ is a positive solution of problem (1.1).

Lemma 8. *Let (C1)-(C6) hold. If conditions*

(C7) $0 < \int_0^1 \delta G(s,s)q(s)ds < \infty$ *and*

(C8) $0 < \int_0^1 q(s)ds < \infty$

hold, then $T_\kappa \colon K \to K$ is completely continuous.

Proof. For any $\vartheta \in K$, from Lemmas 2, 3, 6 and the definition of T_κ, we have $(T_\kappa\vartheta)(t) \geq 0$, $(T_\kappa\vartheta)'(t) \geq 0$, and $(T_\kappa\vartheta)'(t)$ is concave on J. Therefore $T_\kappa(K) \subset K$. Next, with a similar method in Lemma 2.4 in [19], it can be shown that the $T_\kappa \colon K \to K$ is completely continuous. So this part is skipped and thus the Lemma is proved. □

3. Main Results

In this section, we will apply the following Guo-Krasnosel'skii's fixed point theorem [15] to state main results.

Lemma 9 (Theorem 2.3.1 in [15])**.** *Let X be a Banach space, and P be cone in X. Assume that Ω_1 and Ω_2 are two bounded open subsets of X with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. Let $A \colon P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ be a completely continuous operator, satisfying either*

(i) $\|Ax\| \leq \|x\|$, $x \in P \cap \partial\Omega_1$, $\|Ax\| \geq \|x\|$, $x \in P \cap \partial\Omega_2$, *or*
(ii) $\|Ax\| \geq \|x\|$, $x \in P \cap \partial\Omega_1$, $\|Ax\| \leq \|x\|$, $x \in P \cap \partial\Omega_2$.

Then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

For suitability and simplicity in the following discussion, we indicate:

$$h_0 = \liminf_{y\to 0} \min_{t\in[\sigma,1-\sigma]} \frac{h(t,y)}{y}, \qquad h_\infty = \liminf_{y\to\infty} \min_{t\in[\sigma,1-\sigma]} \frac{h(t,y)}{y},$$

$$h^0 = \limsup_{y\to 0} \max_{t\in J} \frac{h(t,y)}{y}, \qquad I_j^0 = \limsup_{y\to 0} \frac{I_j(y)}{y}, \qquad J_j^0 = \limsup_{y\to 0} \frac{J_j(y)}{y}$$

$$h^{\infty}=\limsup_{y\to\infty}\max_{t\in J}\frac{h(t,y)}{y},\qquad I_j^{\infty}=\limsup_{y\to\infty}\frac{I_j(y)}{y},\qquad J_j^{\infty}=\limsup_{y\to\infty}\frac{J_j(y)}{y},$$

$$\begin{aligned}\hat{A}=\frac{1}{\Delta}\Bigg[&\left(-\sum_{j=1}^{n}\beta_i\psi(\xi_i)\right)\cdot\\&\left(\sum_{j=1}^{n}\alpha_i\left(\int_0^1 G(\xi_i,s)q(s)ds+\sum_{j=1}^{n}G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}}q(t_j)+\sum_{j=1}^{n}G(\xi_i,t_j)q(t_j)\right)\right)\\&-\left(\rho-\sum_{j=1}^{n}\alpha_i\psi(\xi_i)\right)\cdot\\&\left(\sum_{j=1}^{n}\beta_i\left(\int_0^1 G(\xi_i,s)q(s)ds+\sum_{j=1}^{n}G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}}q(t_j)+\sum_{j=1}^{n}G(\xi_i,t_j)q(t_j)\right)\right)\Bigg],\end{aligned}$$

$$\begin{aligned}\hat{B}=\frac{1}{\Delta}\Bigg[&\left(-\sum_{j=1}^{n}\alpha_i\varphi(\xi_i)\right)\cdot\\&\left(\sum_{j=1}^{n}\beta_i\left(\int_0^1 G(\xi_i,s)q(s)ds+\sum_{j=1}^{n}G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}}q(t_j)+\sum_{j=1}^{n}G(\xi_i,t_j)q(t_j)\right)\right)\\&-\left(\frac{\rho}{q(1)}-\sum_{j=1}^{n}\beta_i\varphi(\xi_i)\right)\cdot\\&\left(\sum_{j=1}^{n}\alpha_i\left(\int_0^1 G(\xi_i,s)q(s)ds+\sum_{j=1}^{n}G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}}q(t_j)+\sum_{j=1}^{n}G(\xi_i,t_j)q(t_j)\right)\right)\Bigg],\end{aligned}$$

$$\begin{aligned}\dot{A}=\frac{1}{\Delta}\Bigg[&\left(-\sum_{j=1}^{n}\beta_i\psi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^{n}\alpha_i\cdot\\&\left(\int_0^1 G(\xi_i,s)q(s)h^0ds+\sum_{j=1}^{n}G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}}q(t_j)I_j^0+\sum_{j=1}^{n}G(\xi_i,t_j)q(t_j)J_j^0\right)\Bigg)\\&-\left(\rho-\sum_{j=1}^{n}\alpha_i\psi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^{n}\beta_i\cdot\\&\left(\int_0^1 G(\xi_i,s)q(s)h^0ds+\sum_{j=1}^{n}G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}}q(t_j)I_j^0+\sum_{j=1}^{n}G(\xi_i,t_j)q(t_j)J_j^0\right)\Bigg)\Bigg],\end{aligned}$$

$$\dot{B}=\frac{1}{\Delta}\Bigg[\left(-\sum_{j=1}^{n}\alpha_i\varphi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^{n}\beta_i\cdot$$

$$\left(\int_0^1 G(\xi_i,s)q(s)h^0 ds+\sum_{j=1}^{n} G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}} q(t_j)I_j^0+\sum_{j=1}^{n} G(\xi_i,t_j)q(t_j)J_j^0\right)\Bigg)$$
$$-\left(\frac{\rho}{q(1)}-\sum_{j=1}^{n}\beta_i\varphi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^{n}\alpha_i\cdot$$
$$\left(\int_0^1 G(\xi_i,s)q(s)h^0 ds+\sum_{j=1}^{n} G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}} q(t_j)I_j^0+\sum_{j=1}^{n} G(\xi_i,t_j)q(t_j)J_j^0\right)\Bigg)\Bigg],$$

$$\ddot{A}=\frac{1}{\Delta}\Bigg[\left(-\sum_{j=1}^{n}\beta_i\psi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^{n}\alpha_i\Bigg(\int_0^1 G(\xi_i,s)q(s)(h^0+\zeta)ds$$
$$+\sum_{j=1}^{n} G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}} q(t_j)(I_j^0+\zeta)+\sum_{j=1}^{n} G(\xi_i,t_j)q(t_j)(J_j^0+\zeta)\Bigg)\Bigg)$$
$$-\left(\rho-\sum_{j=1}^{n}\alpha_i\psi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^{n}\beta_i\Bigg(\int_0^1 G(\xi_i,s)q(s)(h^0+\zeta)ds$$
$$+\sum_{j=1}^{n} G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}} q(t_j)(I_j^0+\zeta)+\sum_{j=1}^{n} G(\xi_i,t_j)q(t_j)(J_j^0+\zeta)\Bigg)\Bigg)\Bigg],$$

$$\ddot{B}=\frac{1}{\Delta}\Bigg[\left(-\sum_{j=1}^{n}\alpha_i\varphi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^{n}\beta_i\Bigg(\int_0^1 G(\xi_i,s)q(s)(h^0+\zeta)ds$$
$$+\sum_{j=1}^{n} G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}} q(t_j)(I_j^0+\zeta)+\sum_{j=1}^{n} G(\xi_i,t_j)q(t_j)(J_j^0+\zeta)\Bigg)\Bigg)$$
$$-\left(\frac{\rho}{q(1)}-\sum_{j=1}^{n}\beta_i\varphi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^{n}\alpha_i\Bigg(\int_0^1 G(\xi_i,s)q(s)(h^0+\zeta)ds$$
$$+\sum_{j=1}^{n} G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}} q(t_j)(I_j^0+\zeta)+\sum_{j=1}^{n} G(\xi_i,t_j)q(t_j)(J_j^0+\zeta)\Bigg)\Bigg)\Bigg],$$

$$A'=\frac{1}{\Delta}\Bigg[\left(-\sum_{j=1}^{n}\beta_i\psi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^{n}\alpha_i\cdot$$
$$\left(\int_0^1 G(\xi_i,s)q(s)h^\infty ds+\sum_{j=1}^{n} G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}} q(t_j)I_j^\infty+\sum_{j=1}^{n} G(\xi_i,t_j)q(t_j)J_j^\infty\right)\Bigg)$$
$$-\left(\rho-\sum_{j=1}^{n}\alpha_i\psi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^{n}\beta_i\cdot$$

$$\left(\int_0^1 G(\xi_i,s)q(s)h^\infty ds+\sum_{j=1}^n G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}}q(t_j)I_j^\infty+\sum_{j=1}^n G(\xi_i,t_j)q(t_j)J_j^\infty\right)\Bigg)\Bigg],$$

$$B'=\frac{1}{\Delta}\Bigg[\left(-\sum_{j=1}^n \alpha_i\varphi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^n \beta_i\cdot$$
$$\left(\int_0^1 G(\xi_i,s)q(s)h^\infty ds+\sum_{j=1}^n G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}}q(t_j)I_j^\infty+\sum_{j=1}^n G(\xi_i,t_j)q(t_j)J_j^\infty\right)\Bigg)$$
$$-\left(\frac{\rho}{q(1)}-\sum_{j=1}^n \beta_i\varphi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^n \alpha_i\cdot$$
$$\left(\int_0^1 G(\xi_i,s)q(s)h^\infty ds+\sum_{j=1}^n G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}}q(t_j)I_j^\infty+\sum_{j=1}^n G(\xi_i,t_j)q(t_j)J_j^\infty\right)\Bigg)\Bigg],$$

$$A''=\frac{1}{\Delta}\Bigg[\left(-\sum_{j=1}^n \beta_i\psi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^n \alpha_i\Bigg(\int_0^1 G(\xi_i,s)q(s)(h^\infty+\zeta)ds$$
$$+\sum_{j=1}^n G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}}q(t_j)(I_j^\infty+\zeta)+\sum_{j=1}^n G(\xi_i,t_j)q(t_j)(J_j^\infty+\zeta)\Bigg)\Bigg)$$
$$-\left(\rho-\sum_{j=1}^n \alpha_i\psi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^n \beta_i\Bigg(\int_0^1 G(\xi_i,s)q(s)(h^\infty+\zeta)ds$$
$$+\sum_{j=1}^n G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}}q(t_j)(I_j^\infty+\zeta)+\sum_{j=1}^n G(\xi_i,t_j)q(t_j)(J_j^\infty+\zeta)\Bigg)\Bigg)\Bigg]$$

and

$$B''=\frac{1}{\Delta}\Bigg[\left(-\sum_{j=1}^n \alpha_i\varphi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^n \beta_i\Bigg(\int_0^1 G(\xi_i,s)q(s)(h^\infty+\zeta)ds$$
$$+\sum_{j=1}^n G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}}q(t_j)(I_j^\infty+\zeta)+\sum_{j=1}^n G(\xi_i,t_j)q(t_j)(J_j^\infty+\zeta)\Bigg)\Bigg)$$
$$-\left(\frac{\rho}{q(1)}-\sum_{j=1}^n \beta_i\varphi(\xi_i)\right)\cdot\Bigg(\sum_{j=1}^n \alpha_i\Bigg(\int_0^1 G(\xi_i,s)q(s)(h^\infty+\zeta)ds$$
$$+\sum_{j=1}^n G_s(t,s)|_{\substack{t=\xi_i\\ s=t_j}}q(t_j)(I_j^\infty+\zeta)+\sum_{j=1}^n G(\xi_i,t_j)q(t_j)(J_j^\infty+\zeta)\Bigg)\Bigg)\Bigg].$$

Theorem 1. *Assume that (C1)-(C8) hold,*

$$\omega\sigma \int_{\sigma}^{1-\sigma} G(s,s)q(s)dsh_0 > \int_0^1 \delta G(s,s)q(s)dsh^{\infty} + \sum_{j=1}^{n} \delta G_s(t,s)|_{\substack{t=t_j\\ s=t_j}} q(t_j)I_j^{\infty} + \sum_{j=1}^{n} \delta G(t_j,t_j)q(t_j)J_j^{\infty} + (A'+B')\delta \min\{\varphi(0),\psi(1)\},$$

$h^{\infty} + \sum_{j=1}^{n} I_j^{\infty} + \sum_{j=1}^{n} J_j^{\infty} < \infty$ *and* $\omega\sigma \int_{\sigma}^{1-\sigma} G(s,s)q(s)dsh_0 > 0$, *then problem* (1.1) *has at least one positive solution for*

$$\left[\omega\sigma \int_{\sigma}^{1-\sigma} G(s,s)q(s)dsh_0\right]^{-1} < \kappa < \left[\int_0^1 \delta G(s,s)q(s)dsh^{\infty} + \sum_{j=1}^{n} \delta G_s(t,s)|_{\substack{t=t_j\\ s=t_j}} q(t_j)I_j^{\infty} + \sum_{j=1}^{n} \delta G(t_j,t_j)q(t_j)J_j^{\infty} + (A'+B')\delta \min\{\varphi(0),\psi(1)\}\right]^{-1}. \tag{3.1}$$

Proof. By (3.1), there exists $\zeta > 0$ such that

$$\begin{aligned} \left[\omega\sigma \int_{\sigma}^{1-\sigma} G(s,s)q(s)ds(h_0-\zeta)\right]^{-1} &\le \kappa \\ &\le \left[\int_0^1 \delta G(s,s)q(s)ds(h^{\infty}+\zeta) + \sum_{j=1}^{n} \delta G_s(t,s)|_{\substack{t=t_j\\ s=t_j}} q(t_j)(I_j^{\infty}+\zeta) \right. \\ &\quad \left. + \sum_{j=1}^{n} \delta G(t_j,t_j)q(t_j)(J_j^{\infty}+\zeta) + (A''+B'')\delta \min\{\varphi(0),\psi(1)\}\right]^{-1}. \end{aligned} \tag{3.2}$$

Let ζ be fixed. Since $\omega\sigma \int_{\sigma}^{1-\sigma} G(s,s)q(s)dsh_0 > 0$, there exists $\tau_1 > 0$ such that $h(t,y) > (h_0-\zeta)y$, $0 \le y \le \tau_1$, $t \in [\sigma, 1-\sigma]$. Then for $\vartheta \in \partial K_{\tau_1}$, $t \in [\sigma, 1-\sigma]$, we have

$$\begin{aligned} T_{\kappa}\vartheta(t) &\ge \kappa \int_0^1 G(t,s)q(s)h(s,\vartheta(s))ds > \kappa \int_{\sigma}^{1-\sigma} \omega G(s,s)q(s)(h_0-\zeta)\vartheta(s)ds \\ &\ge \kappa\omega\sigma(h_0-\zeta) \int_{\sigma}^{1-\sigma} G(s,s)q(s)ds\|\vartheta\|_{PC} \ge \|\vartheta\|_{PC}. \end{aligned}$$

Hence, $\|T_{\kappa}\vartheta\|_{PC} \ge \|\vartheta\|_{PC}$, $\vartheta \in \partial K_{\tau_1}$. Since $h^{\infty} + \sum_{j=1}^{n} I_j^{\infty} + \sum_{j=1}^{n} J_j^{\infty} < \infty$, there exists $\bar{\tau}_2 > 0$ such that

$$h(t,y) < (h^{\infty}+\zeta)y,\ I_j(y) < (I_j^{\infty}+\zeta)y,\ J_j(y) < (J_j^{\infty}+\zeta)y,\ y \ge \bar{\tau}_2,\ t \in J.$$

Set

$$N = \max\left\{\max_{\substack{t\in J\\ y\in[0,\bar{\tau}_2]}} h(t,y),\ \max_{y\in[0,\bar{\tau}_2]} I_1(y), \dots, \max_{y\in[0,\bar{\tau}_2]} I_n(y),\ \max_{y\in[0,\bar{\tau}_2]} J_1(y), \dots, \max_{y\in[0,\bar{\tau}_2]} J_n(y)\right\},$$

$$\tau_2 > \max\left\{\tau_1, \bar{\tau}_2, \frac{N}{h^\infty+\zeta}, \frac{N}{I_1{}^\infty+\zeta}, \dots, \frac{N}{I_n{}^\infty+\zeta}, \frac{N}{J_1{}^\infty+\zeta}, \dots, \frac{N}{J_n{}^\infty+\zeta}\right\},$$

then

$$h(t,y) < (h^\infty+\zeta)\tau_2,\ I_j(y) < (I_j^\infty+\zeta)\tau_2,\ J_j(y) < (J_j^\infty+\zeta)\tau_2,\ 0 \le y \le \tau_2,\ t \in J.$$

Thus for $\vartheta \in \partial K_{\tau_2}$, we have

$$\begin{aligned}\|T_\kappa \vartheta\|_{PC} &\le \kappa \int_0^1 \delta G(s,s)q(s)h(s,\vartheta(s))ds + \kappa \sum_{j=1}^n \delta G_s(t,s)|_{\substack{t=t_j\\ s=t_j}} q(t_j)I_j(\vartheta(t_j)) \\ &\quad + \kappa \sum_{j=1}^n \delta G(t_j,t_j)q(t_j)J_j(\vartheta(t_j)) + \kappa[A(h)+B(h)]\delta \min\{\varphi(0),\psi(1)\} \\ &\le \kappa \left[\int_0^1 \delta G(s,s)q(s)(h^\infty+\zeta)ds + \delta \sum_{j=1}^n G_s(t,s)|_{\substack{t=t_j\\ s=t_j}} q(t_j)(I_j^\infty+\zeta)\right. \\ &\quad \left. + \delta \sum_{j=1}^n G(t_j,t_j)q(t_j)(J_j^\infty+\zeta) + (A''+B'')\delta \min\{\varphi(0),\psi(1)\}\right] \|\vartheta\|_{PC} \\ &\le \|\vartheta\|_{PC}.\end{aligned}$$

By Lemma 9, the operator T_κ has at least one fixed point ϑ^* with $\tau_1 \le \|\vartheta^*\| \le \tau_2$, which is a positive solution of problem (1.1). □

Theorem 2. *Assume that (C1)-(C8) hold,*

$$\begin{aligned}\omega\sigma \int_\sigma^{1-\sigma} G(s,s)q(s)dsh_\infty &> \int_0^1 \delta G(s,s)q(s)dsh^0 + \sum_{j=1}^n \delta G_s(t,s)|_{\substack{t=t_j\\ s=t_j}} q(t_j)I_j^0 \\ &\quad + \sum_{j=1}^n \delta G(t_j,t_j)q(t_j)J_j^0 + (\dot{A}+\dot{B})\delta \min\{\varphi(0),\psi(1)\},\end{aligned}$$

$h^0 + \sum_{j=1}^n I_j^0 + \sum_{j=1}^n J_j^0 < \infty$ *and* $\omega\sigma \int_\sigma^{1-\sigma} G(s,s)q(s)dsh_\infty > 0$, *then the problem* (1.1) *has at least one positive solution for*

$$\begin{aligned}&\left[\omega\sigma \int_\sigma^{1-\sigma} G(s,s)q(s)dsh_\infty\right]^{-1} < \kappa \\ &< \left[\int_0^1 \delta G(s,s)q(s)dsh^0 + \sum_{j=1}^n \delta G_s(t,s)|_{\substack{t=t_j\\ s=t_j}} q(t_j)I_j^0\right. \\ &\quad \left. + \sum_{j=1}^n \delta G(t_j,t_j)q(t_j)J_j^0 + (\dot{A}+\dot{B})\delta \min\{\varphi(0),\psi(1)\}\right]^{-1}.\end{aligned} \tag{3.3}$$

Proof. By (3.3), there exists $\zeta > 0$ such that

$$\begin{aligned}&\left[\omega\sigma\int_{\sigma}^{1-\sigma} G(s,s)q(s)ds(h_{\infty}-\zeta)\right]^{-1} \leq \kappa \\ &\leq \left[\int_0^1 \delta G(s,s)q(s)ds(h^0+\zeta) + \sum_{j=1}^{n} \delta G_s(t,s)|_{\substack{t=t_j\\ s=t_j}} q(t_j)(I_j^0+\zeta)\right. \\ &\quad \left. + \sum_{j=1}^{n} \delta G(t_j,t_j)q(t_j)(J_j^0+\zeta) + (\ddot{A}+\ddot{B})\delta\min\{\varphi(0),\psi(1)\}\right]^{-1}.\end{aligned} \tag{3.4}$$

Let ζ be fixed. Since $h^0 + \sum_{j=1}^{n} I_j^0 + \sum_{j=1}^{n} J_j^0 < \infty$, there exists $\tau_1 > 0$ such that

$$h(t,y) < (h^0+\zeta)y,\ I_j(y) < (I_j^0+\zeta)y,\ J_j(y) < (J_j^0+\zeta)y, \quad 0 \leq y \leq \tau_1, t \in J. \tag{3.5}$$

Then for $\vartheta \in \partial K_{\tau_1}$, by using (3.4) and (3.5), we have $\|T_\kappa\vartheta\|_{PC} \leq \|\vartheta\|_{PC}$. Since $\omega\sigma\int_{\sigma}^{1-\sigma} G(s,s)q(s)dsh_{\infty} > 0$, there exists $\bar{\tau}_2 > 0$ such that $h(t,y) > (h_{\infty}-\zeta)y$, $y \geq \bar{\tau}_2$, $t \in [\sigma, 1-\sigma]$. Let $\tau_2 \geq \max\{2\tau_1, \frac{\bar{\tau}_2}{\sigma}\}$, then for $\vartheta \in \partial K_{\tau_2}$, $t \in [\sigma, 1-\sigma]$, we can obtain $\|T_\kappa\vartheta\|_{PC} \geq \|\vartheta\|_{PC}$, $\vartheta \in \partial K_{\tau_2}$. It follows from Lemma 9 that T_κ has a fixed point ϑ^* with $\tau_1 \leq \|\vartheta^*\| \leq \tau_2$, which is a positive solution of problem (1.1). □

Theorem 3. *Let (C1)-(C8) hold. If $h_0 = \infty$ or $h_{\infty} = \infty$, then there exists $\kappa_0 > 0$ such that problem (1.1) has a positive solution for $0 < \kappa < \kappa_0$.*

Proof. Choose a number $\tau_1 > 0$ and

$$\begin{aligned}\kappa_0 = \frac{\tau_1}{\overline{N}}\Bigg(&\int_0^1 \delta G(s,s)q(s)ds + \sum_{j=1}^{n} \delta G_s(t,s)|_{\substack{t=t_j\\ s=t_j}} q(t_j) + \sum_{j=1}^{n} \delta G(t_j,t_j)q(t_j) \\ &+ (\hat{A}+\hat{B})\delta\min\{\varphi(0),\psi(1)\}\Bigg)^{-1},\end{aligned}$$

where $\overline{N} = \max\left\{\max_{\substack{t\in J\\ y\in[0,\tau_1]}} h(t,y), \max_{y\in[0,\tau_1]} I_j(y), \max_{y\in[0,\tau_1]} J_j(y)\right\}$. For $\vartheta \in \partial K_{\tau_1}$, $0 < \kappa < \kappa_0$,

$$\begin{aligned}\|T_\kappa\vartheta\|_{PC} \leq \kappa\Bigg[&\int_0^1 \delta G(s,s)q(s)ds \max_{\substack{t\in J\\ y\in[0,\tau_1]}} h(t,y) + \sum_{j=1}^{n} \delta G_s(t,s)|_{\substack{t=t_j\\ s=t_j}} q(t_j) \max_{y\in[0,\tau_1]} I_j(y) \\ &+ \sum_{j=1}^{n} \delta G(t_j,t_j)q(t_j) \max_{y\in[0,\tau_1]} J_j(y) + \overline{N}(\hat{A}+\hat{B})\delta\min\{\varphi(0),\psi(1)\}\Bigg]\end{aligned}$$

$$< \kappa_0 \overline{N}\Big[\int_0^1 \delta G(s,s)q(s)ds + \sum_{j=1}^{n} \delta G_s(t,s)|_{\substack{t=t_j\\ s=t_j}} q(t_j) + \sum_{j=1}^{n} \delta G(t_j,t_j)q(t_j)$$

$$+ (\hat{A}+\hat{B})\delta \min\{\varphi(0),\psi(1)\}\Big]$$

$$= \tau_1 = \|\vartheta\|_{PC}. \tag{3.6}$$

If $h_0 = \infty$, then there exists $\tau_2 \in (0,\tau_1)$ such that $h(t,y) > \eta y$ for all $y \in [0,\tau_2]$, $t \in [\sigma, 1-\sigma]$ where $\eta > 0$ satisfying

$$\kappa\omega\sigma\eta \int_\sigma^{1-\sigma} G(s,s)q(s)ds > 1. \tag{3.7}$$

Then for $\vartheta \in \partial K_{\tau_2}$,

$$T_\kappa \vartheta(t) \geq \kappa \int_\sigma^{1-\sigma} \omega G(s,s)q(s)h(s,\vartheta(s))ds > \kappa \int_\sigma^{1-\sigma} \omega G(s,s)q(s)\eta\vartheta(s)ds$$

$$\geq \kappa\omega\sigma\eta \int_\sigma^{1-\sigma} G(s,s)q(s)ds\|\vartheta\|_{PC} > \|\vartheta\|_{PC}.$$

Hence, $\|T_\kappa\vartheta\|_{PC} \geq \|\vartheta\|_{PC}$, $\vartheta \in \partial K_{\tau_2}$.

If $h_\infty = \infty$, then there exists $\bar{\tau}_3 > 0$ such that $h(t,y) > \gamma y$ for $y \geq \bar{\tau}_3$, $t \in [\sigma, 1-\sigma]$ where $\gamma > 0$ satisfying

$$\kappa\omega\sigma\gamma \int_\sigma^{1-\sigma} G(s,s)q(s)ds > 1. \tag{3.8}$$

Let $\tau_3 > \max\{\tau_1, \dfrac{\bar{\tau}_3}{\sigma}\}$, then for $\vartheta \in \partial K_{\tau_3}$,

$$T_\kappa \vartheta(t) \geq \kappa \int_\sigma^{1-\sigma} \omega G(s,s)q(s)h(s,\vartheta(s))ds > \kappa \int_\sigma^{1-\sigma} \omega G(s,s)q(s)\gamma\vartheta(s)ds$$

$$\geq \kappa\omega\sigma\gamma \int_\sigma^{1-\sigma} G(s,s)q(s)ds\|\vartheta\|_{PC} > \|\vartheta\|_{PC}.$$

Hence, $\|T_\kappa\vartheta\|_{PC} \geq \|\vartheta\|_{PC}$, $\vartheta \in \partial K_{\tau_3}$.
Then, Lemma 9 yields that T_κ has a fixed point in $\overline{K}_{\tau_1}\backslash K_{\tau_2}$ or $\overline{K}_{\tau_3}\backslash K_{\tau_1}$ according to whether $h_0 = \infty$ or $h_\infty = \infty$, respectively. As a result, problem (1.1) has a positive solution for $0 < \kappa < \kappa_0$. □

Theorem 4. *Let (C1)-(C8) hold. If* $h^0 + \sum_{j=1}^{n} I_j^0 + \sum_{j=1}^{n} J_j^0 = 0$ *or* $h^\infty + \sum_{j=1}^{n} I_j^\infty + \sum_{j=1}^{n} J_j^\infty = 0$ *such that problem* (1.1) *has a positive solution for* $\kappa > \kappa_0$.

Proof. The proof is done similarly to the theorem 3. □

Corollary 1. *Let (C1)-(C8) hold. If* $h_0 = h_\infty = \infty$, *then there exists* $\kappa_0 > 0$ *such that problem* (1.1) *has at least two positive solutions for* $0 < \kappa < \kappa_0$.

Corollary 2. *Let (C1)-(C8) hold. If* $h^0+\sum_{j=1}^{n} I_j^0+\sum_{j=1}^{n} J_j^0 = h^\infty+\sum_{j=1}^{n} I_j^\infty+\sum_{j=1}^{n} J_j^\infty = 0$, *then there exists* $\kappa_0>0$ *such that problem* (1.1) *has at least two positive solutions for* $\kappa>\kappa_0$.

Theorem 5. *Let (C1)-(C8) hold. If* $h_0>0$ *and* $h_\infty>0$, *then there exists* $\kappa_0>0$ *such that problem* (1.1) *has no positive solution for* $\kappa>\kappa_0$.

Proof. Since $h_0>0$ and $h_\infty>0$, then there exist positive numbers $\eta_1,\eta_2,\tau_1,\tau_2$ such that $\tau_1<\sigma\tau_2$ and $h(t,y)\geq\eta_1 y$, $y\in[0,\tau_1]$, $t\in[\sigma,1-\sigma]$, $h(t,y)\geq\eta_2 y$, $y\in[\tau_2,+\infty)$, $t\in[\sigma,1-\sigma]$. Let the positive number η_3 be defined by

$$\eta_3=\min\left\{\eta_1,\eta_2,\min_{\substack{t\in[\sigma,1-\sigma]\\ y\in[\tau_1,\tau_2]}}\frac{h(t,y)}{y}\right\}.$$

Then, $h(t,y)\geq\eta_3 y$, $y\in[0,\tau_2]$, $t\in[\sigma,1-\sigma]$ and $h(t,y)\geq\eta_3 y$, $y\in[\tau_1,+\infty)$, $t\in[\sigma,1-\sigma]$.
Assume $\vartheta(t)$ is a positive solution of problem (1.1). We will show that this leads to a contradiction for

$$\kappa>\kappa_0=\frac{1}{\omega\sigma\eta_3\int_\sigma^{1-\sigma}G(s,s)q(s)ds}.$$

Since $T_\kappa\vartheta(t)=\vartheta(t)$ for $t\in J$, then for $\kappa>\kappa_0$ and $\|\vartheta\|_{PC}\in[0,\tau_2]$,

$$\begin{aligned}\|\vartheta\|_{PC}=\|T_\kappa\vartheta\|_{PC}&\geq T_\kappa\vartheta\left(\frac{1}{2}\right)\geq\kappa\int_\sigma^{1-\sigma}G\left(\frac{1}{2},s\right)q(s)h(s,\vartheta(s))ds\\&>\kappa_0\eta_3\int_\sigma^{1-\sigma}\omega G(s,s)q(s)\vartheta(s)ds\\&\geq\kappa_0\omega\sigma\eta_3\int_\sigma^{1-\sigma}G(s,s)q(s)ds\|\vartheta\|_{PC}\geq\|\vartheta\|_{PC},\end{aligned}$$

which is a contradiction. For $\kappa>\kappa_0$ and $\|\vartheta\|_{PC}\in[\frac{\tau_1}{\sigma},+\infty)$,

$$\begin{aligned}\|\vartheta\|_{PC}=\|T_\kappa\vartheta\|_{PC}&\geq T_\kappa\vartheta\left(\frac{1}{2}\right)\geq\kappa\int_\sigma^{1-\sigma}G\left(\frac{1}{2},s\right)q(s)h(s,\vartheta(s))ds\\&\geq\kappa_0\omega\sigma\eta_3\int_\sigma^{1-\sigma}G(s,s)q(s)ds\|\vartheta\|_{PC}\geq\|\vartheta\|_{PC},\end{aligned}$$

which is a contradiction. Hence, problem (1.1) has no positive solution. □

Theorem 6. *Let (C1)-(C8) hold. If* $h^0 + \sum_{j=1}^{n} I_j^0 + \sum_{j=1}^{n} J_j^0 < \infty$ *and* $h^\infty + \sum_{j=1}^{n} I_j^\infty + \sum_{j=1}^{n} J_j^\infty < \infty$, *then there exists* $\kappa_0 > 0$ *such that problem* (1.1) *has no positive solution for* $0 < \kappa < \kappa_0$.

Proof. The proof can be done similarly to the theorem 5. □

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