Miskolc Mathematical Notes
Vol. 23 (2022), No. 2, pp. 881–888

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2022.3626

AN INTEGRAL EQUATION IN PARTIALLY ORDERED SETS

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Received 29 December, 2020

Abstract. Here, some fixed point theorems in partially ordered spaces are proved. As an application, the existence of a solution of an integral equation is obtained.

2010 *Mathematics Subject Classification:* 47H10, 65R20

Keywords: Partially ordered set, Fixed point, Contractive mapping, Integral equation

1. INTRODUCTION

Fixed point problems are interesting problems in Nonlinear Analysis and Engineering Sciences. There is a broad set of applications of fixed point problems, especially in solving differential equations and integral equations. Thus fixed point theory is an interdisciplinary science and many researchers are studying them in different spaces (see [1–6, 9–11, 13–18]).

The existence of fixed points in partially ordered sets was first considered in [12] to solve an matrix equation. This study was continued in [7, 8] by assuming the existence of only a lower solution instead of the usual approach where both lower and upper solutions are assumed to exist. These fixed point theorems were applied to obtain certain uniqueness and existence results for ordinary differential equations in [7, 8].

Definition 1. A partial ordering on a nonempty set X is a relation R on X with the following properties:

(i) xRx for all $x \in X$,
(ii) if xRy and yRx then $x = y$,
(iii) if xRy and yRz then xRz.

Definition 2. Let $(X, \leq)$ be a partially ordered set and $f : X \to X$. f is called monotone nondecreasing if $x \leq y \Rightarrow f(x) \leq f(y)$, where $x, y \in X$.

Ran et al. [12, Theorem 2.1] proved the following theorem.

Theorem 1. *Let X be a partially ordered set such that every pair $x, y \in X$ has a lower and an upper bound. Furthermore, let d be a metric on X such that (X, d) is* *a complete metric space. If $f : X \to X$ is a continuous, monotone (i.e., either order preserving or order-reserving) operator such that*

$$\exists 0 < c < 1 \; d(f(x), f(y) \le cd(x,y), \forall x \ge y,$$

$$\exists x_0 \in X \; x_0 \le f(x_0) \;\; or \;\; x_0 \ge f(x_0),$$

then f has a unique fixed point $\bar{x}$. Moreover, for every $x \in X$, $\lim_{n\to\infty} f^n(x) = \bar{x}$.

Later on, Petruşel et al. [10, Theorem 1.2] by dropping the hypothesis that each pair of points has an upper and a lower bound, proved the following theorem.

Theorem 2. *Let X be a partially ordered set and let d be a metric o X such that the metric space (X,d) is complete and the metric and the ordered structure are compatible. Let $f : X \to X$ be a continuous and monotone (i.e. either order - preserving or order-reversing) operator. Suppose that the following two assertions hold:*

1) *there exits $a \in (0,1)$ such that $d(f(x), f(y)) \le ad(x,y)$ for each $x, y \in X$ with $x \ge y$,*
2) *there exists $x_0 \in X$ such that $x_0 \le f(x_0)$ or $x_0 \ge f(x_0)$.*

Then f has at least a fixed point $x^* \in X$ and for each $x \in X$ and for each $x \in X$ with $x \ge x_0$ (or $x \le x_0$) the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations of f starting from x converges to $x^* \in X$.

Also Nieto et al. [7, 8, Theorem 2.1] proved the following theorem.

Theorem 3. *Let $(X, \le)$ be a partially ordered set. Assume there exists a metric d in X such that (X,d) is a complete metric space. Suppose $f : X \to X$ is a continuous and nondecreasing mapping such that there exists a $k \in [0,1)$ with*

$$d(f(x), f(y)) \le kd(x,y),$$

for all $x \ge y$. If there exists $x_0 \in X$ with $x_0 \le f(x_0)$, then f has a fixed point.

In this paper, we extend the fixed point theorem [7, 8, Theorem 2.1] in partially ordered sets. Finally, an integral equation is solved and an example is stated. The main result of the paper is as follows:

Theorem 4. *Let $(X, \le)$ be a partially ordered set. Assume there exists a metric d in X such that (X,d) is a complete metric space. Suppose $f : X \to X$ is a continuous and nondecreasing mapping satisfying*

$$d(f(x), f(y)) \le \alpha(d(x,y))d(x,y),$$

where $\alpha : [0,\infty) \to [0,1)$ and $\limsup_{s\to t+} \alpha(s) < 1$ for all $t \in [0,\infty)$. If there exists $x_0 \in X$ with $x_0 \le f(x_0)$, then f has a fixed point.

This theorem can be considered as a new version generalization of Theorem 3. Indeed, the contractive constant is replaced by a function.

2. Fixed point theorems in partially ordered set

In this section, we present the proof of the main result and an alternative version of the main result. Then we study the existence of a solution of an integral equation. By induction of proof of Theorem 3 or [7, 8, Theorem 2.1], we state the proof of Theorem 4.

Proof of Theorem 4. . If $x_0 = f(x_0)$, then the proof is finished. If $x_0 \neq f(x_0)$ then $x_0 \leq f(x_0)$ and since f is nondecreasing we have

$$x_0 \leq f(x_0) \leq f^2(x_0) \leq f^3(x_0) \leq \cdots \leq f^n(x_0) \leq f^{n+1}(x_0) \leq \cdots .$$

Now

$$d(f^n(x_0), f^{n+1}(x_0)) \leq \alpha(d(f^{n-1}(x_0), f^n(x_0)))d(f^{n-1}(x_0), f^n(x_0)).$$

If $f^n(x_0) = f^{n+1}(x_0)$ then $f^n(x_0)$ is a fixed point of f. Otherwise, since $\alpha(t) < 1$, we have

$$d(f^n(x_0), f^{n+1}(x_0)) < d(f^{n-1}(x_0), f^n(x_0)).$$

Thus $\{d(f^n(x_0), f^{n+1}(x_0))\}$ is a nonnegative decreasing sequence. Thus it converges to a, where $a \geq 0$, that is

$$d(f^n(x_0), f^{n+1}(x_0)) \to a.$$

Since $\limsup_{s \to t+} \alpha(s) < 1$, then $\alpha(a) < 1$. Therefore, for all $\varepsilon > 0$ there exists $r \in [0,1)$ such that $\alpha(s) < r$ for all $s \in [a, a+\varepsilon)$. Now, choose $v \in \mathbb{N}$ such that

$$a \leq d(f^n(x_0), f^{n+1}(x_0)) \leq a + \varepsilon$$

for all $n \geq v$. Since

$$\begin{aligned} d(f^{n+1}(x_0), f^{n+2}(x_0)) &\leq \alpha(d(f^n(x_0), f^{n+1}(x_0)))d(f^n(x_0), f^{n+1}(x_0)) \\ &\leq rd(f^n(x_0), f^{n+1}(x_0)), \end{aligned}$$

thus by induction

$$d(f^{n+v}(x_0), f^{n+v+1}(x_0)) \leq r^n d(f^v(x_0), f^{v+1}(x_0)). \tag{2.1}$$

In order to show that $\{f^n(x_0)\}$ is a Cauchy sequence, consider

$$\begin{aligned} \sum_{n=1}^{n=\infty} d(f^n(x_0), f^{n+1}(x_0)) \leq \sum_{n=1}^{n=v} & d(f^n(x_0), f^{n+1}(x_0)) \\ & + \sum_{n=v}^{n=\infty} d(f^n(x_0), f^{n+1}(x_0)). \end{aligned}$$

Using (2.1), the second part of the right hand side of the above inequality is estimated as follows:

$$\sum_{n=v}^{n=\infty} d(f^n(x_0), f^{n+1}(x_0)) \leq \sum_{n=1}^{n=\infty} r^n d(f^v(x_0), f^{v+1}(x_0))$$

$$= d(f^v(x_0), f^{v+1}(x_0)) \sum_{n=1}^{n=\infty} r^n \leq \infty.$$

This shows that $\sum_{n=1}^{n=\infty} d(f^n(x_0), f^n(x_0))$ is convergent which implies

$$d(f^n(x_0), f^{n+1}(x_0)) \to 0, \tag{2.2}$$

Notice that the convergence of $\sum_{n=1}^{n=\infty} d(f^n(x_0), f^n(x_0))$ and (2.2) shows that $\{f^n(x_0)\}$ is a Cauchy sequence. Since X is complete, there exists a q in X such that $f^n(x_0) \to q$ as $n \to \infty$. We prove that $q \in X$ is a fixed point of f. Let $\varepsilon > 0$. Using the continuity of f at the point q, given $\varepsilon/2 > 0$, there exists $\delta > 0$ such that $d(z,q) < \delta$ implies that $d(f(z), f(q)) < \varepsilon/2$. Now, by the convergence of $\{f^n(x_0)\}$ to q, assume $\eta = \min\{\varepsilon/2, \delta\}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$d(f^n(x_0), q) < \eta.$$

Then

$$\begin{aligned} d(f(q), q) &\leq d(f(q), f^n(x_0)) + d(f^n(x_0), q) \\ &\leq \varepsilon/2 + \eta = \varepsilon. \end{aligned}$$

This proves that $d(f(q), q) = 0$, and q is a fixed point of f. □

Next theorem is an alternative version of Theorem 4.

Theorem 5. *Let $(X, \leq)$ be a partially ordered set. Assume there exists a metric d in X such that (X, d) is a complete metric space. Suppose $f : X \to X$ is a continuous and nondecreasing mapping satisfying*

$$d(f(x), f(y)) \leq \varphi(d(x,y)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is an increasing and upper semicontinuous function such that $\varphi(t) < t$ for all $t > 0$. If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

Proof. If $x_0 = f(x_0)$ then the proof is finished. If $x_0 \neq f(x_0)$ then $x_0 \leq f(x_0)$ and since f is nondecreasing we have

$$x_0 \leq f(x_0) \leq f^2(x_0) \leq f^3(x_0) \leq \cdots \leq f^n(x_0) \leq f^{n+1}(x_0) \leq \cdots.$$

Using $\varphi(t) < t$,

$$\begin{aligned} d(f^n(x_0, f^{n+1}(x_0)) &\leq \varphi(d(f^{n-1}(x_0), f^n(x_0))) \\ &< d(f^{n-1}(x_0), f^n(x_0)). \end{aligned}$$

Thus $\{d(f^n(x_0), f^{n+1}(x_0))\}$ is a nonnegative decreasing sequence and then it converges to a as $n \to \infty$. We show $a = 0$. If not, since φ is upper semicontinuous, we have

$$a = \lim d(f^n(x_0), f^{n+1}(x_0))$$

$$\leq \lim \varphi(d(f^{n-1}(x_0), f^n(x_0))) \leq \varphi(a).$$

This means $a \leq \varphi(a)$, which is a contradiction. Now, we prove $\{f^n(x_0)\}$ is a Cauchy sequence. Suppose not, there exists $\varepsilon > 0$ such that for all $r \in \mathbb{N}$, there exists $m_r > n_r \geq r$ such that

$$d(f^{m_r}(x_0), f^{n_r}(x_0)) \geq \varepsilon. \tag{2.3}$$

We can suppose that for all r, m_r is the smallest number greater than n_r such that (2.3) holds. Therefore for such k we have

$$\begin{aligned} \varepsilon &\leq d(f^{m_r}(x_0), f^{n_r}(x_0)) \\ &\leq d(f^{m_r}(x_0), f^{m_{r-1}}(x_0)) + d(f^{m_{r-1}}(x_0), f^{n_r}(x_0)) \\ &\leq d(f^{m_r}(x_0), f^{m_{r-1}}(x_0)) + \varepsilon. \end{aligned}$$

Hence

$$\lim_{r\to\infty} d(f^{m_r}(x_0), f^{n_r}(x_0)) = \varepsilon^+,$$

thus we have

$$\begin{aligned} d(f^{m_r}(x_0), f^{n_r}(x_0)) &\leq d(f^{m_r}(x_0), f^{m_{r+1}}(x_0)) + d(f^{m_{r+1}}(x_0), f^{n_{r+1}}(x_0)) \\ &\quad + d(f^{n_{r+1}}(x_0), f^{n_r}(x_0)) \\ &\leq d(f^{m_{r+1}}(x_0), f^{m_r}(x_0)) \\ &\quad + d(f^{n_{r+1}}(x_0), f^{n_r}(x_0)) + \varphi(d(f^{m_r}(x_0), f^{n_r}(x_0))). \end{aligned}$$

It follows that $\varepsilon \leq \varphi(\varepsilon)$, and this a contradiction. Hence $\{f^n(x_0)\}$ is a Cauchy sequence. Since X is complete, there exists a q in X such that $f^n(x_0) \to q$ as $n \to \infty$. The rest of the proof is like as the previous theorem. □

Remark 1. Consider the integral equation

$$y(x) = \int_0^x k(x,s)g(s,y(s))ds, \tag{2.4}$$

where I is a closed interval in $\mathbb{R}$, $k : I \times R \to R^+$ such that $\sup_{x\in I} \int_0^x k(x,s)ds \leq 1$ and $g : I \times R \to R$ is increasing with respect to the second component and $|g(x,y) - g(x,z)| \leq \varphi(d(y,z))$ (for all $x \in I$ and $y \leq z$) where $\varphi : R^+ \to R^+$ is an increasing and upper semicontinuous function such that $\varphi(x) < x$ for all $x > 0$. Assume there exists y_0 such that $y_0(x) \leq \int_0^x k(x,s)g(s,y_0(s))ds$. Then by Theorem 5 the integral equation (2.4) has a solution in $C(I,R)$.

By the main result we are interested in solving the ordinary differential equation

$$y'(x) = 1 + \log(x+1)\sin((\pi y)/x). \tag{2.5}$$

Set $I = [0,1]$ and $X = C(I,R)$. We define $d(y,z) = \sup\{|y(x) - z(x)| : x \in I\}$ for every $y, z \in X$ and an order relation in X as

$$y \leq z \text{ if and only if } y(x) \leq z(x) \text{ for all } x \in I$$

where $y,z \in X$. It is obvious that $(X,\leq)$ is a partially ordered set and (X,d) is a complete metric space. Define $T : X \to X$ by

$$Ty(x) := \int_0^x g(s,y(s))ds, \tag{2.6}$$

where $g : I \to \mathbb{R}$ is

$$g(x,y) := \begin{cases} 1+\log(x+1)\sin((\pi y)/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Since the mapping g is increasing with respect to the second component, T is non-decreasing. Notice that $|g(x,y)-g(x,z)| \leq \log|y-z|$ for every $x \in I$ and $y \leq z$. Also log is an increasing function, $log(x) < x$ for $x > 0$ and an upper semicontinuous function. In addition, for $y \leq z$

$$\begin{aligned} d(T(y),T(z)) &= \sup_{x\in I}|[Ty](x)-[Tz](x)| \\ &\leq \sup_{x\in I}\int_0^x |g(s,y(s))-g(s,z(s))|ds \\ &\leq \sup_{x\in I}\int_0^x \log(|y(s)-z(s)|)ds \\ &\leq \log(\sup|y-z|) \\ &= \log d(y,z)). \end{aligned}$$

If there exists y_0 such that

$$y_0(x) \leq \int_0^x g(s,y_0(s))ds.$$

According to the Theorem 5, T has a fixed point and thus the integral equation (2.6) has a solution in $C(I,R)$. Now consider the following integral equation

$$y(x) = \int_0^x 1+\log(\xi+1)\sin((\pi y)/\xi)d\xi.$$

Notice that since this integral equation has a solution in $[0,1]$ then equivalently (2.5) has a solution. According to numerical method this solution is $y = x$. Notice that this solution satisfies in (2.5).

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