Miskolc Mathematical Notes
Vol. 23 (2022), No. 2, pp. 929–945

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2022.3620

GRAPH WITH RESPECT TO SUPERFLUOUS ELEMENTS IN A LATTICE

TAPATEE SAHOO, HARIKRISHNAN PANACKAL,
KEDUKODI BABUSHRI SRINIVAS, AND SYAM PRASAD KUNCHAM

Received 28 December, 2020

Abstract. We consider superfluous elements in a bounded lattice with 0 and 1, and introduce various types of graphs associated with these elements. The notions such as superfluous element graph ($S(L)$), join intersection graph ($JI(L)$) in a lattice, and in a distributive lattice, superfluous intersection graph ($SI(L)$) are defined. Dual atoms play an important role to find connections between the lattice-theoretic properties and those of corresponding graph-theoretic properties. Consequently, we derive some important equivalent conditions of graphs involving the cardinality of dual atoms in a lattice. We provide necessary illustrations and investigate properties such as diameter, girth, and cut vertex of these graphs.

2010 *Mathematics Subject Classification:* 05C69; 06B99; 06C05

Keywords: lattice, superfluous element, dual atom

1. Introduction

The study of graphs associated with algebraic structures is important to understand their structural aspects. Among possible graphs obtained from various algebraic structures, zero divisor graphs, annihilator graphs, and intersection graphs are the significant ones [5, 6, 12]. Amjadi [2], defined an essential ideal graph with respect to a commutative ring. The notion of the essential submodule and its dualizing concept namely, the superfluous submodule were studied by the authors (Anderson [3], Fluery [13]). Intersection graphs of rings and module over rings were studied by Chakrabarty [12], and in commutative rings, graphs associated with proper non-small ideals were studied by Atani [4]. However, some authors [15, 16] have studied the properties of graphs with respect to lattices obtained from standard substructures of modules over associative rings. Grzeszczuk and Puczyłowski [15] have introduced essential elements and superfluous elements in a lattice. Nimborkar and Vidya [19] have investigated the properties of the essential element graph of a lattice. Alizade and Toksoy [1] have obtained interesting properties and characterization for cofinitely weak supplement lattices. The dualizing submodules notions such as superfluous submodules and hollow submodules are well known in the case of a module over rings. Anderson [3], Fleury [13], and Bhavanari [7] have extensively studied the spanning dimension in modules associated with the notions of superfluous and hollow submodules. The authors Bhavanari and Kuncham [8] studied isomorphism theorems for a directed hypercube, and in Bhavanari et. al [10], graphs with respect to ideal symmetry and related properties were studied in the case of a generalized rings. Indeed, it is interesting to note that several module analogous developments in lattices were found in Calugareanu [11].

The purpose of this paper is to define various types of graphs associated with superfluous elements in a bounded lattice L with 0 and 1. We introduce the notions of superfluous element graph ($S(L)$), weak supplement element graph ($WS_p(L)$), join intersection graph ($JI(L)$) in a lattice, and if a lattice is distributive, we introduce superfluous intersection graph ($SI(L)$). Dual atoms play an important role to find some important connections between the lattice-theoretic properties and those of corresponding graph-theoretic properties. We prove that a non-superfluous element in L is a dual atom if and only if $S(L)$ is complete. We derive important equivalent conditions of graphs involving the cardinality of dual atoms in a lattice. We obtain an equivalent condition for $SI(L)$ to be disconnected if L has exactly two dual atoms. Apart from several properties of $SI(L)$, we show that for any natural number r, $SI(L)$ can not be complete r-partite. Further, we have established some equivalent conditions that yield interrelations between the lattice and graph theoretical properties. We also investigate properties such as diameter, girth, and cut vertex of these graphs.

2. Preliminaries

A lattice is called distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, for all $a,b,c \in L$, and modular if for $a,b,c \in L$ with $a \leq c$, $a \vee (b \wedge c) = (a \vee b) \wedge c$. For any $x,y \in L$ and $x \leq y$, let $[x,y] = \{a \in L \mid x \leq a \leq y\}$ be the interval between x and y. An element a of a lattice L is proper if $a \neq 1$. If L is a bounded lattice, then $a \in L$ is an atom (resp. dual atom), if there does not exists $b \in L$ such that $0 < b < a$ (resp. $a < b < 1$). The set of all dual atoms in L is denoted by $D(L)$.

We consider a simple finite graph G, whose vertex set is $V(G)$ and the edge set is $E(G)$. We denote ab to represent the edge between $a,b \in V(G)$. We denote by $deg(v)$ the number of vertices associated with v. If a vertex is adjacent to all other vertices in G, then we refer to it as a universal vertex. If there is a path between every pair of vertices of G, then G is connected; otherwise, G is called disconnected.

A graph whose vertices set is empty, is called a null graph and a graph whose edge set is empty is called an empty graph. The length of the shortest path between two vertices a,b in G, is denoted by $d(a,b)$, and $d(a,b) = \infty$, if such a path doesn't exist between a and b. Evidently, $d(a,a) = 0$. The diameter of a graph G, denoted by diam(G), is equal to $\sup\{d(a,b) : a,b \in V(G)\}$. The girth of a graph G, denoted by $gr(G)$, is the length of the shortest cycle in G, provided G contains a cycle; otherwise $gr(G) = \infty$. A vertex x of a connected graph G is a cut vertex of G if

$G-\{x\}$ is disconnected. For $r \in \mathbb{N}$, the r-partite graph is a graph whose vertex set is $V(G) = P_1 \cup P_2 \cdots \cup P_r$, $\bigcap_{i=1}^{r} P_i = \varnothing$, and for any $xy \in E(G)$, if $x \in P_i$, $y \in P_j$ for $i \neq j$. A complete r-partite graph is one in which each vertex is adjacent to every vertex that is not the same subset. The complete bipartite (that is, 2-partite) graph with part size m and n is denoted by $K_{m,n}$, and when $m = 1$, we call it a star graph. G is said to be k-regular ($k \in \mathbb{N}$), if every vertex is of degree k. The eccentricity (denoted by, $e(v)$) of a vertex v in a connected graph G is $max\ d(u,v)$ for all $u \in V(G)$. A vertex with minimum eccentricity is called a center of G.

We consider a bounded lattice $(L,\wedge,\vee,0,1)$, where $0,1$ are the smallest and the greatest element respectively. For standard notions and terminologies in lattice theory, we refer to Grätzer [14], and for concepts in graph theory, we refer to Bhavanari and Kuncham [9].

3. SUPERFLUOUS ELEMENT GRAPH: $S(L)$

Definition 1 ([15])**.** An element a of a lattice L is said to be superfluous, denoted by $a \leq_s L$ if for every $1 \neq b \in L$, $a \vee b \neq 1$.

Lemma 1. *If $a \leq_s L$ and $0 \lneq b \leq a$, then $b \leq_s L$.*

Proof. Let $c \in L$ be with $b \vee c = 1$. Now $1 = b \vee c \leq a \vee c$, we get $a \vee c = 1$. Since $a \leq_s L$, we get $c = 1$. Therefore, $b \leq_s L$. □

Definition 2 ([11])**.** (1) An element $x \in L$ is a supplement of $y \in L$, if x is minimal with respect to the property $x \vee y = 1$.
(2) The join of all atoms of L is called socle of L, denoted by $soc(L)$. For an element, a in L, $soc(a)$ is the socle of the sublattice $[0,a]$.

Definition 3 ([11])**.** An element $x \in L$ is a weak supplement of $y \in L$ if $x \vee y = 1$ and $x \wedge y \leq_s L$.

Definition 4 ([18])**.** A proper element $x \in L$ is a soc-weak-supplement of $y \in L$ if $x \vee y = 1$ and $x \wedge y \leq soc(1)$.

Cigdem Bicer et.al [17] studied complete modular lattices and investigated the properties of generalized supplemented lattices. Nebiyev [16] has studied the properties of weak supplement elements.

Now we define the notion of superfluous element graph of a lattice as follows.

Definition 5. A superfluous element graph of L (denoted by, $S(L)$) is a graph with $V(S(L)) = \{a \in L \mid 0 \neq a \neq 1\}$ as its vertex set, and $E(S(L)) = \{ab \mid a \wedge b \leq_s L\}$ as its edge set.

Example 1. Consider $(D_{30},\leq)$, the set of all positive divisors of 30, with $a \leq b \Leftrightarrow a$ divides b, as given in Figure 1. Then $L = (D_{30},\wedge,\vee)$ is a lattice where meet and join are the greatest common divisor and least common multiple respectively. Here, the least element is denoted by 0 and the greatest element is denoted by 1. The corresponding $S(L)$ is given in Figure 1.

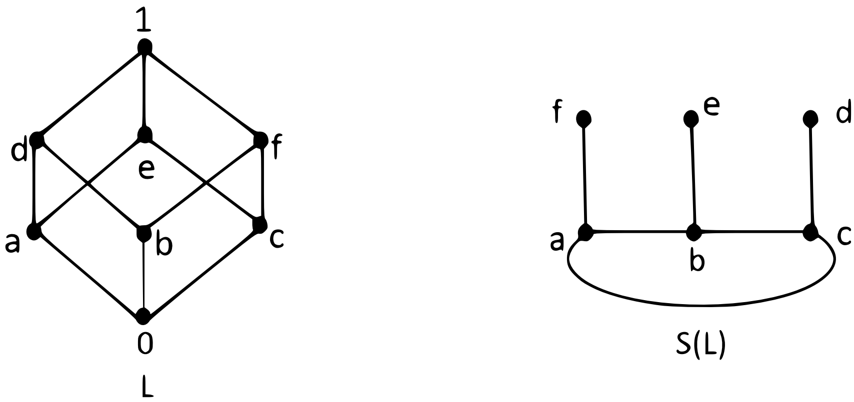

FIGURE 1.

Lemma 2. *If $0 \neq a \in L$ is superfluous, then a is universal in $S(L)$.*

Proof. Suppose $0 \neq a \in L$, and $a \leq_s L$. To show a is a universal vertex, we have to show that for any $b \in V(S(L))$, $ab \in E(S(L))$. Let $b \in V(S(L))$. Then, $b \neq 1$, and since $a \leq_s L$, we have $a \vee b \neq 1$. Since $a \wedge b \leq a \vee b \neq 1$, we get $a \wedge b \in V(S(L))$. Now, for any $c \in L$, if $(a \wedge b) \vee c = 1$, then $1 = (a \wedge b) \vee c \leq a \vee c$. Since $a \leq_s L$, we have $c = 1$. Therefore, $(a \wedge b) \leq_s L$, for all $b \in V(S(L))$, shows that a is a universal vertex. □

Remark 1. The lattices L_1 and L_2 given in Figure 2 are not isomorphic but their superfluous element graph is the same, as shown in Figure 2.

Definition 6. The dual annihilator of an element x in L, denoted by the set, $ann_d(x) = \{y \in L \mid x \vee y = 1\}$.

Proposition 1. *For a proper element x of L, $x \wedge (\bigwedge ann_d(x)) \leq_s L$.*

Proof. Suppose $[x \wedge (\bigwedge ann_d(x))] \vee y = 1$, for some $y \in L$. Clearly,

$$1 = [x \wedge (\bigwedge ann_d(x))] \vee y \leq x \vee y,$$

implies that $x \vee y = 1$. Hence, $y \in ann_d(x)$. Also, since $1 = [x \wedge (\bigwedge ann_d(x))] \vee y \leq y \vee [\bigwedge ann_d(x)]$, it follows that $\bigwedge ann_d(x) \leq y$. This shows that $y \vee (\bigwedge ann_d(x)) = y$. Hence, $1 \leq y \vee (\bigwedge ann_d(x)) = y$, implies $y = 1$. □

Now we define the notions of weak supplement element graph and socle-weak supplement graph of a lattice as follows.

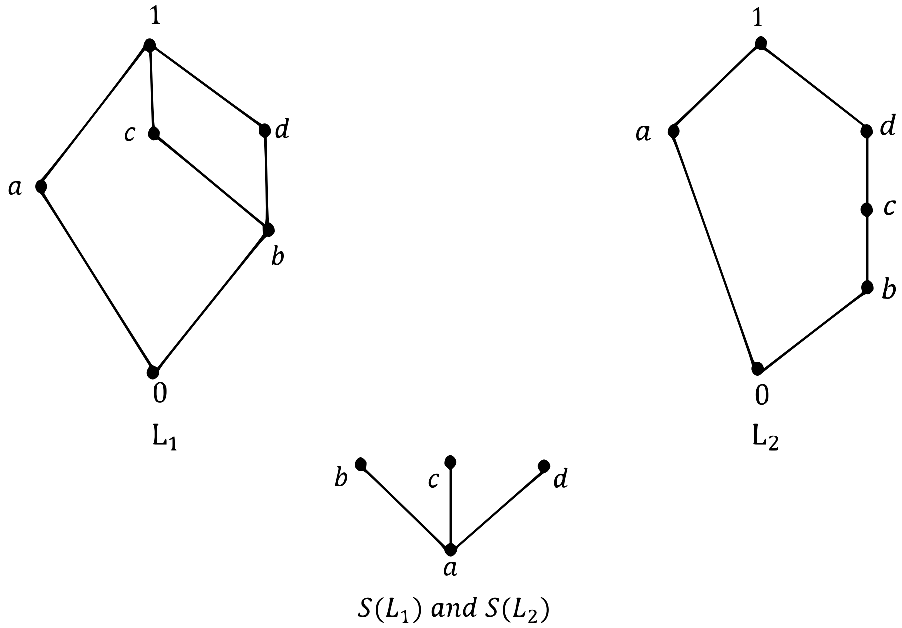

FIGURE 2.

Definition 7.

(1) A weak supplement element graph of L (referred as, $WS_p(L)$), with vertex set $V(WS_p(L)) = \{a \in L \mid 0 \neq a \neq 1\}$, and edge set

$$E(WS_p(L)) = \{ab \mid a \vee b = 1,\ a \wedge b \leq_s L\}.$$

(2) A socle-weak supplement element graph of L (referred as, $SWS_p(L)$), with vertex set $V(SWS_p(L)) = \{a \in L \mid 0 \neq a \neq 1\}$, and edge set

$$E(SWS_p(L)) = \{ab \mid a \vee b = 1,\ a \wedge b \leq soc(1)\}.$$

Example 2. Consider the lattice L given in Figure 3. Then graph $WS_p(L)$ corresponding to L is given in Figure 3.

Proposition 2. *Let $0 \neq a, b \neq 1$ be in L such that a is a weak supplement of b, for all b. Then a is universal in $WS_p(L)$.*

Proof. Suppose $0 \neq a \neq 1 \in L$ and a is a weak supplement of b, for all $0 \neq b \neq 1 \in L$. To show a is a universal vertex, we have to show that for any $b \in V(WS_p(L))$, $ab \in E(WS_p(L))$. Let $b \in V(WS_p(L))$. Then clearly, $0 \neq b \neq 1$, and hence $a \vee b = 1$ and a$\wedge b \leq_s L$. Therefore $ab \in E(WS_p(L))$, for every $b \in V(S(L))$, proves that a is universal. □

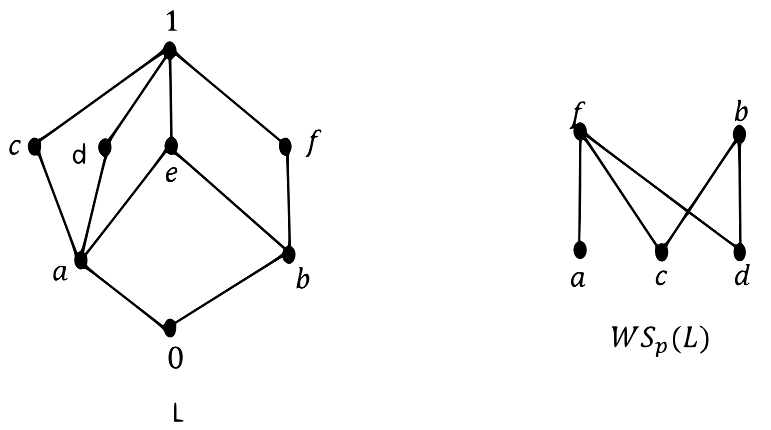

FIGURE 3.

Remark 2. If an element is both an atom as well as a dual atom in L, then it is universal in $SWS_p(L)$. In the lattice L_2 given in Figure 2, a is a universal vertex in $SWS_p(L_2)$.

Lemma 3. *$S(L)$ is a graph with $E(S(L)) = \varnothing$ if and only if L contains a unique non-zero proper element.*

Proof. Clearly, if L contains a unique non-zero proper element, then $E(S(L)) = \varnothing$. Conversely, assume that $E(S(L)) = \varnothing$. Let $a, b \in L$. If L has atoms $c \neq d$ such that $0<c \leq a$ and $0<d \leq b$, then $c \wedge d = 0 \leq_s L$, and so $cd \in E(S(L))$, a contradiction. Thus, $c = d$. Let $1 \neq x \neq c$. If $c \nleq x$, then since c is an atom, x is indifferent from c, showing that $x \wedge c = 0$, and so $xc \in E(S(L))$, a contradiction. Therefore, for every $1 \neq x \in L$, $c \leq x$. Hence, $x \vee c = x \neq 1$, for all $1 \neq x \in L$, implies that $c \leq_s L$. Now $c \leq a$ implies $ac \in E(S(L))$, a contradiction. Therefore, a is the only non-zero element that is proper in L. □

Remark 3. The following example shows that $S(L)$ of a lattice L can be disconnected. Consider the lattice L, given in Figure 3. Then the corresponding $S(L)$ is a disconnected graph shown in Figure 4.

Theorem 1. *Any one of the following conditions implies $S(L)$ is complete.*

(1) *$a \leq_s L$, for all $a \in L$, $0 \neq a \neq 1$.*
(2) *$a \vee b = 1 \Leftrightarrow$ either $a = 1$ or $b = 1$ in L.*
(3) *L has a non-zero superfluous element in which $S(L)$ is k-regular.*

Proof. (1) Let $0 \neq a \in L$ be a superfluous element of L. Then, by Lemma 2, a is a universal vertex in $S(L)$. Hence, a is adjacent to all other vertices in $S(L)$, and this is true for all $0 \neq a \in L$, which shows that $S(L)$ is complete.

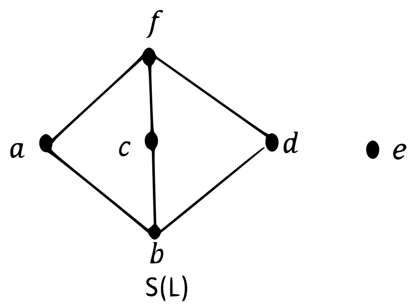

FIGURE 4.

(2) In a contrary way, suppose that $S(L)$ is not complete. Then there exist $a, b \in L$ with $ab \notin E(S(L))$. Then $a \wedge b \not\leq_s L$, implies there exists $1 \neq c \in L$ such that $(a \wedge b) \vee c = 1$. Now by hypothesis, and since $c \neq 1$, we have $a \wedge b = 1$, a contradiction, as a and b are proper. Therefore, $S(L)$ is complete.

(3) Let $S(L)$ be k-regular. Let $0 \neq x \leq_s L$, then by Lemma 2, x is universal in $S(L)$. That is, $xy \in E(S(L))$, for all $y \in V(S(L))$. Therefore, $S(L)$ contains exactly $k+1$ vertices, which implies that $S(L)$ is a complete graph. □

Remark 4. The existence of a non-zero superfluous element in Theorem 1(3) is necessary. However, the condition is not sufficient.

Consider the graph given in Figure 5. Here the non-zero element are $\{a, b\}$ which are non-superfluous in L. But $ab \in E(S(L))$, since $a \wedge b \leq_s L$. Hence, $S(L)$ is complete.

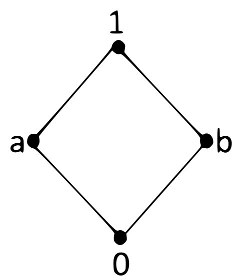

FIGURE 5.

Theorem 2. *$S(L)$ is complete if and only if every $1 \neq a \not\leq_s L$, is a dual atom.*

Proof. Let every $1 \neq a \not\leq_s L$ be a dual atom. Let $0 \neq x, y \neq 1$ be two distinct elements of L. If one of x or y is superfluous, then $x \vee y \neq 1$. Since $x \wedge y \leq x \vee y \neq 1$, we get $x \wedge y \neq 1$. That is, $x \wedge y$ is not a dual atom. Hence, by hypothesis, $x \wedge y \leq_s L$. Therefore, $xy \in E(S(L))$. Suppose neither x nor y is a superfluous element. Then by assumption, $x, y \in D(L)$. Now $(x \wedge y) < x < 1$, shows that $x \wedge y$ is not a dual atom. Therefore, $x \wedge y \leq_s L$ and $xy \in E(S(L))$, which proves $S(L)$ is complete. Conversely, let $1 \neq x \nleq_s L$ and $x < y < 1$. Since $S(L)$ is complete, $xy \in E(S(L))$. Therefore, $x \wedge y \leq_s L$. Now, since $x = x \wedge y$, it follows that $x \leq_s L$, a contradiction. Hence, x is a dual atom. □

Theorem 3. *If $S(L)$ has exactly one universal vertex, then L contains a unique non-zero superfluous element, which is an atom.*

Proof. Let $x \in V(S(L))$ be unique universal. If a and b are two non-zero superfluous elements in L, then by Lemma 2, a and b are universal, a contradiction. Therefore, L contains at most one non-zero superfluous element, say a. Then, $x \wedge a \leq_s L$ and so by Lemma 2, $x \wedge a$ is also universal in $S(L)$. Thus, $x \wedge a = x$. Hence by Lemma 1, we get $x \leq_s L$. Now, by assumption, we get $x = a$. If $0 < y \leq x$, and since $x \leq_s L$, by Lemma 1, $y \leq_s L$. Since x is the only universal vertex, we get $x = y$, and it follows that x is an atom. □

4. JOIN INTERSECTION GRAPH: $JI(L)$

We define the notion of a join intersection graph of a lattice as follows.

Definition 8. The join intersection graph of L, (denoted by $JI(L)$), is the graph with the vertex set $V(JI(L)) = \{a \mid a \in L\}$ and the edge set

$$E(JI(L)) = \{ab \mid a \neq b,\ a \vee b \neq 1\}.$$

Example 3. Consider the Lattice L given in Figure 6. Then graph $JI(L)$ corresponding to L is given in Figure 6.

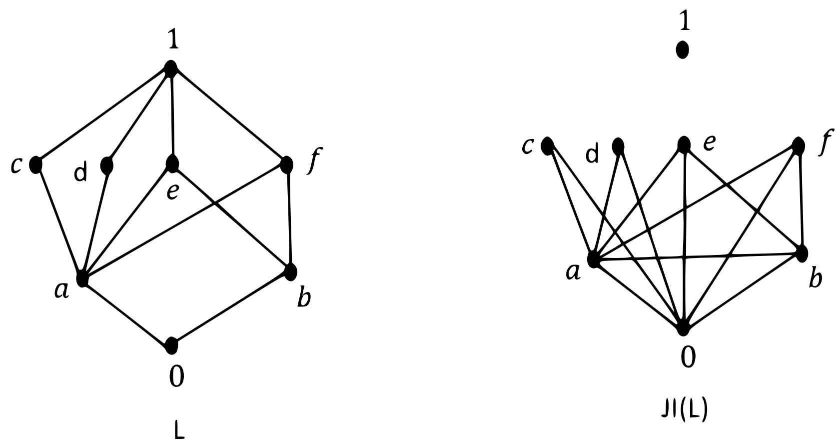

FIGURE 6.

Theorem 4. *The following conditions are equivalent in a join intersection graph of L.*

(1) $a \leq_s L$;
(2) *If* $|V(JI(L))| = n$, *then* $deg(a) = n-2$;
(3) a *is a center in* $JI(L) \setminus \{1\}$.

Proof. (1) ⇒ (2): Suppose $a \leq_s L$ and let $|V(JI(L))| = n$.
Consider $V(JI(L)) = \{x_1, x_2, \cdots, x_n\}$, where $x_1 = 0$, $x_n = 1$ and $x_2 = a$. Since $a \leq_s L$, $a \vee x_i \neq 1$, for all $1 \neq x_i \in V(JI(L))$, shows that $ax_i \in E(JI(L))$. That is, a is adjacent to all in $V(JI(L))$ except 1 and itself. Hence, $deg(a) = n-2$.

(2) ⇒ (3): Let $|V(JI(L))| = n$ and $deg(a) = n-2$. Clearly by definition, $a1 \notin E(JI(L))$ and $aa \notin E(JI(L))$. Since a is adjacent to all other vertices, $d(a,x) = 1$, for all $x \in V(JI(L))$, $x \notin \{0,1\}$. Therefore, a is the center of $JI(L) \setminus \{1\}$.

(3) ⇒ (1): Let a be the center of $JI(L) \setminus \{1\}$. To show $a \leq_s L$, let $1 \neq b \in L$. Then $b \in V(JI(L))$, and $d(a,b)$ is minimum. Hence, $ab \in E(JI(L))$, implies $a \neq b$ and $a \vee b \neq 1$. Since b is arbitrary, we have $a \leq_s L$. □

Theorem 5. *The following conditions are equivalent.*

(1) $a \nleq_s L$, *for all* $0 \neq a \in L$;
(2) $\bigwedge_{d \in D(L)} d = 0$;
(3) 0 *is the only center in* $JI(L) \setminus \{1\}$.

Proof. (1) ⇒ (2): Let $a \nleq_s L$, for all $0 \neq a \in L$. On a contrary, assume that $\bigwedge_{i=1}^{n} a_i = d \neq 0$, where $a_i \in D(L)$, $1 \leq i \leq n$. Then $d \leq a_i$, for all i, implies $d \vee a_i = a_i \neq 1$, for all i, shows that, $d \leq_s L$, a contradiction. Therefore, $\bigwedge_{i=1}^{n} a_i = 0$.

(2) ⇒ (3): Suppose $\bigwedge_{i=1}^{n} a_i = 0$, where $D(L) = \{a_i \mid 1 \leq i \leq n\}$. On the contrary, assume that $a \neq 0$ is a center in $JI(L) \setminus \{1\}$. Then by Theorem 4, $a \leq_s L$, and hence $a \vee a_i \neq 1$, for all i. That is, $aa_i \in E(JI(L))$. Now since a_i's are dual atoms for all i, we have $a \vee a_i \leq a_i$, for all i. Then $a \vee a_i = a_i$, for all i. That is, $a \leq a_i$, for all i, implies that $a \leq \bigwedge_{i=1}^{n} a_i = 0$. Hence, $a = 0$, a contradiction.

(3) ⇒ (1): Suppose 0 is the only center in $JI(L) \setminus \{1\}$. Let $0 \neq b \in L$. Then $b \in V(JI(L))$. Since b is not a center in $JI(L) \setminus \{1\}$, $bx \notin E(JI(L))$, for some $x \in JI(L)$, $0 \neq x \neq 1$. That is, there exists $1 \neq x \in V(JI(L))$ such that $b \vee x = 1$. Hence, $b \nleq_s L$. Since b is arbitrary, we get $b \leq_s L$, for all $0 \neq b \in L$. □

5. SUPERFLUOUS INTERSECTION GRAPH: $SI(L)$

In this section, let L denote a distributive lattice.

Definition 9. An element a of L is said to be non-superfluous, denoted by $a \nleq_s L$, if there exists a proper element b in L such that $a \vee b = 1$.

Lemma 4. *Let a and b be two proper elements of L. If $c \in D(L)$, then $a \wedge b \leq c \Rightarrow a \leq c$ or $b \leq c$.*

Proof. Suppose c is a dual atom and a, b are two proper elements of L such that $a \wedge b \leq c$. Clearly, $c \nless a$, $c \nless b$. If $a \nleq c$ and $b \nleq c$, then there exists $d \in L$ such that $d = a \wedge b \leq c$. This implies that there exists a diamond sublattice in L, a contradiction, as L is distributive. Therefore, $a \leq c$ or $b \leq c$. □

Lemma 5. *Let $D(L) = \{a_i\}_{i \in I}$ and $T \subset I$. Then, $\bigwedge_{i \in T} a_i \nleq_s L$.*

Proof. Assume the contrary, suppose $\bigwedge_{i \in T} a_i \leq_s L$. Then $\bigwedge_{i \in T} a_i \vee x \neq 1$, for all $x \neq 1 \in L$. In particular, $\bigwedge_{i \in T} a_i \vee a_j \neq 1$, for every $a_j \in D(L)$. Since a_j is a dual atom, we have $\bigwedge_{i \in T} a_i \leq a_j$, for each $j \in I \setminus T$. Then by Lemma 4 we have $a_i \leq a_j$, for some $i \in T$, a contradiction as a_i is a dual atom. □

We define the notion of a superfluous intersection graph of a lattice as follows.

Definition 10. The superfluous intersection graph of L (referred as $SI(L)$) is a graph with $V(SI(L)) = \{a \in L \mid 1 \neq a \nleq_s L\}$ as its vertex set and $E(SI(L)) = \{ab \mid a \wedge b \nleq_s L\}$ as its edge set.

Example 4. Consider the Lattice L, the 3-cube given in Figure 1. Then graph $SI(L)$ corresponding to L is given in Figure 7.

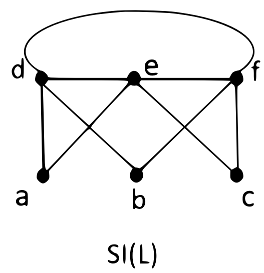

FIGURE 7.

Proposition 3. *$SI(L)$ is a null graph if and only if L has a unique dual atom.*

Proof. Let $SI(L)$ be a null graph. Then, $V(SI(L)) = \varnothing$. That is, there are no proper non-superfluous elements of L. Let L have two dual atoms, say a and b. Then, $0 < a < 1$ and $0 < b < 1$, implies that $a \vee b = 1$, with $b \neq 1$. Hence, $a \nleq_s L$, a contradiction. Therefore, L has a unique dual atom. Conversely, suppose that L has a unique dual atom, say d. To show that $SI(L)$ is a null graph. If there exists an element $1 \neq a \in L$ such that $a \nleq_s L$, then for every $1 \neq b \in L$, $a \vee b = 1$. This shows that $a \vee d = 1$, a contradiction to the hypothesis. Therefore, $V(SI(L)) = \varnothing$. □

Theorem 6. *$SI(L)$ is an empty graph if and only if $D(L) = \{a_1, a_2\}$, where a_1, a_2 are non-superfluous elements.*

Proof. Suppose $SI(L)$ is an empty graph. If $|D(L)| = 1$, then by Proposition 3, $SI(L)$ is a null graph, a contradiction. Let $D(L) = \{a_i\}_{i \in I}$, $|I| \geq 3$. Then by Lemma 5, $a_i \wedge a_j \nleq_s L$ whenever $i \neq j$. Hence $a_i a_j \in E(SI(L))$, a contradiction, since $SI(L)$ is an empty graph. Therefore, $|D(L)| = 2$. Let $D(L) = \{a_1, a_2\}$, with $a_1 \neq a_2$. It remains to show a_1 and a_2 are non-superfluous. Since $a_1 < 1$ and $a_2 < 1$, and $a_1 \vee a_2 = 1$, with $a_2 \neq 1$, it follows that $a_1 \nleq_s L$. Similarly, $a_2 \nleq_s L$. Conversely, let $D(L) = \{a_1, a_2\}$, where $a_1 \nleq_s L$ and $a_2 \nleq_s L$. In order to show that $SI(L)$ is an empty graph, we show that a_1 and a_2 are the only non-superfluous elements of L. Assume the contrary, suppose that $x \in L$ such that $x \nleq_s L$. Then $x < a_1$ or $x < a_2$. Let $x < a_1$. Since $a_1 \wedge a_2$ is maximal such that $a_1 \wedge a_2 < a_1$ and a_2, we get $x \leq a_1 \wedge a_2 < a_2$. Then $x < a_2$. Therefore, $x \leq_s L$, a contradiction to our assumption. Hence, a_1 and a_2 are the only non-superfluous elements of L. This shows that $V(SI(L)) = \{a_1, a_2\}$. Further, since $a_1 \wedge a_2 \leq_s L$, a_1 and a_2 are not adjacent in $SI(L)$. Therefore, $E(SI(L)) = \varnothing$, $SI(L)$ is an empty graph. □

Theorem 7. *The the following conditions are equivalent.*

(1) *$SI(L)$ is a disconnected graph.*
(2) *$|D(L)| = 2$.*
(3) *$SI(L) = SI_1(L) \cup SI_2(L)$, where $SI_1(L)$ and $SI_2(L)$ are two disjoint complete subgraphs of $SI(L)$.*

Proof. (1) ⇒ (2): Suppose that $SI(L)$ is disconnected. Let $SI_1(L)$ and $SI_2(L)$ be two components of $SI(L)$, and a_1, a_2 be two elements of L such that $a_1 \in SI_1(L)$ and $a_2 \in SI_2(L)$. Let x_1 and x_2 be two dual atoms of L such that $a_1 \leq x_1$ and $a_2 \leq x_2$. If $x_1 = x_2$, then $a_1 \leq x_1 < 1$ and $a_2 \leq x_1 < 1$, implies $a_1 \vee a_2 \leq x_1 \neq 1$. Hence $a_1 \leq_s L$ and $a_2 \leq_s L$, shows that $V(SI(L)) = \varnothing$, a contradiction. Therefore, $x_1 \neq x_2$. If $x_1 \wedge x_2 \nleq_s L$, then $x_1 x_2 \in E(SI(L))$, a contradiction. Hence $x_1 \wedge x_2 \leq_s L$. Therefore, $|D(L)| \geq 2$. Let $D(L) = \{a_i\}_{i \in I}$, $|I| \geq 3$. Then by Lemma 5, $a_i \wedge a_j \nleq_s L$ whenever $i \neq j \in I$. This shows that $SI(L)$ is connected, a contradiction. Hence $|D(L)| = 2$.

(2) ⇒ (3): Let $|D(L)| = 2$, say x_1, x_2. Let $SI_1(L) = \{a_j \in L \mid a_j \leq x_i \text{ and } a_j \nleq_s L\}$, for $i = 1, 2$. Let a_1, a_2 be two elements of $SI_1(L)$. If a_1 and a_2 are not adjacent, then $a_1 \wedge a_2 \leq_s L$, which implies $a_1 \wedge a_2 \leq x_1 \wedge x_2 \leq x_2$. Now by Lemma 4, $a_1 \leq x_2$ or $a_2 \leq x_2$. This implies $a_1 \leq_s L$ or $a_2 \leq_s L$, a contradiction. Therefore, $SI_1(L)$ is a complete subgraph of $SI(L)$. In a similar way, we can prove that $SI_2(L)$ is a complete subgraph of $SI(L)$. Next, we show there is no path between $SI_1(L)$ and $SI_2(L)$. On the contrary, suppose a_1 and a_2 are adjacent for some elements $a_1 \in S(L_1)$ and $a_2 \in S(L_2)$. Since $a_1 \wedge a_2 \leq x_1 \wedge x_2$, we have $a_1 \wedge a_2 \leq_s L$, a contradiction. Therefore, none of the

vertices of $SI_1(L)$ and $SI_2(L)$ is adjacent. Hence, $SI(L) = SI_1(L) \cup SI_2(L)$, where $SI_1(L)$ and $SI_2(L)$ are complete subgraphs of $SI(L)$.

(3) $\Rightarrow$ (1): It is clear. □

Remark 5. The following example shows that the graph $SI(L)$ can be totally disconnected, if L is non-distributive having $|D(L)| \geq 3$.

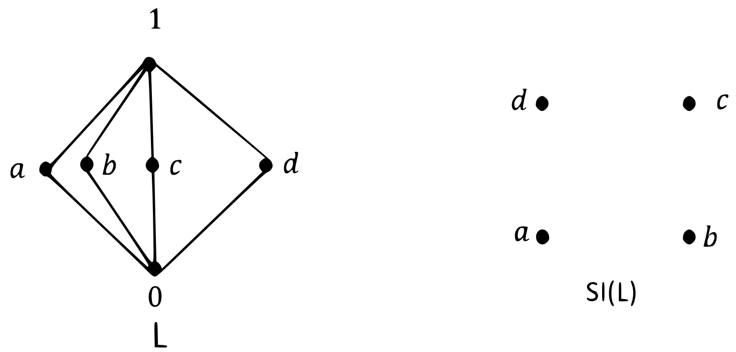

FIGURE 8.

Consider the lattice L given in Figure 8. Here $|D(L)| = 4$, but $SI(L)$ is a disconnected graph.

Theorem 8. *If $SI(L)$ is a connected graph, then $diam(SI(L)) \leq 2$.*

Proof. Let $a,b \in V(SI(L))$. If $ab \in E(SI(L))$, then we are done. Let a and b be two non-adjacent vertices in $SI(L)$. Then, $a \wedge b \leq_s L$. Let $a \leq x_1$ and $b \leq x_2$, for some dual atoms x_1, x_2 of L. If $a \wedge x_2 \nleq_s L$, then $ax_2 \in E(SI(L))$, and since $b \wedge x_2 = b \nleq_s L$, we have $bx_2 \in E(SI(L))$. Therefore, $a - x_2 - b$ is a path in $SI(L)$. Hence, $d(a,b) = 2$. Similarly, if $b \wedge x_1 \nleq_s L$, then $bx_1 \in E(SI(L))$, and since $a \wedge x_1 = a \nleq_s L$, we have $ax_1 \in E(SI(L))$. Therefore, $b - x_1 - a$ is a path in $SI(L)$. Hence $d(a,b) = 2$. Suppose that $a \wedge x_2 \leq_s L$ and $b \wedge x_1 \leq_s L$. Since $SI(L)$ is connected, by Theorem 7, $|D(L)| \geq 3$. Let x_3 be a dual atom in L. Since $a \wedge b \leq_s L$, $(a \wedge b) \vee x_3 \neq 1$. Then $a \wedge b \leq x_3$, which implies $a \leq x_3$ or $b \leq x_3$. Without loss of generality, we assume that $a \leq x_3$. Now we show that $b \wedge x_3 \nleq_s L$. If $b \wedge x_3 \leq_s L$, then $(b \wedge x_3) \vee d \neq 1$, $\forall d \in D(L)$. That is, $b \wedge x_3 \leq d$. Then, by Lemma 4, $b \leq d$ or $x_3 \leq d$. Since x_3 is a dual atom, $x_3 \nleq d$. Therefore, $b \leq d$, for all $d \in D(L)$. Hence, $b \leq_s L$, a contradiction, shows that $b \wedge x_3 \nleq_s L$. Thus, $a - x_3 - b$ is a path in $SI(L)$, and so $d(a,b) = 2$. Hence, $\text{diam}(SI(L)) \leq 2$. □

Theorem 9. *If $SI(L)$ contains a cycle, then $gr(SI(L)) = 3$.*

Proof. Case (i): Suppose $|D(L)| = 2$. Then, by Theorem 7, $SI(L)$ is a union of two disjoint complete subgraphs. Since $SI(L)$ contains a cycle, at least one component should contain a cycle of minimum length 3. Therefore, $gr(SI(L)) = 3$.

Case (ii): If $|D(L)| \geq 3$, say a_1, a_2, a_3, then by Lemma 5, $a_1 - a_2 - a_3 - a_1$ is a cycle in $SI(L)$. So $gr(SI(L)) = 3$. □

Theorem 10. *If $SI(L)$ is connected, then $SI(L)$ has no cut vertex.*

Proof. Let a be a cut vertex of $SI(L)$. Then, $SI(L) \setminus \{a\}$ is not connected. That is; there exists x and y in $SI(L)$ such that a lies on every path from x to y. By Theorem 8, $diam(SI(L)) \leq 2$, and so the shortest path from y to x is of length 2. So $x - a - y$ is a path between x and y. Thus, $x \wedge a \not\leq_s L$ and $a \wedge y \not\leq_s L$. Also since $xy \notin E(SI(L))$, $x \wedge y \leq_s L$. First to show that a is a dual atom of L. If $a \notin D(L)$, then there exists $b \in L$ such that $a<b$. Since $a \not\leq_s L$, we have $b \not\leq_s L$. Since $x \wedge a \leq x \wedge b$ and $x \wedge a \not\leq_s L$, we have $x \wedge b \not\leq_s L$. In a similar way, $y \wedge b \not\leq_s L$. So $x - b - y$ is a path in $SI(L) \setminus \{a\}$, a contradiction. Therefore, a is a dual atom of L. We claim that there exists a dual atom $z \neq a$ of L such that $x \not\leq z$. If $x \leq z$, for each $a \neq z \in D(L)$, then $x \leq \left(\bigwedge_{z \neq a} z \right)$ implies that $x \wedge a \leq \bigwedge_{z \in D(L)} z$. Hence, $x \wedge a \leq_s L$, a contradiction. In a similar way, there exists a dual atom $w \neq a \in L$ such that $y \not\leq w$. Now to show for each $d \in D(L)$, $x \leq d$, or $y \leq d$. Since $x \wedge y \leq_s L$, we have $(x \wedge y) \vee d \neq 1$, for every $d \in D(L)$. Then $(x \wedge y) \leq d$, for every $d \in D(L)$. Thus by Lemma 4, $x \leq d$ or $y \leq d$, for every $d \in D(L)$. Since $SI(L)$ is connected, by Theorem 7, $|D(L)| \geq 3$. Now, let $a \neq t, s \in D(L)$ such that $x \not\leq t$ and $y \not\leq s$. Then $x \leq s$ and $y \leq t$. Hence $x - s - t - y$ is a path in $SI(L) \setminus \{a\}$, a contradiction. Therefore, $SI(L)$ has no cut vertex. □

Theorem 11. *$SI(L)$ can not be complete q-partite for any $q \in \mathbb{N}$.*

Proof. Suppose $SI(L)$ is a complete q-partite graph with q parts $P_1, P_2, \cdots, P_q$.

By Lemma 5, a and b are adjacent for each $a, b \in D(L)$. Therefore, P_i contains at most one dual atom of L. Hence, $|D(L)| \leq q$, by the pigeonhole principle. Next to show $|D(L)| = q$. Suppose, in a contrary, $|D(L)| = \{a_1, a_2, \cdots, a_t\}$, $t<q$. Let $a_i \in P_i$, for $1 \leq i \leq t$. That is, P_{t+1} has no dual atom. Since $|D(L)|$ is finite, by Lemma 5, $\bigwedge_{j \neq i} a_i \not\leq_s L$. Now if $x \in L$ such that $x \neq 1$ and $\bigwedge_{j \neq i} a_i \vee x = 1$, then we have $\left(\bigwedge_{j \neq i} a_i \wedge a_j \right) \vee x < \bigwedge_{j \neq i} a_i \vee x = 1$. Therefore, $\left(\bigwedge_{j \neq i} a_i \wedge a_j \right) \vee x \neq 1$, and so $\bigwedge_{j \neq i} a_i \wedge a_j \leq_s L$. Thus $\bigwedge_{j \neq i} a_i$ and a_j are not adjacent in $SI(L)$. Since $a_i \in P_i$, we have $\bigwedge_{j \neq i} a_i \in P_i$. Let x be a vertex in P_{t+1} and $x \leq a_k$, for some $a_k \in D(L)$. Then $xa_k \in E(SI(L))$. Since $SI(L)$ is a complete q-partite graph and $a_k \in P_k$, so x is adjacent to all elements of P_k. Then x is adjacent to $\bigwedge_{j \neq k} a_j$, a contradiction, as $x \wedge \left(\bigwedge_{j \neq k} a_j \right) \leq a_k \wedge \left(\bigwedge_{j \neq k} a_j \right) \leq_s L$. Hence $|D(L)| = q$. Now, consider $\bigwedge_{i=3}^{q} a_j = d$. By Lemma 5, $d \not\leq_s L$. Since $d \wedge a_1 = \bigwedge_{i \neq 2} a_i \not\leq_s L$, d is adjacent to a_1. Similarly, d

is adjacent to a_2. So $d \notin P_1, P_2$. Since $d \wedge a_i = d \nleq_s L$, for each $3 \leq i \leq q$, we have $da_i \in E(SI(L))$. So $d \notin P_i$, for every $1 \leq i \leq q$, leads to a contradiction. □

Theorem 12. *If L has finitely many dual atoms, then $SI(L)$ has no universal vertex.*

Proof. Let $D(L) = \{a_1, a_2, \cdots, a_t\}$. Suppose, on the contrary, there exists $x \in V(SI(L))$ such that x is a universal vertex. Let $x \leq a_i$. By Lemma 5, $d = \bigwedge\limits_{j \neq i} a_j \nleq_s L$, whereas $x \wedge d \leq a_i \wedge \left(\bigwedge\limits_{j \neq i} a_j \right) = a_i \wedge d \nleq_s L$, a contradiction to x is universal. Thus, no vertex in $SI(L)$ is a universal vertex.

□

Corollary 1. *$SI(L)$ can not be a complete graph.*

Proof. Follows from Theorem 12. □

Theorem 13. *The following statements hold for a lattice L.*

(1) *$SI(L)$ contains a vertex of degree 1 if and only if $|D(L)| = 2$ and $SI(L) = SI_1(L) \cup SI_2(L)$, where $SI_(L)$ and $SI_2(L)$ are two disjoint complete subgraphs of $SI(L)$ and $|V(SI_i(L))| = 2$, for some $i = 1, 2$;*
(2) *$SI(L)$ cannot be a star graph.*

Proof. (1) Let $a \in V(SI(L))$ and $deg(a) = 1$. If $|D(L)| = 1$, by Proposition 3, $SI(L)$ is a null graph, a contradiction. Suppose $|D(L)| \geq 3$. By Lemma 5, for each $x_i \in D(L)$, x_i is adjacent to all other x_j (dual atoms) of L, so $deg(x_i) \geq 2$. Hence, a is not a dual atom. Without loss of generality, assume that $a \leq x_1$. Then, $ax_1 \in E(SI(L))$. As $deg(a) = 1$, we have x_1 as the only vertex adjacent to a in $SI(L)$, and in this case, there is no dual atom $x_i \neq x_1$ of L such that $a \leq x_i$. In particular, $a \wedge x_2 \leq_s L$. Then, $(a \wedge x_2) \vee x_i \neq 1$, for every $x_i \in D(L)$, implies that $a \wedge x_2 \leq x_i$. Now by Lemma 4, $a \leq x_i$ or $x_2 \leq x_i$. Since x_2 is a dual atom, $x_2 \nleq x_i$ for every $i \neq 2$. Therefore, $a \leq x_i$, for all $x_i \in D(L)$, a contradiction. Thus, $|D(L)| = 2$. By Theorem 7, $SI(L) = SI_1(L) \cup SI_2(L)$, where $SI_1(L)$ and $SI_2(L)$ are complete subgraphs of $SI(L)$. Let $a \in SI_i(L)$, for some $i \in \{1, 2\}$. Since $SI_i(L)$ is complete as a subgraph of $SI(L)$ and $deg(a) = 1$, we get $|V(SI_i(L))| = 2$.

The converse is straightforward.

(2) Suppose $SI(L)$ is a star graph. Then $SI(L)$ has an end vertex. So $|D(L)| = 2$, by (1). Now by Theorem 7, $SI(L)$ is disconnected, a contradiction. Therefore, $SI(L)$ cannot be a star graph. □

Definition 11. For every non-negative integer t, the graph G is called t-regular if the degree of each vertex of G is equal to t.

Theorem 14. *The following holds for a lattice L.*

(1) *If a and b are two vertices of $SI(L)$ such that $a \leq b$ in L, then $deg(a) \leq deg(b)$.*
(2) *If $SI(L)$ is a t-regular graph, then $|D(L)| = 2$ and $|V(SI(L))| = 2(t+1)$.*

Proof. (1) Let $a,b \in V(SI(L))$ be such that $a \leq b$ in L. Let $ca \in E(SI(L))$. Since $a \wedge c \nleq_s L$, we have $a \wedge c \leq b \wedge c \nleq_s L$. Thus, $bc \in E(SI(L))$. Therefore, any vertex adjacent to a is also adjacent to b in $SI(L)$. Hence, $deg(a) \leq deg(b)$.

(2) Let $SI(L)$ be t-regular. Clearly, for each $a_i \in D(L)$, $deg(a_i) = t$. By Lemma 5, a_i is adjacent to all other dual atoms of L, hence $|D(L)|$ is finite. If $|D(L)| = 1$, then by Proposition 3, $SI(L)$ is a null graph, a contradiction. Now suppose $|D(L)| \geq 3$. Then by (1) $deg(a_1 \wedge a_2) \leq deg(a_1) = t$.

Case (i): If $deg(a_1 \wedge a_2) < t$, then we get a contradiction to the t-regularity.

Case (ii): We show that $deg(a_1 \wedge a_2) \neq t$. If $d = \bigwedge_{j\neq 2} a_j$, then by Lemma 5, $d \nleq_s L$. Since $(d \wedge a_1) = d \nleq_s L$, we have $d \wedge a_1 \nleq_s L$. Therefore, $a_1 d \in E(SI(L))$. But $d \wedge (a_1 \wedge a_2) \leq_s L$, as $d \wedge (a_1 \wedge a_2) = \bigwedge_i a_i$, and $\bigwedge_i a_i \leq a_j$, for all $a_j \in D(L)$. Therefore, $d \wedge (a_1 \wedge a_2) = \bigwedge_i a_i \leq_s L$, implies that $d(a_1 \wedge a_2) \notin E(SI(L))$. Thus, $deg(a_1 \wedge a_2) \neq t$, a contradiction to the t-regularity. So $|D(L)| = 2$. Hence, by Theorem 7, $SI(L) = SI_1(L) \cup SI_2(L)$, where $SI_1(L)$ and $SI_2(L)$ is the union of two disjoint complete subgraphs. Let $D(L) = \{a_1, a_2\}$ such that $a_1 \in SI_1(L)$ and $a_2 \in SI_2(L)$. Since $deg(a_1) = t$, so $|V(SI_1(L))| = t+1$. Similarly, $|V(SI_2(L))| = t+1$. Hence, $|V(SI(L))| = 2(t+1)$. □

6. Conclusion

In this paper, some properties of superfluous elements in a lattice are considered and investigated corresponding graph-theoretical properties. We have obtained important equivalent conditions of these graphs. These concepts can be extended to study module theoretical analogues spanning dimensional aspects in a lattice, in terms of superfluous and supplement elements.

7. Acknowledgement

The authors express their deep gratitude to the referee(s)/editor(s) for their meticulous reading of the manuscript, and valuable suggestions that have definitely improved the paper. All the authors acknowledge the Manipal Institute of Technology (MIT), Manipal Academy of Higher Education, Manipal, India for their kind encouragement.

References

[1] R. Alizade and S. E. Toksoy, "Cofinitely weak supplemented lattices," *Indian J. Pure and Appl. Math*, vol. 40, no. 5, pp. 337–346, 2009. [Online]. Available: https://hdl.handle.net/11147/2470

[2] J. Amjadi, "The essential ideal graph of a commutative ring," *Asian-European Journal of Mathematics*, vol. 11, no. 04, p. 1850058, 2018, doi: 10.1142/s1793557118500584.

[3] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*. Springer Science & Business Media, 2012, vol. 13, doi: 10.1007/978-1-4612-4418-9.

[4] S. E. Atani, S. Hesari, and M. Khoramdel, “A graph associated to proper non-small ideals of a commutative ring,” *Commentationes Mathematicae Universitatis Carolinae*, vol. 58, no. 1, pp. 1–12, 2017, doi: 10.14712/1213-7243.2015.189.

[5] A. Badawi, “On the annihilator graph of a commutative ring,” *Communications in Algebra*, vol. 42, no. 1, pp. 108–121, 2014, doi: 10.1080/00927872.2012.707262.

[6] I. Beck, “Coloring of commutative rings,” *Journal of algebra*, vol. 116, no. 1, pp. 208–226, 1988, doi: 10.1016/0021-8693(88)90202-5.

[7] S. Bhavanari, “Modules with finite spanning dimension,” *Journal of the Australian Mathematical Society*, vol. 57, no. 2, pp. 170–178, 1994, doi: 10.1017/S1446788700037502.

[8] S. Bhavanari and S. P. Kuncham, “An isomorphism theorem on directed hypercubes of dimension *n*,” *Indian J. pure appl. Math*, vol. 34, no. 10, pp. 1453–1457, 2003.

[9] S. Bhavanari and S. P. Kuncham, *Discrete Mathematics and Graph Theory*. PHI Learning Pvt. Ltd., 2014.

[10] S. Bhavanari, S. P. Kuncham, and B. S. Kedukodi, “Graph of a nearring with respect to an ideal,” *Communications in Algebra*, vol. 38, no. 5, pp. 1957–1967, 2010, doi: 10.1080/00927870903069645.

[11] G. Calugareanu, *Lattice Concepts of Module Theory*. Springer Science & Business Media, 2013, vol. 22, doi: 10.1007/978-94-015-9588-9.

[12] I. Chakrabarty, S. Ghosh, T. Mukherjee, and M. K. Sen, “Intersection graphs of ideals of rings,” *Discrete mathematics*, vol. 309, no. 17, pp. 5381–5392, 2009, doi: 10.1016/j.disc.2008.11.034.

[13] P. Fleury, “A note on dualizing Goldie dimension,” *Canadian Mathematical Bulletin*, vol. 17, no. 4, pp. 511–517, 1974, doi: 10.4153/CMB-1974-090-0.

[14] G. Grätzer, *Lattice theory: foundation*. Springer Science & Business Media, 2011. doi: 10.1007/978-3-0348-0018-1.

[15] P. Grzeszczuk and E. Puczyłowski, “On Goldie and dual Goldie dimensions,” *Journal of Pure and Applied Algebra*, vol. 31, no. 1-3, pp. 47–54, 1984.

[16] C. Nebiyev, “On supplement elements in lattices,” *Miskolc Mathematical Notes*, vol. 20, no. 1, pp. 441–449, 2019, doi: 10.18514/MMN.2019.2844.

[17] C. Nebiyev, Ç. Biçer, and A. Pancar, “Generalized supplemented lattices,” *Miskolc Mathematical Notes*, vol. 19, no. 1, pp. 141–147, 2018, doi: 10.18514/MMN.2018.1974.

[18] S. Nimbhorkar and D. B. Banswal, “Generalizations of supplemented lattices,” *AKCE International Journal of Graphs and Combinatorics*, vol. 16, no. 1, pp. 8–17, 2019, doi: 10.1016/j.akcej.2018.02.005.

[19] S. Nimbhorkar and V. Deshmukh, “The essential element graph of a lattice,” *Asian-European Journal of Mathematics*, vol. 13, no. 01, p. 2050023, 2020, doi: 10.1142/S1793557120500230.

Authors’ addresses

Tapatee Sahoo
Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India
E-mail address: `tapateesahoo96@gmail.com`

Harikrishnan Panackal
Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India
E-mail address: `pk.harikrishnan@manipal.edu`

Kedukodi Babushri Srinivas
Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India
E-mail address: babushrisrinivas.k@manipal.edu

Syam Prasad Kuncham
(**Corresponding author**) Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India
E-mail address: syamprasad.k@manipal.edu
Current address: Manipal Institute of Technology, Manipal Academy of Higher Education, Department of Mathematics, Manipal, India