Constructive Fuzzy Logics

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Abstract

We generalise Kripke's semantics for Intuitionistic logic to Hajek's **BL** and consider the constructive subsystems of $\mathbf{GBL}_{ewf}$ and Intuitionistic Affine logic or **ALi**. The genesis of our semantics is the *Poset Product* construction for GBL-algebras elucidated in a series of papers by Peter Jipsen, Simone Bova, and Franco Montagna. We present natural deduction systems for all of these systems and corresponding deduction theorems for these same. We present the algebraic semantics for each of the logics under consideration, demonstrate their soundness and completeness with respect to these algebraic semantics. We also show how the classical Kripke semantics for Intuitionistic logic can be recast in terms of Poset Products. We then proceed to the main results, showing how a very natural generalisation of the Kripke semantics holds for each of $\mathbf{GBL}_{ewf}$, **ALi** and Hajek's **BL** based on the embedding results of Jipsen and Montagna and the decidability results of Bova and Montagna. We demonstrate soundness and completeness of the logics under our semantics in each case, with the exception of **ALi**, whose robust completeness with respect to the intended models (relational models with frames valued in involutive pocrims) we leave as an open problem for the ambitious reader.

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Finally, I wish to dedicate this thesis to the memory of one Stephanie Gillespie who died tragically in 2019. She was truly a bright light in this world, and the world is much darker without her in it. May her memory gleam in the minds of those who knew her as I did.

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List of abbreviations

ALc	Classical Affine Logic
ALi	Intuitionistic Affine Logic
ALi	Minimal Affine Logic
BA	Boolean Algebras (the variety)
$\{\top,\bot\}_{\mathbf{BA}}$	Boolean Algebra defined over the two-element set
BL	Hajek's Basic Logic
$\vdash_{\mathbf{BL}_H}$	Hilbert-style proof system for **BL**
$\Vdash^{\mathrm{BM}}$	Bova-Montagna semantics for $\mathbf{GBL}_{ewf}$
CL	Classical (Propositional) Logic
C	Urquhart's **C**, alias Gödel-Dummett logic sans contraction
$\mathbf{GBL}_{ewf}$	Intuitionistic Basic Logic, or Basic Logic sans pre-linearity
$\vdash_{\mathbf{GBL}_{ewf}}$	Hilbert-style $\mathbf{GBL}_{ewf}$ provability
GBL	Intuitionistic Basic Logic (Hilbert system)
GBL-CB-QEQ	The quasi-equational theory of Intuitionistic Basic Logic
$\Vdash^{\mathrm{GBM}}$	Generalised Bova-Montagna semantics
$\Vdash^{\mathrm{GGBM}}$	Generalised GBM semantics
GD	Gödel-Dummett logic
$\vdash_{\mathbf{GD}}$	Natural deduction **GD** provability
HA	Heyting Algebras (the variety)
IL	Intuitionistic Logic
$\vdash_{\mathbf{IL}}$	Natural deduction for **IL**
$\vdash_{\mathbf{Int}}$	Hilbert-style proof system for **IL**
$\Vdash^{\mathrm{K}}$	Kripke semantics for Intuitionistic logic
$\Vdash^{\mathrm{LK}}$	Linear Kripke semantics for Gödel-Dummett logic
$\Vdash^{\mathrm{LBM}}$	Linear Bova-Montagna semantics
$\vdash_{\mathbf{BL}}$	Natural deduction **BL** provability
LLc	Classical Lukasiewicz logic
LLi	Intuitionistic Lukasiewicz logic
$\vdash_{\mathbf{LLi}}$	Natural deduction **LLi** provability
LLm	Minimal Lukasiewicz logic
ML	Minimal logic
MV	Many-valued algebra
$[0,1]_{\mathrm{MV}}$	The Many-valued algebra defined over the unit-interval
$\mathbf{P}_\bot$	Bounded Pocrims
$\mathbf{P}_{\mathcal{I}}$	Involutive Pocrims

Chapter 1

Introduction

1.1 Motivation

Fuzzy logics, so-named by Zadeh in his epochal publication [78], are ostensibly logics whose formulae take values in the unit interval $[0, 1]$ rather than the two-valued Boolean algebra $\{\bot, \top\}_{\mathbf{BA}}$ of classical logic. This classification includes logics which historically precede Zadeh's paper, such as Łukasiewicz logic [45], whose infinite-valued presentation is sound and complete for the unit interval endowed with its MV-algebra structure [7].[1] Other widely-studied fuzzy logics in this vein include the Gödel [18] [28] and Product logics [32], each of which have an interval-based semantics and corresponding t-norms [41] [14]. Łukasiewicz, Gödel, and Product logic all share a common core logic known as Hajek's **BL**. This system is the logic of continuous t-norms, and has been thoroughly explored since it's introduction in the late 1990's by Petr Hajek [31], [30].

Work in the aforementioned logics feed the wildly successful research in fuzzy logics-at-large [77], which includes work in fuzzy controllers [8], fuzzy relations [70], fuzzy algebras [64], fuzzy sets [79] and, more generally, fuzzy systems [72]. Work in fuzzy logic *qua* logic (of which Łukasiewicz, Gödel, and Product logic are examples) has resulted in many successful use cases in engineering (e.g. [13] [3]) and continues to inspire deeper work in pure logic.

From the perspectives of proof theory and algebraic logic, fuzzy logics are systems lacking full recourse to structural proof rules, alias *substructural logics*. These particular systems bring principles to bear that are often recalcitrant to

[1] Hereon we will only consider infinite-valued version of this logic, although the finite-valued variant of this system has also been thoroughly studied.

a standard proof-theoretic analysis. Such describes the situation with **BL** [31], the core system mentioned above, as well as $\mathbf{GBL}_{ewf}$ [37], the constructive variant of **BL** obtained by removing the pre-linearity axiom. Yet the smallest subsystem of **BL** and $\mathbf{GBL}_{ewf}$ retaining the structural rule of weakening is Intuitionistic Affine logic (**ALi**) [57] [60], which admits an elegant standard proof-theoretic analysis via sequent calculi. In some sense, **ALi** forms the more restrictive, but not quite minimal, constructive core of the constructive and semi-constructive fuzzy logics $\mathbf{GBL}_{ewf}$ and **BL** (respectively) and their extensions. $\mathbf{GBL}_{ewf}$, **ALi** and **BL** are the focus of this thesis.

What **BL** and $\mathbf{GBL}_{ewf}$ lack in proof theory they compensate for in rich algebraic characterisation. This is appropriate, as the chief motivation behind their development is semantic. But one wonders whether fuzzy systems can be *simplified*, that is, preserve the intuitions that bore them while simultaneously channelling their (logical) resemblance to more proof-theoretically and semantically well-understood logics, such as Intuitionistic (**IL**) and Gödel-Dummett logic (**GD**). These same capture notions of constructivity in mathematics, bear an elegant proof-theoretic presentation via sequents, and offer deep connections to algebra and topology via their semantics, and feature metatheoretic properties generally viewed as desirable on constructive grounds (e.g. disjunction property, analytic proof rules).

Kripke's relational semantics [42], often referred to simply as *Kripke semantics*, bore considerable fruit in the analysis of **IL** and **GD**. Introduced by Saul Kripke as a means for modelling Intuitionistic and Modal logics, the potential for wide application was seized upon by researchers in the 1960's and applied throughout logic. Kripke semantics thus became a mainstay of pure and applied logical research.

What is so distinctive about Kripke semantics, and why should we care? Kripke semantics offers us a paradigm in which the truth of a formula is evaluated distributively (or 'relatively', as one might expect), across nodes in a partial order or *frame*. Each of these nodes, paired with formulae, are in turn valued in an algebra, with the evaluation of a formula distributed across these nodes according to a compositional schematic. The result for the Intuitionistic case is a poset decorated with formulas and their computed values, which *grows* monotonically with the ordering of the poset – with formulas *eventually* being valued true.

This suggests a view of semantics quite different from that of classical model theory or algebraic semantics. Whereas the classical model-theoretic and algebraic view of semantics is *static*, the perspective offered by relational semantics is in some sense *growing* or *evolving*, with the truths or theorems of the logic eventually collecting in the limit, analogous to an idealised process of learning wherein the future is the sum of an agent's knowledge in the past, present and still-to-come.

Kripke's semantics for Intuitionistic logic was indeed intended by Kripke to model Brouwer's ideal mathematician, proceeding through time, collecting lemmata, definitions and other resources as his knowledge grows. This way of thinking is of course appropriate to Intuitionistic logic, which may (or may *not*) have been Brouwer's intended foundational system for carrying out constructive mathematics (he appears to have never used it towards this end).

Contrast this situation with algebraic models in which denotations of formulae are fixed to elements of the intended algebra. While this latter is a safe, standard way of proceeding in semantics, often the relational framework yields insights into a logical system that are not easily detected through non-relational approaches, bringing intimate connections to decidability [75] [29], proof theory [51] [25], topology [59], [66] and the rich model theory that relational methods have borne out in the sixty years since Kripke's original publication.[2]

With this in mind, we have sought to extend Kripke's semantics to fuzzy logics that have a basis in Intuitionistic or Gödel-Dummett logic, thus arriving at our title: *Constructive Fuzzy Logics*. We believe generalising Kripke's relational semantics offers a fresh way of viewing many-valued logical systems. We hope our approach anticipates logical insights imperceptible without the lenses of relational thinking, such as is already appreciated in the work done in modal, Intuitionistic and super-Intuitionistic logics. Analogising the present state of fuzzy logic research with Intuitionistic logic pre-Kripke: the dominant semantic viewpoint as recent as the early 1960's was algebraic (via Heyting and Brouwer Algebras), whereas now the dominant semantic approach for all modal systems – including Intuitionistic and Gödel-Dummett logic – is surely Krip-

[2]To illustrate this point, consider the notion of *Kripke completeness*. Kripke semantics brings an implicit classification of systems: those which can be uniquely captured by a particular class of frames, alias the *Kripke complete* systems, and those which cannot. This suggests a refined notion of completeness only available to relational methods, and induces a hierarchy of interrelated logics, and thus a means of separating the 'good' from the 'bad' in some sense.

kean, or relational. Similarly, sixty years hence fuzzy logics may have a rival point of reference that is not principally algebraic, but relational. We view that eventuality favourably and aspire towards it.[3]

1.2 State of the Art

We briefly recall the state of the art with respect to relational semantics for substructural logics of the sort we study in this thesis.

As suggested above, the default mode of investigation for fuzzy logics is clearly algebraic. Part of this stems from the fact that substructural logics in general can be viewed dually both as proof-theoretic objects resulting from restrictions on the structural rules in the sequent presentation of a logic,[4] or as algebraic objects, typically residuated lattices.[5]

But the real cause for the preference of algebraic methods in fuzzy logics is their semantic origin. These systems were forged with the intent to model continuous behavior and vagueness in natural language[6] and this seems to have forced the unfortunate imbalance of an over-developed semantics and an under-developed proof-theory. The cost of this imbalance is low-esteem in the minds of some luminaries of mainstream mathematical logic. In 1969 W.V. Quine could write-off fuzzy logic [62] as "logic only analogically speaking; it is uninterpreted theory, abstract algebra" asking effectively 'whither the proof theory for fuzzy logic?' – and the question remains as pertinent (and open) as ever.[7]

Quine's perspective is of course informed by an ancient dichotomy between proofs and models – particularly the duality between proofs and algebras, which are what the classical models really are. In some sense the dichotomy is irrelevant, as proofs give way to the natural semantics via term-model constructions. Similarly, we can view sequents and rules as inequations and algebraic laws of a kind. On the other hand there *is* a nontrivial difference between the alge-

[3] Petr Hajek and Melvin Fitting seem to have somewhat anticipated this eventual outcome anyway: Hajek [31] by way of his preliminary exploration of fuzzy modal logics, particularly with an S5 base, whereas Melvin Fitting [22] can be seen as inaugurating the area formally. We will briefly discuss their approaches in the next section.

[4] See [60] for an introduction to substructural logic with a slightly more proof-theoretic cast.

[5] See [26] for an introduction to substructural logics via residuated lattices.

[6] See e.g. the classic Sorites paradox [36] or any of its variants; see [76] for some contemporary coverage of philosophical approaches.

[7] Despite notable exceptions, mostly coming from the Austrian school of Proof Theory, e.g. [1], [10], [12], [24].

braic and proof-theoretic perspectives, as proof-theoretic analogues of algebraic constructions seem to be rare in the literature,[8] and features of decidability found via algebraisation have to do with e.g. computing matrices, or computing the complexity of an algebraic construction (see e.g. [4]). These latter in no way resemble the sort of tried and true methods used in standard theoretical computer science or mathematical logic comprehensible to logicians. This is particularly true for contemporary proof-theory inspired work: The very little work that has been done in fuzzy logic's proof theory does not at all resemble the algorithmic character of proof search in standard analytic calculi that can boast purely syntactic methods.[9] This, of course, does not serve fuzzy logic well and would seem to vindicate Quine's views in our time.

Unfortunately, Quine's views force us to accept the one true logic is classical logic on account of the delicate balance between syntax and semantics. Everything else is deviation. But what he (and Quineans generally) fail to appreciate is that the symmetry we find classically is unique to the classical setting, and therefore not a natural desideratum in constructive, substructural or fuzzy systems. So the tight-connection in the classical setting between proofs and semantics is not necessarily something we should seek to elevate as an ideal at the expense of modelling diverse phenomena, which may require we occasionally deviate from received notions such as how we understand proofs and proof systems or the kinds of models we contemplate. We attempt to illustrate this below, briefly.

Consider classical Linear Logic. The sequent-style presentation is quite natural, bearing all the symmetry of classical logic with the resource control of a substructural system. These are clear aesthetic benefits to the mind of a proof theorist content with symbol-shunting. Yet the semantics of Linear Logic requires considerable technical sophistication to follow. Moreover, all of the semantic models presently on offer (coherence spaces, quantales, phase spaces . . .) appear to be logically 'abhorrent' in a sense: they are incredibly idiosyncratic,

[8] Although see [11] as the hypersequent hierarchy presented there goes some way towards connecting these two worlds, associating each rule expressible via hypersequents an algebraic condition or, what comes to the same thing, Hilbert-style axiom.

[9] Methods such as linear or integer programming are used (see e.g. [43]), and proof theorists find this distasteful, and further evidence of the inability to dispense with semantics in fuzzy logics. On the other hand, for particular cases such as Gödel and Product logic, much is known, the Austrian school has pursued very successful analyses of these cases. But the vast majority of fuzzy logics – Łukasiewicz, MTL, **BL** and **GBL** and extensions thereof – these remain wide open, and what proof theory there is relies heavily on semantic insights, which is of course not palatable for standard proof theorists who prefer to rely on purely syntactical methods of analysis (cut-elimination and so forth).

strangely unique to this setting, and none of the semantics on offer clearly elucidate what the system is *intended* to model in a manner that matches the elegant proof theory (in comparison with classical or modal logic). The sheer diversity of models on offer is enough to repulse the Quinean classical aesthete; but then the Quinean is not resource-conscious anyway (except, oddly, where existential quantifiers are involved).

Hence the intuitions that guide judgment in constructing an elegant proof theory for a logical system (or attract those of a similar mind) do not inevitably yield a transparent or aesthetically palatable algebraic semantics. The dichotomy that arises here between elegant proof theory and rich catalogue of arguably opaque but interesting models is a natural feature of this setting.

On the other hand, take Łukasiewicz logic. Here we find a clear semantic motivation inextricably tied to the most widely adopted semantics for the logic, Chang's **MV**-algebras [6]. And yet there does not appear to be a natural proof-theoretic representation of this system with the desired features of cut-elimination and decidability. Only recently with the advent and development of hypersequents (see e.g. [24]) has this area begun to see improvement, and that with considerable technical challenge: even the 'obvious' alternative of exploiting semantics to obtain a useful proof theory via e.g. analytic tableaux has not been easy [55], and remains in nascent stages of development. Methods of importing semantics into proof theory via labels (e.g [25] or [51]) while promising, have only been tested on cases considerably easier than fuzzy logics, and not the most fundamental, general substructural cases where logics lack distribution of lattice connectives and potentially multiple implications (e.g. Lambek calculi). Hence the path from semantics to proof theory is not easy to trod, and the present state of the art abounds with open problems in the way of extracting meaningful proof theory from semantics.

But thankfully there is more to semantics than algebra. Relational methods of the sort generalising Kripke's insights from Intuitionistic logic along some appropriate dimension are indeed present in the literature, and many of the major figures from the variant schools of substructural logic have dabbled in relational methods for their choice systems. These might lend some hope to the situation with fuzzy logic. We recount some approaches below.

Perhaps the first to generalise Kripke's semantics for substructural logics, specifically of the relevant variety, is Routley and Meyer [65]. They provide a ternary relation semantics, as opposed to Kripke's binary accessibility relation semantics, in which Kripke's worlds or states become resources that can be combined under a ternary relation. Curiously, Alasdair Urquhart happened on his own variant of this same semantics at this time.[10] [11] Meanwhile, Urquhart pursued a relational semantics generalising not the *relation*, but the underlying *algebra*: Urquhart extended Kripke's relational semantics to an operational semantics over additive ordered monoids for a logic he names **C** [74], which is Gödel-Dummett's logic without contraction. In this same article, he extends the semantics to ordered Abelian groups for Łukasiewicz logic.

Urquhart's work on these logics and their semantics have served as a springboard for further development along three principle lines of research in fuzzy logic: via the proof theory (see e.g. [12]) and semantics of Łukasiewicz logic, and in reading his algebraic-relational semantics as a Routley-Meyer ternary semantics [46] and in Urquhart's own terms compared with Dana Scott's very similar approach to Łukasiewicz logic [52].

But the influence of Urquhart's semantics ultimately spread beyond that of the systems considered in [74] to other logics, particularly *substructural* systems, via his work in relevance logic. His work during this period bears a broadly similar character - see e.g. [73] for a semantics via semilattices that expressly generalises Kripke's semantics via semilattices, prefiguring Urquhart's later work with ordered monoids in [74] - even as he examined superficially very different systems (weakening-free vs. contraction-free). One can trace a clear line of development from Urquhart's work from the 1970's to the eventual diffusion of this approach into the 1980's and beyond: Ono and Komori's work on Affine logic in the 1980's [58] and eventually Pym and O'Hearn's work in the

[10] Which unfortunately lives in the annals of unpublished manuscripts: see discussion in Dunn's [20].

[11] A further development in this direction: Jon Michael Dunn [19] and Katalin Bimbo's [2] Gaggle theory continues the trend of generalising insights in Kripke (and Routley and Meyer). They exploit accessibility relations obtaining semantics for operators and connectives of arbitrary arity over algebras that are not always Boolean. This work has seen increasing traction in nonclassical and substructural logic, and has steadfast adherents. Some high-level features of this approach: Gaggle theory seems well-adapted to generalising the representation theorems of Tarski and Johnsson [40] to nonclassical settings, as well as uncovering new operators and fresh connections between these same. This may not lend any more *logical* insight – that is, non-algebraic insight – than is already conferred by approaches from the theory of residuated lattices which bring very strong techniques, e.g. general decomposition theorems for classes of residated lattices, some of which even come with decision procedures in towe. In this way, perhaps Gaggle theory is a very strong stream of work in algebraic logic largely independent of the residuated lattices community, with an organising principle based in relations and distributive vs. non-distributive tonoids. It would seem the connections between decidability, semantics and proof theory in this setting remain under development.

1990's with Bunched Implication [53] and later [49]. This list is by no means exhaustive, either: all of the above share a semantics via partially ordered algebras, typically residuated lattice-ordered monoids, with an eye to modelling weakening-free or contraction-free systems. Urquhart's early work, then, must be credited as the origin of this now standard approach in devising 'Kripke-style' or 'relational' semantics for substructural logic (viewed as a separate stream from Routley-Meyer, of course, which resulted in the Gaggle theory of Dunn and his collaborators).

But in some sense these 'relational' approaches are just *more* algebra. And the point for our present work is to somehow *reduce* where possible the reliance on algebra and algebraic techniques, steering as close to Kripke's original semantics as possible. We consider this to be a step towards making fuzzy logics more amenable to general logical and model-theoretic methods already known in e.g. modal logic, as opposed to methods living purely in the domain of algebra.

There's nothing particularly Kripkean or relational about partially-ordered monoids, in the sense that we have simply taken the Kripkean worlds and order relation of the frame and identified them with the algebra itself. This has the effect of collapsing the insights into constructive mathematics Kripke seeks to model into a static model. And this is indeed the approach in classic papers of substructural logic, e.g. [53] or [58]. While this is certainly technically admissible, it is not necessarily what cognoscenti from modal logic or constructive mathematics *mean* when they say that a logic has a natural Kripke or relational semantics: this is to say, one expects the worlds and their partial order to be *distinct from* the algebra. Indeed, in the classic case for Intuitionistic logic, one has a partially ordered set of worlds, and the formulae together with the worlds are mapped into the Boolean algebra. Now, one can certainly map into Heyting algebras, or indeed take the *whole poset* of worlds *as* a Heyting algebra whose partial order is given by that of the algebra, but these are considered degenerate or trivial cases: the key insight with the Kripkean story is that *locally* the formulae live in a classical setting but *globally* the set up is constructive. Taking the relational set-up as a partially ordered monoid misses this insight, and arguably cheapens the value of the relational framework where available.

As an illustration, take Intuitionistic logic. When one works with Kripke structures in that setting, even though one *could* work with worlds mapped into Heyting algebras, one *doesn't*, as to do so one may as well work directly in the Heyting algebras themselves.

Finally we note the parallel semantic approach initiated by Hajek [31] and Fitting [22] (and their later collaborators). These authors deal with *modal many-valued logics* – fuzzy and many-valued logics equipped with modal operators. These give way to a natural relational semantics generalising the classical modal notions. Fitting outlines two major lines of approach that have since dominated the literature on this subject: one strategy seeks to generalise Kripke semantics by way of the *values* (which become many-valued) where the other generalises Kripke semantics by way of the *accessibility relation* (ditto). This research has been carried on by their students and successors. Of particular interest to the present author, given this thesis, is the work of Cintula and Noguerra [15] and with Rogger [16], in which they look at modal logics over an Intuitionistic Affine base. Besides the logics and modalities themselves, the key difference with our approach is that their semantics is based on *neighbourhood structures*, which are already a generalisation of Kripke semantics, and so already several steps removed in a sense from our goal, which is to obtain a relational semantics that *directly* generalises Kripke's own and keeps the simplicity of Kripke's original presentation, e.g. does not 'fuzzify' the accessibility relation. More, our approach is rooted in the poset product construction, whereas Cintula et. al cannot use the same representation results to obtain completeness.

Our point of departure in this thesis is in the work of Bova and Montagna [4], as well as the work of Jipsen and Montagna [37] , [38] and [39]. This work is deeply algebraic. The authors are primarily concerned with embedding algebras into products of the same, occasionally obtaining full representation theorems. But behind the heavy duty algebra, specifically in [37] and [4], we find behavior that resembles a Kripke semantics in a distant form. It takes quite a bit of cosmetic re-packaging, as well as definitions that suit the task and simplification of the structures so that the relational frames appear as such, but one can extract from the poset product quasi-representations for $\mathbf{GBL}_{ewf}$ and $\mathbf{BL}$ relational semantics generalising Kripke's own to a substructural setting and simultaneously *specialise* to Kripke's original structures when one removes the tensor operation from the language (returning to Intuitionistic logic or one of it's Kripke-complete extensions). This is a desirable end, especially in comparison with the quasi-relational-algebraic tradition in substructural logic outlined above: we map worlds and formulae into involutive algebras of the appropriate sort, analogous to Kripke's original definition, in contrast with the semantics given by Urquhart, Ono, etc. each of whom generalise Kripke's work along some dimension but somehow collapse the distinction between the ordering and the

$$
\begin{array}{ccccc}
\mathbf{ALc} & \longrightarrow & \mathbf{LLc} & \longrightarrow & \mathbf{CL} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{ALi} & \longrightarrow & \mathbf{GBL} & \longrightarrow & \mathbf{IL} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{ALm} & \longrightarrow & \mathbf{LLm} & \longrightarrow & \mathbf{ML}
\end{array}
$$

Figure 1.1: Relationships between systems considered herein, arrows indicating containment: **IL** is contained in **CL**, etc.

algebra maintained in Kripke's original.

1.3 Aim

Our principle aim, as the abstract suggests, is to convince the reader that the relational semantics of Saul Kripke's lifts to fuzzy and substructural logics in a way that simultaneously easily specialises to Intuitionistic logic and takes the intuitions of Kripke's classic semantics into a new, many-valued setting. To buoy this motivation, we prove adequacy of these generalised Kripke semantics for the logics $\mathbf{GBL}_{ewf}$ and **BL**, and **ALi** (this last in albeit in a weak form).

1.4 Contributions

The principle contributions of this thesis are:

- Development of generalised Kripke semantics for fuzzy logics out of the algebraic semantics for these same.
- Soundness and completeness results **GBL**, **BL** for BM-structures and LBM-structures respectively, but only soundness for **ALi** under the GBM-structures.
- As more modest contributions, the natural deduction systems presented in this thesis are, to our knowledge, new (with exception of that presented in our publication [63], which provides the basis for the present work). We relate these to the Hilbert systems of the logics in question via appropriate translation theorems for $\mathbf{GBL}_{ewf}$ (and by extension, **BL**).
- Similarly, we believe our proof of completeness via poset products for Intuitionistic logic represents the first time that proof has appeared in print (following a remark in Jipsen and Montagna in [39]). It seems to be known to cognoscenti. Yet we do not imagine Kripke or his immediate successors thought of his semantics in terms of poset products, so we have given a detailed proof of the theorem using this insight. We find this approach buoys the central claim of the thesis, that our semantics generalises Kripke's original in a manner respecting the insights of [42] appropriate for a 'fuzzy' setting.

Chapter 2

Logics, Algebras and Kripke Structures

In this chapter, we endeavour to present all the required proof-theoretic, algebraic and semantic prerequisites necessary for understanding the work that follows. Since this thesis considers a range of logics defined over an Affine and Intuitionistic base, we present these logics in their classic setting with algebraic semantics (and in the Intuitionistic case, the Kripke semantics). We give the classic deduction theorem for Intuitionistic logic, present a substructural variant appropriate for our considerations, showing that the consequence relation of $\mathbf{GBL}_{ewf}$ and **LLi** coincide. We modify this latter to obtain the same result for **BL**. We prove soundness and completeness for both the algebraic semantics of **IL** and the Kripke semantics of **IL**, this via poset products and canonical extensions. We prove the algebraic soundness and completeness for **ALi**, and provide a sufficiently detailed presentation including various lemmata useful for later. We hope this helps the reader to more easily compare and contrast the later results of this thesis with what is known in Intuitionistic logic.

2.1 Logics and Proof systems

2.1.1 Intuitionistic Logic

Hilbert System for Intuitionistic Logic

Below we present both a Hilbert and natural deduction system in sequent-style for Intuitionistic Logic.

Propositional formulae are built from a countable set of propositional variables $Var = \{p, q, r, ...\}$ and the falsity constant $\perp$ using three binary connectives: $\rightarrow$

(implication), $\wedge$ (conjunction, logical 'and'), $\vee$ (disjunction, 'or'). So here we haven't included negation as an official logical operation; rather, one can *define* $\neg\phi$ ('not ϕ') as $(\phi \to \bot)$. Intuitionistic propositional logic, via a Hilbert-style proof theory, is given as **IL**. This is defined by the following axioms:

1. $\phi \to \phi$
2. $\phi \to (\psi \to \phi)$
3. $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$
4. $(\phi \wedge \psi) \to \phi$
5. $\phi \to (\psi \to (\phi \wedge \psi))$
6. $\phi \to (\phi \vee \psi)$
7. $(\phi \to \chi) \to ((\psi \to \chi) \to ((\phi \vee \psi) \to \chi))$
8. $\bot \to \phi$

and one inference rule:

$$\phi, \phi \to \psi \vdash \psi$$

we call *modus ponens* or MP.

When we wish to emphasize that a sequent $\Gamma \vdash \phi$ is provable in the Hilbert-style system given above, we will write $\Gamma \vdash_{\mathbf{Int}} \phi$, to emphasize the proof system. The natural deduction system will be given below. Finally, we note in passing that we take the contexts to be finite sets of formulas, so that contraction is built in.

Natural Deduction system for Intuitionistic logic

Once again, Intuitionistic logic formulas are inductively defined from atomic formulas (we use $p, q, \ldots$ for propositional variables), including $\bot$, and the binary connectives $\psi \wedge \chi$, $\psi \vee \chi$ and $\psi \to \chi$. We will refer to this language as $\mathcal{L}$. When a formula ϕ is provable in the logic, we will designate this as: $\Gamma \vdash_{\mathbf{IL}} \phi$, although we may write $\Gamma \vdash \phi$ when the context is clear. Finally, we note contraction is built in, as we take the contexts in each sequent to be sets, with the comma separating formulae and contexts naturally taken as conjunction.

Deduction Theorem for Intuitionistic Logic

Herein we present the Deduction Theorem for Intuitionistic Logic, a standard result often presented in classical logic textbooks, where with suitable modification we arrive at the statement below. This not only serves to relate the natural

$$\frac{}{\phi \vdash \phi}\ \text{Ax}$$

$$\frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi}\ \text{W}$$

$$\frac{\Gamma, \phi, \psi, \Delta \vdash \chi}{\Gamma, \psi, \phi, \Delta \vdash \chi}\ \text{Ex}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi} \to \text{I}$$

$$\frac{\Gamma \vdash \phi \qquad \Delta \vdash \phi \to \psi}{\Gamma, \Delta \vdash \psi} \to \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} \wedge \text{I}$$

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2} \vee \text{I}\ (i \in \{1,2\})$$

$$\frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_i} \wedge \text{E}\ (i \in \{1,2\})$$

$$\frac{\Gamma \vdash \phi \vee \psi \qquad \Gamma, \phi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi} \vee \text{E}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}\ \text{ExFalso}$$

$$\frac{\Gamma, \psi, \psi \vdash \phi}{\Gamma, \psi \vdash \phi}\ \text{Contract}$$

Figure 2.1: Intuitionistic Logic

deduction and axiom systems introduced above, providing a sort of origin story for our natural deduction calculus, but also proves essential in the standard completeness argument for Intuitionistic logic with respect to the Kripke semantics. Additionally, we offer the result and the proof of the same as a point of comparison for the reader with respect to the next section's discussion on the translation theorem between Hilbert systems and Natural deduction systems for substructural logics considered in this thesis.

Theorem 2.1.1 (Deduction Theorem for **Int**)**.** *(Folklore[1]) Let* Γ *be an arbitrary finite set of formulas. Then* $\Gamma,\ \phi \vdash_{\mathbf{Int}} \psi$ *if and only if* $\Gamma \vdash_{\mathbf{Int}} \phi \to \psi$*.*

Proof. Right-to-left follows from a single application of Modus Ponens: If from Γ we can prove $\phi \to \psi$, then this together with ϕ gives us ψ. The left-to-right direction proceeds by induction on the derivation of ψ from Γ and $\{\phi\}$ in **Int**, and is quite easy so we give it in full. The possible cases for ψ are as follows: If ψ is an axiom of **Int** or $\psi \in \Gamma$, then clearly $\Gamma \vdash_{\mathbf{Int}} \psi$ and we obtain $\Gamma \vdash_{\mathbf{Int}} \phi \to \psi$ by applying MP to $\Gamma \vdash_{\mathbf{Int}} \psi$ and $\Gamma \vdash_{\mathbf{Int}} \psi \to (\phi \to \psi)$ (this latter an instance

[1] For the *Classical deduction theorem*, we have two standard sources: (i) Godel's PhD thesis (see [21] and [80] for a precis of results and thesis itself) in which he proved the completeness theorem for first-order logic and so would have proven the deduction theorem as a necessary preliminary, and (ii) Jacques Herbrand's thesis [34] from the same year (published 1930, written 1929) in which Herbrand proves both the theorem that bears his name and the deduction theorem. The version of the theorem for Intuitionistic propositional logic is a modification of that for Classical propositional logic, but it is unclear to the present author who, if anyone, could claim priority for this Intuitionistic deduction theorem; so we attribute it to folklore.

of weakening, or the second axiom). If $\psi = \phi$, then we have $\Gamma \vdash_{\mathbf{Int}} \phi \to \phi$, which is of course provable as $\phi \to \phi$ is the first axiom. If ψ is obtained from previously derived χ and $\chi \to \psi$ by MP. Then by induction $\Gamma \vdash_{\mathbf{Int}} \phi \to \chi$ and $\Gamma \vdash_{\mathbf{Int}} \phi \to (\chi \to \psi)$. Then we proceed as follows:

1. $\phi \to \chi$ (by IH)
2. $\phi \to (\chi \to \psi)$ (by IH)
3. $(\phi \to (\chi \to \psi)) \to ((\phi \to \chi) \to (\phi \to \psi))$ an instance of the third axiom, or contraction.
4. $(\phi \to \chi) \to (\phi \to \psi)$ MP from (2) and (3)
5. $\phi \to \psi$ by MP from (1) and (4).

□

Note 1 (Resource Considerations)**.** *A comment on the above theorem: it is not* resource-conscious*, in two different senses or 'levels' syntactically: locally, as an available axiom schema (the third axiom in the system given just earlier), so that one can re-apply a formula as many times as one likes consistent with the use of the contraction axiom (3), but there is also a global insensitivity to resources in that we can reiterate axioms or theorems in a Hilbert-system as often as we like. The goal of substructural logics, and the proof systems employed in such research, is to maintain a tighter control of formulas (alias 'resources'), where one lacks access to the full repository of classically structural rules, and so a suitable modification of the Intuitionistic Deduction Theorem above is required for such calculi.*

Gödel-Dummett logic

One might ask, naturally, whether there are any logics strictly in between classical and Intuitionistic logic. Gödel-Dummett logic, or **GD**, is one such logic. Viewed as a Hilbert system, it can be obtained from Intuitionistic logic by adding the axiom of *pre-linearity*:

- $(\phi \to \psi) \vee (\psi \to \phi)$

The natural deduction system for the logic, again with weakening internalised into the sequent system, is given in figure 2.2. A deduction theorem for this logic can also be given for the logic, and is straightforward modification of the system for Intuitionistic logic given above. This logic (**GD**) can be seen as the tensorless base of **BL**, when one removes contraction from the system.

$$\frac{}{\phi \vdash \phi}\ \text{Ax}$$

$$\frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi}\ \text{W}$$

$$\frac{\Gamma, \phi, \psi, \Delta \vdash \chi}{\Gamma, \psi, \phi, \Delta \vdash \chi}\ \text{Ex}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi}\ \to \text{I}$$

$$\frac{\Gamma \vdash \phi \qquad \Delta \vdash \phi \to \psi}{\Gamma, \Delta \vdash \psi}\ \to \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}\ \wedge \text{I}$$

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2}\ \vee \text{I } (i \in \{1,2\})$$

$$\frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_i}\ \wedge \text{E } (i \in \{1,2\})$$

$$\frac{\Gamma \vdash \phi \vee \psi \qquad \Gamma, \phi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi}\ \vee \text{E}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}\ \bot \text{E}$$

$$\frac{}{\Gamma \vdash (\phi \to \psi) \vee (\psi \to \phi)}\ \text{Pre-Lin}$$

$$\frac{\Gamma, \psi, \psi \vdash \phi}{\Gamma, \psi \vdash \phi}\ \text{Contract}$$

Figure 2.2: Gödel-Dummett logic

Pre-linearity affords an intimate connection with a wide class of fuzzy logics, all viewed as logics of the continuum, or logics of the unit-interval (as they are sometimes called). Many of these have an assumption involving linear structures in order to be axiomatizable, or to be of any use in reflecting logical intuitions about the continuum. **BL** is seen as the sort of core of many fuzzy logics (and was intended to be viewed as such by Hajek [31]), as it is literally the logic of left and right t-norms. So in a sense, **GD** can be seen as the kernel of the core of fuzzy logics captured by **BL**.

Proposition 2.1.2. *The following hold in any calculus with rules Ax,→ I, → E, ∧ I, ∧ E (and so for* $\mathbf{IL}, \mathbf{GD}$*):*

1. $\Gamma, \psi \vdash \chi$ *iff* $\Gamma \vdash \psi \to \chi$.
2. $\Gamma, \phi, \psi \vdash \chi$ *iff* $\Gamma, \phi \wedge \psi \vdash \chi$.

Proof. For the first item, if we have a proof in which $\Gamma, \psi \vdash \chi$, then we apply →I and we have $\Gamma \vdash \psi \to \chi$. For the other direction: supposing $\Gamma \vdash \psi \to \chi$, then since we always have $\psi \vdash \psi$ (as Ax), by →E we have $\Gamma, \psi \vdash \chi$.

For the second item, supposewe we are given $\Gamma, \phi, \psi \vdash \chi$. Then as $\phi \wedge \psi \vdash \psi$ is always provable:

$$\dfrac{\dfrac{\dfrac{}{\phi \land \psi \vdash \phi \land \psi}}{\phi \land \psi \vdash \psi}{\scriptstyle \land \text{ E}} \qquad \dfrac{\dfrac{\overline{\cdots}}{\Gamma, \phi, \psi \vdash \chi}}{\Gamma, \phi \vdash \psi \to \chi}{\scriptstyle \to\text{I}}}{\Gamma, \phi \land \psi, \phi \vdash \chi}{\scriptstyle \to \text{ E}}$$

and once more:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \land \psi \vdash \phi \land \psi}}{\phi \land \psi \vdash \phi}{\scriptstyle \land \text{ E}} \qquad \dfrac{\dfrac{\overline{\cdots}}{\Gamma, \phi \land \psi, \phi \vdash \chi}}{\Gamma, \phi \land \psi \vdash \phi \to \chi}{\scriptstyle \to\text{I}}}{\Gamma, \phi \land \psi, \phi \land \psi \vdash \chi}{\scriptstyle \to \text{ E}}}{\Gamma, \phi \land \psi \vdash \chi}{\scriptstyle \text{Contract}}$$

For the other direction, assume we have a derivation of $\Gamma, \phi \land \psi \vdash \chi$. Then:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \vdash \phi}{\scriptstyle \text{Ax}}}{\psi, \phi \vdash \phi}{\scriptstyle \text{W}} \qquad \dfrac{\dfrac{}{\psi \vdash \psi}{\scriptstyle \text{Ax}}}{\psi, \phi \vdash \psi}{\scriptstyle \text{W}}}{\phi, \psi, \phi, \psi \vdash \phi \land \psi}{\scriptstyle \land \text{ I}} \qquad \dfrac{\dfrac{\overline{\cdots}}{\Gamma, \phi \land \psi \vdash \chi}}{\Gamma \vdash (\phi \land \psi) \to \chi}{\scriptstyle \to\text{I}}}{\Gamma, \phi, \psi, \phi, \psi \vdash \chi}{\scriptstyle \to \text{ E}}}{\Gamma, \phi, \phi, \psi, \psi \vdash \chi}{\scriptstyle \text{Ex}}}{\Gamma, \phi, \phi, \psi \vdash \chi}{\scriptstyle \text{Contract}}}{\Gamma, \phi, \psi \vdash \chi}{\scriptstyle \text{Contract}}$$

□

2.1.2 Extensions of Intuitionistic Affine logic

We have just discussed the proof theory of Intuitionistic logic, and now we consider three extensions of Intuitionistic logic over an Affine base: namely, Intuitionistic Affine, Intuitionistic Łukasiewicz logic or **LLi**, and Hajek's Basic logic or **BL**. These coincide with the logics for which we have devised a relational semantics in this thesis.

We consider below the proof theory of **LLi** in some detail, as this captures simultaneously Intuitionistic Affine logic (as a subsystem) as well as Intuitionistic fragment of Łukasiewicz logic, to which adding the axiom of prelinearity yields **BL**. We present all of these systems in their Hilbert-style and natural deduction renderings for the sake of clarity, but also to serve our later exposition and results (in particular our completeness proofs).

The formulas of Intuitionistic Łukasiewicz (**LLi**) and Intuitionistic Affine logic (**ALi**), are inductively defined from atomic formulas, including $\bot$, and the binary connectives $\psi \land \chi$, $\psi \lor \chi$, $\psi \otimes \chi$ and $\psi \to \chi$. We will refer to this language as $\mathcal{L}_\otimes$, since it extends the language $\mathcal{L}$ of intuitionistic logic with a second form of conjunction $\psi \otimes \chi$.

Figure 2.3 gives a natural deduction system for intuitionistic (propositional) Łukasiewicz logic **LLi** and, by extension, **ALi** (with the omission of divisibility

$$\frac{}{\phi \vdash \phi}\text{Ax} \qquad \frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi}\text{W} \qquad \frac{\Gamma, \phi, \psi, \Delta \vdash \chi}{\Gamma, \psi, \phi, \Delta \vdash \chi}\text{Ex}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi}\to \text{I} \qquad \frac{\Gamma \vdash \phi \to \psi \qquad \Delta \vdash \phi}{\Gamma, \Delta \vdash \psi}\to \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \otimes \psi}\otimes \text{I} \qquad \frac{\Gamma, \phi, \psi \vdash \chi \qquad \Delta \vdash \phi \otimes \psi}{\Gamma, \Delta \vdash \chi}\otimes \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}\wedge \text{I} \qquad \frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_i}\wedge \text{E}$$

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2}(\vee \text{I}) \qquad \frac{\Gamma \vdash \phi \vee \psi \qquad \Delta, \phi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\vee \text{E}$$

$$\frac{\Gamma, \phi, \phi \to \psi \vdash \chi}{\Gamma, \psi, \psi \to \phi \vdash \chi}\text{DIV} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}\bot \text{E}$$

Figure 2.3: Intuitionistic Łukasiewicz logic **LLi**

rule, or axioms A7-A8 below.[2] [3]). When we write a sequent $\Gamma \vdash \phi$ we are always assuming Γ to be a finite sequence of formulas. Note that we have the structural rules of weakening and exchange, but not contraction. Hence, the number of occurrences of a formula in Γ matters, and one could think of the contexts Γ as multisets. In particular, the rule $\to$ I removes one occurrence of ϕ from the context Γ, ϕ, concluding $\phi \to \psi$ from the smaller context Γ. This makes **LLi** a form of Affine logic.

LLi indeed has a deduction theorem, in fact a *resource sensitive* deduction theorem. The connective $\to$ internalises the consequence relation $\vdash$, and $\otimes$ internalises the comma in the sequent:

Proposition 2.1.3. *The following hold in any calculus with rules Ax,$\to$ I, $\to$ E, $\otimes$ I, $\otimes$ E (and so for* **ALi**, **LLi** *and* **BL***:*

1. $\Gamma, \psi \vdash \chi$ *iff* $\Gamma \vdash \psi \to \chi$.

[2]As a **GBL**-algebra is a residuated lattice which satisfies the *divisibility property*, if $x \leq y$ then $y \otimes (y \to x) = x$. This is equivalent to require that the residuated lattice satisfies the equation:

$$x \otimes (x \to y) = y \otimes (y \to x)$$

. Note that since $y \to x \leq y \to x$, it is always the case that $y \otimes (y \to x) \leq x$ (this is the counit of the adjunction defining residuation). The name "divisibility" property makes sense if one interprets $x \otimes y$ as multiplication $x \times y$, and $y \to x$ as division $\frac{x}{y}$. This is saying that if $0 \leq x \leq y \leq 1$ then $y \times \frac{x}{y} = x$. Note that if $y = 0$ then $x = 0$ as well.

[3]From a proof-theoretic perspective (and we credit Paulo Oliva for this insight), divisibility can be read as a kind of "strong" cut rule, in the sense that from ϕ and $\phi \to \psi$ one can get ψ (as in the standard cut rule, modus ponens, or $\to$-elimination), but one also gets $\psi \to \phi$ as an additional conclusion. Hence one loses the subformula property of Affine logic in adopting this rule, but it is perfectly acceptable as long as cut (or modus pones) is acceptable (which it may not be, and much ink has been spilled trying to eliminate cuts where found.

2. $\Gamma, \phi, \psi \vdash \chi$ *iff* $\Gamma, \phi \otimes \psi \vdash \chi$.

Proof. For the first item, if we have a proof in which $\Gamma, \psi \vdash \chi$, then we apply →I and we have $\Gamma \vdash \psi \to \chi$. For the other direction: supposing $\Gamma \vdash \psi \to \chi$, then since we always have $\psi \vdash \psi$ (as Ax), by →E we have $\Gamma, \psi \vdash \chi$.

For the second item, if we are given $\Gamma, \phi, \psi \vdash \chi$, as we always have $\phi \otimes \psi \vdash \phi \otimes \psi$, we apply ⊗E and get $\Gamma, \phi \otimes \psi \vdash \chi$. For the other direction, assume we have a derivation of $\Gamma, \phi \otimes \psi \vdash \chi$. Then:

$$\dfrac{\dfrac{\dfrac{}{\phi \vdash \phi}\text{Ax} \quad \dfrac{}{\psi \vdash \psi}\text{Ax}}{\phi, \psi \vdash \phi \otimes \psi}\otimes\text{E} \quad \dfrac{\dfrac{\overline{\quad \cdots \quad}}{\Gamma, \phi \otimes \psi \vdash \chi}}{\Gamma \vdash (\phi \otimes \psi) \to \chi}\to\text{I}}{\Gamma, \phi, \psi \vdash \chi}\to\text{E}$$

□

Since **LLi** has the exchange rule, we can extend this to $\phi_1, \ldots, \phi_n \vdash \psi$ iff $\phi_{\pi_1} \otimes \ldots \otimes \phi_{\pi_n} \vdash \chi$ iff $\vdash (\phi_{\pi_1} \otimes \ldots \otimes \phi_{\pi_n}) \to \chi$ iff $\vdash \phi_{\pi_1} \to \ldots \to \phi_{\pi_n} \to \chi$, where π is any permutation of $\{1, \ldots, n\}$.

The natural deduction system **LLi** is inspired by, and, as we will see in Proposition 2.1.4, corresponds to, the Hilbert-style system $\mathbf{GBL}_{ewf}$ of ([4]). We shall hereon unscrupulously blur the distinction between $\mathbf{GBL}_{ewf}$ as algebra and the corresponding Hilbert system, referring to the Hilbert system as $\mathbf{GBL}_{ewf}$. The context will always make it clear whether we are referring to the algebra or the Hilbert-system.

(A1) $\phi \to \phi$

(A2) $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$

(A3) $(\phi \otimes \psi) \to (\psi \otimes \phi)$

(A4) $(\phi \otimes \psi) \to \psi$

(A5) $(\phi \to (\psi \to \chi)) \to ((\phi \otimes \psi) \to \chi))$

(A6) $((\phi \otimes \psi) \to \chi)) \to (\phi \to (\psi \to \chi))$

(A7) $(\phi \otimes (\phi \to \psi)) \to (\phi \wedge \psi)$

(A8) $(\phi \wedge \psi) \to (\phi \otimes (\phi \to \psi))$

(A9) $(\phi \wedge \psi) \to (\psi \wedge \phi)$

(A10) $\phi \to (\phi \vee \psi)$

(A11) $\psi \to (\phi \vee \psi)$

(A12) $((\phi \to \psi) \wedge (\chi \to \psi)) \to ((\phi \vee \chi) \to \psi)$

(A13) $\bot \to \phi$

(R1) $\phi, \phi \to \psi \vdash_{\mathbf{GBL}_{ewf}} \psi$

When we wish to stress the precise system in which a sequent $\Gamma \vdash \phi$ is derivable we use the system as a subscript of the provability sign, e.g. $\Gamma \vdash_{\mathbf{LLi}} \phi$.

Proposition 2.1.4. *The natural deduction system* **LLi** *(Figure 4.1) has the same derivable formulas as the Hilbert-style system* $\mathbf{GBL}_{ewf}$ *of [4], and hence corresponds to it in the following sense*[4]

$$\psi_1, \dots, \psi_n \vdash_{\mathbf{LLi}} \phi \quad \textit{iff} \quad \vdash_{\mathbf{GBL}_{ewf}} \psi_1 \to \dots \to \psi_n \to \phi$$

Proof. Left-to-right: The result follows by induction on the structure of the natural deduction proof once we show each instance of a natural deduction rule translates to a theorem of $\mathbf{GBL}_{ewf}$. We translate each sequent $\phi_1, \dots, \phi_n \vdash \chi$ to the formula $[\phi_1, \dots, \phi_n \vdash \chi] = \phi_1 \to \dots \to \phi_n \to \chi$, and each rule to $[\Theta_1] \to \dots \to [\Theta_m] \to [\Psi]$. For example, (Ax) translates to $\phi \to \phi$ (A1) and ($\to$I) to $(\chi_1 \to \dots \chi_n \to \phi \to \psi) \to \chi_1 \to \dots \chi_n \to \phi \to \psi$, which is also a form of (A1).

The analysis of many of the other rules is simplified if we introduce a *provability* relation between formulae:

$$\phi \le \psi \text{ iff } \vdash_{\mathbf{GBL}_{ewf}} \phi \to \psi$$

Forgetting use of (R1), (A1) says that this relation is reflexive, and (A2) that it is transitive. One can see then that this provability relation generates a pre-order on equivalence classes of provable formulae. (A2) also tells us that $\to$ is antitone in its first argument, and (A3), (A5) and (A6) now imply that the relation is monotone in its last argument, and that $\phi \to \psi \to \chi$ is equivalent to $\phi \otimes \psi \to \chi$, $\psi \otimes \phi \to \chi$ and $\psi \to \phi \to \chi$.

One now derives the remaining rules easily. To illustrate, consider the case of (DIV). Using the deduction theorem for **LLi** from Proposition 2.1.4 we can assume that Γ is a single formula θ. We have to derive

$$(\theta \to \phi \to (\phi \to \psi) \to \chi) \to (\theta \to \psi \to (\psi \to \phi) \to \chi)$$

in $\mathbf{GBL}_{ewf}$. We use the provability ordering on formulae introduced above. $\theta \to \phi \to (\phi \to \psi) \to \chi)$ is equivalent to $\theta \to (\phi \otimes (\phi \to \psi)) \to \xi$, and by (A8) plus monotonicity, implies (in fact is equivalent to) $\theta \to (\phi \wedge \psi) \to \xi$. The commutativity of $\wedge$ (A9) allows us to swap ϕ and ψ, i.e. $\theta \to (\psi \wedge \phi) \to \xi$. We now reverse

[4]Note that we use $\psi \otimes \chi$ where many in the algebraic literature use $\phi \cdot \psi$, or as is the case with Bova and Montagna in [4], $\psi \odot \chi$.

the steps, using (A7), we obtain $\theta \to (\psi \otimes (\psi \to \phi)) \to \xi$. Finally, uncurrying (A6), gives us $\theta \to \psi \to (\psi \to \phi) \to \xi$ as desired.

Right-to-left: This follows by induction on the $\mathbf{GBL}_{ewf}$ derivation of $\psi_1 \to \ldots \to \psi_n \to \phi$ once we show that each of the axioms of $\mathbf{GBL}_{ewf}$ are theorems of **LLi**. The only non-trivial case is (A8), which states $(\phi \wedge \psi) \to \phi \otimes (\phi \to \psi)$, and requires an application of DIV. We can show that $\phi \wedge \psi \vdash \phi \otimes (\phi \to \psi)$ is derivable in **LLi** as follows. First we show from ($\wedge$ E) that $\phi \to \phi \wedge \psi \vdash \phi \to \psi$:

$$
\dfrac{
\dfrac{
\dfrac{}{\phi \to \phi \wedge \psi \vdash \phi \to \phi \wedge \psi}\,\text{Ax}
\quad
\dfrac{}{\phi \vdash \phi}\,\text{Ax}
}{\phi \to \phi \wedge \psi, \phi \vdash \phi \wedge \psi}\,\to \text{E}
\quad
\dfrac{
\dfrac{
\dfrac{}{\phi \wedge \psi \vdash \phi \wedge \psi}\,\text{Ax}
}{\phi \wedge \psi \vdash \psi}\,\wedge \text{E}
}{\vdash \phi \wedge \psi \to \psi}\,\to \text{I}
}{
\dfrac{\phi \to \phi \wedge \psi, \phi \vdash \psi}{\phi \to \phi \wedge \psi \vdash \phi \to \psi}\,\to \text{I}
}\,\to \text{E}
$$

Then employ (DIV):

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\phi \wedge \psi \vdash \phi \wedge \psi}\,\text{Ax}
}{\phi \wedge \psi \vdash \phi}\,\wedge \text{E}
}{\vdash \phi \wedge \psi \to \phi}\,\to \text{I}
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\phi \vdash \phi}\,\text{Ax}
\quad
\dfrac{\cdots}{\phi \to \phi \wedge \psi \vdash \phi \to \psi}\,\to \text{I}
}{\phi, \phi \to \phi \wedge \psi \vdash \phi \otimes (\phi \to \psi)}\,\otimes \text{I}
}{\phi \wedge \psi, \phi \wedge \psi \to \phi \vdash \phi \otimes (\phi \to \psi)}\,\text{DIV}
}{\phi \wedge \psi \vdash (\phi \wedge \psi \to \phi) \to \phi \otimes (\phi \to \psi)}\,\to \text{I}
}{\phi \wedge \psi \vdash \phi \otimes (\phi \to \psi)}\,\to \text{E}
$$

□

Proposition 2.1.4 does not suggest that the two separate notions of logical consequence in **LLi** and $\mathbf{GBL}_{ewf}$ coincide, but rather that these two proof systems have the same set of derivable formulas. We have noted already that **LLi** satisfies the deduction theorem in the form $\Gamma, \phi \vdash_{\mathbf{LLi}} \psi$ iff $\Gamma \vdash_{\mathbf{LLi}} \phi \to \psi$. But in the standard notion of consequence for Hilbert-style systems, $\Gamma \vdash \phi$ is interpreted as "ϕ is derivable from axioms (A1)-(A13) + Γ using modus ponens." In this case formulas in Γ can be used multiple times to derive ϕ. This is reflected in the failure of the deduction theorem for $\mathbf{GBL}_{ewf}$ ($\phi \otimes \phi$ is a consequence of ϕ, but we do not have $\vdash_{\mathbf{GBL}_{ewf}} \phi \to \phi \otimes \phi$). Multiple uses of a hypothesis is not allowed in $\Gamma \vdash_{\mathbf{LLi}} \phi$ as **LLi** lacks contraction.

Remark 2.1.5. *We note that the above proposition also holds for Intuitionistic Affine logic (as this is a subsystem of Intuitionistic Łukasiewicz logic) between* **ALi***'s axiomatization and the natural deduction system we provide herein. The proof is that given above, sans consideration of divisibility. Similarly, the above theorem holds for Hajek's* **BL***, where one would only need consider the axiom of pre-linearity. We add the statement of this result below.*

The Hilbert-style system for **BL**, which we formally call $\mathbf{BL}_H$, is Hajek's own ([31]). One simply adds to the axioms given on page 16 that of pre-linearity and the appropriate version of modus ponens:

$$\frac{}{\phi \vdash \phi}\,\text{Ax} \qquad \frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi}\,\text{W} \qquad \frac{\Gamma, \phi, \psi, \Delta \vdash \chi}{\Gamma, \psi, \phi, \Delta \vdash \chi}\,\text{Ex}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi}\,\to \text{I} \qquad \frac{\Gamma \vdash \phi \to \psi \qquad \Delta \vdash \phi}{\Gamma, \Delta \vdash \psi}\,\to \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \otimes \psi}\,\otimes \text{I} \qquad \frac{\Gamma, \phi, \psi \vdash \chi \qquad \Delta \vdash \phi \otimes \psi}{\Gamma, \Delta \vdash \chi}\,\otimes \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}\,\wedge \text{I} \qquad \frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_i}\,\wedge \text{E}$$

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2}\,\vee \text{I)} \qquad \frac{\Gamma \vdash \phi \vee \psi \qquad \Delta, \phi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\,\vee \text{E}$$

$$\frac{\Gamma, \phi, \phi \to \psi \vdash \chi}{\Gamma, \psi, \psi \to \phi \vdash \chi}\,\text{DIV} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}\,\bot \text{E}$$

$$\frac{}{\Gamma \vdash (\phi \to \psi) \vee (\psi \to \phi)}\,\text{Prelin}$$

Figure 2.4: Basic logic **BL**

(A14) $(\phi \to \psi) \vee (\psi \to \phi)$

(R1') $\phi, \phi \to \psi \vdash_{\mathbf{BL}_H} \psi$

Figure 5.1 gives a natural deduction system for Hajek's **BL**. When we wish to stress the precise system in which a sequent $\Gamma \vdash \phi$ is derivable we use the system as a subscript of the provability sign, e.g. $\Gamma \vdash_{\mathbf{BL}} \phi$. Note that Γ here is a multi-set (not a set), as this logic does not have contraction, the number of occurrences of a formula in the context Γ matters. Weakening of contexts is allowed, which is captured in the axiom $\Gamma, \phi \vdash \phi$. This makes **BL** an extension of **LLi** and **ALi**, but also an extension of Intuitionistic logic with pre-linearity sans contraction, alias *Gödel-Dummett* logic. When we wish to stress the precise system in which a sequent $\Gamma \vdash \phi$ is derivable we use the system as a subscript of the provability sign, e.g. $\Gamma \vdash_{\mathbf{BL}_H} \phi$ or $\Gamma \vdash_{\mathbf{BL}} \phi$. Analogising the previous proposition, we have:

Proposition 2.1.6. *The natural deduction system* **BL** *(Figure 5.1) has the same derivable formulas as the Hilbert-style system* $\mathbf{BL}_H$ *of [31], and hence corresponds to it in the following sense.*

$$\psi_1, \ldots, \psi_n \vdash_{\mathbf{BL}} \phi \quad \textit{iff} \quad \vdash_{\mathbf{BL}_H} \psi_1 \to \ldots \to \psi_n \to \phi$$

Proof. As in the proof of the previous proposition. For the left to right, again the result follows by induction on the structure of the natural deduction proof once one shows each instance of a natural deduction rule translates to a theorem of $\mathbf{BL}_H$, and recall the provability ordering from above relevant to the present system:

$$\phi \leq \psi \text{ iff } \vdash_{\mathbf{BL}_H} \phi \to \psi$$

Since $\mathbf{BL}_H$ results from $\mathbf{GBL}_{ewf}$ by adding (A14), all the other cases are as before, except (A14) which simply says that the provability relation is linearly ordered. For the right to left direction of the 'iff', this follows by induction on the $\mathbf{BL}_H$ derivation of $\psi_1 \to \ldots \to \psi_n \to \phi$ once we show that each of the axioms of $\mathbf{BL}_H$ are theorems of **BL**. The only case to consider is (A14), and this is an axiom in the natural deduction calculus **BL**, and so always provable in that calculus. □

2.2 Algebraic Semantics

Before we introduce our Kripke semantics for the principle logics of this thesis, we recall the algebraic semantics for a wide class of substructural logics and Intuitionistic logic. In total we will consider 9 different classes of algebras: Bounded lattices, Heyting algebras, lattice-ordered monoids, residuated lattices, Bounded pocrims, Involutive pocrims, **GBL**-algebras, $\mathbf{GBL}_{ewf}$- algebras, **MV**-algebras.

2.2.1 Heyting Algebras

Definition 2.2.1 (Bounded Lattice)**.** *A* Bounded Lattice L *is a algebraic structure* $(L, \vee, \wedge, \bot, \top)$ *such that* $(L, \vee, \wedge)$ *is a lattice,* $\bot$ *(the lattice's bottom) is the identity element for the join operation* $\vee$*, and* $\top$ *(the lattice's top) is the identity element for the meet operation* $\wedge$*.*

- $a \vee \bot = a,$
- $a \wedge \top = a.$

Definition 2.2.2 (Heyting Algebra)**.** *A Heyting Algebra* **HA** *is a Bounded Lattice* $(H, \vee, \wedge, \bot, \top)$ *such that for all* a, b *in* H *there is a greatest element* x *of* H *such that*

$$a \wedge x \leq b$$

This greatest x *is the relative pseudocomplement of* a *for* b*, which we denote by the residual:*

$$a \to b$$

Definition 2.2.3 (Pseudocomplementation)**.** *The* Pseudocomplement *of* a*, denoted* $\neg a$*, is obtained by taking any element* $a \in H$ *and setting* $\neg a = (a \to \bot)$*.*

Definition 2.2.4 (Boolean Algebra)**.** *A* Boolean Algebra *can be defined simply by setting* $a = \neg\neg a = ((a \to \bot) \to \bot)$.

Lemmata for Heyting Algebras

Herein we record elementary lemmata of the theory of Heyting Algebras useful in the sequel.

Lemma 2.2.5. *In any Heyting Algebra* **HA***:* $\forall x, y, z, u : (x \leq y) \wedge (u \leq (y \to z) \Rightarrow x \wedge u \leq z$.

Proof. Fix x, y, z, u and assume $(x \leq y)$ and $u \leq y \to z$. Then by residuation, $u \wedge y \leq z$ and by monotonicity $x \wedge u \leq y \wedge u$, so that by transitivity $x \wedge u \leq z$.

□

Lemma 2.2.6. *In any Heyting Algebra* **HA***:* $\forall x, y, z : (x \leq y) \Rightarrow x \leq (y \vee z)$.

Proof. Fix x, y, assume $(x \leq y)$. By lattice theory, we have that for any z, $y \leq y \vee z$, and so by transitivity $x \leq y \vee z$. □

Lemma 2.2.7. *In any Heyting Algebra* **HA***:* $\forall x, y : x \leq \bot \Rightarrow x \leq y$.

Proof. As all Heyting algebras are bounded, we have as an axiom: $\forall y : \bot \leq y$. So: assume $x \leq \bot$, and fix x. Then $x \leq y$ for all y, so that $\forall x, y : x \leq \bot \Rightarrow x \leq y$ as desired. □

Note 2. *We note by way of passing that Heyting Algebras are examples of a more general class of algebraic structure, that of (in present case: commutative, integral, bounded and idempotent)* residuated lattice *which we define below and consider in greater depth later:*

Definition 2.2.8 (Commutative Lattice-ordered monoid)**.** *A structure* $\mathcal{A} = \langle A, \wedge, \vee, \otimes, 1 \rangle$ *is a* commutative lattice-ordered monoid *if*

- $\langle A, \wedge, \vee \rangle$ *is a lattice*
- $\langle A, \otimes, 1 \rangle$ *is a commutative monoid*
- $\otimes$ *is monotonic increasing with respect to the lattice order on* A.

There are a number of slightly different definitions of this concept in the literature, varying with the exact relationship required between the lattice and the monoid structure. Our definition is weak. The most common definition has $\otimes$ distributes over $\vee$, and some definitions have that $\otimes$ distributes over both $\vee$ and $\wedge$.

Definition 2.2.9 (Commutative Residuated lattice). *$\mathcal{A} = \langle A, \wedge, \vee, \otimes, 1, \rightarrow \rangle$ is called a* commutative residuated lattice *if*

- *$\langle A, \wedge, \vee, \otimes, 1 \rangle$ is a commutative lattice-ordered monoid*
- *$x \otimes y \le z$ if and only if $x \le y \rightarrow z$*

2.2.2 Heyting Algebras and Validity for IL

Valid Sequents in IL

Definition 2.2.10 (Denotation functions). *Given a Heyting Algebra* **HA** *and a mapping from propositional variables to elements of* **HA***:*

$$p \mapsto [\![p]\!] \in \mathbf{HA}$$

We refer to the denotation of a variable p as $[\![p]\!]_{\mathbf{HA}}$. We can extend that mapping to all formulas in the language of L in a straightforward way:

$$\begin{aligned}
[\![\phi \wedge \psi]\!]_{\mathbf{HA}} &:= [\![\phi]\!]_{\mathbf{HA}} \wedge [\![\psi]\!]_{\mathbf{HA}} \\
[\![\phi \vee \psi]\!]_{\mathbf{HA}} &:= [\![\phi]\!]_{\mathbf{HA}} \vee [\![\psi]\!]_{\mathbf{HA}} \\
[\![\phi \rightarrow \psi]\!]_{\mathbf{HA}} &:= [\![\phi]\!]_{\mathbf{HA}} \rightarrow [\![\psi]\!]_{\mathbf{HA}}
\end{aligned}$$

Definition 2.2.11 (Validity). *A sequent $\phi_1, ..., \phi_n \vdash_{\mathbf{IL}} \psi$ is then said to be valid in* **HA***, if $[\![\phi_1]\!] \wedge ... \wedge [\![\phi_n]\!] \le [\![\psi]\!]$ holds in* **HA***. A sequent is said to be* valid *if it is valid in* all *Heyting Algebras. We represent this thus:*

$$\Gamma \vDash_{\mathbf{HA}} \phi$$

When we wish to say that ϕ is valid, if it's valid in all Heyting Algebras, we write:

$$\vDash_{\mathbf{HA}} \phi$$

ItIt It is easy to show that the valid sequents, in the sense above, are precisely the ones provable in Intuitionistic logic, and this indeed what we show in Section 2.3.9.

2.3 Kripke Semantics: Intuitionistic and Gödel-Dummett logic

2.3.1 Kripke Semantics for Intuitionistic logic

The Kripke semantics for **IL** is based on Kripke's classical construction, which as we've discussed earlier, is intended to reflect the intuitions of Brouwer's philosophy of mathematics. We first need to define a particular class of functions from the set of worlds W to involutive, Heyting Algebras alias Boolean algebras. specifically the characteristic Boolean Algebra $\{\top, \bot\}_{\mathbf{BA}}$.

Definition 2.3.1 (Step-functions)**.** *Let* $\mathcal{W} = \langle W, \geq \rangle$ *be a partial order and* $\{\top, \bot\}_{\mathbf{BA}}$ *the characteristic Boolean Algebra over* $\top$ *and* $\bot$*,* $\wedge, \vee$ *as the meet and join of the same lattice, respectively. A function* $f{:}\,\mathcal{W} \to \{\top, \bot\}_{\mathbf{BA}}$ *is said to be a* step-function *if* $f(w) = \top \Rightarrow \forall v : v \geq w(f(v) = \top)$.[5]

Lemma 2.3.2. *If* $f{:}\,W \to \{\top, \bot\}_{\mathbf{BA}}$ *and* $g{:}\,W \to \{\top, \bot\}_{\mathbf{BA}}$ *are step-functions, then the following functions are also step functions:*

$$
\begin{aligned}
(f \wedge g)(w) &:= f(w) \wedge g(w) \\
(f \vee g)(w) &:= f(w) \vee g(w)
\end{aligned}
$$

and moreover if $w \geq v \in W$*,* f, g *are step-functions, then* $((f(w) \wedge g(w)) \geq (f(v) \wedge g(v))$*,* $(f(w) \vee g(w)) \geq (f(v) \vee g(v))$*.*

Proof. Let $w \geq v \in W$ and f, g be step-functions. Let us consider each case:

- $f \wedge g$. Then because f, g are step functions and therefore monotone functions, $f(w) \geq f(v)$ and $g(w) \geq g(v)$. By order theory, $f(w) \geq f(w) \wedge f(v)$ and $g(w) \geq g(w) \wedge g(v)$, and so $f(w) \wedge g(w) \geq f(v) \wedge g(v)$.
- $f \vee g$. Then because f, g are step functions and therefore monotone functions, $f(w) \geq f(v)$ and $g(w) \geq g(v)$. By order theory, $f(w) \vee f(v) \geq f(v)$ and $g(w) \vee g(v) \geq g(v)$, and so $f(w) \vee g(w) \geq f(v) \vee g(v)$.

□

Definition 2.3.3 (Kripke structure)**.** *Let* $\{\top, \bot\}_{\mathbf{BA}}$ *be the characteristic Boolean Algebra. A* Kripke structure *for* $\{\top, \bot\}_{\mathbf{BA}}$ *(or K-structure) is a pair* $\mathcal{M}_{\mathcal{P}} = \langle \mathcal{W}, \Vdash^{\mathrm{K}} \rangle$ *where* $\mathcal{W} = \langle W, \geq \rangle$ *is a poset, and* $\Vdash^{\mathrm{K}}$ *is an infix operator (on worlds and propositional variables) taking values in* $\{\top, \bot\}_{\mathbf{BA}}$*, i.e.* $(w \Vdash^{\mathrm{K}} p) \in \{\top, \bot\}_{\mathbf{BA}}$*,*

[5] In view of the content that is to follow, a more obvious definition here would be $f(w) > \bot \Rightarrow \forall v : v \geq w(f(v) = \top)$, but at the suggestion of the examination panel we have opted for the simpler presentation given above.

such that for any propositional variable p the function $\lambda w.(w \Vdash^{K} p){:}W \to \{\top,\bot\}_{\mathbf{BA}}$ *is a step-function. Thus* $w \Vdash^{K} p$ *is a binary relation between worlds* $w \in W$ *and propositional variables p satisfying the following conditions:*

(M) If $w \Vdash^{K} p$ *and* $v \geq w$ *then* $v \Vdash^{K} p$

(⊥) $\neg(w \Vdash^{K} \bot)$

Given a Kripke structure

$$\mathcal{K} = \langle \mathcal{W}, \Vdash^{K} \rangle$$

we extend the relation $\Vdash^{K}$ *to a relation between worlds and arbitrary* $\mathcal{L}$*-formulas as*

$$\begin{aligned} w \Vdash^{K} \psi \wedge \chi &:= (w \Vdash^{K} \psi) \wedge (w \Vdash^{K} \chi) \\ w \Vdash^{K} \psi \vee \chi &:= (w \Vdash^{K} \psi) \vee (w \Vdash^{K} \chi) \\ w \Vdash^{K} \psi \to \chi &:= \forall v \geq w((v \Vdash^{K} \psi) \to (v \Vdash^{K} \chi)) \end{aligned}$$

Lemma 2.3.4. *(Existence of Infs for Kripke structures) Let* $f : W \to \{\top,\bot\}_{\mathbf{BA}}$ *as above, noting that* $\{\top,\bot\}_{\mathbf{BA}}$ *is order-complete as the characteristic Boolean algebra, and f inducing* $\{f(v)|w \leq v\}$ *in* $\{\top,\bot\}_{\mathbf{BA}}$*. Then* $\inf\{f(v)|w \leq v\}$ *exists in* $\{\top,\bot\}_{\mathbf{BA}}$*, as does* $\sup\{f(v)|w \leq v\}$*.*

Proof. Let W be a poset under $\leq$, let $\{\top,\bot\}_{\mathbf{BA}}$ be the characteristic Boolean algebra, and let $f : W \to \{\top,\bot\}_{\mathbf{BA}}$ with $\{f(v)|w \leq v\}$ in $\{\top,\bot\}_{\mathbf{BA}}$. By 2.3.42 a complete, lattice-ordered involutive pocrim must have suprema and infima for all subsets X of $\{\top,\bot\}_{\mathbf{BA}}$; but $\{f(v)|w \leq v\} \subseteq \{\top,\bot\}_{\mathbf{BA}}$, so that $\inf\{f(v)|w \leq v\}$ and $\sup\{f(v)|w \leq v\}$ exist in $\{\top,\bot\}_{\mathbf{BA}}$. □

Note 3. *We use* $\bigwedge$ *to refer to the standard operation on an arbitrary poset or Boolean Algebra, but we use* $\mathbf{inf}_{v \geq w}$ *to refer to the operation on specifically on a set of formulas evaluated* in *a Boolean algebra. The following makes the relationship clear.*

Note 4. *(Inf of a set of valuations.) Let* $\{\top,\bot\}_{\mathbf{BA}}$ *be the complete characteristic Boolean algebra, such that* $v \Vdash^{K} \psi$ *and* $v \Vdash^{K} \chi$ *are valuations in the characteristic Boolean algebra. Then one can define* $\mathbf{inf}_{v \geq w}((v \Vdash^{K} \psi) \to (v \Vdash^{K} \chi))$ *as follows, by 4.3.5:*

$$\begin{aligned} \bigwedge\{(v \Vdash^{K} \psi) \to (v \Vdash^{K} \chi)|v : w \leq v\} &\Leftrightarrow \forall v \geq w(((v \Vdash^{K} \psi) \to (v \Vdash^{K} \chi))) \\ &\Leftrightarrow \mathbf{inf}_{v \geq w}((v \Vdash^{K} \psi) \to (v \Vdash^{K} \chi)) \end{aligned}$$

Lemma 2.3.5. *For any formula* ϕ *the function* $\lambda w.(w \Vdash^{K} \phi){:}W \to \{\top,\bot\}_{\mathbf{BA}}$ *is a step-function.*

Proof. By induction on the complexity of the formula ϕ. The cases for $\psi \vee \xi, \psi \wedge \xi$ follow directly from Lemma 2.3.2. The case for $\psi \to \xi$ follows from the fact that, given w, v such that $w \leq v$ and $w \Vdash^{\text{GBM}} \psi \to \chi$, $\mathbf{inf}_{v \geq w}(w \Vdash^{\text{K}} \psi \to \chi) \leq \mathbf{inf}_{v \geq w}((v \Vdash^{\text{K}} \psi) \to (v \Vdash^{\text{K}} \chi))$. □

Using the above, one obtains an essential (and well-known) property characterising the satisfaction of formulas in intuitionistic logic.

Proposition 2.3.6. *The monotonicity property (M) holds for all $\mathcal{L}$-formulas ϕ, i.e.*

$$\text{if } w \Vdash^{\text{K}} \phi \text{ and } v \geq w \text{ then } v \Vdash^{\text{K}} \phi$$

2.3.2 Validity under the Kripke semantics

Definition 2.3.7. *Let $\Gamma = \psi_1, \ldots, \psi_n$. Consider the following definitions:*

- *We say that a sequent $\Gamma \vdash \phi$* holds *in a Kripke-structure $\mathcal{M}$ (written $\Gamma \Vdash^{\text{K}}_{\mathcal{M}} \phi$) if for all $w \in W$ we have*

$$(w \Vdash^{\text{K}} \psi_1 \wedge \ldots \wedge \psi_n) \leq (w \Vdash^{\text{K}} \phi)$$

We will, for space considerations, sometimes abbreviate this as

$$\forall w \in W (w \Vdash^{\text{K}} \Gamma \leq w \Vdash^{\text{K}} \phi)$$

or, when we wish to emphasize the closure of the context under 'meet', we will write:

$$\forall w \in W (w \Vdash^{\text{K}} \bigwedge \Gamma \leq w \Vdash^{\text{K}} \phi)$$

Otherwise (i.e. if $\Gamma \nvdash \phi$), we say that the sequent fails *in a structure $\mathcal{M}$ (written $\Gamma \nVdash^{\text{K}}_{\mathcal{M}} \phi$) and this means:*

$$\exists w \in W : (w \Vdash^{\text{K}} \psi_1 \wedge \ldots \wedge \psi_n) > (w \Vdash^{\text{K}} \phi)$$

- *A sequent $\Gamma \vdash \phi$ is said to be valid under the Kripke semantics for $\mathcal{L}_\wedge$ (written $\Gamma \Vdash^{\text{K}} \phi$) if $\Gamma \Vdash^{\text{K}}_{\mathcal{M}_Q} \phi$ for all Kripke-structures $\mathcal{M}_Q$.*

Note 5. *We note in passing that the condition*

$$\forall w \in W (w \Vdash^{\text{K}} \Gamma \leq w \Vdash^{\text{K}} \phi)$$

Boils down to the following, given the Kripke semantics given earlier (and residuation):

$$\forall w \in W((w \Vdash^{\mathrm{K}} \Gamma) \to (w \Vdash^{\mathrm{K}} \phi))$$

2.3.3 Kripke semantics for Gödel-Dummett logic

Here we quickly define Kripke structures appropriate to **GD**, and give some of their properties analogous to the case for Intuitionistic logic.

Definition 2.3.8 (Linear Kripke structure). *A* Linear Kripke structure *consists of a pair $\mathcal{K} = \langle \mathcal{W}, \Vdash^{\mathrm{LK}} \rangle$, as above in the Intuitionistic case (via Step-functions into $\{\top, \bot\}_{\mathbf{BA}}$), with $w \Vdash^{\mathrm{LK}} p$ defined as above, except that $\mathcal{W} = \langle W, \geq \rangle$ is a* linear *order on the set of worlds; all else is as above.*

Definition 2.3.9 (Linear Kripke semantics for $\mathcal{L}$). *Given a Kripke structure*

$$\mathcal{LK} = \langle \mathcal{W}, \Vdash^{\mathrm{LK}} \rangle$$

we extend the relation $\Vdash^{\mathrm{LK}}$ to a relation between worlds and arbitrary $\mathcal{L}$-formulas as

$$\begin{aligned} w \Vdash^{\mathrm{LK}} \psi \wedge \chi &:= (w \Vdash^{\mathrm{LK}} \psi) \wedge (w \Vdash^{\mathrm{LK}} \chi) \\ w \Vdash^{\mathrm{LK}} \psi \vee \chi &:= (w \Vdash^{\mathrm{LK}} \psi) \vee (w \Vdash^{\mathrm{LK}} \chi) \\ w \Vdash^{\mathrm{LK}} \psi \to \chi &:= \forall v \geq w((v \Vdash^{\mathrm{LK}} \psi) \to (v \Vdash^{\mathrm{LK}} \chi)) \end{aligned}$$

Lemma 2.3.10. *(Existence of Infs for Linear Kripke structures) Let $f : W \to \{\top, \bot\}_{\mathbf{BA}}$ as above, noting that $\{\top, \bot\}_{\mathbf{BA}}$ is order-complete as the characteristic Boolean algebra, and f inducing $\{f(v) | w \leq v\}$ in $\{\top, \bot\}_{\mathbf{BA}}$. Then $\inf\{f(v) | w \leq v\}$ exists in $\{\top, \bot\}_{\mathbf{BA}}$, as does $\sup\{f(v) | w \leq v\}$.*

Proof. Follows from the intuitionistic case given in the previous section above, as every linearly ordered poset is indeed a poset. ☐

Lemma 2.3.11. *For any formula ϕ the function $\lambda w.(w \Vdash^{\mathrm{LK}} \phi) : W \to \{\top, \bot\}_{\mathbf{BA}}$ is a step-function.*

Proof. By induction on the complexity of the formula ϕ. The cases for $\psi \vee \xi, \psi \wedge \xi$ follow directly from Lemma 2.3.2. The case for $\psi \to \xi$ follows from the fact that, given w, v such that $w \leq v$ and $w \Vdash^{\mathrm{LK}} \psi \to \chi$, $\mathbf{inf}_{v \geq w}(w \Vdash^{\mathrm{LK}} \psi \to \chi) \leq \mathbf{inf}_{v \geq w}((v \Vdash^{\mathrm{LK}} \psi) \to (v \Vdash^{\mathrm{LK}} \chi))$. ☐

And of course, following from the generality of Kripke semantics for Intuitionistic logic, we have:

Proposition 2.3.12. *The monotonicity property (M) holds for all $\mathcal{L}$-formulas ϕ, i.e.*

$$\text{if } w \Vdash^{\mathrm{LK}} \phi \text{ and } v \geq w \text{ then } v \Vdash^{\mathrm{LK}} \phi$$

2.3.4 Adequacy for IL under the Kripke semantics

Soundness

Note 6. *We now prove the soundness of Intuitionistic logic under the Kripke semantics. We note that the proof relies on algebraic lemmata proved in a later section. All such lemmata are referenced in the body of the proof.*

Theorem 2.3.13 (Soundness)**.** *If $\Gamma \vdash_{\mathbf{IL}} \phi$ then $\Gamma \Vdash^{\mathrm{K}} \phi$.*

Proof. By induction on the derivation of $\Gamma \vdash_{\mathbf{IL}} \phi$. Assume $\Gamma = \psi_1, \ldots, \psi_n$ and let $\wedge\Gamma := \psi_1 \wedge \ldots \psi_n$. Fix a Kripke-structure $\mathcal{M} = \langle \mathcal{W}, \Vdash^{\mathrm{K}} \rangle$ with $\mathcal{W} = \langle W, \geq \rangle$, and let $w \in W$.

(Axiom) $\Gamma, \phi \vdash_{\mathbf{IL}} \phi$. By Definition 2.3.7, we need to show, for all $w \in W$:

$$\begin{aligned} w \Vdash^{\mathrm{K}} (\wedge\Gamma) \wedge \phi &\equiv (w \Vdash^{\mathrm{K}} \psi_1) \wedge \ldots (w \Vdash^{\mathrm{K}} \psi_n) \wedge (w \Vdash^{\mathrm{K}} \phi) \\ &\overset{(\mathrm{L.2.3.48})}{\leq} w \Vdash^{\mathrm{K}} \phi \end{aligned}$$

($\wedge$I) By IH we have $\forall w : (w \Vdash^{\mathrm{K}} \wedge\Gamma) \leq (w \Vdash^{\mathrm{K}} \phi)$ and $\forall w : (w \Vdash^{\mathrm{K}} \wedge\Gamma) \leq (w \Vdash^{\mathrm{K}} \psi)$. Now fix w. By 2.3.54

$$(w \Vdash^{\mathrm{K}} \wedge\Gamma) \leq (w \Vdash^{\mathrm{K}} \phi) \wedge (w \Vdash^{\mathrm{K}} \psi) \equiv (w \Vdash^{\mathrm{K}} \phi \wedge \psi)$$

($\wedge$E) By IH we have $\forall w : (w \Vdash^{\mathrm{K}} \wedge\Gamma) \leq (w \Vdash^{\mathrm{K}} \phi \wedge \psi)$. Fix w. By 2.3.52 this implies both $(w \Vdash^{\mathrm{K}} \wedge\Gamma) \leq (w \Vdash^{\mathrm{K}} \phi)$ and $(w \Vdash^{\mathrm{K}} \wedge\Gamma) \leq (w \Vdash^{\mathrm{K}} \psi)$.

($\vee$I) By IH we have $\forall w : (w \Vdash^{\mathrm{K}} \wedge\Gamma) \leq (w \Vdash^{\mathrm{K}} \phi)$. Now fix w. Therefore by 2.3.55:

$$(w \Vdash^{\mathrm{K}} \wedge\Gamma) \leq (w \Vdash^{\mathrm{K}} \phi \vee \psi)$$

($\vee$E) By IH we have, for all $w \in W$:

- $w \Vdash^{\mathrm{K}} \wedge\Gamma \leq \{w \Vdash^{\mathrm{K}} \phi\} \vee \{w \Vdash^{\mathrm{K}} \psi\}$
- $(w \Vdash^{\mathrm{K}} (\bigwedge\Gamma) \wedge \phi) \leq (w \Vdash^{\mathrm{K}} \chi)$
- $(w \Vdash^{\mathrm{K}} (\bigwedge\Gamma) \wedge \psi) \leq (w \Vdash^{\mathrm{K}} \chi)$

Now fix w. By Lemma 2.3.50, these imply $(w \Vdash^{\mathrm{K}} (\wedge\Gamma) \wedge (\bigwedge\Delta)) \leq (w \Vdash^{\mathrm{K}} \chi)$.

(→I) By IH we have, for all $w \in W$:

$$(w \Vdash^{\mathrm{K}} \bigwedge \Gamma \wedge \phi) \leq (w \Vdash^{\mathrm{K}} \psi) \tag{I}$$

We must show

$$(w \Vdash^{\mathrm{K}} \Gamma) \leq \forall v : v \geq w(v \Vdash^{\mathrm{K}} \phi \rightarrow v \Vdash^{\mathrm{K}} \psi)) \tag{II}$$

Assuming, for all w,

$$(w \Vdash^{\mathrm{K}} \wedge\Gamma) \wedge (w \Vdash^{\mathrm{K}} \phi) \leq (w \Vdash^{\mathrm{K}} \psi) \tag{III}$$

we have (by residuation), for all w,

$$(w \Vdash^{\mathrm{K}} \wedge\Gamma) \leq (w \Vdash^{\mathrm{K}} \phi) \rightarrow (w \Vdash^{\mathrm{K}} \psi) \tag{IV}$$

Hence, assuming

$$(w \Vdash^{\mathrm{K}} \Gamma) \tag{V}$$

by monotonicity we can conclude

$$(v \Vdash^{\mathrm{K}} \Gamma), \quad \text{for} \quad \text{all} \quad v \geq w \tag{VI}$$

Using (IV) and (VI) we have

$$\forall v \geq w((v \Vdash^{\mathrm{K}} \phi) \rightarrow (v \Vdash^{\mathrm{K}} \psi)) \tag{VII}$$

and hence, for all w,

$$(w \Vdash^{\mathrm{K}} \wedge\Gamma) \leq \forall v : v \geq w((v \Vdash^{\mathrm{K}} \phi) \rightarrow (v \Vdash^{\mathrm{K}} \psi)) \tag{VIII}$$

(→E) By IH we have, for $\forall w \in W$:

$$(w \Vdash^{\mathrm{K}} \bigwedge \Gamma) \leq (w \Vdash^{\mathrm{K}} \phi) \tag{IX}$$

And:

$$(w \Vdash^{\mathrm{K}} \bigwedge \Gamma) \leq \forall v : v \geq w : ((v \Vdash^{\mathrm{K}} \phi) \rightarrow (v \Vdash^{\mathrm{K}} \psi)) \tag{X}$$

We want to show:

$$(w \Vdash^{\mathrm{K}} \bigwedge \Gamma) \leq (v \Vdash^{\mathrm{K}} \psi) \tag{XI}$$

Now recall that (XII):

$$(w \Vdash^{K} \bigwedge \Gamma) \leq \mathbf{inf}((v \Vdash^{K} \phi) \to (v \Vdash^{K} \psi)) \tag{XII}$$

is equivalent to (XIII):

$$\forall w, v \geq w((w \Vdash^{K} \Gamma) \leq ((v \Vdash^{K} \phi) \to (v \Vdash^{K} \psi))) \tag{XIII}$$

So set $v := w$ in (X). Then we have

- $(w \Vdash^{K} \bigwedge \Gamma) \leq (w \Vdash^{K} \phi)$
- $(w \Vdash^{K} \bigwedge \Gamma) \leq ((w \Vdash^{K} \phi) \to (w \Vdash^{K} \psi))$

After applying 2.2.5 we have

$$(w \Vdash^{K} (\bigwedge \Gamma) \wedge (\bigwedge \Gamma)) \leq (w \Vdash^{K} \psi)$$

which is equivalent to:

$$(w \Vdash^{K} \bigwedge \Gamma) \leq (w \Vdash^{K} \psi)$$

as desired.

($\bot$E) By IH we have $\forall w : (w \Vdash^{K} \wedge\Gamma) \leq (w \Vdash^{K} \bot)$. By 2.3.49 $(w \Vdash^{K} \wedge\Gamma) \leq (w \Vdash^{K} \phi)$, for any ϕ.

□

2.3.5 Completeness

Theorem 2.3.14. *If a formula is true in every possible world of any Kripke model, then it is derivable in Intuitionistic logic.*

We do not give the classic proof here, as there are several authoritative references to which we can refer the reader, e.g. [47] or [42] or even [56]. Instead, we give a proof via an embedding into poset products, in keeping with the theme of this thesis (and thus preparing the reader for the completeness proofs that follow). This is inspired by remark of Peter Jipsen and Franco Montagna in [39]. We first explore the notion of a poset product, and then relate the Kripke semantics in terms of poset products.

Poset products

We introduce and briefly discuss an important algebraic construction due to Peter Jipsen and Franco Montagna [39]. Here we follow both Jipsen and Montagna and Fussner's [23] presentation of the idea.

Let $X = (X, \leq)$ be a poset and let $(A_x : x \in X)$ be a collection of residuated lattices indexed by the poset X. We assume that all A_x share the same neutral element $\top$ and that all A_x that are bounded share the same minimum element $\bot$. Suppose that if x is not minimal, then A_x is integral and if x is not maximal then A_x is bounded. The *poset product* $\prod_{x \in (X, \leq)} A_x$ is the algebra defined as follows[6]:

Definition 2.3.1. The domain of $\prod_{x \in (X, \leq)} A_x$ is the set of all maps h on X such that for all $x \in X$:

1. $h(x) \in A_x$ and

2. if $h(x) \neq \top$ then for all $y < x$, $h(y) = \bot$

3. The monoid operation and the lattice operations are defined pointwise.

4. The residuals are defined by:

$$(h \to g)(x) = \begin{cases} h(x) \to g(x) & \text{if} \quad g(y) \leq h(y) \quad \text{for all} \quad x > y \\ \bot & \text{otherwise .} \end{cases}$$

Here x, y denote residuals and order in A_x.

The dual poset product is also of some use, and is defined as expected: that is, the poset product $\prod_{x \in X} A_x$ of the same algebras, with a dual poset, denoted X^d. Note that in the dual poset product condition (2) must be replaced by the following condition:

$$(2)' \quad \text{if} \quad h(x) \neq \top, \quad \text{then} \quad \text{for} \quad \text{all} \quad y > x, \quad h(y) = \bot$$

The definition of residuals is then:

$$(h \to g)(x) = \begin{cases} h(x) \to g(x) & \text{if} \quad h(y) \leq g(y) \quad \text{for all} \quad x > y \\ \bot & \text{otherwise .} \end{cases}$$

Now let $(X, \leq)$ be a poset and $\{A_x : x \in X\}$ is an indexed collection of integral bounded residuated lattices. We set $B = \prod_{x \in (X, \leq)} A_x$ and the map $\sigma \colon B \to B$ we

[6] NB. the note on notation in 7. We provide here the standard notation from the literature and inform the reader where we occasionally depart from that standard.

define by

$$\sigma(f)(x) = \begin{cases} f(x) & \text{if } f(y) = 1 \text{ for all } y > x \\ 0 & \text{if there exists } y > x \text{ with } f(y) \neq 1. \end{cases}$$

The map σ is a conucleus on B by [39], and the *poset product* of this indexed family $\{A_x : x \in X\}$ is the algebra B_σ, which in the literature (see e.g. [23]) is sometimes denoted

$$B_\sigma = \prod_{(X,\leq)} A_x.$$

Thus one can view a Poset Products as a direct products with an indexing set that is a poset $(X, \leq)$ rather than a set.

Now, in the present literature there are a few differences of notation. The presentation given above is that preferred by the algebraists, and has priority. In our own work ([44] and in this thesis) we have used infima either by way of $\lfloor \mathbf{inf} \rfloor$, or inf as a means to emphasise the numeric character of the functions in the products of MV-chains and pocrims. In recent discussions with Fussner, we have decided to side with the algebraists morally, as the real behavior we seek to model is that of a box modality, analogous to that present in modal logic (and already guiding the translation from Intuitionistic logic to S4 modal logic, which we will not discuss further here). However, for the presentation in the current thesis, we have opted to proceed with our precedent in the publication [44], preferring $\lfloor \mathbf{inf} \rfloor$ as opposed to σ.

This σ or $\lfloor \mathbf{inf} \rfloor$ resembles a box modality (as is demonstrated in Fussner's [23]), and in keeping with the tradition of translation theory (in modal and Intuitionistic logic), is a *conucleus operator*. The σ-fixed points are called *antichain labelings* in [39] and other papers in this tradition, whereas we refer to these same as *sloping functions* (due to their 'sloping' numeric behavior in the poset). These are all distinctions without a difference in a sense (as they come to the same thing for us) but represent the present state of the art.

From [39] and [23], one can piece together the following results (listed in Fussner's Lemma 3.5 and Jipsen and Montagna's Theorem 6.2):

Theorem 2.3.15. *Let $X = (X, \leq)$ be a poset and let $(A_x : x \in X)$ be a collection of residuated lattices indexed by the poset X. Again, set:*

$$B = \prod_{x \in X} A_x$$

Then the following hold:

1. *If A_x is an MV-algebra for each $x \in X$, then B is a GBL-algebra.*
2. *If A_x is a Boolean algebra $\{\bot, \top\}$ for each $x \in X$, then B is a Heyting-algebra.*
3. *If $(X, \leq)$ is a root system and A_x is MV-chain for each $x \in X$, then B is a BL-algebra.*
4. *If $(X, \leq)$ is a chain and A_x is an MV-chain for each $x \in X$, then B is a linear BL-algebra.*

Note 7. *We note in passing that we will freely interchange notation for the poset product: instead of using $\prod_{x\in(X,\leq)} A_x$, or $\prod_{(X,\leq)} A_x$ when it is clear X is a poset, we write the poset product: $\prod_{x\in X} A_x$ or $\prod_X A_x$; when we wish to emphasize the resemblance to classic Kripke structures (with possible worlds as partial order), we will simply write $(X, \leq)$ as $\langle W, \preceq\rangle$ and instead of $\prod_{w\in\langle W,\preceq\rangle} A_w$ or $\prod_{w\in W} A_w$ or $\prod_W A_w$ we write $\mathbf{A}_{\mathcal{W}}$.*

Kripke-structures and Poset Products

Recall that a *Poset Product* (cf. [4, Def. 2] and [37] but also see above) is defined over a poset $\mathcal{W} = \langle W, \succeq\rangle$, as the algebra $\mathbf{A}_{\mathcal{W}}$ of signature $\mathcal{L}$ whose elements are sloping functions $f : W \to [0,1]$ (see our chapter on $\mathbf{GBL}_{ewf}$). We note, however, that since all step-functions *are* sloping functions, we can adapt that construction to the current case: $f : W \to \{\bot, \top\}_{\mathbf{BA}}$ and operations are defined as

$$
\begin{array}{lcl}
(\bot)(w) & := & \bot \\
(f_1 \wedge f_2)(w) & := & f_1(w) \wedge f_2(w) \\
(f_1 \vee f_2)(w) & := & f_1(w) \vee f_2(w) \\
(f_1 \to f_2)(w) & := & \begin{cases} f_1(w) \to f_2(w) & \text{if} \quad \forall v \succ w \quad (f_1(v) \leq f_2(v)) \\ \bot & \text{if} \quad \exists v \succ w \quad (f_1(v) > f_2(v)) \end{cases}
\end{array}
$$

Since f_1 and f_2 are step-functions, we have that

$$\forall v \succ w(f_1(v) \leq f_2(v)) \quad \Leftrightarrow \quad \forall v \succ w((f_1(v) \to f_2(v)) = \top)$$

Therefore, this last clause of the definition can be simplified to

$$(f_1 \to f_2)(w) \quad := \quad \mathbf{inf}_{v \succeq w}(f(v) \to f(v))$$

For the rest of the current section, we take poset products to be the algebra $\mathbf{A}_{\mathcal{W}}$ of signature $\mathcal{L}$ whose elements are step functions $f : W \to \{\bot, \top\}_{\mathbf{BA}}$.

Definition 2.3.16 (Poset Product semantics for $\mathcal{L}$)**.** *Let $\mathcal{W} = \langle W, \geq \rangle$ be a fixed poset, and $\mathbf{A}_{\mathcal{W}}$ be the poset product described above. Given $h : Atom \to \mathbf{A}_{\mathcal{W}}$ an assignment of atomic formulas to elements of $\mathbf{A}_{\mathcal{W}}$, any formula ϕ can be mapped to an element $[\![\phi]\!]_h \in \mathbf{A}_{\mathcal{W}}$ as follows:*

$$\begin{array}{lcl} [\![p]\!]_h & := & h(p) \quad (\textit{for atomic formulas } p) \\ [\![\bot]\!]_h & := & \bot \\ [\![\phi \wedge \psi]\!]_h & := & [\![\phi]\!]_h \wedge [\![\psi]\!]_h \\ [\![\phi \vee \psi]\!]_h & := & [\![\phi]\!]_h \vee [\![\psi]\!]_h \\ [\![\phi \to \psi]\!]_h & := & [\![\phi]\!]_h \to [\![\psi]\!]_h \end{array}$$

A formula ϕ is said to be valid in $\mathbf{A}_{\mathcal{W}}$ under h *if for every $w \in W$*

$$[\![\phi]\!]_h^{\mathbf{A}_{\mathcal{W}}}(w) = \top$$

(which is $\top$ in $\{\bot, \top\}_{\mathbf{BA}}$). A formula ϕ is said to be valid in $\mathbf{A}_{\mathcal{W}}$ *if it is valid in $\mathbf{A}_{\mathcal{W}}$ under h for any possible mapping $h : Atom \to \mathbf{A}_{\mathcal{W}}$.*

Observe that given a poset product $\mathbf{A}_{\mathcal{W}}$ (for a poset $\mathcal{W} = \langle W, \geq \rangle$) and a mapping $h : Atom \to \mathbf{A}_{\mathcal{W}}$ of atomic formulas to elements of $\mathbf{A}_{\mathcal{W}}$, we can obtain a Kripke structure $\mathcal{M}^{\mathbf{A}_{\mathcal{W}}} = \langle \mathcal{W}, \Vdash_h^{\mathrm{K}} \rangle$, by taking

$$w \Vdash_h^{\mathrm{K}} p := h(p)(w)$$

recalling that $h(p) : W \to \{\bot, \top\}$ is a step-function.

Proposition 2.3.17. *Let $\mathbf{A}_{\mathcal{W}}$ be the poset product over the partially ordered set $\mathcal{W} = \langle W, \geq \rangle$, and $h : Atom \to \mathbf{A}_{\mathcal{W}}$ be a fixed mapping of atomic formulas to elements of $\mathcal{W}$. Let $\mathcal{M}^{\mathbf{A}_{\mathcal{W}}}$ be the Kripke-structure defined above. Then, for any formula ϕ*

$$w \Vdash_h^{\mathrm{K}} \phi = [\![\phi]\!]_h^{\mathbf{A}_{\mathcal{W}}}(w)$$

Proof. By induction on the complexity of ϕ. □

So we can transform an interpretation of $\mathcal{L}$ formulas in a poset product $\mathbf{A}_{\mathcal{W}}$ into a Kripke semantics (on the Kripke frame $\mathcal{W}$) for $\mathcal{L}$ formulas.

Poset Products and Canonical Kripke structures

Let $X = (X, \leq)$ be a poset, and let $P\uparrow(X)$ be the set of upwards-closed subsets of X, where a subset U of X is upwards-closed iff for all $x, y \in X$: $x \leq y$, if

$x \in U$ then $y \in U$. Then $P \uparrow (X)$ becomes a Heyting algebra with respect to the constants $\perp$ (bottom) and X $(= \top)$ and with respect to the operations $\cup$, $\cap$, and $\rightarrow$, where for all $Y, Z \in P \uparrow (X)$, $Y \rightarrow Z = \{x : \forall y \geq x \quad (\text{if} \quad y \in Y, \quad \text{then} \quad y \in Z)\}$. We denote this Heyting algebra by $P \uparrow (X)$. This latter is known as the canonical Kripke frame for **IL**, with the poset X being the Kripke frame associated with the algebra $P \uparrow (X)$. We first note some basic definitions and facts about filters, including the prime filter theorem. The proof of this latter we leave to a standard textbook, e.g. [17].

Definition 2.3.18. *Let* **HA** *be a Heyting Algebra. We call a subset F a filter on* **HA** *if:*

1. $F \neq \varnothing$
2. $x, y \in F$ *then* $x \wedge y \in F$ *($\wedge$-closure)*
3. *if* $x \in F$ *and* $x \leq y$ *then* $y \in F$ *(upwards-closure)*

Proposition 2.3.19. $F \subseteq$ **HA** *is a filter iff*

- $\top \in F$
- $x \in F$ *and* $x \rightarrow y \in F$ *then* $y \in F$.

Proof. Suppose $F \subseteq$ **HA** a filter. Then as for all $x \in$ **HA**: $x \leq \top$, then $\top \in F$ by upwards closure. Suppose $x \in F$ and $x \rightarrow y \in F$. By $\wedge$-closure, we have that $x \wedge (x \rightarrow y) \in F$. Now $F \subseteq$ **HA**, and always $x \wedge (x \rightarrow y) \leq y$ in **HA**, so that by upwards closure of F we have $y \in F$.

Suppose that $F \subseteq$ **HA**, $\top \in F$ and if $x \in F$ and $x \rightarrow y \in F$ then $y \in F$. We must show F is a filter. $F \neq \varnothing$ since $\top \in F$, and $\wedge$-closure holds since if $x, y \in F$ but $x \wedge y \notin F$, then as $F \subseteq$ **HA**, $x, y \in$ **HA** and therefore $x \wedge y \in$ **HA** (as all lattices are closed under $\wedge$), thus we would have $x \wedge y \notin$ **HA** and $x \wedge y \in$ **HA**, a contradiction. Now to show if $x \in F$ and $x \leq y$ then $y \in F$ from our hypothesis that if $x \in F$ and $x \rightarrow y \in F$, as $F \subseteq$ **HA** we have $x \in$ **HA** and $(x \rightarrow y) \in$ **HA** and by $\wedge$-closure of **HA**, $x \wedge (x \rightarrow y) \in$ **HA**. Now $x \wedge (x \rightarrow y) \leq y \in$ **HA** always (as an axiom), and this is equivalent by lattice theory to $(x \wedge (x \rightarrow y)) \vee y = y = \top$; now for any $x \in$ **HA**, $x \leq \top$ in **HA** and $\top \in F$ (by assumption), so that $x \leq y$ and $y \in F$. ☐

Proposition 2.3.20. *For any filter F of* **HA**, $\forall x, y \in$ **HA**

- $x \wedge y \in F$ *iff* $x \in F$ *and* $y \in F$

Proof. Suppose that $x \wedge y \in F$. Then as $x \wedge y \leq y$ and $x \wedge y \leq x$ in **HA**, we have $y \in F$ and $x \in F$ by upwards-closure. For the other direction of the iff, if $x, y \in F$ then $x \wedge y \in F$ by $\wedge$-closure. ☐

Definition 2.3.21. *For any filter F of* $\mathbf{HA}$ *such that $F \neq \mathbf{HA}$, we say F is* prime *if $\forall x, y \in \mathbf{HA}$ if $x \vee y \in F$ then $x \in F$ or $y \in F$.*

Proposition 2.3.22. *For any prime filter F of* $\mathbf{HA}$*, $\forall x, y \in \mathbf{HA}$*

- *$x \vee y \in F$ iff $x \in F$ or $y \in F$*

Proof. Suppose $x \vee y \in F$ with F prime. Then either $x \in F$ or $y \in F$ by definition. On the other hand, suppose $x \in F$. Then $x \in \mathbf{HA}$ (since $F \subseteq \mathbf{HA}$) and as $x \leq (x \vee y)$ holds in $\mathbf{HA}$, $(x \vee y) \in F$ by upwards-closure. □

Definition 2.3.23. *For each $x \in \mathbf{HA}$, let $F_x = \{y \in \mathbf{HA} \colon x \leq y\}$. Then F_x is the smallest filter containing x, and we call this the* principle filter *generated by x.*

Theorem 2.3.24 (Prime Filter Theorem)**.** *Let F be a filter of a Heyting algebra* $\mathbf{HA}$ *with $x \notin F$ for $x \in \mathbf{HA}$. Then there exists a prime filter G of* $\mathbf{HA}$ *such that $x \notin G$ and $F \subseteq G$.*

We note a weaker form of Stone's representation theorem from [69] which we prove, following [56]:

Theorem 2.3.25 (Weak Stone's Representation of Heyting Algebras)**.** *Every Heyting algebra* $\mathbf{HA}$ *embeds into one of the form $P \uparrow (X)$.*

Proof. Let $\mathbf{HA}$ be a Heyting Algebra, $\mathcal{D}(\mathbf{HA}) = \langle \mathbf{D}(\mathbf{HA}), \subseteq \rangle$ where $D(\mathbf{HA})$ is is the set of all prime filters of $\mathbf{HA}$ partially-ordered by subset inclusion. Take $P \uparrow (X)$ (which we show in the next theorem is a Heyting Algebra), and take $X = \mathcal{D}(\mathbf{HA})$, so that we have $P \uparrow (\mathcal{D}(\mathbf{HA}))$, which is now the set of upwards-closed subsets of the poset of prime filters of $\mathbf{HA}$, alias the *canonical extension* of $\mathbf{HA}$. Define a function $f : \mathbf{HA} \to \mathbf{P} \uparrow (\mathcal{D}(\mathbf{HA}))$ as follows: $f(x) = \{F \in D(\mathbf{HA}) \colon x \in F\}$ for each $x \in \mathbf{HA}$. This set is an upwards-closed subset of $D(\mathbf{HA})$, since for all filters F, G we have if $x \in F$ and $F \subseteq G$ then we have $x \in G$.

We must show f is an embedding, i.e.

1. f is injective, and
2. f is a homomorphism.

To the first point, suppose that $x \neq y$, wolog further assume that $y \nleq x$ Take the principle filter F_y generated by the element y. Then we have $x \notin F_y$ by hypothesis. Using the prime filter theorem we have a prime filter G of $\mathbf{HA}$ where $x \notin G$ with $F_y \subseteq G$. As $x \notin G$ the filter G does not belong to $f(x)$ while $G \in f(y)$ as $y \in F_y \subseteq G$. Hence $f(x) \neq f(y)$.

To the second item, $f(\bot) = \varnothing$ as there is no prime filter $F : \bot \in F$; Since for any filter F we have $x \wedge y \in F$ iff $x \in F$ and $y \in F$, we have $f(x \wedge y) = f(x) \cap f(y)$, and similarly since F is prime, we have that $x \vee y$ iff $x \in F$ or $y \in F$, we have $f(x \vee y) = f(x) \cup f(y)$. To show $f(x \to y) = f(x) \Rightarrow f(y)$ (using the definition of $\Rightarrow$ given below in the next theorem) we must show:

$$x \to y \in F \quad \text{iff} \quad \forall G : F \subseteq G \quad G \quad \text{is} \quad \text{prime} \quad \text{and} \quad x \in G \quad \text{then} \quad y \in G$$

First we show the right-to-left direction of the iff: Suppose F is a prime filter such that $x \to y \notin F$. We must show that there is a prime filter $G : F \subseteq G$, $x \in G$ and $y \notin G$. Let $F' = F \cup \{x\}$. Suppose for a contradiction that $y \in F'$. Then $x \wedge z \leq y$ for some $z \in F$ and so $z \leq x \to y$ by residuation. So $x \to y \in F$ by upwards-closure of a filter. But this contradicts our assumption that F is a prime filter such that $x \to y \notin F$. Hence $y \notin F'$. By the prime filter theorem we get a prime filter G: $F' \subseteq G$ and $y \notin G$, but $x \in G$ and $F \subseteq G$.

For the left-to-right direction of the iff: Assume $x \to y \in F$ and take $G : F \subseteq G$ and prime with $x \in G$. Then $x \to y \in G$ by inclusion and since filters are closed under implication, $y \in G$ as well. □

We can strengthen this latter result in the finite case [69], again following [56]:

Theorem 2.3.26. *If* $\mathbf{HA}$ *is finite, then* $f : \mathbf{HA} \to \mathbf{P}\uparrow(\mathcal{D}(\mathbf{HA}))$ *is an isomorphism.*

Proof. We show that f is surjective when we assume $\mathbf{HA}$ finite. So take $X \in P\uparrow(\mathcal{D}(\mathbf{HA}))$. We must show there is a $y \in \mathbf{HA}$ such that for any prime filter

$$F : F \in X \quad \text{iff} \quad y \in F$$

Well, $\mathbf{HA}$ is finite, and so X is a finite set of prime filters, say $\{x_1, x_2, \ldots x_n\}$. Since every filter of a finite Heyting algebra is principal, we can represent these: $\{F_{x_1}, F_{x_2}, \ldots F_{x_n}\}$. These will be ordered by inclusion, as we know the members of $P\uparrow(\mathcal{D}(\mathbf{HA}))$ are ordered by inclusion; in fact, $F_1 \subseteq F_2$ iff $F_2 \leq F_1$. Call a filter F_z *minimal* in X if for all $z \in X$: $F_{z'} \subseteq F_z$ then $F_{z'} = F_z$. Let $\{F_{z_1}, F_{z_2}, \ldots F_{z_m}\}$ be the minimal members of X. Then for any $F_{x_i} \in X$ (with $1 \leq i \leq n$) there is a minimal F_{z_j} (with $1 \leq j \leq m$) such that $F_{z_j} \subseteq F_{x_i}$, and so by the definition $F_{x_i} \leq F_{z_j} \in \mathbf{HA}$. Now take $y \in \mathbf{HA}$ as follows: $y = z_1 \vee z_2 \ldots \vee z_m$. Suppose that $F \in X$, with F a prime filter of $\mathbf{HA}$. Then there is an $i : F_{z_i} \subseteq F$, hence $z_i \in F$ and so $y \in F$. For the other direction of the 'iff', suppose $y \in F$, then for some i $z_i \in F$ as F is a prime filter. This means that $F_{z_i} \subseteq F$, and as $F_{z_i} \in X$ and X upwards-closed, $F \in X$. □

The next result stems from a remark of Peter Jipsen and Franco Montagna in [39], the proof of which they do not provide. We believe this result is interesting in it's own right, further solidifying the importance of the poset product construction as a technique deeply entwined with Kripke semantics and generalisations thereof. The result prefigures the later results of this thesis, especially the completeness arguments. To our knowledge this is the first explicit proof of the result appearing in print, although of course the result is known to cognoscenti.

Theorem 2.3.27 (Jipsen and Montagna [39]). *For every $x \in X$ let A_x denote the two-element Boolean algebra (understood as a residuated lattice). Then the poset product $\prod_{x \in X} A_x$ is isomorphic to $P\uparrow(X)$ under the isomorphism Φ defined, for all $Y \in P\uparrow(X)$ and for all $x \in X$, by:*

$$\Phi(Y)(x) = \begin{cases} \top & \textit{if } x \in Y \\ \bot & \textit{if } x \notin Y. \end{cases}$$

Proof. We break this into four items.

1. Φ above is a step-function (as defined on page 26).
2. $P\uparrow(X)$ is a Heyting Algebra **HA**, and that the image of Φ preserves the constants and operations of $P\uparrow(X)$, so that the range of Φ is a Heyting subalgebra of $\prod_{x \in X} A_x$.
3. Φ is an embedding of $P\uparrow(X)$ into $\prod_{x \in X} A_x$.
4. Φ is onto.

Subproof (1). We must show the function Φ defined above, is a step-function. Recall from page 26: f is a *step-function* if $f(w) > \bot \Rightarrow \forall v : v \geq w(f(v) = \top)$. In the current setting, this translates to: if $\Phi(Y)(x) = \top \Rightarrow \forall y : y \geq x \quad (\Phi(Y)(y) = \top)$.

We know that

$$\Phi(Y)(x) = \top \quad \text{iff} \quad x \in Y \quad (*)$$

So, assume Y is an upwards-closed set, i.e.

$$x \in Y \quad \text{and} \quad (x \leq y) \quad \Rightarrow \quad y \in Y \quad (**)$$

.

(*) and (**) gives

$$\Phi(Y)(x) = \top \quad \text{and} \quad (x \leq y) \Rightarrow \Phi(Y)(y) = \top$$

which gives that Φ is a step-function.

Subproof (2). We show that $P\uparrow(X)$ is a Heyting Algebra **HA**, and that the image of Φ preserves the constants and operations of $P\uparrow(X)$, so that the range of Φ is a Heyting subalgebra of $\prod_{x\in X} A_x$.

Subproof (2).i First, we show that $P\uparrow(X)$ is a Heyting Algebra **HA**. Well, the members of $P\uparrow(X)$ are the up-sets, or upwards-closed subsets, of X. Recall definition of upwards-closed subset of X: a subset U of X is upwards-closed iff for all $x, y \in X$: $x \leq y$, if $x \in U$ then $y \in U$. $\varnothing$ and X are trivially upwards-closed, and if U, V are upwards-closed, so is $U \cup V$ and $U \cap V$. We must define the residual operation, $\rightarrow$: for subsets U, V of X, define $U \rightarrow V$ of X:

$$U \rightarrow V = \{x \in X : \forall y(x \leq y) \quad \text{if} \quad y \in U \quad \text{then} \quad y \in V\}$$

This is upwards-closed by definition. We now show that $P\uparrow(X)$, which is the upwards-closed subsets with the above operations defined gives us a Heyting Algebra **HA**. First, inclusion gives a partial order on the upwards-closed subsets. More, $P\uparrow(X)$ is bounded from above by X under inclusion, and bounded from below by $\varnothing$ also under inclusion, and $\cap, \cup$ are lattice operations. It remains to show $P\uparrow(X)$ satisfies residuation, i.e. for $U, V, W \in P\uparrow(X)$:

$$U \cap V \subseteq W \quad \text{iff} \quad U \subseteq V \rightarrow W$$

So let U, V, W be upwards-closed subsets of X. For left to right of the iff, assume $U \cap V \subseteq W$. We wish to show $U \subseteq V \rightarrow W$. To this end, take $u \in U$ and take $v \in V : u \leq v$. U is upwards-closed, so that $v \in U$, thus $v \in U \cap V$. By assumption $U \cap V \subseteq W$ and so $v \in W$, hence $U \subseteq V \rightarrow W$. For the right to left direction of the iff, assume $U \subseteq V \rightarrow W$ and take $u \in U \cap V$. Then $u \in U$, and so $u \in V \rightarrow W$, and as $u \in U \cap V$ also gives $u \in V$, yielding $u \in W$. This means $U \cap V \subseteq W$, as desired.

Subproof (2).ii Now we know that $P\uparrow(X)$ is a **HA**, we show Φ preserves the constants and operations, making the range of Φ a Heyting subalgebra of $\prod_{x\in X} A_x$.

For constants, we note that by the definition of Φ, we have Φ is defined for all $Y \in P\uparrow(X)$ and for all $x \in X$, hence Φ must be defined on X, as of course $X \in P\uparrow(X)$, hence $\Phi(X)(x) = \top$ iff $x \in X$; since $X = \top$ in $P\uparrow(X)$, we have $\Phi(\top)(x) = \top$ iff $x \in X$, and similarly $\Phi(\bot)(x) = \bot$ iff $x \notin X$.

For the operations, we have two subcases each, depending on whether Φ evaluates to $\top$ or $\bot$.

Case 1. $U \cap V$. We must show $\Phi(U \cap V)(x) = \Phi(U)(x) \wedge \Phi(V)(x)$.

1. $\Phi(U \cap V)(x) = \top$ iff $x \in U \cap V$ iff $x \in U$ and $x \in V$ iff $\Phi(U)(x) = \top$ and $\Phi(V)(x) = \top$ iff $\Phi(U)(x) \wedge \Phi(V)(x) = \top$.

2. $\Phi(U \cap V)(x) = \bot$ iff $x \notin U \cap V$ iff $x \notin U$ or $x \notin V$ iff $\Phi(U)(x) = \bot$ or $\Phi(V)(x) = \bot$ iff $\Phi(U)(x) \wedge \Phi(V)(x) = \bot$.

Case 2. $U \cup V$. We must show $\Phi(U \cup V)(x) = \Phi(U)(x) \vee \Phi(V)(x)$.

1. $\Phi(U \cup V)(x) = \top$ iff $x \in U \cup V$ iff $x \in U$ or $x \in V$ iff $\Phi(U)(x) = \top$ or $\Phi(V)(x) = \top$ iff $\Phi(U)(x) \vee \Phi(V)(x) = \top$.

2. $\Phi(U \cup V)(x) = \bot$ iff $x \notin U \cup V$ iff $x \notin U$ and $x \notin V$ iff $\Phi(U)(x) = \bot$ and $\Phi(V)(x) = \bot$ iff $\Phi(U)(x) \vee \Phi(V)(x) = \bot$.

Case 3. $U \to V$: We must show $\Phi(U \to V)(x) = \Phi(U)(y) \to \Phi(V)(y)$.

1. $\Phi(U \to V)(x) = \top$; this holds iff $x \in (U \to V)$. Now suppose $x \in U$, and let $y : x \leq y$. Then $y \in U$ by upward closure, and so $\Phi(U)(y) = \top$, and by definition of $\to$ above between upwards-closed subsets, we have $y \in V$, and so $\Phi(V)(y) = \top$. So $\Phi(U)(y) \to \Phi(V)(y) = \top$, and so $\Phi(U \to V)(x) = \Phi(U)(y) \to \Phi(V)(y) = \top$.
2. $\Phi(U \to V)(x) = \bot$; this holds iff $x \notin (U \to V)$. So $\exists y : x \leq y \wedge y \in U \wedge y \notin V$. This gives that $\Phi(U)(y) = \top$ and $\Phi(V)(y) = \bot$, so that $\Phi(U)(y) \to \Phi(V)(y) = \bot$, and hence $\Phi(U \to V)(x) = \Phi(U)(y) \to \Phi(V)(y) = \bot$, as desired.

Subproof (3). We must show Φ is an embedding of $P\uparrow(X)$ into $\prod_{x \in X} A_x$. The above argument proved preservation of operations and therefore homomorphism, so it remains to show that Φ is injective. Take $U, V \in P\uparrow(X) : U \neq V$. Then if $U \neq \varnothing$ and $V \neq \varnothing$ either $x \in U$ and $x \notin V$ or $x \notin U$ and $x \in V$. In the first case, $\Phi(U)(x) = \top$ and $\Phi(V)(x) = \bot$; in the second case, $\Phi(V)(x) = \top$ and $\Phi(U)(x) = \bot$; in either case, $\Phi(V)(x) \neq \Phi(U)(x)$. The case in which either

$U \neq \varnothing$ or $V \neq \varnothing$ is analogous.

Subproof (4). We need to show Φ is onto, i.e. for any step-function $f(x)$ in $\prod_{x \in X} A_x$ there exists $Y \in P\uparrow(X) : \Phi(Y)(x) = f(x)$. So take a step-function $f(x)$ in $\prod_{x \in X} A_x$, and assume $f(x) > \bot$; then $f(x) = \top$ and by definition of a step-function $\forall y : y \geq x (f(y) = \top)$. Now fix $y \geq x$ and suppose for a contradiction there is no $Y \in P\uparrow(X) : \Phi(Y)(y) = \top$. Then for all $Y \in P\uparrow(X) : \Phi(Y)(y) = \bot$ and $y \notin Y$ by definition of Φ. As $f(x) = \top$ is in a poset product, x is the index for a set A_x ordered by the partial order $X = \langle X, \geq \rangle$ in the collection $(A_x : x \in X)$. By assumption $x \leq y$; by upwards closure, $y \in X$, so that by definition of Φ we have $\Phi(X)(y) = \top$, contradicting the assumption that there is no $Y \in P\uparrow(X) : \Phi(Y)(y) = \top$. Hence Φ is onto.

□

Completeness

We conclude this section by arguing that the Kripke semantics above is also complete, referring to Proposition 2.3.17 which relates Poset Products and Kripke-structures, the classic Representation results of Stone, and the remark of Jipsen and Montagna which we proved in Theorem 2.3.27.

Theorem 2.3.28 (Completeness). *If* $\Gamma \Vdash^{\mathrm{K}} \phi$ *then* $\Gamma \vdash_{\mathbf{IL}} \phi$.

Proof. Let $\Gamma \equiv \psi_1, \ldots, \psi_n$. Suppose

$$\Gamma \nvdash_{\mathbf{IL}} \phi$$

By Proposition 2.1.4 it follows that

$$\nvdash_{\mathbf{IL}_H} \psi_1 \to \ldots \to \psi_n \to \phi$$

By the algebraic completeness result for **HA** algebras with respect to the proof system $\mathbf{IL}_H$ (Theorem 2.3.63), it follows that for some **HA** algebra $\mathcal{G}$ and some mapping $h : \mathit{Atom} \to G$ from propositional variables to elements of $\mathcal{G}$, we have

$$[\![\psi_1 \to \ldots \to \psi_n \to \phi]\!]^{\mathcal{G}}_h \neq \top$$

By Harrop's Theorem [33], we know Intuitionistic logic has the finite model property, and thus we can take $\mathcal{G}$ to be finite. By Theorem 2.3.26, $\mathcal{G}$ is isomorphic to the canonical extension $P\uparrow(\mathcal{D}(\mathcal{G}))$, and by Theorem 2.3.27 $P\uparrow(\mathcal{D}(\mathcal{G}))$ is isomorphic to the poset product $\prod_{d(g) \in \mathcal{D}(\mathcal{G})} A_{d(g)}$. Hence there exists a finite

$$\frac{}{\Gamma, \phi \vdash \phi}\ \text{Ax} \qquad \frac{\Gamma \vdash \phi \qquad \Delta \vdash \phi \to \psi}{\Gamma, \Delta \vdash \psi}\ \to \text{E}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi}\ \to \text{I} \qquad \frac{\Gamma \vdash \phi \otimes \psi \qquad \Delta, \phi, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\ \otimes \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \otimes \psi}\ \otimes \text{I} \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi}\ \wedge\text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}\ \wedge \text{I} \qquad \frac{\Gamma \vdash \phi \vee \psi \qquad \Delta, \phi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\ \vee \text{E}$$

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi}\ \vee \text{I} \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma \vdash \neg\phi, \Delta}\ \neg\text{R}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}\ \bot \text{E} \qquad \frac{\Gamma \vdash \phi}{\Gamma, \neg\phi, \vdash}\ \neg\text{L}$$

Figure 2.5: Intuitionistic Affine logic **ALi**

poset $\mathcal{W} = \langle P \uparrow \mathcal{D}(\mathcal{G}), \subseteq \rangle$ and an assignment $h' : Atom \to \{\bot, \top\}_{\mathbf{BA}}$ of atomic formulas to elements of the Poset Product $\prod_{d(g) \in \mathcal{D}(\mathcal{G})} A_{d(g)}$, which we now write $\mathbf{A}_{\mathcal{W}}$, such that for some $w \in P \uparrow \mathcal{D}(\mathcal{G})$:

$$[\![\psi_1 \to \ldots \to \psi_n \to \phi]\!]_{h'}^{\mathbf{A}_{\mathcal{W}}}(w) \neq \top$$

By Proposition 2.3.17, we have a Kripke-structure $\mathcal{M}^{\mathbf{A}_{\mathcal{W}}}$ such that for some $w \in P \uparrow \mathcal{D}(\mathcal{G})$:

$$(w \Vdash_{h'}^{\mathrm{K}} \psi_1 \to \ldots \to \psi_n \to \phi) \neq \top$$

so that:

$$(w \Vdash_{h'}^{\mathrm{K}} \psi_1 \wedge \ldots \wedge \psi_n) \nleq (w \Vdash^{\mathrm{K}} \phi)$$

and $\psi_1, \ldots, \psi_n \nVdash^{\mathrm{K}} \phi$. □

2.3.6 Pocrims

Here we discuss the algebraic semantics appropriate for Intuitionistic Affine logic and Classical Affine logic, namely bounded pocrims and Involutive pocrims. We generalise these to lattice-ordered and Complete pocrims in this work. We first discuss residuated lattices and lattice-ordered monoids, as our algebraic semantics for the logics considered in this thesis fall into this larger category of residuated structures.

Lattice-ordered Monoids and Residuated Lattices

Definition 2.3.29 (Commutative Lattice-ordered monoid)**.** *A structure* $\mathcal{A} = \langle A, \wedge, \vee, \otimes, 1\rangle$ *is a* commutative lattice-ordered monoid *if*

- $\langle A, \wedge, \vee\rangle$ *is a lattice*
- $\langle A, \otimes, 1\rangle$ *is a commutative monoid*
- $\otimes$ *is monotonic increasing with respect to the lattice order on* A*.*

There are a number of slightly different definitions of this concept in the literature, varying with the exact relationship required between the lattice and the monoid structure. Our definition is weak. The most common definition has $\otimes$ distributes over $\vee$, and some definitions have that $\otimes$ distributes over both $\vee$ and $\wedge$.

Definition 2.3.30 (Commutative Residuated lattice)**.** $\mathcal{A} = \langle A, \wedge, \vee, \otimes, 1, \rightarrow\rangle$ *is called a* commutative residuated lattice *if*

- $\langle A, \wedge, \vee, \otimes, 1\rangle$ *is a commutative lattice-ordered monoid*
- $x \otimes y \leq z$ *if and only if* $x \leq y \rightarrow z$

Note 8. *We prove one of the essential preconditions of a residuated lattice, namely that it is monotone with respect to the monoidal operation. Indeed, the following lemma only requires the assumption that the algebra is a partially ordered monoid, and reflects our expectation of the tensor operation that it should, in some sense, make a claim logically "stronger" with respect to the ordering.*

Lemma 2.3.31. *In a partially-ordered monoid (and therefore a bounded or involutive pocrim)* $\mathcal{P}$*,* $\forall x, y, z, z' : x \leq y$ *and* $z \leq z' \Rightarrow x \otimes z \leq y \otimes z'$*.*

Proof. Assume $x \leq y$ and $z \leq z'$. By monotonicity of the monoid w.r.t partial ordering, $x \otimes z \leq x \otimes z'$ (using $z \leq z'$ and adding x on both sides of the inequality). Similarly, $x \otimes z' \leq y \otimes z'$ (by using $x \leq y$ and adding z' on both sides of the inequality). By transitivity we have $x \otimes z \leq y \otimes z'$. □

Note 9. *Standard definitions of residuated lattice do not include the fact that* $\otimes$ *preserves the lattice order in a weaker sense: namely that if* $x' \leq x$ *then* $x' \otimes y \leq x \otimes y$*. This, however follows from the residuation property: Suppose* $x' \leq x$*, then since* $x \otimes y \leq x \otimes y$*,* $x \leq y \rightarrow (x \otimes y)$*. Therefore* $x' \leq y \rightarrow (x \otimes y)$ *and hence* $x' \otimes y \leq x \otimes y$*.*

Lemma 2.3.32. *In any residuated lattice* $\mathcal{A}$*,* $\forall x, y, z \in \mathcal{A} : x \leq y \Rightarrow y \rightarrow z \leq x \rightarrow z$*.*

Proof. Suppose that $x \leq y$. Then

$$x \otimes (y \to z) \leq y \otimes (y \to z)$$

by lemma 2.3.46. Now, in a residuated lattice one always has:

$$y \otimes (y \to z) \leq z$$

by residuation, since one always has the identity $x \otimes (y \to z) \leq y \otimes (y \to z)$ in a residuated lattice. Then by transitivity we have:

$$x \otimes (y \to z) \leq z$$

and thus by residuation again:

$$(y \to z) \leq (x \to z)$$

as desired. □

In the following lemma we prove an algebraic analogue of the Cut rule from sequent calculus in the context of residuated lattices.

Lemma 2.3.33. *In any residuated lattice $\mathcal{A}$, $\forall x, y, z, w \in \mathcal{A} : x \leq y$ and $z \leq w$ then*

$$x \otimes (y \to z) \leq w.$$

Proof. Assume $x \leq y$ and $z \leq w$. Then by 2.3.32 we have that

$$y \to z \leq x \to z$$

Applying residuation, this gives us:

$$x \otimes (y \to z) \leq z$$

and by transitivity of $\leq$ we have:

$$x \otimes (y \to z) \leq w$$

as desired. □

Definition 2.3.34. *(Bounded Pocrim) A Bounded pocrim, that is, a partially-ordered-commutative-residuated-integral-monoid, is*

$$\mathbf{P}_{\bot} = \langle P, \leq, \top, \bot, \otimes, \to \rangle$$

P is a poset ordered by $\leq$, partial order given by

$$a \leq b \Leftrightarrow a \to b = \top$$

bounded by $\bot, \top$ with $\bot$ as the bottom and $\top$ as the top and identity. Here $\otimes$ is a monoidal operation, with residual given by $\to$.

More specifically:

- *$\langle P, \otimes, \top \rangle$ is a commutative monoid with neutral element $\top$;*
- *$\langle P, \leq \rangle$ is a partially ordered set such that $\leq$ is compatible with $\otimes$ (i.e., $a \leq b$ implies $a \otimes c \leq b \otimes c$ and $\top$ is the maximum of $\langle P, \leq \rangle$;*
- *$\langle P, \leq \rangle$ has the residuum property, that is $a \otimes c \leq b$ if and only if $c \leq a \to b$.*

Definition 2.3.35. *(Involutive Pocrim) An* Involutive Pocrim*, is a bounded Pocrim satisfying $\neg\neg a = a$ for all $a \in P$; we can take $((a \to \bot) \to \bot) = \neg\neg a$.*

Definition 2.3.36. *(Lattice-ordered Involutive Pocrim) A* lattice-ordered involutive pocrim *is an involutive pocrim with a lattice ordering, with greatest lower bound and least upper bound given by meet and join respectively i.e. $\wedge, \vee$.*

Definition 2.3.37. *(Complete lattice-ordered Involutive Pocrim) A* Complete lattice-ordered involutive pocrim *is a lattice-ordered involutive pocrim such that arbitrary meet and arbitrary join are defined. More formally: $\forall X \subseteq \mathcal{P} : \bigwedge X$, $\bigvee X$ exists.*

Definition 2.3.38. *(Definition of Lower Bound) Let $Y = \langle Y, \leq \rangle$, i.e. a set partially ordered by $\leq$, and $X \subseteq Y$. Then a* lower bound *of a subset X of Y is an element $u \in Y : u \leq x$ for all $x \in X$.*

Definition 2.3.39. *(Definition of Infima) Let X, Y be as above. Then an* infimum*, or* inf *of a subset X of Y, denoted* $\inf(X)$*, is an element u' such that:*

- *$u' \in Y : u' \leq x$ for all $x \in X$ i.e. u' is a lower bound (i.e. u' is a lower bound of a subset X of Y)*
- *$\forall y \in X$ such that y is a lower bound of X in Y, $y \leq u'$ (i.e. u' is the greatest such lower bound)*

From this latter it follows that:

Corollary 2.3.40. *Let X, Y be as above, and take $x \in Y$. Then $x \leq \inf(X) \Leftrightarrow \forall y \in X : (x \leq y)$.*

We will reuse this latter in the sequel.

Definition 2.3.41. *(Inf of a set in the lattice is the Meet, and vice versa.) Let X, Y be as above, with $X \subseteq Y$. Then* $\inf(X) = \bigwedge X$. *Similarly, the* $\sup(X) = \bigvee X$.

Lemma 2.3.42. *(Existence of Suprema and Infima in Complete pocrims.) Let $\mathcal{P}$ be a complete, lattice-ordered, involutive pocrim. Then for any $X \subseteq \mathcal{P}$, we have that* $\inf(X)$,*and therefore $\bigwedge \mathcal{P}$, exists (and similarly for $\bigvee(X) = \sup(X)$).*

Proof. By 2.3.37 and 2.3.40. □

Note 10. *We note in passing that every involutive pocrim is a bounded pocrim. We also note that hereon we say 'bounded pocrim', from now on we mean 'bounded, lattice-ordered pocrim.'*

2.3.7 Valid Sequents in ALi

Definition 2.3.43 (Denotation functions)**.** *Given an bounded, lattice-ordered, complete pocrim $\mathcal{P}_\bot$, and a mapping from propositional variables to elements of $\mathcal{P}_\bot$:*

$$p \mapsto [\![p]\!] \in \mathcal{P}_\bot$$

We thus refer to the denotation of a variable p as $[\![p]\!]_{\mathcal{P}_\bot}$. We can extend that mapping to all formulas in the language of L in a straightforward way:

$$\begin{array}{rcl} [\![\phi \otimes \psi]\!]_{\mathcal{P}_\bot} & := & [\![\phi]\!]_{\mathcal{P}_\bot} \otimes [\![\psi]\!]_{\mathcal{P}_\bot} \\ [\![\phi \wedge \psi]\!]_{\mathcal{P}_\bot} & := & [\![\phi]\!]_{\mathcal{P}_\bot} \wedge [\![\psi]\!]_{\mathcal{P}_\bot} \\ [\![\phi \vee \psi]\!]_{\mathcal{P}_\bot} & := & [\![\phi]\!]_{\mathcal{P}_\bot} \vee [\![\psi]\!]_{\mathcal{P}_\bot} \\ [\![\phi \to \psi]\!]_{\mathcal{P}_\bot} & := & [\![\phi]\!]_{\mathcal{P}_\bot} \to [\![\psi]\!]_{\mathcal{P}_\bot} \end{array}$$

Definition 2.3.44 (Validity)**.** *A sequent $\phi_1, ..., \phi_n \vdash_{\mathbf{ALi}} \psi$ is then said to be valid in $\mathcal{P}_\bot$, alias $\mathcal{P}_\bot$-valid, if $[\![\phi_1]\!] \otimes ... \otimes [\![\phi_n]\!] \leq [\![\psi]\!]$ holds in $\mathcal{P}_\bot$. A sequent is said to be* valid *in* **ALi** *if it is valid in* all *bounded, lattice-ordered pocrims. We sometimes represent this thus:*

$$\Gamma \vDash_{\mathbf{ALi}} \phi$$

In the case where ϕ is valid in all bounded lattice-ordered pocrims, we write

$$\vDash_{\mathbf{ALi}} \phi$$

One can show that the valid sequents, in the sense above, are precisely the ones provable in Intuitionistic Affine logic. Indeed, we show this in a later section.

Note 11. *We may sometimes abbreviate the statement '$\phi_1, ..., \phi_n \vdash \psi$ is valid in $\mathcal{P}_\perp$', or '$[\![\phi_1]\!] \otimes ... \otimes [\![\phi_n]\!] \leq [\![\psi]\!]$ holds in $\mathcal{P}_\perp$' with $\otimes\Gamma \leq [\![\psi]\!]$ is valid in $\mathcal{P}_\perp$.*

Proposition 2.3.45. *A sequent $\Gamma \vdash \psi$ is* **ALi***-valid iff it is provable in* **ALi**.

Algebraic Lemmata for Complete, involutive Pocrims

Here we record some useful lemmata for the theory of lattice ordered pocrims for the sake of ease in the sequel. The reader is encouraged to skim these, as they mostly are included for the sake of completeness of exposition.

Lemma 2.3.46. *In a partially-ordered monoid (and therefore a bounded or involutive pocrim) $\mathcal{P}$, $\forall x, y, z, z' : x \leq y$ and $z \leq z' \Rightarrow x \otimes z \leq y \otimes z'$.*

Proof. Assume $x \leq y$ and $z \leq z'$. By monotonicity of the monoid w.r.t partial ordering, $x \otimes z \leq x \otimes z'$ (using $z \leq z'$ and adding x on both sides of the inequality). Similarly, $x \otimes z' \leq y \otimes z'$ (by using $x \leq y$ and adding z' on both sides of the inequality). By transitivity we have $x \otimes z \leq y \otimes z'$. □

Lemma 2.3.47. *In any bounded pocrim $\mathbf{P}_\perp$: $\forall x, y : x \otimes y \leq x$.*

Proof. As all pocrims are integral, we have as an axiom: $\forall y : y \leq \top$. So:

$$\begin{aligned} y \leq \top \quad &\Rightarrow \quad x \otimes y \leq x \otimes \top (monotonicity) \\ &\Rightarrow \quad x \otimes y \leq x (neutrality) \end{aligned}$$

so $\forall x, y : x \otimes y \leq x$, as desired. □

One can generalise this last lemma:

Lemma 2.3.48. *In any bounded pocrim $\mathbf{P}_\perp$: $x_1 \otimes x_2 ... x_n \leq x_i$ for $1 \leq i \leq n$.*

Proof. By induction on n. □

Lemma 2.3.49. *In any bounded pocrim $\mathbf{P}_\perp$: $\forall x, y : x \leq \perp \Rightarrow x \leq y$.*

Proof. As all bounded pocrims are bounded below by $\perp$, we have as an axiom: $\forall y : \perp \leq y$. So: assume $x \leq \perp$, and fix x. Then $x \leq y$ for all y, so that $\forall x, y : x \leq \perp \Rightarrow x \leq y$ as desired. □

Lemma 2.3.50. *In a lattice-ordered pocrim: $\mathcal{P}$, $\forall x, x', y, z, z' : x \leq (y \vee z) \wedge (x' \otimes y) \leq z' \wedge (x' \otimes z) \leq z' \Rightarrow x \otimes x' \leq z'$.*

Proof. Assume $x \leq (y \vee z)$, $(x' \otimes y) \leq z'$ and $(x' \otimes z) \leq z'$. By residuation, $y \leq x' \to z'$ and $z \leq x' \to z'$, and so $y \vee z \leq x' \to z'$; so we have $x \leq y \vee z \leq x' \to z'$, hence $x \leq x' \to z'$ and after applying residuation again $x' \otimes x \leq z'$ and commutativity gives $x \otimes x' \leq z'$. □

Lemma 2.3.51. *In a lattice-ordered pocrim:* $\mathcal{P}$*,* $\forall x, y, z, x', z' : x \leq (y \otimes z) \wedge (z' \otimes (y \otimes z)) \leq x' \Rightarrow x \otimes z' \leq x'$*.*

Proof. Assume $x \leq (y \otimes z)$ and $z' \otimes (y \otimes z) \leq x'$.

$$\begin{aligned} x \leq y \otimes z : \quad & \Rightarrow \quad x \otimes z' \leq (y \otimes z) \otimes z' (monotonicity) \\ & \Rightarrow \quad x \otimes z' \leq z' \otimes (y \otimes z) (commutativity) \\ z' \otimes (y \otimes z) \leq x' : \quad & \Rightarrow \quad x \otimes z' \leq x' (transitivity) \end{aligned}$$

□

Lemma 2.3.52. *In a lattice-ordered pocrim:* $\mathcal{P}$*,* $\forall x, y, z, w : (w \leq x) \wedge (z \leq x \to y) \Rightarrow w \otimes z \leq y$*.*

Proof. Assume $(w \leq x) \wedge (z \leq x \to y)$. Then by monotonicity $(w \otimes z) \leq (x \otimes z)$ and residuation $z \otimes x \leq y$. By commutativity, $x \otimes z \leq y$. By transitivity, $w \otimes z \leq y$ as desired. □

Lemma 2.3.53. *In a lattice-ordered pocrim:* $\mathcal{P}$*,* $\forall x, y : x \leq (y \wedge z) \Rightarrow (x \leq y) \wedge (x \leq z)$*.*

Proof. Suppose $x \leq (y \wedge z)$. Then $(y \wedge z) \leq y$ for any y, and $(y \wedge z) \leq z$ for any z. By transitivity, $x \leq y$ and $x \leq z$ by transitivity. □

Lemma 2.3.54. *In a lattice-ordered pocrim:* $\mathcal{P}$*,* $\forall x, y : (x \leq y) \wedge (x \leq z) \Rightarrow x \leq (y \wedge z)$*.*

Proof. Suppose $(x \leq y) \wedge (x \leq z)$. We have that $(y \wedge z) \leq z$ for any z, and similarly $(y \wedge z) \leq y$ for any y. x cannot be greater than $y \wedge z$ without being greater than either, which would contradict our assumption, so $x \leq (y \wedge z)$. □

Lemma 2.3.55. *In a lattice-ordered pocrim:* $\mathcal{P}$*,* $\forall x, y : (x \leq y) \Rightarrow x \leq (y \vee z)$*.*

Proof. Suppose $(x \leq y)$. Then since $y \leq y \vee z$ for any z, $x \leq y \vee z$ by transitivity. □

Lemma 2.3.56. *In an involutive, lattice-ordered pocrim:* $\mathcal{P}$*,* $\forall x, y, z, u : (x \leq y) \wedge (u \leq (y \to z) \Rightarrow x \otimes u \leq z$*.*

Proof. Fix x, y, z, u and assume $(x \leq y)$ and $u \leq y \to z$. Then by residuation, $u \otimes y \leq z$ and by monotonicity $x \otimes u \leq y \otimes u$, so that by transitivity $x \otimes u \leq z$. □

2.3.8 GBL, BL, and MV-algebras

Definition 2.3.57 (**GBL**-algebras)**.** *A* **GBL**-algebra *is a residuated lattice which satisfies the* divisibility property[7]*: if $x \leq y$ then $y \otimes (y \to x) = x$. This is equivalent to requiring that the residuated lattice satisfies the equation:*

$$x \otimes (x \to y) = y \otimes (y \to x)$$

A **GBL**-*algebra is said to be* commutative *if $\otimes$ is a commutative operation.*

A **GBL**-*algebra is said to be* integral *if 1 is the top element of the lattice, i.e. $x \leq 1$ for all $x \in A$. In this case we also denote 1 by $\top$.*

A **GBL**-*algebra is said to be* bounded *if the lattice has a bottom element $\bot$, i.e. $\bot \leq x$ for all $x \in A$.*

Note 12. *We abbreviate 'commutative, integral and bounded* **GBL**-*algebras' by* $\mathbf{GBL}_{ewf}$-*algebras.*

Note 13. $\mathbf{GBL}_{ewf}$-*algebras provide an algebraic semantics for both logical systems* $\mathbf{GBL}_{ewf}$*, alias* **LLi***. For* $\mathbf{GBL}_{ewf}$*, this is mentioned in various papers of Montagna et al, e.g. [4], wherein the authors say that "*$\mathbf{GBL}_{ewf}$ *is strongly algebraizable ... Its equivalent algebraic semantics is the variety of commutative, integral and bounded GBL algebras."*

Definition 2.3.58. *A* **BL**-*algebra is a* **GBL**-*algebra that additionally satisfies prelinearity, i.e. $(x \to y) \vee (y \to x)$.*

Definition 2.3.59 (**MV**-algebra)**.** *A bounded* **GBL**-*algebra is called a* **MV**-*algebra if the negation map ($\neg x = x \to \bot$) is an involution, i.e. $(x \to \bot) \to \bot = x$, for all x.*

MV-algebras provide an algebraic semantics for classical Łukasiewicz logic. Here we are interested in a particular MV algebra which we will use in our Kripke semantics for **BL**:

Definition 2.3.60 (Standard **MV**-chain)**.** *For $x \in [0,1]$, let $\overline{x} := 1 - x$. The* standard **MV**-chain*, denoted $[0,1]_{\mathrm{MV}}$, is the* **MV**-*algebra defined as follows: The domain of $[0,1]_{\mathrm{MV}}$ is the unit interval $[0,1]$, with the constants and binary*

[7] Note that since $y \to x \leq y \to x$, it is always the case that $y \otimes (y \to x) \leq x$ (this is the counit of the adjunction defining residuation). The name "divisibility" property makes sense if one interprets $x \otimes y$ as multiplication $x \times y$, and $y \to x$ as division $\frac{x}{y}$. This is cancellation: i.e. if $0 \leq x \leq y \leq 1$ then $y \times \frac{x}{y} = x$. Note that if $y = 0$ then $x = 0$ as well.

operations defined as

$$\begin{array}{lcl} \top & := & 1 \\ \bot & := & 0 \\ x \wedge y & := & \min\{x, y\} \\ x \vee y & := & \max\{x, y\} \\ x \otimes y & := & \max\{0, \overline{\overline{x} + \overline{y}}\} \\ x \to y & := & \min\{1, \overline{\overline{y} - \overline{x}}\} \end{array}$$

Note 14. *$x \otimes y$ is equivalent to $\max\{0, x + y - 1\}$, and $x \to y$ is equivalent to $\min\{1, y - x + 1\}$.*[8]

Lemma 2.3.61. *Recall that we are using the abbreviation $\overline{x} := 1 - x$. The following hold in the standard MV-chain $[0,1]_{MV}$*

(i) *For all $n \geq 2$, $x_1 \otimes \ldots \otimes x_n = \max\{0, \overline{\overline{x_1} + \ldots + \overline{x_n}}\}$.*

(ii) *$x_1 \otimes \ldots \otimes x_n \leq x_i$, for $i \in \{1, \ldots, n\}$*

(iii) *if $x \leq y \vee z$ and $u \otimes y \leq v$ and $u \otimes z \leq v$ then $u \otimes x \leq v$.*

(iv) *If $x \leq y$ and $u \leq y \to z$ then $x \otimes u \leq z$.*

(v) *If $x \leq y$ and $z \leq w$ then $x \otimes z \leq y \otimes w$.*

(vi) *If $x \leq y$ and $v \otimes y \leq z$ then $v \otimes x \leq z$.*

(vii) *$x \otimes (x \to y) = y \otimes (y \to x)$.*

Proof. We prove (i) in detail, the other properties follow easily from the fact that $[0,1]_{MV}$ is an MV algebra. By induction on n.

Basis: $n = 2$. By Definition 2.3.60.

Induction Step: Assume the result holds for $n > 2$, we show

$$x_1 \otimes \ldots \otimes x_n \otimes x_{n+1} = \max\{0, \overline{\overline{x_1} + \ldots + \overline{x_n} + \overline{x_{n+1}}}\}$$

[8] The examiners have queried our choice of definition, as the standard definition in this note is appears simpler. This is really just a matter of presentation: we find it more intuitive to work with $\overline{x}$ so that one can see that $x \otimes y$ is essentially $x + y$ (but with the bars). We also find that it helps trim down the proofs of certain lemmata, and makes for a cleaner presentation (no excess 1's flanking formulae, etc.).

Indeed

$$
\begin{aligned}
x_1 \otimes \ldots \otimes x_n \otimes x_{n+1} &= (x_1 \otimes \ldots \otimes x_n) \otimes x_{n+1} \\
&\overset{\{D\}2.3.60}{=} \max\{0, \overline{\overline{\overline{x_1 \otimes \ldots \otimes x_n}} + \overline{x_{n+1}}}\} \\
&\overset{\text{IH}}{=} \max\{0, \overline{\overline{\max\{0, \overline{\overline{x_1} + \ldots + \overline{x_n}}\}} + \overline{x_{n+1}}}\} \\
&\overset{(*)}{=} \max\{0, \overline{\min\{1, \overline{x_1} + \ldots + \overline{x_n}\} + \overline{x_{n+1}}}\} \\
&\overset{(\dagger)}{=} \max\{0, \overline{\overline{x_1} + \ldots + \overline{x_n} + \overline{x_{n+1}}}\}
\end{aligned}
$$

using that

(∗) $\overline{\max\{0, \overline{\{}\phi\}\}} = \min\{1, a\}$

(†) $\max\{0, \overline{\min\{1, a\} + b}\} = \max\{0, \overline{a + b}\}$.

□

2.3.9 Adequacy for IL under Heyting Algebras

Soundness

Lemma 2.3.62. *(Soundness) Let* **HA** *be a Heyting algebra and let* $[\![]\!] : P \to \mathbf{HA}$ *be a valuation function from the parameters of* $\mathcal{L}$*, the language of* **IL***, to* **HA***. If* $\Gamma \vdash_{\mathbf{IL}} \phi$ *then* $\Gamma \vDash_{\mathbf{HA}} \phi$*.*

Proof. The proof is by induction on the height of derivations in Intuitionistic logic. This means we check that the rules of the logic preserve the order of **HA**. Assume $\Gamma = \psi_1, ..., \psi_n$ and let $\bigwedge \Gamma := \psi_1 \wedge ... \wedge \psi_n$. Here we write $[\![\phi]\!]$ for $[\![\phi]\!]_{\mathbf{HA}}$. We ignore the structural rules in the proof that follows, as they can be seen to follow straightforwardly.

(Ax). $\Gamma, \phi \vdash_{\mathbf{IL}} \phi$. By 2.2.2 we must show $[\![\bigwedge \Gamma \wedge \phi]\!] \leq [\![\phi]\!]$, or by definition of the valuation, $[\![\bigwedge \Gamma]\!] \wedge [\![\phi]\!] \leq [\![\phi]\!]$. But this latter follows directly from the definition of lattice $\wedge$, i.e. $x \wedge y \leq y$ for any $y \in \mathbf{HA}$.

($\wedge$I). Suppose the last rule in the derivation $\Gamma \vdash_{\mathbf{IL}} \phi$ was ($\wedge$I). Then $\phi = \psi \wedge \chi$ and we have derivations $\Gamma_1 \vdash_{\mathbf{IL}} \psi$ and $\Gamma_2 \vdash_{\mathbf{IL}} \chi$ for subsets $\Gamma_1, \Gamma_2 \subseteq \Gamma$. By induction hypothesis we then have $[\![\bigwedge \Gamma_1]\!] \leq [\![\psi]\!]$ and $[\![\bigwedge \Gamma_2]\!] \leq [\![\chi]\!]$, whence $[\![\bigwedge \Gamma]\!] \leq [\![\bigwedge \Gamma_1]\!] \wedge [\![\bigwedge \Gamma_2]\!] \leq [\![\psi]\!] \wedge [\![\chi]\!] = [\![\psi \wedge \chi]\!]$ by the definition of valuation and meet in a Heyting algebra.

($\wedge$E). Suppose the last rule applied in the derivation of $\Gamma \vdash_{\mathbf{IL}} \phi$ was ($\wedge$E). Then $\phi = \psi$ or $\phi = \chi$ and we have a derivation of $\Gamma \vdash_{\mathbf{IL}} \psi \wedge \chi$. By induction hypothesis we have that $[\![\bigwedge \Gamma]\!] \leq [\![\psi \wedge \chi]\!]$, and by definition of valuation and meet we have that $[\![\psi]\!] \wedge [\![\chi]\!] = [\![\psi \wedge \chi]\!]$, so that $[\![\bigwedge \Gamma]\!] \leq [\![\psi]\!] \wedge [\![\chi]\!]$ and by lattice theory we either have $[\![\bigwedge \Gamma]\!] \leq [\![\psi]\!]$ or $[\![\bigwedge \Gamma]\!] \leq [\![\chi]\!]$.

($\to$I). Suppose the last rule applied in the derivation of $\Gamma \vdash_{\mathbf{IL}} \phi$ was ($\to$I). Then $\phi = \psi \to \chi$ and $\Gamma, \psi \vdash_{\mathbf{IL}} \chi$ and so $[\![\bigwedge \Gamma]\!] \wedge [\![\psi]\!] = [\![\bigwedge \Gamma]\!] \wedge [\![\psi]\!] \leq [\![\chi]\!]$ by the induction hypothesis. But then by the definition of $\to$ in **HA** we have $[\![\bigwedge \Gamma]\!] \leq [\![\psi]\!] \to [\![\chi]\!] = [\![\psi \to \chi]\!]$, where the equality holds by the definition of valuation.

($\to$E). Suppose the last rule applied in the derivation of $\Gamma \vdash_{\mathbf{IL}} \phi$ was ($\to$E). Then $\phi = \chi$ with the end sequent is of the form $\Gamma, \Delta \vdash_{\mathbf{IL}} \chi$, and we have derivations of $\Gamma \vdash_{\mathbf{IL}} \psi$ and $\Delta \vdash_{\mathbf{IL}} \psi \to \chi$. By inductive hypothesis we have $[\![\bigwedge \Gamma]\!] \leq [\![\psi]\!]$ and $[\![\bigwedge \Delta]\!] \leq [\![\psi \to \chi]\!]$. By the definition of the valuation, this last gives us $[\![\bigwedge \Delta]\!] \leq [\![\psi]\!] \to [\![\chi]\!]$. By lemma 2.2.5, this gives us $[\![\bigwedge \Gamma]\!] \wedge [\![\bigwedge \Delta]\!] \leq [\![\psi]\!]$, or $[\![\Gamma \wedge \bigwedge \Delta]\!] \leq [\![\psi]\!]$, as desired.

($\vee$I) Suppose that $\Gamma \vdash_{\mathbf{IL}} \phi$ is derived with last rule ($\vee$I), such that $\phi = \psi \vee \chi$ so that there is a derivation of ψ, i.e. $\Gamma \vdash_{\mathbf{IL}} \psi$ or a derivation of χ, i.e. $\Gamma \vdash_{\mathbf{IL}} \chi$. By induction hypothesis this means that we have $[\![\bigwedge \Gamma]\!] \leq [\![\psi]\!]$ or we have $[\![\bigwedge \Gamma]\!] \leq [\![\chi]\!]$. In either case, by lemma 2.2.6 we have $[\![\bigwedge \Gamma)]\!] \leq [\![\psi]\!] \vee [\![\chi]\!]$, which is equivalent by the definition of valuation to $[\![\bigwedge \Gamma]\!] \leq [\![\psi \vee \chi]\!]$.

($\vee$E) Suppose $\Gamma \vdash_{\mathbf{IL}} \phi$ is derived with last rule ($\vee$E) so that there is a derivation $\Gamma \vdash_{\mathbf{IL}} \psi \vee \chi$ with $\Gamma \cup \{\psi\} \vdash_{\mathbf{IL}} \phi$ and $\Gamma \cup \{\chi\} \vdash_{\mathbf{IL}} \phi$. Then by induction hypothesis we have $[\![\bigwedge \Gamma]\!] \leq [\![\psi \vee \chi]\!] = [\![\psi]\!] \vee [\![\chi]\!]$, and $[\![\bigwedge \Gamma]\!] \wedge [\![\psi]\!] \leq [\![\phi]\!], [\![\bigwedge \Gamma]\!] \wedge [\![\chi]\!] \leq [\![\phi]\!]$. That is, $[\![\bigwedge \Gamma]\!] \leq [\![\bigwedge \Gamma]\!] \wedge ([\![\psi]\!] \vee [\![\chi]\!]) = ([\![\bigwedge \Gamma]\!] \wedge [\![\psi]\!]) \vee ([\![\bigwedge \Gamma]\!] \wedge [\![\chi]\!]) \leq [\![\phi]\!]$.

($\perp$E). Suppose $\Gamma \vdash_{\mathbf{IL}} \phi$ is derived with last rule ($\perp$E) so that there is a derivation $\Gamma \vdash_{\mathbf{IL}} \perp$. Then by induction hypothesis we have $[\![\bigwedge \Gamma]\!] \leq [\![\perp]\!]$. Then as $[\![\perp]\!] \leq [\![\phi]\!]$ for any ϕ, by 2.2.7 we have that $[\![\bigwedge \Gamma]\!] \leq [\![\phi]\!]$ for any ϕ.

□

Note 15. *We do not prove soundness for Gödel-Dummett logic under linearly ordered Heyting algebras, but this can be carried out as above, only adding to our considerations the axiom of pre-linearity.*

Completeness

The completeness of the algebraic semantics via the *Lindenbaum-Tarski method* [71], which takes an equivalence class of provable formulae and produces a canonical model of the logic. The Lindenbaum-Tarski algebra that results is then shown to be the target algebra, in this case the Heyting algebra **HA**. The result however follows from the proof given in 2.7.3, by taking $\otimes = \wedge$ (and thereby regaining contraction on the left of the turnstyle). The proof is essentially the same, with the residuation case in that setting (2.7.3) being modified to residuation of $\wedge$. We give this proof for completeness of exposition, but recall we also use the result itself in our earlier proof of completeness for the Kripke semantics via poset products. It is useful to compare the term-model construction with the Kripke model built there out of a poset product of step-functions, so we provide details of this classic argument to facilitate that comparison.

Theorem 2.3.63. *If ϕ is **HA**-valid, for each Heyting algebra **HA**, then ϕ is provable in **IL**.*

Construction 2.3.64. *Construct a Heyting algebra **HA** as follows.* [9] *Let $\mathcal{L}$ be the set of **IL**-formulas. Define an equivalence relation on $\mathcal{L}$ by*

$$\phi \sim \psi \quad \textit{iff} \quad \vdash_{\mathbf{IL}} \phi \leftrightarrow \psi$$

Let $H = \mathcal{L}/\sim$ be the set of equivalence classes $[\![\phi]\!] = \{\psi \in \mathcal{L} : \phi \sim \psi\}$ with respect to $\sim$. Partially order H by $[\![\phi]\!] \leq [\![\psi]\!]$ iff $\vdash_{\mathbf{IL}} \phi \to \psi$. Set $\bot_H = [\![\bot]\!]$ and $\top_H = [\![\top]\!]$. Define operations by $[\![\phi]\!] \wedge_H [\![\psi]\!] = [\![\phi \wedge \psi]\!]$, $[\![\phi]\!] \vee_H [\![\psi]\!] = [\![\phi \vee \psi]\!]$, $[\![\phi]\!] \to_H [\![\psi]\!] = [\![\phi \to \psi]\!]$.

Now, check that $(H, \leq, \wedge_H, \vee_H, \to_H, \bot_H, \top_H)$ is a Heyting algebra via deductions in the proof calculus. For instance, we can verify residuation in our natural deduction system:

$$[\![\phi]\!] \wedge_H [\![\psi]\!] \leq [\![\chi]\!] \quad \textit{iff} \quad [\![\phi]\!] \leq [\![\psi]\!] \to_H [\![\chi]\!]$$

which is

$$\phi \wedge \psi \vdash_{\mathbf{IL}} \chi \quad \textit{iff} \quad \phi \vdash_{\mathbf{IL}} \psi \to \chi$$

First, the left to right direction, or:

$$\textit{if} \quad [\![\phi]\!] \wedge_H [\![\psi]\!] \leq [\![\chi]\!] \quad \textit{then} \quad [\![\phi]\!] \leq ([\![\psi]\!] \to_H [\![\chi]\!])$$

Now suppose $[\![\phi]\!] \wedge_H [\![\psi]\!] \leq [\![\chi]\!]$, or $\phi \wedge \psi \vdash_{\mathbf{IL}} \chi$. Then by 2.1.2 we have $\phi, \psi \vdash_{\mathbf{IL}} \chi$

[9] This is a special Heyting Algebra constructed from the terms of the calculus, hence our denotation H, instead of **HA**, but of course it ends up coinciding with **HA** in a certain sense, i.e. it is the canonical Heyting Algebra characterising the logic, as the proof demonstrates.

and by ($\to$ *I*) *we get* $\phi \vdash_{\mathbf{IL}} \psi \to \chi$. *Hence* $\phi \vdash_{\mathbf{IL}} (\psi \to \chi)$ *or if* $[\![\phi]\!] \wedge_H [\![\psi]\!] \leq [\![\chi]\!]$ *then* $[\![\phi]\!] \leq ([\![\psi]\!] \to_H [\![\chi]\!])$ *as desired.*

The converse direction of residuation, i.e.

$$\textit{if} \quad [\![\phi]\!] \leq ([\![\psi]\!] \to [\![\chi]\!]) \quad \textit{then} \quad [\![\phi]\!] \wedge_H [\![\psi]\!] \leq [\![\chi]\!]$$

proceeds similarly, also using 2.1.2.

Now, define from parameters to the Heyting Algebra constructed above $[\![]\!] : P \to H$ by

$$[\![p]\!] = [\![p]\!]_H$$

Then we observe (by induction): For any formula ϕ of **IL**, $[\![\phi]\!] = [\![\phi]\!]_H$.

Now suppose ϕ is a formula of **IL** such that $\nvdash_{\mathbf{IL}} \phi$. Then as ϕ is not provable, $\nvdash_{\mathbf{IL}} \phi \leftrightarrow \top$ and so $[\![\phi]\!] \neq \top_H$. Hence ϕ is not valid in the Heyting algebra H. This ends the proof of completeness.

Note 16. *Again, as noted in the previous section on soundness, we do not prove completeness for Gödel-Dummett logic here with respect to the algebraic semantics, but the proof is a straightforward extension of the preceding argument, where the relevant Lindenbaum-Tarski algebra is a* prelinear *Heyting algebra, i.e. a Heyting algebra with* $(x \to y) \vee (y \to x)$ *for all* $x, y \in H$.

2.3.10 Adequacy for ALi under Bounded Pocrims

Below we give the proofs for the soundness and completeness of Intuitionistic Affine logic for the algebraic semantics of bounded pocrims. We do this because the result appears to be folklore, that is, has not been given explicitly in print (to our knowledge), and to some extent the plausibility of our work in the next chapter depends upon the reader believing the bounded pocrims are indeed the adequate semantics for Intuitionistic Affine logic.

Soundness

Theorem 2.3.65. *(Soundness) Let* $\mathbf{P}_\perp$ *be a Bounded pocrim and let* $[\![]\!] : P \to \mathbf{P}_\perp$ *be the denotation function from the parameters of* $\mathcal{L}_\otimes$*, the language of* **ALi***, to* $\mathbf{P}_\perp$*. If* $\Gamma \vdash_{\mathbf{ALi}} \phi$ *then* $\Gamma \vDash_{\mathbf{ALi}} \phi$*.*

Proof. The proof is by induction on the height of derivations in Intuitionistic Affine logic. This means we check that the rules of the logic preserve the order of $\mathbf{P}_\perp$. Assume $\Gamma = \psi_1, ..., \psi_n$ and let $\bigotimes \Gamma := \psi_1 \otimes ... \otimes \psi_n$. We write $[\![\phi]\!]$ for $[\![\phi]\!]_{\mathbf{P}_\perp}$. We ignore the exchange structural rule although it can be seen to be sound under the semantics.

(Ax). $\Gamma, \phi \vdash_{\mathbf{ALi}} \phi$. By 2.2.2 we must show $[\![\bigotimes \Gamma \otimes \phi]\!] \leq [\![\phi]\!]$, or by definition of the valuation, $[\![\bigotimes \Gamma]\!] \otimes [\![\phi]\!] \leq [\![\phi]\!]$. But this latter follows directly from integrality in the algebra or 2.3.47, i.e. $x \otimes y \leq y$ for any $y \in \mathbf{P}_\perp$.

($\wedge$I). Suppose the last rule in the derivation $\Gamma \vdash_{\mathbf{ALi}} \phi$ was ($\wedge$I). Then $\phi = \psi \wedge \chi$ and we have derivations $\Gamma_1 \vdash_{\mathbf{ALi}} \psi$ and $\Gamma_2 \vdash_{\mathbf{ALi}} \chi$ for subsets $\Gamma_1, \Gamma_2 \subseteq \Gamma$. By induction hypothesis we then have $[\![\bigotimes \Gamma_1]\!] \leq [\![\psi]\!]$ and $[\![\bigotimes \Gamma_2]\!] \leq [\![\chi]\!]$, whence $[\![\bigotimes \Gamma]\!] \leq [\![\bigotimes \Gamma_1]\!] \wedge [\![\bigotimes \Gamma_2]\!] \leq [\![\psi]\!] \wedge [\![\chi]\!] = [\![\psi \wedge \chi]\!]$ by the definition of valuation of meet and 2.3.54 in a lattice-ordered bounded pocrim.

($\wedge$E). Suppose the last rule applied in the derivation of $\Gamma \vdash_{\mathbf{ALi}} \phi$ was ($\wedge$E). Then $\phi = \psi$ or $\phi = \chi$ and we have a derivation of $\Gamma \vdash_{\mathbf{ALi}} \psi \wedge \chi$. By induction hypothesis we have that $[\![\bigotimes \Gamma]\!] \leq [\![\psi \wedge \chi]\!]$, and by definition of valuation and meet we have that $[\![\psi]\!] \wedge [\![\chi]\!] = [\![\psi \wedge \chi]\!]$, so that $[\![\bigotimes \Gamma]\!] \leq [\![\psi]\!] \wedge [\![\chi]\!]$ and by 2.3.53 we either have $[\![\bigotimes \Gamma]\!] \leq [\![\psi]\!]$ or $[\![\bigotimes \Gamma]\!] \leq [\![\chi]\!]$.

($\rightarrow$I). Suppose the last rule applied in the derivation of $\Gamma \vdash_{\mathbf{ALi}} \phi$ was ($\rightarrow$I). Then $\phi = \psi \rightarrow \chi$ and $\Gamma, \psi \vdash_{\mathbf{ALi}} \chi$ and so $[\![\bigotimes \Gamma \otimes \psi]\!] = [\![\bigotimes \Gamma]\!] \otimes [\![\psi]\!] \leq [\![\chi]\!]$ by the induction hypothesis. But then by the definition of $\rightarrow$ in $\mathbf{P}_\perp$ we have $[\![\bigotimes \Gamma]\!] \leq [\![\psi]\!] \rightarrow [\![\chi]\!] = [\![\psi \rightarrow \chi]\!]$, where the equality holds by the definition of valuation.

($\rightarrow$E). Suppose the last rule applied in the derivation of $\Gamma \vdash_{\mathbf{ALi}} \phi$ was ($\rightarrow$E). Then $\phi = \chi$ with the end sequent is of the form $\Gamma, \Delta \vdash_{\mathbf{ALi}} \chi$, and we have derivations of $\Gamma \vdash_{\mathbf{ALi}} \psi$ and $\Delta \vdash_{\mathbf{ALi}} \psi \rightarrow \chi$. By inductive hypothesis we have $[\![\bigotimes \Gamma]\!] \leq [\![\psi]\!]$ and $[\![\bigotimes \Delta]\!] \leq [\![\psi \rightarrow \chi]\!]$. By the definition of the valuation, this last gives us $[\![\bigotimes \Delta]\!] \leq [\![\psi]\!] \rightarrow [\![\chi]\!]$. By lemma 2.3.56, this gives us $[\![\bigotimes \Gamma]\!] \otimes [\![\bigotimes \Delta]\!] \leq [\![\psi]\!]$ (or $[\![\bigotimes \Gamma \otimes \bigotimes \Delta]\!] \leq [\![\psi]\!]$) as desired.

($\vee$I) Suppose that $\Gamma \vdash_{\mathbf{ALi}} \phi$ is derived with last rule ($\vee$I), such that $\phi = \psi \vee \chi$ so that there is a derivation of ψ, i.e. $\Gamma \vdash_{\mathbf{ALi}} \psi$ or a derivation of χ, i.e. $\Gamma \vdash_{\mathbf{ALi}} \chi$. By induction hypothesis this means that we have $[\![\bigotimes \Gamma]\!] \leq [\![\psi]\!]$ or we have $[\![\bigotimes \Gamma]\!] \leq [\![\chi]\!]$. In either case, by lemma 2.3.55 we have $[\![\bigotimes \Gamma]\!] \leq [\![\psi]\!] \vee [\![\chi]\!]$, which is equivalent by the definition of valuation to $[\![\bigotimes \Gamma]\!] \leq [\![\psi \vee \chi]\!]$.

($\vee$E) Suppose $\Gamma \vdash_{\mathbf{ALi}} \phi$ is derived with last rule ($\vee$E) so that there is a derivation $\Gamma \vdash_{\mathbf{ALi}} \psi \vee \chi$ with $\Gamma \cup \{\psi\} \vdash_{\mathbf{ALi}} \phi$ and $\Gamma \cup \{\chi\} \vdash_{\mathbf{ALi}} \phi$. Then by induction hypothesis we have:

- $[\![\otimes \Gamma]\!] \leq \{[\![\phi]\!]\} \vee \{[\![\psi]\!]\}$

- $[\![\otimes\Delta]\!] \otimes [\![\phi]\!] \leq [\![\chi]\!]$
- $[\![\otimes\Delta]\!] \otimes [\![\psi]\!] \leq [\![\chi]\!]$

By Lemma 2.3.50, these imply $[\![\otimes\Gamma]\!] \otimes [\![\otimes\Delta]\!] \quad \leq \quad [\![\chi]\!]$.

($\bot$E). Suppose $\Gamma \vdash_{\mathbf{ALi}} \phi$ is derived with last rule ($\bot$E) so that there is a derivation $\Gamma \vdash_{\mathbf{ALi}} \bot$. Then by induction hypothesis we have $[\![\bigotimes \Gamma]\!] \leq [\![\bot]\!]$. Then as $[\![\bot]\!] \leq [\![\phi]\!]$ for any ϕ, by 2.3.49 we have that $[\![\bigotimes \Gamma]\!] \leq [\![\phi]\!]$ for any ϕ.

□

Note 17. *We note here in passing that we can obtain soundness for* $\mathbf{GBL}_{ewf}$ *(alias* **LLi***) and* **BL** *by making suitable adjustments to the algebraic semantics, i.e. using* $\mathbf{GBL}_{ewf}$ *and* **BL***-algebras respectively and considering as additional cases the divisibility rule and pre-linearity respectively.*

Completeness

Note 18. *We prove completeness of the algebraic semantics via the Lindenbaum -Tarski method, as in the Intuitionistic case with Heyting algebras, this time with bounded, lattice-ordered pocrims* $\mathbf{P}_\bot$*. The proof is standard, and imitates the one we gave before for* **IL***. Nonetheless, we give the proof, as the result is folklore and does not have a source and we shall need it in following chapter.*

Theorem 2.3.66. *If* ϕ *is* $\mathbf{P}_\bot$*-valid, for each for each bounded lattice-ordered pocrim* $\mathbf{P}_\bot$*, then* ϕ *is provable in* **ALi***.*

Construction 2.3.67. *Construct the following bounded, lattice-ordered pocrim. Let* $\mathcal{L}_\otimes$ *be the set of* **ALi***-formulas. Define an equivalence relation on* $\mathcal{L}_\otimes$ *by*

$$\phi \sim \psi \quad \textit{iff} \quad \vdash_{\mathbf{ALi}} \phi \leftrightarrow \psi$$

Let $P_\bot = \mathcal{L}_\otimes / \sim$ *be the set of equivalence classes* $[\![\phi]\!] = \{\psi \in \mathcal{L}_\otimes : \phi \sim \psi\}$ *with respect to* $\sim$*. Partially order* $P_\bot$ *by* $[\![\phi]\!] \leq [\![\psi]\!]$ *iff* $\vdash_{\mathbf{ALi}} \phi \to \psi$*. Set* $\bot_{P_\bot} = [\![\bot]\!]$ *and* $\top_{P_\bot} = [\![\top]\!]$*. Define operations by* $[\![\phi]\!] \wedge_{P_\bot} [\![\psi]\!] = [\![\phi \wedge \psi]\!]$, $[\![\phi]\!] \otimes_{P_\bot} [\![\psi]\!] = [\![\phi \otimes \psi]\!]$, $[\![\phi]\!] \vee_{P_\bot} [\![\psi]\!] = [\![\phi \vee \psi]\!]$, $[\![\phi]\!] \to_{P_\bot} [\![\psi]\!] = [\![\phi \to \psi]\!]$*. Now, check that* $(P, \leq, \wedge_{P_\bot}, \otimes_{P_\bot}, \vee_{P_\bot}, \to_{P_\bot}, \bot_{P_\bot}, \top_{P_\bot})$ *is a bounded lattice-ordered pocrim. For example, we can check that the tensor is residuated:*

$$[\![\phi]\!] \otimes_{P_\bot} [\![\psi]\!] \leq [\![\chi]\!] \quad \textit{iff} \quad [\![\phi]\!] \leq [\![\psi]\!] \to_{P_\bot} [\![\chi]\!]$$

which is

$$\phi \otimes \psi \vdash_{\mathbf{ALi}} \chi \quad \textit{iff} \quad \phi \vdash_{\mathbf{ALi}} \psi \to \chi$$

First, the left to right direction, or:

$$\textit{if} \quad [\![\phi]\!] \otimes_{P_\perp} [\![\psi]\!] \le [\![\chi]\!] \quad \textit{then} \quad [\![\phi]\!] \le ([\![\psi]\!] \to_{P_\perp} [\![\chi]\!])$$

Now suppose $[\![\phi]\!] \otimes_{P_\perp} [\![\psi]\!] \le [\![\chi]\!]$*, or* $\phi \otimes \psi \vdash_{\mathbf{ALi}} \chi$*. Then by 2.1.3 we have* $\phi, \psi \vdash_{\mathbf{ALi}} \chi$ *and by (*$\to$ *I) we get* $\phi \vdash_{\mathbf{ALi}} \psi \to \chi$*. Hence* $\phi \vdash_{\mathbf{ALi}} (\psi \to \chi)$ *or if* $[\![\phi]\!] \wedge_{P_\perp} [\![\psi]\!] \le [\![\chi]\!]$ *then* $[\![\phi]\!] \le ([\![\psi]\!] \to_{P_\perp} [\![\chi]\!])$ *as desired.*

The converse direction of residuation, i.e.

$$\textit{if} \quad [\![\phi]\!] \le ([\![\psi]\!] \to_{P_\perp} [\![\chi]\!]) \quad \textit{then} \quad [\![\phi]\!] \otimes_{P_\perp} [\![\psi]\!] \le [\![\chi]\!]$$

proceeds similarly, also using 2.1.3.

Now, define from parameters to the bounded lattice-ordered pocrim constructed above $[\![]\!] : P \to P_\perp$ by

$$[\![p]\!] = [\![p]\!]_{P_\perp}$$

Then we observe (by induction): For any formula ϕ of **ALi**, $[\![\phi]\!] = [\![\phi]\!]_{P_\perp}$.

Now suppose ϕ is a formula of **ALi** such that $\nvdash_{\mathbf{ALi}} \phi$. Then as ϕ is not provable, $\nvdash_{\mathbf{ALi}} \phi \leftrightarrow \top$ and so $[\![\phi]\!] \ne \top_{P_\perp}$. Hence ϕ is not valid in the bounded lattice-ordered pocrim $P_\perp$. This ends the proof of completeness.

Note 19. *We note that the proofs of completeness for the algebraic semantics of* **BL** *with respect to* **BL***-algebras, and* $\mathbf{GBL}_{ewf}$ *with respect to* $\mathbf{GBL}_{ewf}$*-algebras are easily obtained from this proof by making suitable adjustments: e.g. in the case of* $\mathbf{GBL}_{ewf}$*-algebras, add to the above proof consideration of the divisibility axiom, and then proceed otherwise as in the case of bounded pocrims. Similarly with* **BL***, we merely consider the added axiom of prelinearity.*

Chapter 3

$\mathbf{GBL}_{ewf}$

3.1 Introduction

In [4], Bova and Montagna study the computational complexity of the propositional logic $\mathbf{GBL}_{ewf}$. We will refer to this logic as *Intuitionistic Łukasiewicz logic* **LLi**. The original name $\mathbf{GBL}_{ewf}$ derives from the fact that this logic has a sound and complete algebraic semantics based on commutative (exchange), integral (weakening) and bounded (ex-falsum) **GBL**-algebras. Bova and Montagna have shown that the consequence problem for **LLi** is PSPACE complete, by showing that the equational and the quasiequational theories of commutative, integral and bounded GBL algebras are PSPACE complete.

Their decision procedure relies on the construction of a particular commutative, integral and bounded GBL algebra which the authors called *poset sum*, now known as poset product (presented herein) for any given poset. The elements of the poset product are particular monotone functions, which are called antichain labellings there (also in [37], [38] and other papers in the area) assigning real values (in $[0, 1]_{\mathbf{MV}}$) to each element of the poset. In this chapter we demonstrate how this can be viewed as a generalization of Kripke semantics adequate for Intuitionistic Łukasiewicz logic **LLi**.

The semantics developed herein generalises Kripke's own in several respects. First, it generalises the valuation functions, which take formulas and the partial order of worlds and returns values in the unit interval endowed with the MV-algebra. This is to be contrasted with the classic case [42] in which the codomain of the valuation is the two-element Boolean algebra. Our semantics also generalises the notion of monotonicity from Kripke's semantics, in a fashion we call 'sloping': the idea is that the formula goes from trivial values to non-trivial values, and then immediately to truth. This contrasts with the classic Kripkean case in which a formula is assigned no value at all until it's valued true, and then in the future, forever true. However, it turns out that by removing the tensor, and restricting to the top and bottom elements of the unit interval we return to the classic Kripke semantics for Intuitionistic logic, hence we truly have a 'fuzzy' generalisation of Kripke's semantics for Intuitionistic logic. **LLi**, viewed as a fragment of Łukasiewicz logic, acts as the nexus of several important nominally 'fuzzy' logics, such as basic logic (**BL**), Gödel logic, and the product logic. The Hilbert-style presentation of **LLi** coincides with Hajek's **BL** [30] minus pre-linearity, and therefore behaves as the constructive (or Intuitionistic) kernel of **BL**. As such, **LLi** can also be viewed as a generalisation of Intuitionistic propositional logic. **GBL**-algebras, introduced by Montagna and Jipsen in [37], can be regarded as a generalization of Heyting algebras, the standard algebraic model for **IL**. In general **GBL** lacks the commutativity present in **BL** and Heyting algebras, although finite **GBL**-algebras are commutative (see [37]).

First we present the standard algebraic semantics for propositional $\mathbf{GBL}_{ewf}$ alias **LLi**, then define validity for the system. Following this, we present the generalisation of Kripke semantics which we shall call *Bova-Montagna*[1] *semantics*, for **LLi**. In Section 3.5 we outline how this semantics is a natural generalization of the Kripke semantics for intuitionistic logic. The following sections prove the soundness and completeness of the semantics.

To refresh the reader's mind and fix notation for the present section (and minimise returning to the second chapter), we give the logic as a sequent-style natural deduction system (already shown equivalent to the Hilbert-style system $\mathbf{GBL}_{ewf}$ in chapter 2) just above.

3.2 $\mathbf{GBL}_{ewf}$ and LLi and Valid Sequents

Here we give a definition of validity based on the algebraic semantics for already given in 2.3.8.

Definition 3.2.1 (Algebraic semantics for $\mathcal{L}_{\otimes}$)**.** *Given a* $\mathbf{GBL}_{ewf}$ *algebra* $\mathcal{A} = \langle A, \wedge, \vee, \otimes, \top, \bot, \to \rangle$ *and a mapping* $h : Atom \to A$ *from propositional variables to elements of* $\mathcal{A}$ *we can extend that mapping to mapping* $[[\phi]]_h^{\mathcal{A}} \in A$ *on all formulas*

[1] The semantics we present has been extracted from Bova and Montagna's definition of poset products ([4], Definition 2), which itself is based on the work of Jipsen and Montagna on ordinal sum constructions [37].

$$\frac{}{\phi \vdash \phi}\text{ Ax} \qquad \frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi}\text{ W} \qquad \frac{\Gamma, \phi, \psi, \Delta \vdash \chi}{\Gamma, \psi, \phi, \Delta \vdash \chi}\text{ Ex}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi} \to \text{I} \qquad \frac{\Gamma \vdash \phi \to \psi \qquad \Delta \vdash \phi}{\Gamma, \Delta \vdash \psi} \to \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \otimes \psi} \otimes \text{I} \qquad \frac{\Gamma, \phi, \psi \vdash \chi \qquad \Delta \vdash \phi \otimes \psi}{\Gamma, \Delta \vdash \chi} \otimes \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} \wedge \text{I} \qquad \frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_i} \wedge \text{E}$$

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2} \vee \text{I)} \qquad \frac{\Gamma \vdash \phi \vee \psi \qquad \Delta, \phi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi} \vee \text{E}$$

$$\frac{\Gamma, \phi, \phi \to \psi \vdash \chi}{\Gamma, \psi, \psi \to \phi \vdash \chi}\text{ DIV} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \phi} \bot \text{E}$$

Figure 3.1: Intuitionistic Łukasiewicz logic **LLi**

ϕ *as:*

$$\begin{aligned}
[\![p]\!]_h^{\mathcal{A}} &:= h(p) \\
[\![\bot]\!]_h^{\mathcal{A}} &:= \bot \\
[\![\phi \wedge \psi]\!]_h^{\mathcal{A}} &:= [\![\phi]\!]_h^{\mathcal{A}} \wedge [\![\psi]\!]_h^{\mathcal{A}} \\
[\![\phi \vee \psi]\!]_h^{\mathcal{A}} &:= [\![\phi]\!]_h^{\mathcal{A}} \vee [\![\psi]\!]_h^{\mathcal{A}} \\
[\![\phi \otimes \psi]\!]_h^{\mathcal{A}} &:= [\![\phi]\!]_h^{\mathcal{A}} \otimes [\![\psi]\!]_h^{\mathcal{A}} \\
[\![\phi \to \psi]\!]_h^{\mathcal{A}} &:= [\![\phi]\!]_h^{\mathcal{A}} \to [\![\psi]\!]_h^{\mathcal{A}}
\end{aligned}$$

A sequent $\phi_1, \ldots \phi_n \vdash \psi$ *is then said to be* $\mathbf{GBL}_{ewf}$*-valid in* $\mathcal{A}$*, if for any mapping* $h : Atom \to A$

$$[\![\phi_1]\!]_h^{\mathcal{A}} = \top \quad \textit{and} \quad \ldots \quad \textit{and} \quad [\![\phi_n]\!]_h^{\mathcal{A}} = \top \quad \textit{implies} \quad [[\psi]]_h^{\mathcal{A}} = \top$$

A sequent $\phi_1, \ldots, \phi_n \vdash \psi$ *is said to be* $\mathbf{GBL}_{ewf}$*-valid if it is* $\mathbf{GBL}_{ewf}$*-valid in all* $\mathbf{GBL}_{ewf}$*-algebras.*

A sequent $\phi_1, \ldots, \phi_n \vdash \psi$ *is said to be* **LLi***-valid in* $\mathcal{A}$*, if for any* $h : Atom \to A$

$$[\![\phi_1]\!]_h^{\mathcal{A}} \otimes \ldots \otimes [\![\phi_n]\!]_h^{\mathcal{A}} \leq [\![\psi]\!]_h^{\mathcal{A}}$$

with the understanding that if the context $\Gamma = \phi_1, \ldots, \phi_n$ *is empty* $(n = 0)$ *then* $[\![\phi_1]\!]_h^{\mathcal{A}} \otimes \ldots \otimes [\![\phi_n]\!]_h^{\mathcal{A}} = \top$.

A sequent is said to be **LLi***-valid if it is* **LLi***-valid in all* $\mathbf{GBL}_{ewf}$*-algebras.*

Proposition 3.2.2. *A sequent* $\Gamma \vdash \psi$ *is* **LLi***-valid iff it is provable in* **LLi***.*

We note again (see discussion around deduction theorem for $\mathbf{GBL}_{ewf}$ in Proposition 2.1.3), even though the sequents of $\mathbf{GBL}_{ewf}$ and **LLi** are given different interpretations, these interpretations coincide for theorems, i.e. for sequents with empty context $\vdash \psi$.

3.3 Kripke Semantics for LLi

Note 20. *The Kripke semantics for* **LLi** *that we propose is based on the poset sum alias poset product construction of [4] (see Section 5.7 for more details). We first need to define a particular class of functions a partial order* $\mathcal{W} = \langle W, \geq \rangle$ *to the standard MV-chain:*

Definition 3.3.1 (Sloping functions)**.** *Let* $\mathcal{W} = \langle W, \geq \rangle$ *be a partial order, and let* $v \succ w := v \geq w$ *and* $v \neq w$*. A function* $f : W \to [0,1]$ *is said to be a* sloping function *if* $f(w) > \bot$ *implies* $\forall v \succ w(f(v) = \top)$*.*

The above implies that if $f : W \to [0,1]$ is a sloping function and $f(w) < \top$ then $\forall v \prec w(f(v) = \bot)$. That is, along any increasing chain $w_1 \prec w_2 \prec \ldots \prec w_n$, there can only be at most one point i such that $\bot < f(w_i) < \top$, and for $j < i$ we must have $f(w_j) = \bot$, and for $j > i$ we must have $f(w_j) = \top$.[2]

Lemma 3.3.2. *If* $f : W \to [0,1]$ *and* $g : W \to [0,1]$ *are sloping functions, then the following functions are also sloping functions:*

$$\begin{array}{rcl} (f \wedge g)(w) & := & \min\{fw, gw\} \\ (f \vee g)(w) & := & \max\{fw, gw\} \\ (f \otimes g)(w) & := & \max\{0, \overline{\overline{fw} + \overline{gw}}\} \end{array}$$

Proof. Let f, g be sloping functions. Let us consider each case:

- $f \wedge g$. Assume $(f \wedge g)(w) > \bot$, i.e. $\min\{fw, gw\} > \bot$. This implies that we have both $fw > \bot$ and $gw > \bot$. But since f and g are assumed to be sloping functions, we get that $\forall v \succ w(f(v) = \top)$ and $\forall v \succ w(g(v) = \top)$, from which it follows that $\forall v \succ w(\min\{f(v), g(v)\} = \top)$.
- $f \vee g$. Assume $(f \vee g)(w) > \bot$ i.e. $\max\{fw, gw\} > \bot$. This implies that we have at least one of $fw > \bot$ or $gw > \bot$. In case $fw > \bot$, f is a sloping

[2] One might ask, as the reviewers of this thesis have done, why there is only one point such that $\bot < f(w) < \top$, and what would happen/break if there were more than one such point? The answer to the first question is given above in the derivation, i.e. this is a consequence of the original definition of poset products, which uses antichain labellings. The answer to the second question: is while relaxing the current sloping functions to more general monotonic functions works for Affine (i.e. is sound), it is unsound once one adds divisibility to the picture.

function by hypothesis, so we have $\forall v \succ w(f(v) = \top)$ from which it follows $\forall v \succ w(\max\{f(v), g(v)\} = \top)$. The case of $gw > \bot$ is similar.

- $f \otimes g$. Assume $(f \otimes g)(w) > \bot$ i.e. $\max\{0, \overline{\overline{fw} + \overline{gw}}\} > 0$. This means $\max\{0, \overline{\overline{fw} + \overline{gw}}\} = \max\{0, f(w)+g(w)-1\} > 0$; and hence $f(w)+g(w)-1 > 0$. This implies that neither $f(w) = \bot$ nor $g(w) = \bot$, i.e. we have both $f(w) > \bot$ and $g(w) > \bot$. Since both $f(w), g(w)$ are sloping functions by hypothesis $\forall v \succ w(f(v) = \top)$ and $\forall v \succ w(g(v) = \top)$. So $\forall v \succ w$ $\max\{0, f(v) + g(v) - 1\} = \max\{0, \top + \top - 1\} = \max\{0, \top + 0\} = \top$, as desired.

□

Definition 3.3.3. *A* Bova-Montagna structure *(or BM-structure) is a pair* $\mathcal{M} = \langle \mathcal{W}, \Vdash^{\mathrm{BM}} \rangle$ *where* $\mathcal{W} = \langle W, \succeq \rangle$ *is a poset, and* $\Vdash^{\mathrm{BM}}$ *is an infix operator (on worlds and propositional variables) taking values in* $[0,1]_{\mathrm{MV}}$*, i.e.* $(w \Vdash^{\mathrm{BM}} p) \in [0,1]_{\mathrm{MV}}$*, such that for any propositional variable* p *the function* $\lambda w.(w \Vdash^{\mathrm{BM}} p) : W \to [0,1]$ *is a sloping function.*

Definition 3.3.4. *Let* $\lfloor \cdot \rfloor$ *be the usual "floor" operation on the standard MV-chain* $[0,1]_{\mathrm{MV}}$*, corresponding to the case distinction*

$$\lfloor x \rfloor := \begin{cases} \top & \text{if } \quad x = \top \\ \bot & \text{if } \quad x < \top \end{cases}$$

which is known as the "Monteiro-Baaz Δ*-operator." Given a (not necessarily sloping) function* $f : W \to [0,1]$ *and a* $w \in W$*, let us write* $\lfloor \inf \rfloor_{v \succeq w}$ *for the following construction:*

$$\lfloor \inf \rfloor_{v \succeq w} f(v) := \min\{f(w), \inf_{v \succ w} \lfloor f(v) \rfloor\}$$

where $\inf_{v \succ w} \lfloor f(v) \rfloor$ *is the infimum of the set* $\{\lfloor f(v) \rfloor : v \succ w\} \subseteq [0,1]$.

Lemma 3.3.5. *This definition of* $\lfloor \inf \rfloor_{v \succeq w}$ *can also be equivalently written as*

$$\lfloor \inf \rfloor_{v \succeq w} f(v) := \begin{cases} f(w) & \text{if} \forall v \succ w(f(v) = \top) \\ \bot & \text{if} \exists v \succ w(f(v) < \top) \end{cases}$$

and for any $f : W \to [0,1]$ *the function* $\lambda w. \lfloor \inf \rfloor_{v \succeq w} f(v)$ *is a sloping function.*

Proof. First let us show that this is an equivalent definition. Consider two cases:
Case 1. $\forall v \succ w(f(v) = \top)$. In this case $\inf_{v \succ w} \lfloor f(v) \rfloor = \top$ and hence

$$\lfloor \inf \rfloor_{v \succeq w} f(v) = \min\{f(w), \top\} = f(w)$$

Case 2. $\exists v > w(f(v) < \top)$. In this case $\inf_{v>w}\lfloor f(v) \rfloor = \bot$

$$\lfloor \inf \rfloor_{v \geq w} f(v) = \min\{f(w), \bot\} = \bot$$

In order to see that $\lambda w.\lfloor \inf \rfloor_{v\geq w} f(v)$ is a sloping function, assume that for some w we have $\lfloor \inf \rfloor_{v\geq w} f(v) > \bot$, and let $w' > w$. By definition we have that $\forall v > w(f(v) = \top)$, and hence $f(w') = \top$ and $\forall v > w'(f(v) = \top)$, which implies $\lfloor \inf \rfloor_{v\geq w'} f(v) = \top$. □

Definition 3.3.6 (Kripke Semantics for $\mathcal{L}_\otimes$). *Given a BM-structure*

$$\mathcal{M} = \langle \mathcal{W}, \Vdash^{\mathrm{BM}} \rangle$$

the valuation function $w \Vdash^{\mathrm{BM}} p$ on propositional variables p can be extended to all $\mathcal{L}_\otimes$-formulas as:

$$\begin{array}{lcl} w \Vdash^{\mathrm{BM}} \bot & := & \bot \\ w \Vdash^{\mathrm{BM}} \phi \wedge \psi & := & (w \Vdash^{\mathrm{BM}} \phi) \wedge (w \Vdash^{\mathrm{BM}} \psi) \\ w \Vdash^{\mathrm{BM}} \phi \vee \psi & := & (w \Vdash^{\mathrm{BM}} \phi) \vee (w \Vdash^{\mathrm{BM}} \psi) \\ w \Vdash^{\mathrm{BM}} \phi \otimes \psi & := & (w \Vdash^{\mathrm{BM}} \phi) \otimes (w \Vdash^{\mathrm{BM}} \psi) \\ w \Vdash^{\mathrm{BM}} \phi \to \psi & := & \lfloor \inf \rfloor_{v\geq w}((v \Vdash^{\mathrm{BM}} \phi) \to (v \Vdash^{\mathrm{BM}} \psi)) \end{array}$$

where the operations on the right-hand side are the operations on the standard MV-chain $[0,1]_{MV}$, and $\lfloor \inf \rfloor_{v\geq w}$ as in Definition 3.3.4.

Lemma 3.3.7. *For any formula ϕ the function $\lambda w.(w \Vdash^{\mathrm{BM}} \phi) : W \to [0,1]$ is a sloping function.*

Proof. By induction on the complexity of the formula ϕ. The cases for $\psi \vee \xi, \psi \wedge \xi$ and $\psi \otimes \xi$ follow directly from Lemma 3.3.2. The case for $\psi \to \xi$ follows from Lemma 5.3.4. □

We can now generalise the monotonicity property of intuitionistic logic to intuitionistic Łukasiewicz logic **LLi**:

Corollary 3.3.8 (Monotonicity). *The following monotonicity property holds for all $\mathcal{L}_\otimes$-formulas ϕ, i.e.*

$$\text{if} \quad v \geq w \quad \text{then} \quad (v \Vdash^{\mathrm{K}} \phi) \geq (w \Vdash^{\mathrm{K}} \phi)$$

Proof. This follows from the observation that sloping functions are in particular monotone functions. □

3.4 Validity under BM-structures

Definition 3.4.1. *Let $\Gamma = \psi_1, \ldots, \psi_n$. Consider the following definitions:*

- *We say that a sequent $\Gamma \vdash \phi$ holds in a BM-structure $\mathcal{M}$ (written $\Gamma \Vdash^{\mathrm{BM}}_{\mathcal{M}} \phi$) if for all $w \in W$ we have*

$$(w \Vdash^{\mathrm{BM}} \psi_1 \otimes \ldots \otimes \psi_n) \leq (w \Vdash^{\mathrm{BM}} \phi)$$

- *A sequent $\Gamma \vdash \phi$ is said to be valid under the Kripke semantics for $\mathcal{L}_{\otimes}$ (written $\Gamma \Vdash^{\mathrm{BM}} \phi$) if $\Gamma \Vdash^{\mathrm{BM}}_{\mathcal{M}} \phi$ for all BM-structures $\mathcal{M}$.*

We will prove that this semantics is sound and complete for **LLi**, i.e. a sequent $\Gamma \vdash \phi$ is provable in **LLi** iff it is valid in all BM-structures. But first let us show that the semantics presented above is a direct generalisation of Kripke's original semantics.

3.5 BM-structures generalise Kripke structures

Bova-Montagna structures generalise Kripke structures, i.e. Kripke structures are a particular case of BM-structures, when the valuations $w \vDash p \in [0,1]_{\mathrm{MV}}$ are always in the finite set $\{\top, \bot\}$. These can then be identified with the Booleans. Therefore, any Kripke structure $\mathcal{K} = \langle \mathcal{W}, \Vdash^{\mathrm{K}} \rangle$ can be seen as a BM-structure $\mathcal{M} = \langle \mathcal{W}, \Vdash^{\mathrm{BM}} \rangle$, by defining

$$w \Vdash^{\mathrm{BM}} p = \begin{cases} \top & \text{if } w \Vdash^{\mathrm{K}} p \\ \bot & \text{if } w \nVdash^{\mathrm{K}} p \end{cases}$$

for all $w \in W$ and propositional variables p. Recall that $\mathcal{L} \subset \mathcal{L}_{\otimes}$, so any $\mathcal{L}$-formula is also an $\mathcal{L}_{\otimes}$-formula.

Theorem 3.5.1. *For any Kripke structure $\mathcal{K} = \langle \mathcal{W}, \Vdash^{\mathrm{K}} \rangle$ and corresponding BM-structure $\mathcal{M} = \langle \mathcal{W}, \Vdash^{\mathrm{BM}} \rangle$ we have that*

$$w \Vdash^{\mathrm{K}} \phi \quad \textit{iff} \quad (w \Vdash^{\mathrm{BM}} \phi) = \top$$

for all $\mathcal{L}$-formula ϕ.

Proof. It is easy to check that, when restricted to Kripke structures, we have $(v \Vdash^{\mathrm{BM}} \phi) \in \{\top, \bot\}$ for all formulas ϕ. Hence, the result above can be proven by a simple induction on the complexity of the formula ϕ.

Basis: If ϕ is an atomic formulas the result is immediate.

Induction step: Suppose the result holds for all sub-formulas of ϕ:

Case 1. $\phi = \psi \wedge \chi$. We have:

$$\begin{aligned} w \Vdash^{\mathrm{K}} \psi \wedge \chi &\equiv (w \Vdash^{\mathrm{K}} \psi) \quad \text{and} \quad (w \Vdash^{\mathrm{K}} \chi) \\ &\overset{\text{(IH)}}{\Leftrightarrow} (w \Vdash^{\mathrm{BM}} \psi) = \top \quad \text{and} \quad (w \Vdash^{\mathrm{BM}} \chi) = \top \\ &\Leftrightarrow \min\{w \Vdash^{\mathrm{BM}} \psi, w \Vdash^{\mathrm{BM}} \chi\} = \top \\ &\equiv (w \Vdash^{\mathrm{BM}} \psi \wedge \chi) = \top \end{aligned}$$

Case 2. $\phi = \psi \vee \chi$. We have:

$$\begin{aligned} w \Vdash^{\mathrm{K}} \psi \vee \chi &\equiv (w \Vdash^{\mathrm{K}} \psi) \quad \text{or} \quad (w \Vdash^{\mathrm{K}} \chi) \\ &\overset{\text{(IH)}}{\Leftrightarrow} (w \Vdash^{\mathrm{BM}} \psi) = \top \quad \text{or} \quad (w \Vdash^{\mathrm{BM}} \chi) = \top \\ &\Leftrightarrow \max\{w \Vdash^{\mathrm{BM}} \psi, w \Vdash^{\mathrm{BM}} \chi\} = \top \\ &\equiv (w \Vdash^{\mathrm{BM}} \psi \vee \chi) = \top \end{aligned}$$

Case 3. $\phi = \psi \to \chi$. We have

(i) $(v \Vdash^{\mathrm{BM}} \psi) = \top \text{implies} (v \Vdash^{\mathrm{BM}} \chi) = \top$ iff $(v \Vdash^{\mathrm{BM}} \psi) \to (v \Vdash^{\mathrm{BM}} \chi) = \top$

(ii) $\lfloor (v \Vdash^{\mathrm{BM}} \psi) \to (v \Vdash^{\mathrm{BM}} \chi) \rfloor = (v \Vdash^{\mathrm{BM}} \psi) \to (v \Vdash^{\mathrm{BM}} \chi)$, i.e. the "floor operation" is unnecessary, and $\lfloor \inf \rfloor_{v \geq w}$ becomes the standard $\inf_{v \geq w}$ operation.

Therefore:

$$\begin{aligned} w \Vdash^{\mathrm{K}} \psi \to \chi &\equiv \forall v \geq w ((v \Vdash^{\mathrm{K}} \psi) \text{implies} (v \Vdash^{\mathrm{K}} \chi)) \\ &\overset{\text{(IH)}}{\Leftrightarrow} \forall v \geq w ((v \Vdash^{\mathrm{BM}} \psi) = \top \text{implies} (v \Vdash^{\mathrm{BM}} \chi) = \top) \\ &\overset{(i)}{\Leftrightarrow} \forall v \geq w ((v \Vdash^{\mathrm{BM}} \psi) \to (v \Vdash^{\mathrm{BM}} \chi) = \top) \\ &\overset{(ii)}{\Leftrightarrow} \lfloor \inf \rfloor_{v \geq w} ((v \Vdash^{\mathrm{BM}} \psi) \to (v \Vdash^{\mathrm{BM}} \chi)) = \top \\ &\equiv (w \Vdash^{\mathrm{BM}} \psi \to \chi) = \top \end{aligned}$$

which concludes the proof. □

3.6 BM-structures and Poset Products

Recall that a *Poset Product* (cf. [4, Def. 2] and [37]) is defined over a poset $\mathcal{W} = \langle W, \geq \rangle$, as the algebra $\mathbf{A}_{\mathcal{W}}$ of signature $\mathcal{L}_{\otimes}$ whose elements are sloping functions $f : W \to [0,1]$ and operations are defined as

$$
\begin{array}{lll}
(\bot)(w) & := & \bot \\
(f_1 \wedge f_2)(w) & := & \min\{f_1 w, f_2 w\} \\
(f_1 \vee f_2)(w) & := & \max\{f_1 w, f_2 w\} \\
(f_1 \otimes f_2)(w) & := & \max\{0, \overline{\overline{f_1 w} + \overline{f_2 w}}\} \\
(f_1 \to f_2)(w) & := & \begin{cases} f_1(w) \to f_2(w) & \text{if} \quad \forall v > w \quad (f_1(v) \leq f_2(v)) \\ \bot & \text{if} \quad \exists v > w \quad (f_1(v) > f_2(v)) \end{cases}
\end{array}
$$

Since f_1 and f_2 are sloping functions, we have that

$$\forall v > w(f_1(v) \leq f_2(v)) \quad \Leftrightarrow \quad \forall v > w((f_1(v) \to f_2(v)) = \top)$$

Therefore, this last clause of the definition can be simplified to

$$(f_1 \to f_2)(w) \quad := \quad \lfloor \inf \rfloor_{v \geq w}(f_1(v) \to f_2(v))$$

Definition 3.6.1 (Poset Product semantics for $\mathcal{L}_\otimes$)**.** *Let $\mathcal{W} = \langle W, \geq \rangle$ be a fixed poset, and $\mathbf{A}_\mathcal{W}$ be the poset sum described above. Given $h : Atom \to \mathbf{A}_\mathcal{W}$ an assignment of atomic formulas to elements of $\mathbf{A}_\mathcal{W}$, any formula ϕ can be mapped to an element $[\![\phi]\!]_h \in \mathbf{A}_\mathcal{W}$ as follows:*

$$
\begin{array}{lll}
[\![p]\!]_h & := & h(p) \quad (\textit{for atomic formulas } p) \\
[\![\bot]\!]_h & := & \bot \\
[\![\phi \wedge \psi]\!]_h & := & [\![\phi]\!]_h \wedge [\![\psi]\!]_h \\
[\![\phi \vee \psi]\!]_h & := & [\![\phi]\!]_h \vee [\![\psi]\!]_h \\
[\![\phi \otimes \psi]\!]_h & := & [\![\phi]\!]_h \otimes [\![\psi]\!]_h \\
[\![\phi \to \psi]\!]_h & := & [\![\phi]\!]_h \to [\![\psi]\!]_h
\end{array}
$$

A formula ϕ is said to be valid in $\mathbf{A}_\mathcal{W}$ under h *if for every $w \in W$*

$$[\![\phi]\!]_h^{\mathbf{A}_\mathcal{W}}(w) = \top$$

(which is 1 in $[0,1]_{\mathrm{MV}}$). A formula ϕ is said to be valid in $\mathbf{A}_\mathcal{W}$ *if it is valid in $\mathbf{A}_\mathcal{W}$ under h for any possible mapping $h : Atom \to \mathbf{A}_\mathcal{W}$.*

We end this section by noting that given a poset product $\mathbf{A}_\mathcal{W}$ (for a poset $\mathcal{W} = \langle W, \geq \rangle$) and a mapping $h : Atom \to \mathbf{A}_\mathcal{W}$ of atomic formulas to elements of

$\mathbf{A}_{\mathcal{W}}$, we can obtain a BM structure $\mathcal{M}^{\mathbf{A}_{\mathcal{W}}} = \langle \mathcal{W}, \Vdash_h^{\text{BM}} \rangle$, by taking

$$w \Vdash_h^{\text{BM}} p := h(p)(w)$$

recalling that $h(p) : W \to [0,1]$ is a sloping function.

Proposition 3.6.2. *Let $\mathbf{A}_{\mathcal{W}}$ be the poset product over $\mathcal{W}$, and $h : Atom \to \mathbf{A}_{\mathcal{W}}$ be a fixed mapping of atomic formulas to elements of $\mathcal{W}$. Let $\mathcal{M}^{\mathbf{A}_{\mathcal{W}}}$ be the BM-structure defined above. Then, for any formula ϕ*

$$w \Vdash_h^{\text{BM}} \phi = [\![\phi]\!]_h^{\mathbf{A}_{\mathcal{W}}}(w)$$

Proof. The above can be shown by a straightforward induction on the complexity of ϕ. ☐

Therefore, one can always transform an interpretation of $\mathcal{L}_{\otimes}$ formulas in a poset product $\mathbf{A}_{\mathcal{W}}$ into a Kripke semantics (on the Kripke frame $\mathcal{W}$) for $\mathcal{L}_{\otimes}$ formulas.

3.7 Soundness

Let us now prove the soundness of the Kripke semantics for **LLi**.

Theorem 3.7.1 (Soundness)**.** *If $\Gamma \vdash_{\mathbf{LLi}} \phi$ then $\Gamma \Vdash^{\text{BM}} \phi$.*

Proof. By induction on the derivation of $\Gamma \vdash \phi$. Assume $\Gamma = \psi_1, \ldots, \psi_n$ and let $\otimes\Gamma := \psi_1 \otimes \ldots \otimes \psi_n$. Fix a BM-structure $\mathcal{M} = \langle \mathcal{W}, \Vdash^{\text{BM}} \rangle$ with $\mathcal{W} = \langle W, \geq \rangle$, and let $w \in W$.

(Axiom) $\Gamma, \phi \vdash \phi$. By Definition 5.4.1, we need to show:

$$\begin{array}{rcl} w \Vdash^{\text{BM}} (\otimes\Gamma) \otimes \phi & \overset{(\text{D.}5.3.6)}{\equiv} & (w \Vdash^{\text{BM}} \psi_1) \otimes \ldots (w \Vdash^{\text{BM}} \psi_n) \otimes (w \Vdash^{\text{BM}} \phi) \\ & \overset{(\text{L.}2.3.61\ (ii))}{\leq} & w \Vdash^{\text{BM}} \phi \end{array}$$

($\wedge$I) By IH we have $(w \Vdash^{\text{BM}} \otimes\Gamma) \leq (w \Vdash^{\text{BM}} \phi)$ and $(w \Vdash^{\text{BM}} \otimes\Gamma) \leq (w \Vdash^{\text{BM}} \psi)$. Hence

$$(w \Vdash^{\text{BM}} \otimes\Gamma) \quad \leq \quad \min\{w \Vdash^{\text{BM}} \phi, w \Vdash^{\text{BM}} \psi\} \quad \equiv \quad w \Vdash^{\text{BM}} \phi \wedge \psi$$

($\wedge$E) By IH we have $(w \Vdash^{\text{BM}} \otimes\Gamma) \leq (w \Vdash^{\text{BM}} \phi \wedge \psi)$, i.e.

$$(w \Vdash^{\text{BM}} \otimes\Gamma) \quad \leq \quad \min\{w \Vdash^{\text{BM}} \phi, w \Vdash^{\text{BM}} \psi\}$$

This implies both $(w \Vdash^{\text{BM}} \otimes\Gamma) \leq (w \Vdash^{\text{BM}} \phi)$ and $(w \Vdash^{\text{BM}} \otimes\Gamma) \leq (w \Vdash^{\text{BM}} \psi)$.

($\vee$I) By IH we have $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{BM}} \phi)$. Therefore

$$(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \quad \leq \quad \max\{w \Vdash^{\mathrm{BM}} \phi, w \Vdash^{\mathrm{BM}} \psi\} \quad \equiv \quad w \Vdash^{\mathrm{BM}} \phi \vee \psi$$

($\vee$E) By IH we have

- $w \Vdash^{\mathrm{BM}} \otimes\Gamma \leq \max\{w \Vdash^{\mathrm{BM}} \phi, w \Vdash^{\mathrm{BM}} \psi\}$
- $(w \Vdash^{\mathrm{BM}} (\otimes\Delta) \otimes \phi) \leq (w \Vdash^{\mathrm{BM}} \chi)$
- $(w \Vdash^{\mathrm{BM}} (\otimes\Delta) \otimes \psi) \leq (w \Vdash^{\mathrm{BM}} \chi)$

By Lemma 2.3.61 (iii), these imply $(w \Vdash^{\mathrm{BM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq w \Vdash^{\mathrm{BM}} \chi$.

($\to$I) By IH we have $(w \Vdash^{\mathrm{BM}} (\otimes\Gamma) \otimes \phi) \leq (w \Vdash^{\mathrm{BM}} \psi)$, for all $w \in W$. By the adjointness property we get

$$(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{BM}} \phi) \to (w \Vdash^{\mathrm{BM}} \psi)$$

for all $w \in W$. Fix $w \in W$, and let us consider two cases. First, if for some $v > w$ we have $(v \Vdash^{\mathrm{BM}} \phi) \to (v \Vdash^{\mathrm{BM}} \psi) < \top$, then we must have that $(v \Vdash^{\mathrm{BM}} \otimes\Gamma) < \top$, and hence $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) = \bot$, and trivially

$$(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq \lfloor\inf\rfloor_{v \geq w}((v \Vdash^{\mathrm{BM}} \phi) \to (v \Vdash^{\mathrm{BM}} \psi))$$

If on the other hand, $(v \Vdash^{\mathrm{BM}} \phi) \to (v \Vdash^{\mathrm{BM}} \psi) = \top$ for all $v > w$, then

$$\lfloor\inf\rfloor_{v \geq w}((v \Vdash^{\mathrm{BM}} \phi) \to (v \Vdash^{\mathrm{BM}} \psi)) = (w \Vdash^{\mathrm{BM}} \phi) \to (w \Vdash^{\mathrm{BM}} \psi)$$

and we indeed have $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{BM}} \phi) \to (w \Vdash^{\mathrm{BM}} \psi)$.

($\to$E) By IH we have

- $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq w \Vdash^{\mathrm{BM}} \phi$
- $(w \Vdash^{\mathrm{BM}} \otimes\Delta) \leq \lfloor\inf\rfloor_{v \geq w}((v \Vdash^{\mathrm{BM}} \phi) \to (v \Vdash^{\mathrm{BM}} \psi))$

We again consider two cases. First, if for some $v > w$ we have $(v \Vdash^{\mathrm{BM}} \phi) \to (v \Vdash^{\mathrm{BM}} \psi) < \top$, then

$$\lfloor\inf\rfloor_{v \geq w}((v \Vdash^{\mathrm{BM}} \phi) \to (v \Vdash^{\mathrm{BM}} \psi)) = \bot$$

and hence $(w \Vdash^{\mathrm{BM}} \otimes\Delta) = \bot$ and $(w \Vdash^{\mathrm{BM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq w \Vdash^{\mathrm{BM}} \psi$. If on the other hand, $(v \Vdash^{\mathrm{BM}} \phi) \to (v \Vdash^{\mathrm{BM}} \psi) = \top$ for all $v > w$, then

$$\lfloor\inf\rfloor_{v \geq w}((v \Vdash^{\mathrm{BM}} \phi) \to (v \Vdash^{\mathrm{BM}} \psi)) = (w \Vdash^{\mathrm{BM}} \phi) \to (w \Vdash^{\mathrm{BM}} \psi)$$

so that our assumption is $(w \Vdash^{\mathrm{BM}} \otimes\Delta) \leq (w \Vdash^{\mathrm{BM}} \phi) \to (w \Vdash^{\mathrm{BM}} \psi)$. By Lemma 2.3.61 (iv) we obtain $(w \Vdash^{\mathrm{BM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq w \Vdash^{\mathrm{BM}} \psi$.

$(\bot\mathrm{E})$ By IH we have $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq w \Vdash^{\mathrm{BM}} \bot$. Since $(w \Vdash^{\mathrm{BM}} \bot) = 0$, we have that $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) = 0$, which implies $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{BM}} \phi)$, for any ϕ.

$(\otimes\mathrm{I})$ By IH $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq w \Vdash^{\mathrm{BM}} \phi$ and $(w \Vdash^{\mathrm{BM}} \otimes\Delta) \leq w \Vdash^{\mathrm{BM}} \psi$. By Lemma 2.3.61 (v) we have

$$(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \otimes (w \Vdash^{\mathrm{BM}} \otimes\Delta) \leq (w \Vdash^{\mathrm{BM}} \phi) \otimes (w \Vdash^{\mathrm{BM}} \psi)$$

and hence

$$(w \Vdash^{\mathrm{BM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq (w \Vdash^{\mathrm{BM}} \phi \otimes \psi)$$

$(\otimes\mathrm{E})$ By IH we have

- $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{BM}} \phi) \otimes (w \Vdash^{\mathrm{BM}} \psi)$
- $(w \Vdash^{\mathrm{BM}} \otimes\Delta) \otimes (w \Vdash^{\mathrm{BM}} \phi) \otimes (w \Vdash^{\mathrm{BM}} \psi) \leq w \Vdash^{\mathrm{BM}} \chi$

By Lemma 2.3.61 (vi), we have

$$(w \Vdash^{\mathrm{BM}} \otimes\Gamma) \otimes (w \Vdash^{\mathrm{BM}} \otimes\Delta) \leq w \Vdash^{\mathrm{BM}} \chi$$

i.e. $(w \Vdash^{\mathrm{BM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq w \Vdash^{\mathrm{BM}} \chi$.

(DIV) It is sufficient to show that

$$w \Vdash^{\mathrm{BM}} (\phi \to \psi) \otimes \phi \quad \leq \quad w \Vdash^{\mathrm{BM}} (\psi \to \phi) \otimes \psi$$

i.e.

$$(w \Vdash^{\mathrm{BM}} \phi \to \psi) \otimes (w \Vdash^{\mathrm{BM}} \phi) \leq (w \Vdash^{\mathrm{BM}} \psi \to \phi) \otimes (w \Vdash^{\mathrm{BM}} \psi)$$

We consider two cases:

Case 1. $w \Vdash^{\mathrm{BM}} \phi = \bot$. In this case the result is immediate.

Case 2. $w \Vdash^{\mathrm{BM}} \phi > \bot$. This implies that $\forall v \succ w(w \Vdash^{\mathrm{BM}} \phi = \top)$, and hence $\forall v \succ w((w \Vdash^{\mathrm{BM}} \psi \to w \Vdash^{\mathrm{BM}} \phi) = \top)$, so

$$w \Vdash^{\mathrm{BM}} \psi \to \phi \quad = \quad (w \Vdash^{\mathrm{BM}} \psi) \to (w \Vdash^{\mathrm{BM}} \phi)$$

Since

$$w \Vdash^{\mathrm{BM}} \phi \to \psi, \phi \leq ((w \Vdash^{\mathrm{BM}} \phi) \to (w \Vdash^{\mathrm{BM}} \psi)) \otimes (w \Vdash^{\mathrm{BM}} \phi)$$

it remains to show that

$$((w \Vdash^{\mathrm{BM}} \phi) \to (w \Vdash^{\mathrm{BM}} \psi)) \otimes (w \Vdash^{\mathrm{BM}} \phi)$$
$$\leq ((w \Vdash^{\mathrm{BM}} \psi) \to (w \Vdash^{\mathrm{BM}} \phi)) \otimes (w \Vdash^{\mathrm{BM}} \psi)$$

which follows from Lemma 2.3.61 (vii). □

3.8 Completeness

We conclude this section by arguing that the Kripke semantics above is also complete, referring to Proposition 3.6.2 which relates poset sums and BM-structures, and the completeness results of Jipsen and Montagna.

Theorem 3.8.1 (Completeness)**.** *If* $\Gamma \Vdash^{\mathrm{BM}} \phi$ *then* $\Gamma \vdash_{\mathbf{LLi}} \phi$.

Proof. Let $\Gamma \equiv \psi_1, \ldots, \psi_n$. Suppose

$$\Gamma \nvdash_{\mathbf{LLi}} \phi$$

By Proposition 2.1.4 it follows that

$$\nvdash_{\mathbf{GBL}_{ewf}} \psi_1 \to \ldots \to \psi_n \to \phi$$

By the algebraic completeness result for $\mathbf{GBL}_{ewf}$ algebras with respect to the proof system $\mathbf{GBL}_{ewf}$ (cf. Section 3.2), it follows that for some $\mathbf{GBL}_{ewf}$ algebra $\mathcal{G}$ and some mapping $h : \mathit{Atom} \to G$ from propositional variables to elements of $\mathcal{G}$, we have

$$[\![\psi_1 \to \ldots \to \psi_n \to \phi]\!]^{\mathcal{G}}_h \neq \top$$

By ([4, Theorem 1] – see also [37]) there exists a finite poset $\mathcal{W} = \langle W, \succeq \rangle$ and an assignment $h' : \mathit{Atom} \to [0,1]$ of atomic formulas to elements of the Poset Product $\mathbf{A}_{\mathcal{W}}$, such that for some $w \in W$

$$[\![\psi_1 \to \ldots \to \psi_n \to \phi]\!]^{\mathbf{A}_{\mathcal{W}}}_{h'}(w) \neq \top$$

By Proposition 3.6.2, we have a BM-structure $\mathcal{M}^{\mathbf{A}_{\mathcal{W}}}$ such that for some $w \in W$

$$(w \Vdash^{\mathrm{BM}}_{h'} \psi_1 \to \ldots \to \psi_n \to \phi) \neq \top$$

and hence

$$(w \Vdash^{\mathrm{BM}}_{h'} \psi_1 \otimes \ldots \otimes \psi_n) \nleq (w \Vdash^{\mathrm{BM}} \phi)$$

and $\psi_1, \ldots, \psi_n \nVdash^{\mathrm{BM}} \phi$. □

Note 21. *It would be interesting to be able to prove this completeness result directly, by constructing a BM-structure directly from the logic (term model), as is done for Intuitionistic logic. However, we have not been able to find such direct proof, and hence have appealed to Bova and Montagna results on the completeness of* $\mathbf{GBL}_{ewf}$ *for Poset Products.*

3.9 Open Problem

In ending this chapter, we note the following open problem for the ambitious reader:

Open Problem 1. *Is there a more direct completeness construction which does not require a detour thru Poset Products?*

Chapter 4

Intuitionistic Affine Logic

4.1 Introduction

In the present chapter we continue our discussion Intuitionistic Affine logic, alias **ALi**, from the second chapter. **ALi** is the base system for the logics considered in this thesis.

ALi has been studied under various guises. For instance, one arrives at **ALi** by adding to Intuitionistic Linear Logic the structural rule of weakening [54], or extending Full Lambek calculus with the rules of exchange and weakening [26]. More, Intuitionistic Affine logic encompasses a wide class of systems independent of the logic defined herein, including $\mathbf{GBL}_{ewf}$ and **BL** but also the combinator-based systems BCI and BCK logics [35]. Thus it seems **ALi** is often rediscovered and re-packaged from another setting. The proof theory of **ALi** is indeed elegant, having a natural reading via the Curry-Howard correspondence [61], where the same cannot be said for **GBL** and **BL** on account of the divisibility axiom.[1]

In what follows we provide two generalisations of Kripke semantics for **ALi** by mapping into bounded (GGBM) and involutive (GBM) pocrims. The idea behind our semantics is that it generalises Kripke's semantics for **IL** by mapping to involutive algebras corresponding to Affine logic (involutive pocrims), with the sole restriction on the valuations being that they are monotone increasing with respect to the partial order of worlds. This shows the intuitions of Kripke's semantics can be lifted from Boolean and Heyting algebras to other

[1] And indeed, construction of an analytic proof theory for $\mathbf{GBL}_{ewf}$ or **BL** is presently an open problem.

$$\frac{}{\phi \vdash \phi}\,\text{Ax} \qquad \frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi}\,\text{W} \qquad \frac{\Gamma, \phi, \psi, \Delta \vdash \chi}{\Gamma, \psi, \phi, \Delta \vdash \chi}\,\text{Ex}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi}\,\rightarrow \text{I} \qquad \frac{\Gamma \vdash \phi \rightarrow \psi \qquad \Delta \vdash \phi}{\Gamma, \Delta \vdash \psi}\,\rightarrow \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \otimes \psi}\,\otimes \text{I} \qquad \frac{\Gamma, \phi, \psi \vdash \chi \qquad \Delta \vdash \phi \otimes \psi}{\Gamma, \Delta \vdash \chi}\,\otimes \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}\,\wedge \text{I} \qquad \frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_i}\,\wedge \text{E}$$

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2}\,\vee \text{I)} \qquad \frac{\Gamma \vdash \phi \vee \psi \qquad \Delta, \phi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi}\,\vee \text{E}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}\,\bot \text{E}$$

Figure 4.1: Intuitionistic Affine logic **ALi**

abstract algebras in some way appropriate to Intuitionistic Affine, which is the core subsystem of $\mathbf{GBL}_{ewf}$, **BL**, and other fuzzy logic systems.

The present structures we call *GBM-structures* as they also generalise the Bova-Montagna structures (BM structures) for $\mathbf{GBL}_{ewf}$ (alias **LLi**) introduced in the next chapter. Those BM-structures map worlds and formulae to MV-chains, which are involutive MV-algebras over the unit-interval, in analogy with the Kripke semantics for **IL** in which one maps worlds and formulae to the characteristic Boolean algebra $\{\top, \bot\}_{\mathbf{BA}}$.

We have already defined the sequent-based natural deduction system (in section 2.4), we have already said quite a bit about pocrims as algebraic items via many lemmata, and have proven adequacy for the logic under the algebraic semantics. We proceed now directly to the definition of validity and defining the relational semantics, after reminding the reader the of the presentation of the system (see figure 3.1).

4.2 Valid Sequents in ALi

Definition 4.2.1 (Denotation functions)**.** *Given an involutive, lattice-ordered, complete pocrim $\mathcal{P}$, and a mapping from propositional variables to elements of $\mathcal{P}$:*

$$p \mapsto [\![p]\!] \in \mathcal{P}$$

We thus refer to the denotation of a variable p as $[\![p]\!]_{\mathcal{P}}$. We can extend that mapping to all formulas in the language of L in a straightforward way:

$$\begin{aligned}
[\![\phi \otimes \psi]\!]_{\mathcal{P}} &:= [\![\phi]\!]_{\mathcal{P}} \otimes [\![\psi]\!]_{\mathcal{P}} \\
[\![\phi \wedge \psi]\!]_{\mathcal{P}} &:= [\![\phi]\!]_{\mathcal{P}} \wedge [\![\psi]\!]_{\mathcal{P}} \\
[\![\phi \vee \psi]\!]_{\mathcal{P}} &:= [\![\phi]\!]_{\mathcal{P}} \vee [\![\psi]\!]_{\mathcal{P}} \\
[\![\phi \rightarrow \psi]\!]_{\mathcal{P}} &:= [\![\phi]\!]_{\mathcal{P}} \rightarrow [\![\psi]\!]_{\mathcal{P}}
\end{aligned}$$

Definition 4.2.2 (Validity)**.** *A sequent $\phi_1, ..., \phi_n \vdash_{\mathbf{ALi}} \psi$ is then said to be valid in $\mathcal{P}_\perp$, alias $\mathcal{P}_\perp$-valid, if $[\![\phi_1]\!] \otimes ... \otimes [\![\phi_n]\!] \leq [\![\psi]\!]$ holds in $\mathcal{P}_\perp$. A sequent is said to be* valid *in* **ALi** *if it is valid in* all *bounded, lattice-ordered pocrims. We sometimes represent this thus:*

$$\Gamma \vDash_{\mathbf{ALi}} \phi$$

In the case where ϕ is valid in all bounded lattice-ordered pocrims, we write

$$\vDash_{\mathbf{ALi}} \phi$$

It is easy to show that the valid sequents, in the sense above, are precisely the ones provable in Intuitionistic Affine logic. Indeed, we have already done so in the previous chapter with the proof of adequacy with respect to algebraic semantics of bounded pocrims.

Proposition 4.2.3. *A sequent $\Gamma \vdash \psi$ is* **ALi***-valid iff it is provable in* **ALi***.*

4.3 Kripke Semantics for ALi

The Kripke semantics for **ALi** that we propose is based on the Bova-Montagna construction of poset sums introduced in (see Section for more details). We first need to define a particular class of functions from the set of worlds W to involutive, lattice-ordered complete pocrims.

Definition 4.3.1 (Monotone functions)**.** *Let $\mathcal{W} = \langle W, \geq \rangle$ be a partial order and $\mathcal{P}$ an involutive, lattice-ordered complete pocrim with $\wedge, \vee$ as the meet and join of the lattice respectively. A function $f : W \rightarrow \mathcal{P}$ is said to be a* monotone function *if $\forall w, v \in W : w \geq v$ implies $f(w) \geq f(v)$.*

Lemma 4.3.2. *If $f : W \rightarrow \mathcal{P}$ and $g : W \rightarrow \mathcal{P}$ are monotone functions, then the* *following functions are also monotone functions:*

$$(f \wedge g)(w) := f(w) \wedge g(w)$$
$$(f \vee g)(w) := f(w) \vee g(w)$$
$$(f \otimes g)(w) := f(w) \otimes g(w)$$

and moreover if $w \geq v \in W$, f, g *are monotone, then* $((f(w) \wedge g(w)) \geq (f(v) \wedge g(v))$, $(f(w) \vee g(w)) \geq (f(v) \vee g(v))$ *and* $(f(w) \otimes g(w)) \geq (f(v) \otimes g(v))$.

Proof. Let $w \geq v \in W$ and f, g be monotone functions. Let us consider each case:

- $f \wedge g$. Then because f, g are monotone, $f(w) \geq f(v)$ and $g(w) \geq g(v)$. By order theory, $f(w) \geq f(w) \wedge f(v)$ and $g(w) \geq g(w) \wedge g(v)$, and so $f(w) \wedge g(w) \geq f(v) \wedge g(v)$.
- $f \vee g$. Then because f, g are monotone, $f(w) \geq f(v)$ and $g(w) \geq g(v)$. By order theory, $f(w) \vee f(v) \geq f(v)$ and $g(w) \vee g(v) \geq g(v)$, and so $f(w) \vee g(w) \geq f(v) \vee g(v)$.
- $f \otimes g$. Assuming f, g are monotone, $f(w) \geq f(v)$ and $g(w) \geq g(v)$. Then $f(w) \otimes g(w) \geq f(v) \otimes g(v)$ follows from 2.3.46.

□

Definition 4.3.3. *Let* $\mathcal{P}$ *be an involutive, lattice-ordered, complete pocrim. A* Generalised Bova-Montagna structure *for* $\mathcal{P}$ *(or GBM-structure) is a pair* $\mathcal{M}_{\mathcal{P}} = \langle \mathcal{W}, \Vdash^{\text{GBM}} \rangle$ *where* $\mathcal{W} = \langle W, \geq \rangle$ *is a poset, and* $\Vdash^{\text{GBM}}$ *is an infix operator (on worlds and propositional variables) taking values in* $\mathcal{P}$*, i.e.* $(w \Vdash^{\text{GBM}} p) \in \mathcal{P}$*, such that for any propositional variable* p *the function* $\lambda w.(w \Vdash^{\text{GBM}} p) : W \to \mathcal{P}$ *is a monotone function.*

Definition 4.3.4 (GBM Kripke Semantics for $\mathcal{L}_{\otimes}$)**.** *Given a GBM-structure*

$$\mathcal{M}_{\mathcal{P}} = \langle \mathcal{W}, \Vdash^{\text{GBM}} \rangle$$

the valuation function $w \Vdash^{\text{GBM}} p$ *on propositional variables* p *can be extended* *to all $\mathcal{L}_\otimes$-formulas as:*

$$\begin{array}{lcl}
w \Vdash^{\text{GBM}} \top & := & \top \\
w \Vdash^{\text{GBM}} \bot & := & \bot \\
w \Vdash^{\text{GBM}} \phi \wedge \psi & := & (w \Vdash^{\text{GBM}} \phi) \wedge (w \Vdash^{\text{GBM}} \psi) \\
w \Vdash^{\text{GBM}} \phi \vee \psi & := & (w \Vdash^{\text{GBM}} \phi) \vee (w \Vdash^{\text{GBM}} \psi) \\
w \Vdash^{\text{GBM}} \phi \otimes \psi & := & (w \Vdash^{\text{GBM}} \phi) \otimes (w \Vdash^{\text{GBM}} \psi) \\
w \Vdash^{\text{GBM}} \phi \to \psi & := & \mathbf{inf}_{v \geq w}((v \Vdash^{\text{GBM}} \phi) \to (v \Vdash^{\text{GBM}} \psi))
\end{array}$$

where the operations on the right-hand side are the operations on an involutive, complete pocrim $\mathcal{P}$.

We note the following, which in view of 2.3.40, completely characterises the $\mathbf{inf}_{v \geq w}$ operation in the GBM structures in terms of the partial order of the pocrim. It is in this sense that we can see our definition really generalises the Kripke semantics of **IL**.

Lemma 4.3.5. *(Existence of Infs for GBM structures) Let $f : W \to \mathcal{P}$ as above, with $\mathcal{P}$ a complete, lattice-ordered pocrim, and f inducing $\{f(v)|w \leq v\}$ in $\mathcal{P}$. Then $\inf\{f(v)|w \leq v\}$ exists in $\mathcal{P}$, as does $\sup\{f(v)|w \leq v\}$.*

Proof. Let W be a poset under $\leq$, let $\mathcal{P}$ be a complete, lattice-ordered involutive pocrim, and let $f : W \to \mathcal{P}$ with $\{f(v)|w \leq v\}$ in $\mathcal{P}$. By 2.3.42 a complete, lattice-ordered involutive pocrim must have suprema and infima for all subsets X of $\mathcal{P}$; but $\{f(v)|w \leq v\} \subseteq \mathcal{P}$, so that $\inf\{f(v)|w \leq v\}$ and $\sup\{f(v)|w \leq v\}$ exist in $\mathcal{P}$. □

Note 22. *We use* inf *to refer to the standard operation on an arbitrary poset or pocrim, but we use* $\mathbf{inf}_{v \geq w}$ *to refer to the operation on specifically on a set of formulas evaluated* in *a pocrim. The following makes the relationship clear.*

Note 23. *(Inf of a set of valuations.) Let $\mathcal{P}$ be a complete lattice-ordered pocrim, such that $v \Vdash^{\text{GBM}} \psi$ and $v \Vdash^{\text{GBM}} \chi$ are valuations in the pocrim. Then one can define $\mathbf{inf}_{v \geq w}((v \Vdash^{\text{GBM}} \psi) \to (v \Vdash^{\text{GBM}} \chi))$ as follows, by 4.3.5 (for $v : w \leq v$ in the below left):*

$$\begin{array}{rcl}
\inf\{(v \Vdash^{\text{GBM}} \psi) \to (v \Vdash^{\text{GBM}} \chi)\} & \Leftrightarrow & \forall v \geq w((v \Vdash^{\text{GBM}} \psi) \to (v \Vdash^{\text{GBM}} \chi)) \\
& \Leftrightarrow & \mathbf{inf}_{v \geq w}((v \Vdash^{\text{GBM}} \psi) \to (v \Vdash^{\text{GBM}} \chi))
\end{array}$$

Lemma 4.3.6. *For any formula ϕ the function $\lambda w.(w \Vdash^{\text{GBM}} \phi) : W \to \mathcal{P}$ is a monotone function.*

Proof. By induction on the complexity of the formula ϕ. The cases for $\psi \vee \xi, \psi \wedge \xi$ and $\psi \otimes \xi$ follow directly from Lemma 5.3.2. The case for $\psi \to \xi$ follows from the fact that, given w, v such that $w \leq v$ and $w \Vdash^{\mathrm{GBM}} \psi \to \chi$, $\mathbf{inf}_{v \geq w}(w \Vdash^{\mathrm{GBM}} \psi \to \chi) \leq \mathbf{inf}_{v \geq w}((v \Vdash^{\mathrm{GBM}} \psi) \to (v \Vdash^{\mathrm{GBM}} \chi))$. □

We can now generalise the monotonicity property of intuitionistic logic to intuitionistic Affine logic **ALi**:

Corollary 4.3.7 (Monotonicity)**.** *The following monotonicity property holds for all $\mathcal{L}_{\otimes}$-formulas ϕ, i.e.*

$$\textit{if } w \leq v \textit{ then } (w \Vdash^{\mathrm{GBM}} \phi) \leq (v \Vdash^{\mathrm{GBM}} \phi)$$

Proof. This follows from the observation that the valuations are monotone functions. □

4.4 Validity under GBM-structures

Definition 4.4.1. *Let $\Gamma = \psi_1, \ldots, \psi_n$. Consider the following definitions:*

- *We say that a sequent $\Gamma \vdash \phi$* holds *in a GBM-structure $\mathcal{M}$ (written $\Gamma \Vdash^{\mathrm{GBM}}_{\mathcal{M}} \phi$) if for all $w \in W$ we have*

$$(w \Vdash^{\mathrm{GBM}} \psi_1 \otimes \ldots \otimes \psi_n) \leq (w \Vdash^{\mathrm{GBM}} \phi)$$

  Otherwise (i.e. if $\Gamma \nvdash \phi$), we say that the sequent fails *$\mathcal{M}$ (written $\Gamma \nVdash^{\mathrm{GBM}}{}_{\mathcal{M}} \phi$) and this means:*

$$\exists w \in W : (w \Vdash^{\mathrm{GBM}} \psi_1 \otimes \ldots \otimes \psi_n) > (w \Vdash^{\mathrm{GBM}} \phi)$$

- *A sequent $\Gamma \vdash \phi$ is said to be valid under the GBM Kripke semantics for $\mathcal{L}_{\otimes}$ (written $\Gamma \Vdash^{\mathrm{GBM}} \phi$) if $\Gamma \Vdash^{\mathrm{GBM}}_{\mathcal{M}} \phi$ for all GBM-structures $\mathcal{M}$.*

Note 24. *We will prove that this semantics is sound and complete for* **ALi**, *i.e. a sequent $\Gamma \vdash \phi$ is provable in* **ALi** *iff it is valid in all GBM-structures. But first let us show that the semantics presented above is a direct generalisation of Kripke's original semantics.*

4.5 GBM-structures generalise Kripke structures

We note the following:

Proposition 4.5.1. *(GBM structures generalise BM structures) GBM structures also generalise BM structures, i.e. For any Bova-Montagna structure* $\mathcal{BM} = \langle \mathcal{BM}, \Vdash^{\mathrm{BM}} \rangle$ *and* $\mathcal{L}_{\otimes}$*-formula* ϕ*, we have* $\forall w$*:*

$$(w \Vdash^{\mathrm{BM}} \phi) \Rightarrow (w \Vdash^{\mathrm{GBM}} \phi)$$

Proof. The result follows from the fact that all sloping functions are monotone. □

From this last result and 5.5.1 we obtain:

Corollary 4.5.2. *GBM's generalise Kripke structures.*

4.6 Generalising GBM-structures

Definition 4.6.1. *Let* $\mathcal{Q}$ *be a lattice-ordered, complete pocrim (and therefore a bounded pocrim). A* Generalised GBM *for* $\mathcal{Q}$ *(or GGBM-structure) is a pair* $\mathcal{M}_{\mathcal{Q}} = \langle \mathcal{W}, \Vdash^{\mathrm{GGBM}} \rangle$ *where* $\mathcal{W} = \langle W, \leq \rangle$ *is a poset, and* $\Vdash^{\mathrm{GGBM}}$ *is an infix operator (on worlds and propositional variables) taking values in* $\mathcal{Q}$*, i.e.* $(w \Vdash^{\mathrm{GGBM}} p) \in \mathcal{Q}$*, such that for any propositional variable* p *the function* $\lambda w.(w \Vdash^{\mathrm{GGBM}} p) : W \to \mathcal{Q}$ *is a monotone function.*

Definition 4.6.2 (GGBM Kripke Semantics for $\mathcal{L}_{\otimes}$)**.** *Given a GGBM*

$$\mathcal{M}_{\mathcal{Q}} = \langle \mathcal{W}, \Vdash^{\mathrm{GGBM}} \rangle$$

the valuation function $w \Vdash^{\mathrm{GGBM}} p$ *on propositional variables* p *can be extended to all* $\mathcal{L}_{\otimes}$*-formulas as:*

$$\begin{array}{lcl}
w \Vdash^{\mathrm{GGBM}} \top & := & \top \\
w \Vdash^{\mathrm{GGBM}} \bot & := & \bot \\
w \Vdash^{\mathrm{GGBM}} \phi \wedge \psi & := & (w \Vdash^{\mathrm{GGBM}} \phi) \wedge (w \Vdash^{\mathrm{GGBM}} \psi) \\
w \Vdash^{\mathrm{GGBM}} \phi \vee \psi & := & (w \Vdash^{\mathrm{GGBM}} \phi) \vee (w \Vdash^{\mathrm{GGBM}} \psi) \\
w \Vdash^{\mathrm{GGBM}} \phi \otimes \psi & := & (w \Vdash^{\mathrm{GGBM}} \phi) \otimes (w \Vdash^{\mathrm{GGBM}} \psi) \\
w \Vdash^{\mathrm{GGBM}} \phi \to \psi & := & \mathbf{inf}_{w \leq v}((v \Vdash^{\mathrm{GGBM}} \phi) \to (v \Vdash^{\mathrm{GGBM}} \psi))
\end{array}$$

where the operations on the right-hand side are the operations on an complete, lattice-ordered pocrim (and therefore bounded pocrim) $\mathcal{Q}$*.*

Definition 4.6.3. *Let* $\Gamma = \psi_1, \ldots, \psi_n$*. Consider the following definitions:*

- *We say that a sequent* $\Gamma \vdash \phi$ holds *in a GGBM-structure* $\mathcal{M}_{\mathcal{Q}}$ *(written* $\Gamma \Vdash^{\text{GGBM}}_{\mathcal{M}_{\mathcal{Q}}} \phi$*) if for all* $w \in W$ *we have*

$$(w \Vdash^{\text{GGBM}} \psi_1 \otimes \ldots \otimes \psi_n) \leq (w \Vdash^{\text{GGBM}} \phi)$$

We will, for space considerations, sometimes abbreviate this as

$$\forall w \in W : \Gamma \leq (w \Vdash^{\text{GGBM}} \phi)$$

or even:

$$\forall w \in W : \otimes\Gamma \leq (w \Vdash^{\text{GGBM}} \phi)$$

To emphasize the 'tensoring' of the members of the context. Otherwise (i.e. if $\Gamma \nvdash \phi$*), we say that the sequent* fails *in a structure* $\mathcal{M}_{\mathcal{Q}}$ *(written* $\Gamma \nVdash^{\text{GGBM}}_{\mathcal{M}_{\mathcal{Q}}} \phi$*) and this means:*

$$\exists w \in W : (w \Vdash^{\text{GGBM}} \psi_1 \otimes \ldots \otimes \psi_n) > (w \Vdash^{\text{GGBM}} \phi)$$

- *A sequent* $\Gamma \vdash \phi$ *is said to be valid under the Kripke semantics for* $\mathcal{L}_{\otimes}$ *(written* $\Gamma \Vdash^{\text{GGBM}} \phi$*) if* $\Gamma \Vdash^{\text{GGBM}}_{\mathcal{M}_{\mathcal{Q}}} \phi$ *for all GGBM-structures* $\mathcal{M}_{\mathcal{Q}}$*.*

- *The definition of validity for GBM's is as above but with appropriate subscripts added. When we wish to emphasize the comparison between GGBM's and GBM's, we will always make reference to the underlying algebras of bounded pocrims and involutive pocrims respectively, by subscripting the turnstyle.*

4.7 Completeness under GGBM-structures

Note 25. *We briefly note* **ALi** *is complete with respect to the GGBM semantics introduced above.*

Proposition 4.7.1. *If* $\Gamma \Vdash^{\text{GGBM}} \phi$ *then* $\Gamma \vdash_{\mathbf{ALi}} \phi$*.*

Proof. Let $\Gamma \equiv \psi_1, ..., \psi_n$. Suppose $\Gamma \nvdash_{\mathbf{ALi}} \phi$ (i.e. $\Gamma \vdash_{\mathbf{ALi}} \phi$ fails). We must show $\Gamma \nVdash^{\text{GGBM}} \phi$. Using the fact that Intuitionistic Affine logic is complete for the algebraic semantics of lattice-ordered bounded pocrims 2.3.10, there exists a bounded pocrim $\mathcal{A}$ such that $\mathcal{A} \nvDash_{\mathbf{ALi}} \phi$. Now take $\mathcal{M}_Q$ such that $\mathcal{M}_Q = \langle \mathcal{A}^S, \Vdash^{\text{GGBM}} \rangle$ i.e. where the frame of the model or $\mathcal{W} = \langle \mathcal{A}^S, \preceq \rangle$, consists of one world that is a bounded pocrim, and $\Gamma \nVdash^{\text{GGBM}}_{\mathcal{M}_{\mathcal{Q}}} \phi$. So $\Gamma \nVdash^{\text{GGBM}} \phi$, as desired. ☐

Note 26. *A word of comment is in order here. This is not the sort of robust completeness one desires in comparison with the standard presentation of Kripke semantics, where the formulae are mapped into a poset decorated with involutive algebras. However, for many substructural logics, the sort of completeness argument given above* is *the standard expected for a completeness argument with respect to a somewhat broader sense of relational semantics in which the frame is identified with a partially-ordered algebra, e.g. [53] or [58].*

4.8 Soundness under GBM-structures

Let us now prove the soundness of the Kripke semantics for **ALi**.

Theorem 4.8.1 (Soundness)**.** *If* $\Gamma \vdash_{\mathbf{ALi}} \phi$ *then* $\Gamma \Vdash^{\mathrm{GBM}} \phi$*.*

Proof. By induction on the derivation of $\Gamma \vdash \phi$. Assume $\Gamma = \psi_1, \ldots, \psi_n$ and let $\otimes\Gamma := \psi_1 \otimes \ldots \psi_n$. Fix a GBM-structure $\mathcal{M} = \langle \mathcal{W}, \Vdash^{\mathrm{GBM}} \rangle$ with $\mathcal{W} = \langle W, \geq \rangle$, and let $w \in W$.

(Axiom) $\Gamma, \phi \vdash \phi$. By Definition 5.4.1, we need to show, for all $w \in W$, $(w \Vdash^{\mathrm{GBM}} (\otimes\Gamma) \otimes \phi) \leq (w \Vdash^{\mathrm{GBM}} \phi)$, and this latter holds by Lemma 2.3.48 after unwinding using Definition 5.3.6.

($\wedge$I) By IH we have $\forall w : (w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{GBM}} \phi)$ and $\forall w : (w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{GBM}} \psi)$. Now fix w. By 2.3.54

$$(w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{GBM}} \phi) \wedge (w \Vdash^{\mathrm{GBM}} \psi) \equiv w \Vdash^{\mathrm{GBM}} \phi \wedge \psi$$

($\wedge$E) By IH we have $\forall w : (w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{GBM}} \phi \wedge \psi)$. Fix w. By 2.3.52 this implies both $(w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{GBM}} \phi)$ and $(w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{GBM}} \psi)$.

($\vee$I) By IH we have $\forall w : (w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{GBM}} \phi)$. Now fix w. Therefore by 2.3.55:

$$(w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \leq w \Vdash^{\mathrm{GBM}} \phi \vee \psi$$

($\vee$E) By IH we have, for all $w \in W$:

- $w \Vdash^{\mathrm{GBM}} \otimes\Gamma \leq \{w \Vdash^{\mathrm{GBM}} \phi\} \vee \{w \Vdash^{\mathrm{GBM}} \psi\}$
- $(w \Vdash^{\mathrm{GBM}} (\otimes\Delta) \otimes \phi) \leq (w \Vdash^{\mathrm{GBM}} \chi)$
- $(w \Vdash^{\mathrm{GBM}} (\otimes\Delta) \otimes \psi) \leq (w \Vdash^{\mathrm{GBM}} \chi)$

Now fix w. By Lemma 2.3.50, these imply $(w \Vdash^{\mathrm{GBM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq w \Vdash^{\mathrm{GBM}} \chi$.

($\to$I) By IH we have, for all $w \in W$:

$$(w \Vdash^{\text{GBM}} (\otimes\Gamma) \otimes \phi) \leq (w \Vdash^{\text{GBM}} \psi) \tag{I}$$

We must show

$$\forall w (w \Vdash^{\text{GBM}} \Gamma \leq \mathbf{inf}_{v \geq w} (v \Vdash^{\text{GBM}} \phi \to v \Vdash^{\text{GBM}} \psi)) \tag{II}$$

Now II is equivalent to III:

$$\forall w, v \geq w ((w \Vdash^{\text{GBM}} \Gamma) \leq ((v \Vdash^{\text{GBM}} \phi) \to (v \Vdash^{\text{GBM}} \psi))) \tag{III}$$

So fix w, v in III. Set $w := v$ in I. By definition of tensor we get $(v \Vdash^{\text{GBM}} (\otimes\Gamma)) \otimes (v \Vdash^{\text{GBM}} \phi) \leq (v \Vdash^{\text{GBM}} \psi)$

Applying residuation we get

$$(w \Vdash^{\text{GBM}} \otimes\Gamma) \leq (w \Vdash^{\text{GBM}} \phi) \to (w \Vdash^{\text{GBM}} \psi)$$

By monotonicity we have $(w \Vdash^{\text{GBM}} \Gamma) \leq (v \Vdash^{\text{GBM}} \Gamma)$; By inductive hypothesis $(v \Vdash^{\text{GBM}} \Gamma) \leq ((v \Vdash^{\text{GBM}} \phi) \to (v \Vdash^{\text{GBM}} \psi))$; and finally by transitivity of the ordering we have

$$(w \Vdash^{\text{GBM}} \Gamma) \leq ((v \Vdash^{\text{GBM}} \phi) \to (v \Vdash^{\text{GBM}} \psi))$$

as desired.

($\to$E) By IH we have, for $\forall w \in W$:

- $(w \Vdash^{\text{GBM}} \otimes\Gamma) \leq w \Vdash^{\text{GBM}} \phi$
- $(w \Vdash^{\text{GBM}} \otimes\Delta) \leq \mathbf{inf}((v \Vdash^{\text{GBM}} \phi) \to (v \Vdash^{\text{GBM}} \psi))$

We want to show:

$$(w \Vdash^{\text{GBM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq (v \Vdash^{\text{GBM}} \psi)$$

Now recall that IV:

$$(w \Vdash^{\text{GBM}} \otimes\Delta) \leq \mathbf{inf}((v \Vdash^{\text{GBM}} \phi) \to (v \Vdash^{\text{GBM}} \psi)) \tag{IV}$$

is equivalent to V:

$$\forall w, v \geq w ((w \Vdash^{\text{GBM}} \Delta) \leq ((v \Vdash^{\text{GBM}} \phi) \to (v \Vdash^{\text{GBM}} \psi))) \tag{V}$$

So fix w, v in V, and set $v := w$ in IV. Then we have

- $(w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \leq w \Vdash^{\mathrm{GBM}} \phi$
- $(w \Vdash^{\mathrm{GBM}} \otimes\Delta) \leq ((w \Vdash^{\mathrm{GBM}} \phi) \rightarrow (w \Vdash^{\mathrm{GBM}} \psi))$

Hence:

- $((w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \leq w \Vdash^{\mathrm{GBM}} \phi)$
- $((w \Vdash^{\mathrm{GBM}} \otimes\Delta) \leq ((w \Vdash^{\mathrm{GBM}} \phi) \rightarrow (w \Vdash^{\mathrm{GBM}} \psi)))$

After applying 2.3.56 we have

$$(w \Vdash^{\mathrm{GBM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq (w \Vdash^{\mathrm{GBM}} \psi)$$

as desired.

($\perp$E) By IH we have $\forall w : (w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \leq w \Vdash^{\mathrm{GBM}} \perp$. By 2.3.49 $(w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{GBM}} \phi)$, for any ϕ.

($\otimes$I) By IH $\forall w : (w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \leq w \Vdash^{\mathrm{GBM}} \phi$ and $(w \Vdash^{\mathrm{GBM}} \otimes\Delta) \leq w \Vdash^{\mathrm{GBM}} \psi$. By Lemma 2.3.46 we have

$$(w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \otimes (w \Vdash^{\mathrm{GBM}} \otimes\Delta) \leq (w \Vdash^{\mathrm{GBM}} \phi) \otimes (w \Vdash^{\mathrm{GBM}} \psi)$$

and hence

$$(w \Vdash^{\mathrm{GBM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq (w \Vdash^{\mathrm{GBM}} \phi \otimes \psi)$$

($\otimes$E) By IH we have, $\forall w \in W :$

- $(w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{GBM}} \phi) \otimes (w \Vdash^{\mathrm{GBM}} \psi)$
- $(w \Vdash^{\mathrm{GBM}} \otimes\Delta) \otimes (w \Vdash^{\mathrm{GBM}} \phi) \otimes (w \Vdash^{\mathrm{GBM}} \psi) \leq w \Vdash^{\mathrm{GBM}} \chi$

Fix w. By Lemma 2.3.51, we have

$$(w \Vdash^{\mathrm{GBM}} \otimes\Gamma) \otimes (w \Vdash^{\mathrm{GBM}} \otimes\Delta) \leq w \Vdash^{\mathrm{GBM}} \chi$$

i.e. $(w \Vdash^{\mathrm{GBM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq w \Vdash^{\mathrm{GBM}} \chi$.

□

4.9 Open Problem

Note 27. *We should like to do better than the trivial completeness result of 4.7: we should like completeness for GBM-structures, not merely the GGBM's. This is a very weak result indeed, and is not desirable in that the robust sense* *of completeness that Intuitionistic logic enjoys (i.e. completeness for Heyting algebras and Kripke models where the world and formulae are valued in the Boolean algebras) eludes us in the present case. We end this chapter by noting, at the time of writing, an open problem:*

Open Problem 2. *Is* **ALi** *complete for our GBM-structures?*

Chapter 5

BL

5.1 Introduction

In the present chapter we examine Hajek's **BL**, a system which occupies a central place in contemporary research on fuzzy logic. The semantics we devise for **BL** restricts that of $\mathbf{GBL}_{ewf}$ in that the relational structures operate over linear frames, hence our designation 'Linear Bova Montagna structure' or LBM structure.

As one would expect, these LBM structures stand to GBM's as Linearly ordered Kripke structures for Gödel-Dummett logic stand to Kripke structures for Intuitionistic logic. However, **BL** has a very rich catalogue of semantic models and constructions which yield completeness, and the LBM's arguably represent a small portion of what's possible in the construction of generalised Kripke structures for **BL**. For instance, Wesley Fussner has recently shown in an unpublished note (generalising the approach of our own [44]) our semantics for $\mathbf{GBL}_{ewf}$ can be exported to a wide class of logical systems whose varieties are closed under the poset product construction of Jipsen and Montagna. These include structures for **BL** in which the frames are forests, trees, or linear orders valued in MV-algebras.

It remains to be seen how the many different ordinal sum constructions present in the literature of **BL** [1] can be accommodated into an amenable relational semantics: the present author has attempted one such semantics for **BL**, and our conjecture, borne out by trial and error, is that poset product construction is essential for the kind of relational semantics obtained here.

[1]See [5] for relevant discussion in connection to poset products.

$$\frac{}{\phi \vdash \phi}\ \text{Ax} \qquad \frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi}\ \text{W} \qquad \frac{\Gamma, \phi, \psi, \Delta \vdash \chi}{\Gamma, \psi, \phi, \Delta \vdash \chi}\ \text{Ex}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi} \to \text{I} \qquad \frac{\Gamma \vdash \phi \to \psi \qquad \Delta \vdash \phi}{\Gamma, \Delta \vdash \psi} \to \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \otimes \psi} \otimes \text{I} \qquad \frac{\Gamma, \phi, \psi \vdash \chi \qquad \Delta \vdash \phi \otimes \psi}{\Gamma, \Delta \vdash \chi} \otimes \text{E}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} \wedge \text{I} \qquad \frac{\Gamma \vdash \phi_1 \wedge \phi_2}{\Gamma \vdash \phi_i} \wedge \text{E}$$

$$\frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2} \vee \text{I)} \qquad \frac{\Gamma \vdash \phi \vee \psi \qquad \Delta, \phi \vdash \chi \qquad \Delta, \psi \vdash \chi}{\Gamma, \Delta \vdash \chi} \vee \text{E}$$

$$\frac{\Gamma, \phi, \phi \to \psi \vdash \chi}{\Gamma, \psi, \psi \to \phi \vdash \chi}\ \text{DIV} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \phi} \bot \text{E}$$

$$\frac{}{\Gamma \vdash (\phi \to \psi) \vee (\psi \to \phi)}\ \text{Prelin}$$

Figure 5.1: Basic logic **BL**

The structure of this chapter is as before, with a brief recall from chapter 2 of **BL**'s natural deduction system, followed by suitable definitions of validity and relational semantics.

5.2 Valid Sequents in BL

Definition 5.2.1 (Denotation functions)**.** *Given a* **MV***-chain* $[0,1]_{\mathbf{MV}}$*, and a mapping from propositional variables to elements of* $[0,1]_{\mathbf{MV}}$*:*

$$p \mapsto [\![p]\!] \in [0,1]_{\mathbf{MV}}$$

We thus refer to the denotation of a variable p *as* $[\![p]\!]_{\mathbf{MV}}$*. We can extend that mapping to all formulas in the language of* L *in a straightforward way:*

$$\begin{aligned}
[\![\phi \otimes \psi]\!]_{\mathbf{MV}} &:= [\![\phi]\!]_{\mathbf{MV}} \otimes [\![\psi]\!]_{\mathbf{MV}} \\
[\![\phi \wedge \psi]\!]_{\mathbf{MV}} &:= [\![\phi]\!]_{\mathbf{MV}} \wedge [\![\psi]\!]_{\mathbf{MV}} \\
[\![\phi \vee \psi]\!]_{\mathbf{MV}} &:= [\![\phi]\!]_{\mathbf{MV}} \vee [\![\psi]\!]_{\mathbf{MV}} \\
[\![\phi \to \psi]\!]_{\mathbf{MV}} &:= [\![\phi]\!]_{\mathbf{MV}} \to [\![\psi]\!]_{\mathbf{MV}}
\end{aligned}$$

Definition 5.2.2 (Validity)**.** *A sequent* $\phi_1, ..., \phi_n \vdash_{\mathbf{BL}} \psi$ *is then said to be valid* *in* **BL**-*algebras, if* $[\![\phi_1]\!] \otimes ... \otimes [\![\phi_n]\!] \leq [\![\psi]\!]$ *holds in* **BL**-*algebras. A sequent is said to be* valid *if it is valid in* all **BL**-*algebras. We can write this:*

$$\Gamma \vDash_{\mathbf{BL}} \phi$$

In the case where ϕ *is valid in all* **BL**-*algebras, we write*

$$\vDash_{\mathbf{BL}} \phi$$

It is easy to show that the valid sequents, in the sense above, are precisely the ones provable in Basic Logic.

Proposition 5.2.3. *A sequent* $\Gamma \vdash \psi$ *is* **BL**-*valid iff it is provable in* **BL**.

5.3 Kripke Semantics for BL

Note 28. *The Kripke semantics for* **BL** *that we propose is based on the Bova-Montagna construction of poset sums introduced in (see Section for more details). We first need to define a particular class of functions from the set of worlds* W *to MV-chains.*

Definition 5.3.1 (Sloping functions)**.** *Let* $\mathcal{W} = \langle W, \geq \rangle$ *be a linear order and* $[0,1]_{\mathbf{MV}}$ *a* **BL**-*algebra. A function* $f: W \to [0,1]_{\mathbf{MV}}$ *is said to be a* sloping function for **BL** *(hereon* sloping function*, or* sloping*) if* $f(w) > \bot$ *implies* $\forall v > w(f(v) = \top)$.

Lemma 5.3.2. *If* $f: W \to [0,1]_{\mathbf{MV}}$ *and* $g: W \to [0,1]_{\mathbf{MV}}$ *are sloping, then the following functions are also sloping:*

$$\begin{aligned} (f \wedge g)(w) &:= \min\{fw, gw\} \\ (f \vee g)(w) &:= \max\{fw, gw\} \\ (f \otimes g)(w) &:= \max\{0, \overline{\overline{fw} + \overline{gw}}\} \end{aligned}$$

Proof. Let f, g be sloping functions. Let us consider each case:

- $f \wedge g$. Assume $(f \wedge g)(w) > \bot$, i.e. $\min\{fw, gw\} > \bot$. This implies that we have both $fw > \bot$ and $gw > \bot$. But since f and g are assumed to be sloping functions, we get that $\forall v > w(f(v) = \top)$ and $\forall v > w(g(v) = \top)$, from which it follows that $\forall v > w(\min\{f(v), g(v)\} = \top)$.

- $f \vee g$. Assume $(f \vee g)(w) > \bot$ i.e. $\max\{fw, gw\} > \bot$. This implies that we have at least one of $fw > \bot$ or $gw > \bot$. In case $fw > \bot$, f is a sloping function by hypothesis, so we have $\forall v > w(f(v) = \top)$ from which it follows $\forall v > w(\max\{f(v), g(v)\} = \top)$. The case of $gw > \bot$ is similar.

- $f \otimes g$. Assume $(f \otimes g)(w) > \bot$ i.e. $\max\{0, \overline{\overline{fw} + \overline{gw}}\} > 0$. This means $\max\{0, \overline{\overline{fw} + \overline{gw}}\} = \max\{0, f(w)+g(w)-1\} > 0$; and hence $f(w)+g(w)-1 > 0$. This implies that neither $f(w) = \bot$ nor $g(w) = \bot$, i.e. we have both $f(w) > \bot$ and $g(w) > \bot$. Since both $f(w), g(w)$ are sloping functions by hypothesis $\forall v > w(f(v) = \top)$ and $\forall v > w(g(v) = \top)$. So $\forall v > w$ $\max\{0, f(v) + g(v) - 1\} = \max\{0, \top + \top - 1\} = \max\{0, \top + 0\} = \top$, as desired.

□

Definition 5.3.3. *The "floor" function, as in 3.3.4. Let $\lfloor \cdot \rfloor$ be the usual "floor" operation on the standard MV-chain $[0,1]_{\text{MV}}$, corresponding to the case distinction*

$$\lfloor x \rfloor := \begin{cases} \top & \textit{if} \quad x = \top \\ \bot & \textit{if} \quad x < \top \end{cases}$$

As in 3.3.4, we write $\inf_{v \geq w}$ for the following construction:

$$\lfloor \mathbf{inf} \rfloor_{v \geq w} f(v) := \min\{f(w), \inf_{v > w} \lfloor f(v) \rfloor\}$$

where the same conditions apply as in 3.3.4.

Lemma 5.3.4. *This definition of $\inf_{v \geq w}$ can also be equivalently written as*

$$\lfloor \mathbf{inf} \rfloor_{v \geq w} f(v) := \begin{cases} f(w) & \textit{if} \quad \forall v > w(f(v) = \top) \\ \bot & \textit{if} \quad \exists v > w(f(v) < \top) \end{cases}$$

and for any $f : W \to [0,1]$ the function $\lambda w. \lfloor \mathbf{inf} \rfloor_{v \geq w} f(v)$ is a sloping function.

Proof. As in 3.3.2. □

Definition 5.3.5. *Let $[0,1]_{\mathbf{MV}}$ be a* **MV***-algebra. A* Linear Bova-Montagna structure *for $[0,1]_{\mathbf{MV}}$ (or LBM-structure) is a pair $\mathcal{M}_{[0,1]_{\mathbf{MV}}} = \langle \mathcal{W}, \Vdash^{\text{LBM}} \rangle$ where $\mathcal{W} = \langle W, \geq \rangle$ is a linear order, and $\Vdash^{\text{LBM}}$ is an infix operator (on worlds and propositional variables) taking values in $[0,1]_{\mathbf{MV}}$, i.e. $(w \Vdash^{\text{LBM}} p) \in [0,1]_{\mathbf{MV}}$, such that for any propositional variable p the function $\lambda w.(w \Vdash^{\text{LBM}} p) : W \to [0,1]_{\mathbf{MV}}$ is a sloping function.*

Definition 5.3.6 (LBM Kripke Semantics for $\mathcal{L}_{\otimes}$)**.** *Given a LBM-structure*

$$\mathcal{M}_{[0,1]_{\mathbf{MV}}} = \langle \mathcal{W}, \Vdash^{\text{LBM}} \rangle$$

the valuation function $w \Vdash^{\text{LBM}} p$ on propositional variables p can be extended to *all $\mathcal{L}_{\otimes}$-formulas as:*

$$\begin{array}{lcl} w \Vdash^{\mathrm{LBM}} \top & := & \top \\ w \Vdash^{\mathrm{LBM}} \bot & := & \bot \\ w \Vdash^{\mathrm{LBM}} \phi \wedge \psi & := & (w \Vdash^{\mathrm{LBM}} \phi) \wedge (w \Vdash^{\mathrm{LBM}} \psi) \\ w \Vdash^{\mathrm{LBM}} \phi \vee \psi & := & (w \Vdash^{\mathrm{LBM}} \phi) \vee (w \Vdash^{\mathrm{LBM}} \psi) \\ w \Vdash^{\mathrm{LBM}} \phi \otimes \psi & := & (w \Vdash^{\mathrm{LBM}} \phi) \otimes (w \Vdash^{\mathrm{LBM}} \psi) \\ w \Vdash^{\mathrm{LBM}} \phi \rightarrow \psi & := & \lfloor \mathbf{inf} \rfloor_{v \geq w}((v \Vdash^{\mathrm{LBM}} \phi) \rightarrow (v \Vdash^{\mathrm{LBM}} \psi)) \end{array}$$

where the operations on the right-hand side are the operations on $[0,1]_{\mathbf{BL}}$.

Lemma 5.3.7. *For any formula ϕ the function $\lambda w.(w \Vdash^{\mathrm{LBM}} \phi): W \rightarrow [0,1]_{\mathbf{MV}}$ is a sloping function.*

Proof. By induction on the complexity of the formula ϕ, in fact, as in 3.3.7. □

We can generalise the preceding results for LBM stuctures:

Lemma 5.3.8. *(Existence of General Infima and Suprema for LBM structures) Let $f : W \rightarrow [0,1]_{\mathbf{MV}}$ as above, with $[0,1]_{\mathbf{MV}}$ a* **MV**-*algebra, and f inducing $\{f(v)|w \leq v\}$ in $[0,1]_{\mathbf{MV}}$. Then $\inf\{f(v)|w \leq v\}$ exists in $[0,1]_{\mathbf{MV}}$, as does $\sup\{f(v)|w \leq v\}$.*

Proof. Let W be a linear order under $\leq$, let $[0,1]_{\mathbf{MV}}$ be a **MV**-chain, and let $f : W \rightarrow [0,1]_{\mathbf{MV}}$ with $\{f(v)|w \leq v\}$ in $[0,1]_{\mathbf{MV}}$. Now by 5.3.2 and 5.3.4 we have that f is monotone from W to $[0,1]_{\mathbf{MV}}$, and this together with the fact that $[0,1]_{\mathbf{MV}}$ is complete as a lattice means $\inf\{f(v)|w \leq v\}$ and $\sup\{f(v)|w \leq v\}$ always exists in $[0,1]_{\mathbf{MV}}$. □

Lemma 5.3.9. *(The sloping functions are linearly ordered in LBM's.) Let $f, g : W \rightarrow [0,1]_{\mathbf{MV}}$ be sloping for* **BL**. *Then:*

$$\forall v \geq w : (f(v) \geq g(v)) \vee \forall v \geq w : (g(v) \geq f(v))$$

Proof. We prove $\neg\forall v \geq w : (f(v) \geq g(v)) \Rightarrow \forall v \geq w : (g(v) \geq f(v))$ as this is classically equivalent to the above statement. So assume that $\neg\forall v \geq w : (f(v) \geq g(v))$. Then $\exists v \geq w : (f(v) < g(v))$. But then $g(v) > \bot$; and since f, g are sloping, this means for any $v\prime > v$ we have $g(v\prime) = \top$ and so $g(v\prime) \geq f(v\prime)$. On the other hand, for any $v\prime < v$, $f(v\prime) = \bot$ as f is sloping, and since this is the least element of the ordering, in particular we have $g(v\prime) \geq f(v\prime)$. In either case, we have $\forall v \geq w : (g(v) \geq f(v))$ as desired. □

Note 29. *We use* inf *to refer to the standard operation on $[0,1]_{\mathbf{MV}}$, but we use $\mathbf{inf}_{v \geq w}$ to refer to the operation on a set of formulas evaluated* in *a linear* *ordering; the former refers to a point in the linear ordering induced by the frame* $\langle \mathcal{W}, \leq \rangle$*, while the latter is an operation on the frame itself. The following makes this relationship clear.*

We can now generalise the monotonicity property of intuitionistic logic and Gödel-Dummett logic to **BL**:

Corollary 5.3.10 (Monotonicity)**.** *The following (generalised) monotonicity property holds for all* $\mathcal{L}_{\otimes}$*-formulas* ϕ*, i.e.*

$$\text{if } w \leq v \text{ then } (w \Vdash^{\text{LBM}} \phi) \leq (v \Vdash^{\text{LBM}} \phi)$$

Proof. This follows from the observation that the valuations are sloping functions, which are in turn monotone functions. □

5.4 Validity under LBM structures

Definition 5.4.1. *Let* $\Gamma = \psi_1, \ldots, \psi_n$*. Consider the following definitions:*

- *We say that a sequent* $\Gamma \vdash \phi$ holds *in a LBM-structure* $\mathcal{M}$ *(written* $\Gamma \Vdash^{\text{LBM}}_{\mathcal{M}} \phi$*) if for all* $w \in W$ *we have*

$$(w \Vdash^{\text{LBM}} \psi_1 \otimes \ldots \otimes \psi_n) \leq (w \Vdash^{\text{LBM}} \phi)$$

  Otherwise (i.e. if $\Gamma \nvdash \phi$*), we say that the sequent* fails $\mathcal{M}$ *(written* $\Gamma \Vdash\!\!\!/^{\text{GBM}}{}_{\mathcal{M}} \phi$*) and this means:*

$$\exists w \in W : (w \Vdash^{\text{LBM}} \psi_1 \otimes \ldots \otimes \psi_n) > (w \Vdash^{\text{LBM}} \phi)$$

- *A sequent* $\Gamma \vdash \phi$ *is said to be valid under the LBM Kripke semantics for* $\mathcal{L}_{\otimes}$ *(written* $\Gamma \Vdash^{\text{LBM}} \phi$*) if* $\Gamma \Vdash^{\text{LBM}}_{\mathcal{M}} \phi$ *for all LBM-structures* $\mathcal{M}$*.*

Note 30. *We will prove that this semantics is sound and complete for* **BL***, i.e. a sequent* $\Gamma \vdash \phi$ *is provable in* **BL** *iff it is valid in all LBM-structures. But first let us show that the semantics presented above is a direct generalisation of Kripke's original semantics.*

5.5 LBMs and Linear Kripke structures

Note 31. *Linear Bova-Montagna structures generalise linear Kripke structures, i.e. Kripke structures where the frame has a linear ordering. This is because Kripke structures merely require the valuations* $(w \Vdash^{\text{LBM}} p) \in [0,1]_{\mathbf{MV}}$

are always in the finite set $\{0,1\}$ *or* $\{\bot,\top\}$*. These can then be identified with the Booleans. Therefore, any Linear Kripke structure can be seen as a LBM-structure, by defining*

$$w \Vdash^{\text{LBM}} \phi = \begin{cases} \top & \text{if } w \Vdash^{\text{LK}} \phi \\ \bot & \text{if } w \nVdash^{\text{LK}} \phi \end{cases}$$

Note 32. *Recall that* $\mathcal{L} \subset \mathcal{L}_{\otimes}$*, so any* $\mathcal{L}$*-formula is also an* $\mathcal{L}_{\otimes}$*-formula.*

Theorem 5.5.1. *For any Linear Kripke structure* $\mathcal{LK} = \langle \mathcal{W}, \Vdash^{\text{K}} \rangle$ *and* $\mathcal{L}$*-formula* ϕ*, we have* $\forall w$*:*

$$w \Vdash^{\text{LK}} \phi \quad \textit{iff} \quad (w \Vdash^{\text{LBM}} \phi) = \top$$

Proof. By induction on the complexity of the formula ϕ.

Basis: If ϕ is an atomic formulas the result is immediate.

Induction step: Suppose the result holds for all sub-formulas of ϕ:

Case 1. $\phi = \psi \wedge \chi$. We have:

$$\begin{aligned} w \Vdash^{\text{LK}} \psi \wedge \chi \quad &\equiv \quad (w \Vdash^{\text{LK}} \psi) \wedge (w \Vdash^{\text{LK}} \chi) \\ &\overset{\text{(IH)}}{\Leftrightarrow} \quad (w \Vdash^{\text{LBM}} \psi) = \top \wedge (w \Vdash^{\text{LBM}} \chi) = \top \\ &\equiv \quad (w \Vdash^{\text{LBM}} \psi \wedge \chi) = \top \end{aligned}$$

Case 2. $\phi = \psi \vee \chi$. We have:

$$\begin{aligned} w \Vdash^{\text{LK}} \psi \vee \chi \quad &\equiv \quad (w \Vdash^{\text{LK}} \psi) \vee (w \Vdash^{\text{LK}} \chi) \\ &\overset{\text{(IH)}}{\Leftrightarrow} \quad (w \Vdash^{\text{LBM}} \psi) = \top \vee (w \Vdash^{\text{LBM}} \chi) = \top \\ &\equiv \quad (w \Vdash^{\text{LBM}} \psi \vee \chi) = \top \end{aligned}$$

Case 3. $\phi = \psi \to \chi$. Here we use that, when restricted to Linear Kripke structures, $(v \Vdash^{\text{LBM}} \psi) \in \{\top, \bot\}$ and $(v \Vdash^{\text{LBM}} \chi) \in \{\top, \bot\}$, and hence

(i) $\forall v \geq w(((v \Vdash^{\text{LBM}} \psi) = \top) \to ((v \Vdash^{\text{LBM}} \chi) = \top)) \Leftrightarrow \forall v \geq w((v \Vdash^{\text{LBM}} \psi) \to (v \Vdash^{\text{LBM}} \chi)) = \top$

(ii) $\forall v \geq w(((v \Vdash^{\text{LBM}} \psi) \to (v \Vdash^{\text{LBM}} \chi)) = \top) \Leftrightarrow \mathbf{inf}_{v \geq w}((v \Vdash^{\text{LBM}} \psi) \to (v \Vdash^{\text{LBM}} \chi)) = \top$, i.e. the $\lfloor \mathbf{inf} \rfloor_{v \geq w}$ translates directly into a universally quantifed expression, i.e. it is (again) a standard $\inf_{v \geq w}$ operation (on a set).

Therefore:

$$\begin{aligned}
w \Vdash^{\mathrm{LK}} \psi \to \chi \quad &\equiv \quad \forall v \geq w((v \Vdash^{\mathrm{LK}} \psi) \to (v \Vdash^{\mathrm{LK}} \chi)) \\
&\overset{(\mathrm{IH})}{\Leftrightarrow} \quad \forall v \geq w((v \Vdash^{\mathrm{LBM}} \psi) = \top \to (v \Vdash^{\mathrm{LBM}} \chi) = \top) \\
&\overset{(5.5)}{\Leftrightarrow} \quad \forall v \geq w((v \Vdash^{\mathrm{LBM}} \psi) \to (v \Vdash^{\mathrm{LBM}} \chi) = \top) \\
&\overset{(5.5)}{\Leftrightarrow} \quad \lfloor \mathbf{inf} \rfloor_{v \geq w}((v \Vdash^{\mathrm{LBM}} \psi) \to (v \Vdash^{\mathrm{LBM}} \chi)) = \top \\
&\equiv \quad (w \Vdash^{\mathrm{LBM}} \psi \to \chi) = \top
\end{aligned}$$

which concludes the proof. □

We also note the following:

Proposition 5.5.2. *BM structures generalise LBM structures.*

Proof. This follows from the fact that all linear orders are partial orders. □

Note 33. *Proposition 5.5.2, along with the result of our previous paper that BM structures generalise Kripke structures, gives an alternative proof of 5.5.1. We have opted for a direct proof so that the reader may directly observe the generalisation and compare with the earlier paper.*

5.6 Soundness

Let us now prove the soundness of the Kripke semantics for **BL**.

Theorem 5.6.1 (Soundness)**.** *If* $\Gamma \vdash_{\mathbf{BL}} \phi$ *then* $\Gamma \Vdash^{\mathrm{LBM}} \phi$.

Proof. By induction on the derivation of $\Gamma \vdash \phi$. Assume $\Gamma = \psi_1, \ldots, \psi_n$ and let $\otimes\Gamma := \psi_1 \otimes \ldots \otimes \psi_n$. Fix a LBM-structure $\mathcal{M} = \langle \mathcal{W}, \Vdash^{\mathrm{LBM}} \rangle$ with $\mathcal{W} = \langle W, \geq \rangle$, and let $w \in W$.

(Axiom) $\Gamma, \phi \vdash \phi$. By Definition 5.4.1, we need to show $(w \Vdash^{\mathrm{LBM}} (\otimes\Gamma) \otimes \phi) \leq (w \Vdash^{\mathrm{LBM}} \phi)$, which of course follows by applying 5.3.6 and Lemma 2.3.61 (*ii*).
($\wedge$I) By IH we have $(w \Vdash^{\mathrm{LBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{LBM}} \phi)$ and $(w \Vdash^{\mathrm{LBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{LBM}} \psi)$. Hence

$$(w \Vdash^{\mathrm{LBM}} \otimes\Gamma) \leq \min\{w \Vdash^{\mathrm{LBM}} \phi, w \Vdash^{\mathrm{LBM}} \psi\} \equiv w \Vdash^{\mathrm{LBM}} \phi \wedge \psi$$

($\wedge$E) By IH we have $(w \Vdash^{\mathrm{LBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{LBM}} \phi \wedge \psi)$, i.e.

$$(w \Vdash^{\mathrm{LBM}} \otimes\Gamma) \leq \min\{w \Vdash^{\mathrm{LBM}} \phi, w \Vdash^{\mathrm{LBM}} \psi\}$$

This implies both $(w \Vdash^{\mathrm{LBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{LBM}} \phi)$ and $(w \Vdash^{\mathrm{LBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{LBM}} \psi)$.

(∨I) By IH we have $(w \Vdash^{\mathrm{LBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{LBM}} \phi)$. Therefore

$$(w \Vdash^{\mathrm{LBM}} \otimes\Gamma) \leq \max\{w \Vdash^{\mathrm{LBM}} \phi, w \Vdash^{\mathrm{LBM}} \psi\} \equiv w \Vdash^{\mathrm{LBM}} \phi \vee \psi$$

(∨E) By IH we have

- $w \Vdash^{\mathrm{LBM}} \otimes\Gamma \leq \max\{w \Vdash^{\mathrm{LBM}} \phi, w \Vdash^{\mathrm{LBM}} \psi\}$
- $(w \Vdash^{\mathrm{LBM}} (\otimes\Delta) \otimes \phi) \leq (w \Vdash^{\mathrm{LBM}} \chi)$
- $(w \Vdash^{\mathrm{LBM}} (\otimes\Delta) \otimes \psi) \leq (w \Vdash^{\mathrm{LBM}} \chi)$

By Lemma 2.3.61 (iii), these imply $(w \Vdash^{\mathrm{LBM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq w \Vdash^{\mathrm{LBM}} \chi$.
(→I) By IH we have $(w \Vdash^{\mathrm{LBM}} (\otimes\Gamma) \otimes \phi) \leq (w \Vdash^{\mathrm{LBM}} \psi)$, for all $w \in W$. By the adjointness property we get

$$(w \Vdash^{\mathrm{LBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{LBM}} \phi) \to (w \Vdash^{\mathrm{LBM}} \psi)$$

for all $w \in W$. Fix $w \in W$, and let us consider two cases. First, if for some $v > w$ we have $(v \Vdash^{\mathrm{BM}} \phi) \to (v \Vdash^{\mathrm{BM}} \psi) < \top$, then we must have that $(v \Vdash^{\mathrm{BM}} \otimes\Gamma) < \top$, and hence $(w \Vdash^{\mathrm{BM}} \otimes\Gamma) = \bot$, and trivially

$$(w \Vdash^{\mathrm{LBM}} \otimes\Gamma) \leq \lfloor \mathbf{inf} \rfloor_{v \geq w}((v \Vdash^{\mathrm{LBM}} \phi) \to (v \Vdash^{\mathrm{LBM}} \psi))$$

If on the other hand, $(v \Vdash^{\mathrm{LBM}} \phi) \to (v \Vdash^{\mathrm{LBM}} \psi) = \top$ for all $v > w$, then

$$\lfloor \mathbf{inf} \rfloor_{v \geq w}((v \Vdash^{\mathrm{LBM}} \phi) \to (v \Vdash^{\mathrm{LBM}} \psi)) = (w \Vdash^{\mathrm{LBM}} \phi) \to (w \Vdash^{\mathrm{LBM}} \psi)$$

and we indeed have $(w \Vdash^{\mathrm{LBM}} \otimes\Gamma) \leq (w \Vdash^{\mathrm{LBM}} \phi) \to (w \Vdash^{\mathrm{LBM}} \psi)$.
(→E) By IH we have

- $(w \Vdash^{\mathrm{LBM}} \otimes\Gamma) \leq w \Vdash^{\mathrm{LBM}} \phi$
- $(w \Vdash^{\mathrm{LBM}} \otimes\Delta) \leq \lfloor \mathbf{inf} \rfloor_{v \geq w}((v \Vdash^{\mathrm{LBM}} \phi) \to (v \Vdash^{\mathrm{LBM}} \psi))$

We again consider two cases. First, if for some $v > w$ we have $(v \Vdash^{\mathrm{LBM}} \phi) \to (v \Vdash^{\mathrm{LBM}} \psi) < \top$, then

$$\lfloor \mathbf{inf} \rfloor_{v \geq w}((v \Vdash^{\mathrm{LBM}} \phi) \to (v \Vdash^{\mathrm{LBM}} \psi)) = \bot$$

and hence $(w \Vdash^{\mathrm{LBM}} \otimes\Delta) = \bot$ and $(w \Vdash^{\mathrm{LBM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq w \Vdash^{\mathrm{LBM}} \psi$. If on the other hand, $(v \Vdash^{\mathrm{LBM}} \phi) \to (v \Vdash^{\mathrm{LBM}} \psi) = \top$ for all $v > w$, then

$$\lfloor \mathbf{inf} \rfloor_{v \geq w}((v \Vdash^{\mathrm{LBM}} \phi) \to (v \Vdash^{\mathrm{LBM}} \psi)) = (w \Vdash^{\mathrm{LBM}} \phi) \to (w \Vdash^{\mathrm{LBM}} \psi)$$

so that our assumption is $(w \Vdash^{\text{LBM}} \otimes\Delta) \leq (w \Vdash^{\text{LBM}} \phi) \to (w \Vdash^{\text{LBM}} \psi)$. By Lemma 2.3.61 (iv) we obtain $(w \Vdash^{\text{LBM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq w \Vdash^{\text{LBM}} \psi$.

($\bot$E) By IH we have $(w \Vdash^{\text{LBM}} \otimes\Gamma) \leq w \Vdash^{\text{LBM}} \bot$. Since $(w \Vdash^{\text{LBM}} \bot) = 0$, we have that $(w \Vdash^{\text{LBM}} \otimes\Gamma) = 0$, which implies $(w \Vdash^{\text{LBM}} \otimes\Gamma) \leq (w \Vdash^{\text{LBM}} \phi)$, for any ϕ.

($\otimes$I) By IH $(w \Vdash^{\text{LBM}} \otimes\Gamma) \leq w \Vdash^{\text{LBM}} \phi$ and $(w \Vdash^{\text{LBM}} \otimes\Delta) \leq w \Vdash^{\text{LBM}} \psi$. By Lemma 2.3.61 (v) we have

$$(w \Vdash^{\text{LBM}} \otimes\Gamma) \otimes (w \Vdash^{\text{LBM}} \otimes\Delta) \leq (w \Vdash^{\text{LBM}} \phi) \otimes (w \Vdash^{\text{LBM}} \psi)$$

and hence

$$(w \Vdash^{\text{LBM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq (w \Vdash^{\text{LBM}} \phi \otimes \psi)$$

($\otimes$E) By IH we have

- $(w \Vdash^{\text{LBM}} \otimes\Gamma) \leq (w \Vdash^{\text{LBM}} \phi) \otimes (w \Vdash^{\text{LBM}} \psi)$
- $(w \Vdash^{\text{LBM}} \otimes\Delta) \otimes (w \Vdash^{\text{LBM}} \phi) \otimes (w \Vdash^{\text{LBM}} \psi) \leq w \Vdash^{\text{LBM}} \chi$

By Lemma 2.3.61 (vi), we have

$$(w \Vdash^{\text{LBM}} \otimes\Gamma) \otimes (w \Vdash^{\text{LBM}} \otimes\Delta) \leq w \Vdash^{\text{LBM}} \chi$$

i.e. $(w \Vdash^{\text{LBM}} (\otimes\Gamma) \otimes (\otimes\Delta)) \leq w \Vdash^{\text{LBM}} \chi$.

(DIV) It is sufficient to show that

$$w \Vdash^{\text{LBM}} (\phi \to \psi) \otimes \phi \quad \leq \quad w \Vdash^{\text{LBM}} (\psi \to \phi) \otimes \psi$$

i.e.

$$(w \Vdash^{\text{LBM}} \phi \to \psi) \otimes (w \Vdash^{\text{LBM}} \phi) \leq (w \Vdash^{\text{LBM}} \psi \to \phi) \otimes (w \Vdash^{\text{BM}} \psi)$$

We consider two cases:

Case 1. $w \Vdash^{\text{LBM}} \phi = \bot$. In this case the result is immediate.

Case 2. $w \Vdash^{\text{LBM}} \phi > \bot$. This implies that $\forall v \succ w(w \Vdash^{\text{LBM}} \phi = \top)$, and hence $\forall v \succ w((w \Vdash^{\text{LBM}} \psi \to w \Vdash^{\text{LBM}} \phi) = \top)$, so

$$w \Vdash^{\text{LBM}} \psi \to \phi \quad = \quad (w \Vdash^{\text{LBM}} \psi) \to (w \Vdash^{\text{LBM}} \phi)$$

Since

$$w \Vdash^{\text{LBM}} \phi \to \psi, \phi \leq ((w \Vdash^{\text{LBM}} \phi) \to (w \Vdash^{\text{LBM}} \psi)) \otimes (w \Vdash^{\text{LBM}} \phi)$$

it remains to show that

$$((w \Vdash^{\text{LBM}} \phi) \to (w \Vdash^{\text{LBM}} \psi)) \otimes (w \Vdash^{\text{LBM}} \phi)$$
$$\leq ((w \Vdash^{\text{LBM}} \psi) \to (w \Vdash^{\text{LBM}} \phi)) \otimes (w \Vdash^{\text{LBM}} \psi)$$

which follows from Lemma 2.3.61 (vii).

(PRELIN) $\Gamma \vdash (\phi \to \psi) \vee (\psi \to \phi)$. By Definition 5.4.1, we need to show:

$$w \Vdash^{\text{LBM}} (\otimes\Gamma) \quad \overset{(\text{L.5.4.1})}{\leq} \quad (w \Vdash^{\text{LBM}} (\phi \to \psi) \vee (\psi \to \phi)) = \top$$

which is equivalent to:

$$w \Vdash^{\text{LBM}} (\otimes\Gamma) \quad \overset{(\text{L.2.3.60})}{\leq} \quad \max\{(w \Vdash^{\text{LBM}} (\phi \to \psi)), (w \Vdash^{\text{LBM}} (\psi \to \phi))\} = \top$$

where the right of the inequality means: Either $(w \Vdash^{\text{LBM}} (\phi \to \psi)) = \top$ or $(w \Vdash^{\text{LBM}} (\psi \to \phi)) = \top$. Here we break into cases.

Case 1. $(w \Vdash^{\text{LBM}} (\phi \to \psi)) = \top$. We have:

$$\begin{aligned}(w \Vdash^{\text{LBM}} (\phi \to \psi)) = \top \quad &\equiv \quad \lfloor \mathbf{inf} \rfloor_{v \geq w}((v \Vdash^{\text{LBM}} \phi) \to (v \Vdash^{\text{LBM}} \psi)) = \top \\ &\Leftrightarrow \quad \forall v : v \geq w((v \Vdash^{\text{LBM}} \phi) \leq (v \Vdash^{\text{LBM}} \psi))\end{aligned}$$

Case 2. $(w \Vdash^{\text{LBM}} (\psi \to \phi)) = \top$. We have:

$$\begin{aligned}(w \Vdash^{\text{LBM}} (\psi \to \phi)) = \top \quad &\equiv \quad \lfloor \mathbf{inf} \rfloor_{v \geq w}((v \Vdash^{\text{LBM}} \psi) \to (v \Vdash^{\text{LBM}} \phi)) = \top \\ &\Leftrightarrow \quad \forall v : v \geq w((v \Vdash^{\text{LBM}} \psi) \leq (v \Vdash^{\text{LBM}} \phi))\end{aligned}$$

These latter cases show that we must then prove:

$$\forall v : v \geq w((v \Vdash^{\text{LBM}} \phi) \leq (v \Vdash^{\text{LBM}} \psi)) \vee \forall v : v \geq w((v \Vdash^{\text{LBM}} \psi) \leq (v \Vdash^{\text{LBM}} \phi)) \tag{I}$$

But by 5.3.7, $\lambda w.(v \Vdash^{\text{LBM}} \phi) \colon W \to [0,1]_{\mathbf{MV}}$ and $\lambda w.(v \Vdash^{\text{LBM}} \psi) \colon W \to [0,1]_{\mathbf{MV}}$ are sloping functions; and by 5.3.9 the sloping functions for LBM's are linearly ordered, so that indeed I above holds. □

5.7 LBM's and Poset Products

Recall that a *poset product* (cf. [4] and [37]) is defined over a poset $\mathcal{W} = \langle W, \leq \rangle$, as the algebra $\mathbf{A}_{\mathcal{W}}$ of signature $\mathcal{L}_{\otimes}$ whose elements are sloping functions $f \colon W \to [0,1]_{\mathbf{MV}}$ and operations are defined as below:

$$
\begin{array}{lcl}
(\bot)(w) & := & \bot \\
(f_1 \wedge f_2)(w) & := & \min\{f_1 w, f_2 w\} \\
(f_1 \vee f_2)(w) & := & \max\{f_1 w, f_2 w\} \\
(f_1 \otimes f_2)(w) & := & \max\{0, \overline{\overline{f_1 w} + \overline{f_2 w}}\} \\
(f_1 \to f_2)(w) & := & \begin{cases} f_1(w) \to f_2(w) & \text{if} \quad \forall v > w(f_1(v) \leq f_2(v)) \\ \bot & \text{otherwise.} \end{cases}
\end{array}
$$

Since f_1 and f_2 are sloping functions, we have that

$$
\forall v > w(f_1(v) \leq f_2(v)) \quad \Leftrightarrow \quad \forall v > w((f_1(v) \to f_2(v)) = \top)
$$

Therefore, this last clause of the definition can be simplified to

$$
(f_1 \to f_2)(w) \quad := \quad \lfloor \mathbf{inf} \rfloor_{v \geq w} (f_1(v) \to f_2(v))
$$

Definition 5.7.1 (Poset Product semantics for $\mathcal{L}_\otimes$)**.** *Let $\mathcal{W} = \langle W, \preceq \rangle$ be a fixed linearly ordered poset, and $\mathbf{A}_\mathcal{W}$ be the poset product described above. Given h : Atoms $\to \mathbf{A}_\mathcal{W}$ an assignment of atomic formulas to elements of $\mathbf{A}_\mathcal{W}$, any formula ϕ can be mapped to an element $[\![\phi]\!]_h \in \mathbf{A}_\mathcal{W}$ as follows:*

$$
\begin{array}{lcl}
[\![p]\!]_h & := & h(p) \quad (\textit{for atomic formulas } p) \\
[\![\bot]\!]_h & := & \bot \\
[\![\phi \wedge \psi]\!]_h & := & [\![\phi]\!]_h \wedge [\![\psi]\!]_h \\
[\![\phi \vee \psi]\!]_h & := & [\![\phi]\!]_h \vee [\![\psi]\!]_h \\
[\![\phi \otimes \psi]\!]_h & := & [\![\phi]\!]_h \otimes [\![\psi]\!]_h \\
[\![\phi \to \psi]\!]_h & := & [\![\phi]\!]_h \to [\![\psi]\!]_h
\end{array}
$$

A formula ϕ is said to be valid in $\mathbf{A}_\mathcal{W}$ under h *if for every $w \in W$*

$$
[\![\phi]\!]_h^{\mathbf{A}_\mathcal{W}}(w) = \top
$$

(which is 1 in $[0,1]_{\mathrm{MV}}$). A formula ϕ is said to be valid in $\mathbf{A}_\mathcal{W}$ *if it is valid in $\mathbf{A}_\mathcal{W}$ under h for any possible mapping h: Atoms $\to \mathbf{A}_\mathcal{W}$.*

Note 34. *We note in passing that while the poset product construction of Jipsen and Montagna [37] and [38] produces posets of a certain kind (as the name suggests), we can restrict their construction to give us our desired LBM-structures. The following proposition (whose proof we credit here to discussions with Dr.*

Wesley Fussner) edges us towards this desired result.

Proposition 5.7.2 (W. Fussner)**.** *Let $(X, \leq)$ be a linearly-ordered poset and $\{\mathbf{A}_x : x \in X\}$ an indexed collection of integral, bounded, linearly-ordered residuated lattices. Then:*

$$\prod_{x \in (X, \leq)} \mathbf{A}_x$$

or the poset product of this collection is linearly ordered; i.e. linearly ordered factors indexed by a chain in turn yields a linearly ordered product.

Proof. If $X = (X, \leq)$ is a chain or linearly ordered poset, and each A_x is also a linearly ordered poset, then an element f of the corresponding poset product has at most one index $x \in X$ for which $f(x) \neq 0$ or $f(x) \neq 1$. Suppose that f and g are in the poset product. To compare f and g, we just need to know:

$$U_f = \{x \in X : f(x) \neq 0\}$$

$$U_g = \{x \in X : g(x) \neq 0\}$$

and the values for which they are not 0 or 1 (if any; one can call these non-trivial values). Now X is a chain, so we can assume U_f and U_g are comparable under inclusion. If U_f is strictly contained in U_g, then $f < g$. If $U_f = U_g$ and $f(x) \neq 0, 1$, $g(x) \neq 0, 1$, then $f \leq g$ iff $f(x) \leq g(x)$. This is enough to show that f and g are comparable. □

From specialising this latter we have as a corollary:

Corollary 5.7.3. *Let $\mathcal{W} = \langle W, \leq \rangle$ be a linearly-ordered poset and $\{\mathbf{A}_w : w \in W\}$ an indexed collection of MV-chains. Then:*

$$\mathbf{A}_{\mathcal{W}} = \prod_{w \in \langle W, \leq \rangle} \mathbf{A}_w$$

or the poset product of this collection is a linearly ordered **BL***-algebra, i.e. a* **BL***-chain.*

This of course is a restatement of one of the results in 2.3.15.

Note 35. *We conclude this section by observing that given a poset product $\mathbf{A}_{\mathcal{W}}$ over a linearly-ordered poset $\mathcal{W} = \langle W, \geq \rangle$ and a mapping $h\colon Atoms \to \mathbf{A}_{\mathcal{W}}$ of atomic formulas to elements of $\mathbf{A}_{\mathcal{W}}$, we can obtain a LBM structure $\mathcal{M}^{\mathbf{A}_{\mathcal{W}}} = \langle \mathcal{W}, \Vdash_h^{\mathrm{LBM}} \rangle$, by taking*

$$(w \Vdash_h^{\mathrm{LBM}} p) := h(p)(w)$$

recalling that $h(p)\colon W \to [0,1]_{\mathbf{MV}}$ is a sloping function.

Proposition 5.7.4. *Let $\mathbf{A}_{\mathcal{W}}$ be the poset product over a linearly ordered poset $\mathcal{W} = \langle W, \geq \rangle$, and $h\colon Atoms \to \mathbf{A}_{\mathcal{W}}$ be a fixed mapping of atomic formulas to elements of $\mathcal{W}$. Let $\mathcal{M}^{\mathbf{A}_{\mathcal{W}}}$ be the LBM-structure defined above. Then, for any formula ϕ*

$$(w \Vdash_h^{\text{LBM}} \phi) = [\![\phi]\!]_h^{\mathbf{A}_{\mathcal{W}}}(w)$$

Proof. By induction on the complexity of ϕ. □

Hence, again, we can always transform an interpretation of $\mathcal{L}_{\otimes}$ formulas in the poset product $\mathbf{A}_{\mathcal{W}}$ into a general Kripke semantics (on the Kripke frame $\mathcal{W}$) for $\mathcal{L}_{\otimes}$ formulas.

5.8 Completeness of LBM-semantics

Theorem 5.8.1 (Completeness)**.** *If $\Gamma \Vdash^{\text{LBM}} \phi$ then $\Gamma \vdash_{\mathbf{BL}} \phi$.*

Lemma 5.8.2. *If a formula fails in a* **BL***, then it fails in a linear BM.*[2]

Proof. Let $\Gamma \equiv \psi_1, \ldots, \psi_n$. Suppose $\Gamma \vdash \phi$ fails in **BL**. By Proposition 2.1.6 and the remark 2.1.5, it follows that

$$\nvdash_{\mathbf{BL}_H} \psi_1 \to \ldots \to \psi_n \to \phi$$

By the algebraic completeness result for **BL** algebras with respect to the Hilbert-style proof system $\mathbf{BL}_H$ (obtained from that of $\mathbf{GBL}_{ewf}$, or see [30], with completeness stated in 5.2.3), it follows that for some **BL**-algebra $\mathcal{G}$ and some mapping $h\colon Atom \to \mathcal{G}$ from propositional variables to elements of $\mathcal{G}$, we have

$$[\![\psi_1 \to \ldots \to \psi_n \to \phi]\!]_h^{\mathcal{G}} \neq \top$$

By ([30, Theorem 1]) we can take $\mathcal{G}$ to be a **BL**-chain, and indeed by [48] we can take this same $\mathcal{G}$ to be finite as the variety of **BL** is generated by finite **BL** chains. By Montagna's Theorem 3 of [48], we can take $\mathcal{G}$ as the ordinal sum of finitely many copies of $[0,1]_{\mathbf{MV}}$; and by Busaniche's [5] this is isomorphic to a poset product of finitely many copies of $[0,1]_{\mathbf{MV}}$ (which product will therefore also be linearly ordered and finite; see also Corollary 5.7.3). Let $\chi = \psi_1 \to \ldots \to \psi_n \to \phi$, let $\mathcal{G}$ be a finite **BL**-chain invalidating χ, $\mathcal{H}$ Montagna's

[2] We wish to note the argument given above differs from that presented in Wesley Fussner's manuscript on Kripke Completeness [23]. Therein Dr. Fussner gives a very general *algebraic* argument in which he proves soundness and completeness for what he calls 'relational semantics' for various extensions of **GBL**, including **BL**. We wish to note his priority. The specific argument we give above differs from his by way of generality, but also, appeals solely to facts known from the literature on **BL**-chains. We also note that for Dr. Fussner soundness is a much harder issue to resolve (contrasted with our argument for soundness).

ordinal sum and $\mathcal{A}$ be a poset product. Then, piecing together the cited results of Montagna and Busaniche:

$$\begin{aligned} \mathcal{G} \not\models \chi \quad &\Leftrightarrow \quad \exists \text{ finite ordinal sum } \mathcal{H} \not\models \chi \\ &\Leftrightarrow \quad \exists \text{ finite poset product } \mathcal{A} \not\models \chi \text{ and } \mathcal{A} \cong \mathcal{H} \end{aligned}$$

Hence by Proposition 5.7.4 there exists a finite linear order $\langle W, \preceq \rangle$ and a map $h\prime : Atoms \to [0,1]_{\mathbf{MV}}$ from atoms to elements of the poset product $\mathbf{A}_{\mathcal{W}}$ such that for some $w \in W$:

$$[\![\psi_1 \to \ldots \to \psi_n \to \phi]\!]_{h'}^{\mathbf{A}_{\mathcal{W}}}(w) \neq \top$$

By Proposition 5.7.4, we have a LBM-structure $\mathcal{M}^{\mathbf{A}_{\mathcal{W}}}$ such that for some $w \in W$

$$(w \Vdash_{h'}^{\mathrm{LBM}} \psi_1 \to \ldots \to \psi_n \to \phi) \neq \top$$

and hence

$$(w \Vdash_{h'}^{\mathrm{LBM}} \psi_1 \otimes \ldots \otimes \psi_n) \nleq (w \Vdash^{\mathrm{LBM}} \phi)$$

i.e. $\psi_1, \ldots, \psi_n \nVdash^{\mathrm{LBM}} \phi$, so the sequent fails in a LBM. □

5.9 Open Problem

In closing this section, we leave the following as an open problem:

Open Problem 3. *Can one devise a more direct completeness argument that does not require a detour through Poset Products?*

Chapter 6

Conclusions and further work

6.1 Summary of contributions

In the foregoing, we have generalised Kripke's semantics for Intuitionistic logic in a way appropriate to the logics **LLi** or $\mathbf{GBL}_{ewf}$, **ALi** and Hajek's **BL**. We have shown, with some caveats (given in the next section below), the semantics presented for these logics herein are adequate to the same.

Further, in each case, we have shown that the semantics does actually generalise the classic Kripke semantics for **IL**, and shares the essential properties typifying that semantics. As we have discussed in the Introduction, our semantics truly attempts to de-emphasise the algebra, as in the classic case for **IL**, whereas the substructural logic literature preceding our contribution features generalisations of Kripke semantics that do not specialise in an obvious way to the semantics for **IL**. This is a philosophical point, but also a technical one, because, as discussed in the Introduction and articulated throughout the technical chapters foregoing the present chapter, the proposals stemming from e.g. Urquhart [9], Ono [58], [49] and even Routley and Meyer [65] are only Kripkean in a distant sense: they are fundamentally algebraic semantics, where the partial order in the worlds and monotonicity of the valuations are merged into the algebra via a semantics over an ordered, typically residuated, monoid. Similarly the relational semantics present in the literature on fuzzy modal logics relies on generalising the accessibility relation or else starts from a premise in which they further generalise from what is already a generalisation of Kripke semantics (e.g. the neighbourhood semantics of Montague [50] and Scott [67]).

We hope to impart unto the reader our confidence that the generalised Kripke semantics given in this thesis can be applied to a wider range of fuzzy and substructural logics, particularly extensions of Intuitionistic Affine logic.

6.2 Further work

This thesis leaves several potential directions for future research open. We have already noted some of these in passing.

- We do not have the robust completeness result for GBM's in the case of **ALi**. Indeed we have completeness for GGBM's, but these are really (to our mind) a trivial variant of the algebraic completeness obtained for the logic with bounded pocrim semantics. There is nothing especially Kripkean about it.

- Related to this last, we should find a way to relate the GBM's and GGBM's to the Phase Space semantics of Girard [27], for which **ALi** is complete. It would be interesting to know to what extent we can transform our generalised Kripke semantics in this case, and in other cases, to the Phase Spaces. We have already started towards this goal, but it is not easy. We conjecture their equivalence (and we already have some partial results in this direction, not included here in this thesis). If this conjecture does hold, that fact would be somewhat perplexing, as the general intuition behind the Kripke semantics adopted here (as in the classic case) is that in some sense the models reflect a 'growing' feature of the logic (theorems of the logic, or the knowledge of the mathematician, grows with traversal up the poset).

  Compare this 'growing' feature with Girard's Phase Spaces, which are comparably 'static': this makes sense if one considers that the underlying structure of the Kripkean semantics is a partial order with valuations defined as monotone over the structure, whereas the standard Phase Space semantics is not defined over a poset but a residuated monoid. Nonetheless, such an equivalence would further generalise the equivalence between, say, Kripke structures valued in Heyting Algebras and Kripke structures valued in Boolean Algebras, and would go some way towards completing the analogy we have sought exploit in this thesis – that the kind of robust completeness between Kripke structures valued in Heyting and Boolean Algebras respectively in the classic Intuitionistic case can be lifted in the case of $\mathbf{GBL}_{ewf}$ and **BL**.

- The completeness results we have in the cases of **LLi** or $\mathbf{GBL}_{ewf}$ and **BL** are entirely reliant on features unique to the Poset Product construction. We wonder whether one can prove completeness based on another method, hopefully de-emphasising the algebra. This is possible in most other logics (e.g. Classical logic, modal logics and Intuitionistic logic, as illustrated in

Chapter 2), but the systems extending Intuitionistic Affine logic considered herein (and others in the family) seem recalcitrant to more or less non-algebraic approaches. We seek to understand this better.

- Another potential direction is to leverage the relational semantics herein to construct semantic tableaux [68]. There are already tableaux for **BL** [43] and Classical Łukasiewicz logic **LLc** [6], but to our knowledge none for logics in this family (considered herein) in which the relational framework is exploited. This semantic approach might be best, as currently there are, as far as we know, no syntactic methods adequate to the proof-theoretic questions for logics in this area (sans **ALi**, which is well-understood proof-theoretically). We anticipate this can be done on the heels of the work in this thesis.

  Further to improved proof-theoretic analysis (such as this area requires), one might exploit the relational insights considered herein and extract appropriate sequent systems, as in e.g. [25] or [51]. Such labelled systems as Negri's, for instance, have been successfully applied to a host of logics whose proof theory has traditionally been considered intractable, principally via their relational semantics, which she imports into the calculi via labels. A related, but distinct approach, also pursued by Negri, is her use of non-logical axioms in a sequent setting. This approach could be useful in modelling logical systems whose sequent presentation is otherwise analytic yet whose full analysis is marred by the presence of a single non-analytic axiom, e.g. $\mathbf{GBL}_{ewf}$, where the axiom of Divisibility represents a non-analytic, purely algebraic principle sitting astride an otherwise well-understood and proof-theoretically well-behaved (Intuitionistic Affine) base.

- Finally, we note that our work might be extended from the propositional systems considered here to the first-order case. The main challenge is this requires poset product, or otherwise some suitable equivalent, representation of the corresponding algebraic semantics for extensions for e.g. First-order **GBL**.

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