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Research article

Optimal variational iteration method for parametric boundary value problem

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Abstract: Mathematical applications in engineering have a long history. One of the most well-known analytical techniques, the optimal variational iteration method (OVIM), is utilized to construct a quick and accurate algorithm for a special fourth-order ordinary initial value problem. Many researchers have discussed the problem involving a parameter c. We solve the parametric boundary value problem that can't be addressed using conventional analytical methods for greater values of c using a new method and a convergence control parameter h. We achieve a convergent solution no matter how huge c is. For the approximation of the convergence control parameter h, two strategies have been discussed. The advantages of one technique over another have been demonstrated. Optimal variational iteration method can be seen as an effective technique to solve parametric boundary value problem.

Keywords: boundary value problem; h-curves; residual error method; optimal variational iteration method
Mathematics Subject Classification: 34B05, 65K05

1. Introduction

In Engineering, the interpretation and solution of some problems necessitates the use of mathematical models directly. It is usually necessary to apply components of statistics, linear algebra, or differential and integral calculus to understand and analyze these mathematical models. When it comes to observing natural phenomena, nothing in mathematics is more significant than differential equations. To come up with an effective control technique for an engineering problem, mathematical modeling in different dynamical systems, such as fractional and stochastic modeling, might be applied. Simultaneously, to solve these models, various methods have been developed. It has been noticed that fractional differential equations elaborate the natural phenomenon more accurately than ordinary differential equations [1–4]. Applications of inverse problems, boundary value problems, and integral equations in industry have been extensively studied by the researchers studied [19,20].

In the fields of applied mathematics, higher order ordinary initial and boundary value problems are often encountered. An effective method is required to analyze the mathematical model which provides solutions conforming to physical reality. Therefore, we must be able to solve nonlinear higher order ordinary differential equations. In this paper, we aimed to solve a fourth order parametric initial value problem as [18]

$$u''''(t) = 1 + cu'' - cu(t) + \frac{1}{2}ct^2 - 1 \tag{1}$$

with initial and boundary conditions given as

$$u(0) = 1, u'(0) = 1, \text{ and } u(1) = \frac{3}{2} + sinh(1), u'(1) = 1 + cosh(1).$$

The problem was initially reported by Scott and Watts in 1973. Interesting thing about this problem is that its exact solution is independent of parameter c, that is

$$u_{exact}(t) = 1 + \frac{1}{2}t^2 + sinh(t). \tag{2}$$

Although itself does. Here u is used to model many types of problems arise in the mathematical modeling of viscoelastic and inelastic flows, deformation of beams and plate deflection theory branches of mathematical, physical, and engineering sciences, t is showing time and c is independent parameter. To obtain the exact solution of nonlinear differential equations, one of the most common semi-analytical methods, such as the Optimal Variational Iteration Method (OVIM), is considered. The original idea of VIM is introduced by He in order to solve different autonomous ordinary differential equations as well as fractional differential equations. When it comes to the analytical solution of differential equation, the Homotopy perturbation method (HPM) is one of the top-rated methods [5,11]. Proposed by He, the approach is highly accepted and now it is used as a tool for efficiently solving mathematical and engineering problems accurately. In this approach, the solution is seen as the summation of an infinite series that converges to the exact solution easily.

The above-mentioned analytical methods, on the other hand, all yield solutions that are reliant on the parameter c. Researcher [5] discovered that the variational iteration method's approximate solution to the problem (1) is valid for small values of c. Researchers [12] used the Homotopy perturbation method to study the same problem (1) and discovered that the approximate solution achieved is viable for small values of c. For solving the boundary value problem (1), Momani and Noor [17] examined the Homotopy perturbation approach, Adomian's decomposition method, and differential transformation method. They discovered that the approximate solution provided by the Adomian's decomposition method is the same as the solution provided by the homotopy perturbation method, and thus agrees well with the exact solution only for small values of c, whereas the approximate solution provided by the differential transformation method is valid for a wide range of values. Other related studies by Momani et al can be seen in [14,15].

The above discussed techniques are quite good but it takes too much iterations to calculate for the larger value of c, for example, in case of $c = 100$, it takes 20 iterations for VIM. We solve it with OVIM and check the accuracy of method for these types of unusual problems.

The article is divided in to following sections. Section 2 gives an overview of the methodologies used in the current research. Section 3 discusses the problem formulation and its solution. We will see that OVIM needs only 10 iterations for a quite accurate solutions up to $c = 10^{20}$. Section 4 discusses the results using the h-curve and residual error method. In Section 4, we conclude our results.

2. Methodology

2.1. Variational iteration method

The definition of this technique [5–11] is based on the construction of a functional correction via a general Lagrange multiplier. The multiplier is selected in such a way that, with respect to the initial approximation or the trial function, its correction solution is improved. Consider the following nonlinear equation to explain the basic concept of the variational iteration method,

$$Lv(x,t) + Nv(x,t) = f(x,t), \tag{3}$$

Here linear operator is denoted by L, nonlinear operator is denoted by N, and a known analytical function is denoted by $f(x,t)$. We can create the following correction function according to the variational iteration method,

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda(s)[Lv_n(x,s) + N\tilde{v}_n(x,s) - f(x,s)]\,ds, \tag{4}$$

where λ is a general Lagrange multiplier that can be optimally described by variational theory. Generally, it can be calculated as,

$$\lambda = \frac{(-1)^m(s-t)^{m-1}}{(m-1)!}. \tag{5}$$

$U_0(x,t)$ is an initial approximation with uncertain possibilities. First, we evaluate the Lagrange multiplier which, through integration by sections, will be defined optimally. Many approximations immediately follow as,

$$u(t) = \lim_{n\to\infty} u_n(t). \tag{6}$$

2.2. Optimal variational iteration method

It is possible to write Eq (3) in the following manner [13,16],

$$L(v) = h\{L(v) + N(v) - g(t)\} + L(v), \tag{7}$$

where h is a constant, which we name in what follows as the convergence control parameter. After inverting the linear operator L, Eq (7) can be converted into the following iterative formula,

$$v_{k+1}(t) = hL^{-1}\{L(v_k) + N(v_k) - g(t)\} + v_k(t). \tag{8}$$

In the case of optimal variational iteration technique, the presence of the parameter h plays a critical role. When the parameter h is set to minus or plus unity, it is evident that Eq (8) is the classical variational iteration method.

3. Problem formulation

The Optimal variational iteration method, in which a convergence control parameter h is added, provides a considerably more flexible manner of generating the consecutive iterations of the problem. Because the exact choice of the convergence control parameter h is not aimed to alter the solution, the region of validity of the parameter h can be found by sketching constant h curves for specific values of the solution. This can be accomplished for any physical problem by selecting a non-zero fixed value of the solution, plotting it against the parameter h, and watching the interval of h for which only a minor change in the value is noticed. At the order of approximation, the residual error can be used to determine a better and optimal value for the convergence control parameter h using following relationship,

$$Res(h) = \int_a^b \{L(v_k(t)) + N(v_k(t)) - g(t)\}^2 \, dt, \tag{9}$$

in which an assumption is imposed that we wish to find the solution of Eq (3) in the interval [a,b]. One can easily minimize Eq (9) by imposing the requirement

$$\frac{dRes}{da} = 0. \tag{10}$$

It should be recalled here that if the exact square residual error Res(a) defined needs too much CPU time in practice. To avoid the time-consuming computation, the constant h-curves idea introduced above can be made use of.

According to OVIM, taking the linear operator $L = \frac{d^4}{dt^4}$ source, we can write Eq (1) in the following manner,

$$u_{k+1}(t) = u_k(t) + \text{h}\int_0^t \left[u''''(\tau) - (1+c)u''(\tau) + cu(\tau) - \frac{1}{2}c(\tau)^2 + 1\right] d\tau. \tag{11}$$

Using $u_0(t)=1$, we can write

$$u_1(t) = 1 + \text{h}\int_0^t \left[u''''(\tau) - (1+c)u''(\tau) + cu(\tau) - \frac{1}{2}c(\tau)^2 + 1\right] d\tau. \tag{12}$$

For $c = 10$, the first iteration using OVIM can be calculated as,

$$u_1(t) = 1 + t - \frac{3}{2}t^2 + 3t^2 sinh(1) - t^2 cosh(1) + t^3 - 2t^3 sinh(1) + t^3 cosh(1) - \frac{1}{42}ht^7 sinh(1) + \frac{1}{84}ht^7 cosh(1) + \frac{1}{84}ht^7 - \frac{1}{18}ht^6 + \frac{1}{12}ht^6 sinh(1) - \frac{1}{36}ht^6 cosh(1) - \frac{7}{15}ht^5 + \frac{11}{10}ht^5 sinh(1) - \frac{11}{20}ht^5 cosh(1) + \frac{11}{6}ht^4 - \frac{11}{4}ht^4 sinh(1) + \frac{11}{12}ht^4 cosh(1) \tag{13}$$

Now, to reach the solution, we will calculate the convergence control parameter *h*. We will use both approaches to calculate *h*. Initially, we will discuss residual error method to evaluate *h*, and then will discuss *h*-curves method.

4. Results and discussion

4.1. Residual error method

Residual error method gives reliable values of *h* for small values of *c*. But for larger values of *c*, this method fails to converge. Using the concept given in Eqs (9) and (10), the values of *h* can be calculated easily.

Here we see that these results are only valid for small values of *c* as shown in Figure 1a and b (Figure 1). As c becomes greater, *h* also becomes larger, effecting the solution very badly which consequently fails to converge. Residual error method follows the same results for negative values of *c*. To get a nearly convergent solution, one need to run too many iterations which cost time and CPU labor.

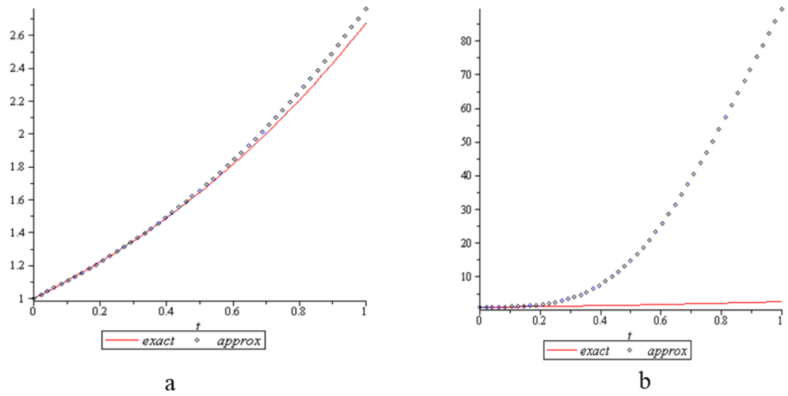

Figure 1. HPM solution for Eq (1) using residual error method. (a: $c = 10^5$, $h = 0.00257$; b: $c = 10^8$, $h = 0.002578$)

4.2. h-curves method

Now we solve same problem with h-curve method. We notice that in Eq (12), h = −1 corresponds to He's standard variational iteration method. We choose 10th iteration calculated by OVIM for t = 1 and t = 2 in Figure 2 to determine the region of validity of the convergence control parameter *h*. We, then drawing constant *h*-curves and obtain a region of optimal value for convergence control parameter *h*. It appears from Figure 2, that at the very least, *h* should fall within the interval [−1,1].

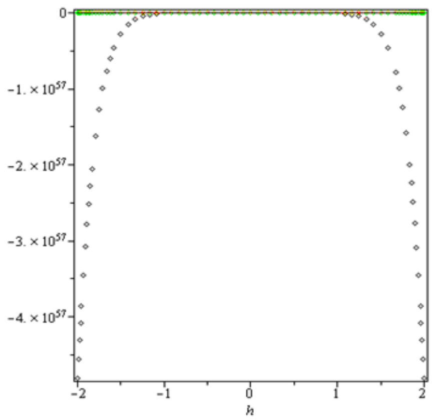

Figure 2. h-curves drawn for Eq (1).

When contrasted to the solution produced using the optimal variational iteration method, Figure 3a and b shows that the optimal variational iteration approximation really converges sharply to the exact solution.

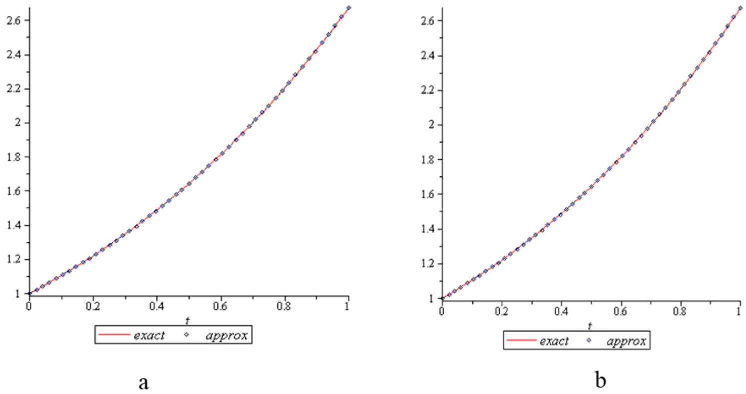

Figure 3. HPM solution for Eq (1) using h-curves method. (a: c = 100, h = 0.0022; b: c = 1000, h =0.0025)

Optimally choosing $h = 0.51 \times 10^{-12}$, we can get a convergent solution for any value of c, no matter how large or how small up to 10^{20} as shown in Figure 4.

This provides plausible proof for our new optimal variational iterative method's validity and precision.

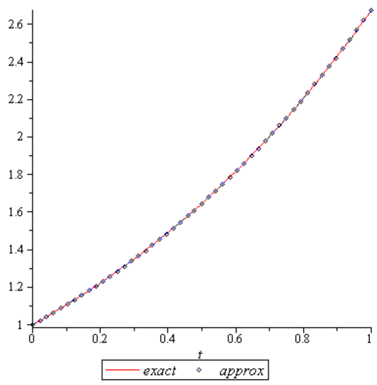

Figure 4. $c = 10^{20}$, $h = 0.51 \times 10^{-12}$.

5. Results and conclusions

The Optimal Variational Iteration Method is very effective for solving nonlinear differential equations. A few numbers of iterations are enough to obtain highly accurate solution. In this paper, a fourth order boundary value problem is solved with OVIM.

OVIM uses the concept of convergence control parameter h. Two ideas are given to calculate the optimal value of convergence controlling parameter h, that is, h-curve method and residual error method. It can be seen that residual error method solution is divergent for greater value of c. The approximated solution calculated with h-curve approach is not only convergent for higher value of c but also found very closed to exact solution. It is experienced that h-curve method is a flexible method than that of residual error method. It provides an interval rather than a fixed value which is given in residual error method. Moreover, h-curve method reduces the labor and CPU time consumption which is inherited in residual error method.

Conflict of interest

The authors declare no conflict of interests.

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