Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(9): 16706–16718.
DOI: 10.3934/math.2022916
Received: 03 May 2022
Revised: 28 June 2022
Accepted: 06 July 2022
Published: 12 July 2022

Theory article

Fuzzifying completeness and compactness in fuzzifying bornological linear spaces

Zhenyu Jin[1] **and Conghua Yan**[2,*]

[1] School of Mathematical Sciences, Suzhou University of Science and Technology, Jiangsu 215009, China

[2] School of Mathematical Sciences, Nanjing Normal University, Jiangsu 210023, China

* **Correspondence:** Email: chyan@njnu.edu.cn.

Abstract: The notions of completeness and compactness play important role in classical functional analysis. The main purpose of this paper is to generalize these notions to the setting of fuzzifying bornological linear spaces. At first, the concepts of fuzzifying Cauchy sequences and fuzzifying completeness are introduced and some interesting properties of them are studied. The relationships among fuzzifying completeness, separation axiom and fuzzifying bornological closed set are discussed. Then the notions of fuzzifying compactness and precompactness are presented, several properties of them are discussed. Particularly, it is demonstrated that a subset is fuzzifying bornological compact if and only if it is fuzzifying bornological precompact and bornological complete.

Keywords: fuzzifying bornological Cauchy sequence; fuzzifying bornological completeness; fuzzifying bornological compactness; fuzzifying bornological precompactness
Mathematics Subject Classification: 46S40, 54A40

1. Introduction

It is well known that the concept of bornological spaces is one class of those spaces that form the general and precise framework in which the fundamental theorems and techniques of functional analysis hold. Hu [11, 12] first introduced the concepts of bornologies and bornological spaces to define the concept of boundedness in topological spaces. Hogle-Nled [10] presented the definition of bornological linear spaces and established the theory of bornological linear spaces. After that, general bornological linear spaces play a key role in many fields. Meyer [17] developed the basic theory of smooth representations of locally compact groups on bornological linear spaces and pointed out that bornological linear spaces provide an ideal setting for noncommutative geometry and representation theory. Mesón and Vericat [18] investigated the topological entropy in bornological linear spaces, whose results indicate that bornological linear spaces are useful tools for the research of optimization theory. Additionally, in the seminal paper [15], Lechicki, Levi and Spakowski studied a family of uniform bornological convergences in the hyperspace of a metric space. Di Concilio and Guadagni [8] showed that the hyperspace of all non-empty closed subsets of a local proximity space carries a very appropriate Fell-type topology, which admits a formulation as hit and far-miss topology and also characterizes as the topology of a two-sided uniform bornological convergence. The theory of general bornological linear spaces acts as an important part of the research of convergence structures on hyperspaces [2, 3] as well as of topologies on function spaces [4, 6, 7, 9, 19, 23].

In [1], Abel and Šostak first studied the theory of bornological spaces in the context of fuzzy sets. They discussed bornologies over an infinitely distributive complete lattice L and gave the concept of an L-bornology as an extension of that of crisp bornologies, specially $L = [0, 1]$. In particular, they showed that the category *L-Born* of L-bornological spaces is a topological construct in [1]. Afterwards, Šostak and Uijane [25] proposed an alternative approach to the fuzzification of the bornologies and developed a construction of an L-valued bornology on a set from a family of crisp bornologies on the same set. Meanwhile, they investigated the initial and final L-valued bornologies. It should be mentioned that, in 2014, Paseka et al. [21,22] investigated the L-bornological linear spaces and got that for certain complete lattices, the category *L-VBorn* of bornological linear spaces is topological over the category *Vec* of vector spaces. Jin and Yan [13] provided the specific description of L-bornological linear spaces, and presented the equivalent characterization of the separation axiom in L-bornological vector spaces in terms of L-Mackey convergence. In 2022, Jin and Yan [14] introduced the notion of fuzzifying bornological linear spaces after considering the necessary and sufficient condition for fuzzifying bornologies to be compatible with linear structures.

As important concepts of topological spaces, completeness and compactness have been widely extended in fuzzy topological space. Among them, the degree approach to Cauchy sequence and compactness are the most general. Li and Shi [16] proposed the degree of fuzzy compactness in L-fuzzy topological spaces, Pang and Shi [20] introduced the degree of compactness in (L, M)-fuzzy convergence spaces, and Qiu [24] gave the degree of Cauchy sequence and compactness in fuzzifying topological linear spaces. The motivation of this paper is to apply the degree approach to Cauchy sequence and compactness in the study of fuzzifying bornological linear spaces. The results we obtained may provide some assistance to establish a systematic theory of fuzzifying bornological linear spaces. This paper is organized as follows. Section 2 is devoted to recall some preliminary results in bornological linear spaces. The main results are presented in Sections 3 and 4. At first, we give the notions of fuzzifying bornological Cauchy sequences and fuzzifying completeness of fuzzifying bornological linear spaces, and show some important properties of these notions in Section 3. Next, fuzzifying bornological compactness and precompactness are defined and some related properties are demonstrated in Section 4. It is presented that a subset is fuzzifying bornological compact if and only if it is bornological precompact and bornological complete.

2. Preliminaries

In this section, we recall some necessary notions and fundamental results which are used in the sequel.

Throughout this paper, X always denotes a universe of discourse. 2^X and $\mathcal{F}(X)$ denote the classes of all crisp and fuzzy subsets of X, respectively. The notation $N(X)$ denotes the set of all sequences in X. $\mathbb{K}$ represents a field of real or complex numbers, the symbol $*$ means a continuous t-norm and $\rightarrow_L$ stands for the Łukasiewicz implication, i.e., $a \rightarrow_L b = \min\{1, 1 - a + b\}$.

Definition 2.1. (Birkhoff [5]) A cl-monoid is a tuple $(L, \leq, \wedge, \vee, *)$, where $(L, \leq, \wedge, \vee)$ is a complete lattice and operation $* : L \times L \rightarrow L$ satisfies conditions:

(1) for all $\alpha, \beta, \gamma \in L$, if $\alpha \leq \beta$, then $\alpha * \gamma \leq \beta * \gamma$;

(2) for all $\alpha, \beta \in L$, $\alpha * \beta = \beta * \alpha$ holds;

(3) for all $\alpha, \beta, \gamma \in L$, $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$ holds;

(4) $\alpha * 1_L = \alpha$ and $\alpha * 0_L = 0_L$ hold for all $\alpha \in L$;

(5) operation $*$ distributes over arbitrary joins: $\alpha * \left(\bigvee_{i\in J} \beta_i\right) = \bigvee_{i\in J} (\alpha * \beta_i)$ for all $\alpha \in L$ and for all $\{\beta_i : i \in J\} \subseteq L$.

Definition 2.2. (Šostak and Uijane [25]) Given a cl-monoid $(L, \leq, \wedge, \vee, *)$, an $(L, *)$-valued bornology on a set X is a mapping $\mathcal{B} : 2^X \rightarrow L$ satisfying the following conditions:

(B1) for all $x \in X$, $\mathcal{B}(\{x\}) = 1_L$ holds;

(B2) if $U \subseteq V \subseteq X$, then $\mathcal{B}(V) \leq \mathcal{B}(U)$;

(B3) for all $U, V \subset X$, $\mathcal{B}(U \cup V) \geq \mathcal{B}(U) * \mathcal{B}(V)$ holds.

Then $(X, \mathcal{B})$ is said to be a L-valued bornological linear space.

Definition 2.3. (Šostak and Uijane [25]) A mapping $f : (X, \mathcal{B}_X) \rightarrow (Y, \mathcal{B}_Y)$ of $(L, *)$-valued bornological spaces is called bounded if $\mathcal{B}_X(A) \leq \mathcal{B}_Y(f(A))$ for all $A \in 2^X$.

Moreover, we can define the degree to which f is bounded as $[Bd(f)] = \bigwedge_{A\subseteq X} \min\{1, 1 - \mathcal{B}_X(A) + \mathcal{B}_Y(f(A))\}$.

Theorem 2.4. (Jin and Yan [14]) Let $\mathcal{B}$ be a fuzzifying bornology on X. Then $\mathcal{B}$ is linear if it satisfies the following conditions:

(B4) $\mathcal{B}(U + V) \geq \mathcal{B}(U) * \mathcal{B}(V)$;

(B5) $\mathcal{B}(\lambda U) \geq \mathcal{B}(U)$, for all $\lambda \in \mathbb{K}$;

$\mathcal{B}\left(\bigcup_{|\alpha|\leq 1} \alpha U\right) \geq \mathcal{B}(U)$.

Then $(X, \mathcal{B})$ is said to be a fuzzifying bornological linear space.

Theorem 2.5. (Jin and Yan [14]) Let $\{(X_j, \mathcal{B}_j)\}_{j\in J}$ be a family of fuzzifying bornological linear spaces indexed by a non-empty set J and $X = \prod_{j\in J} X_j$ be the product of the sets X_j. For all $j \in J$, $f_j : X \rightarrow X_j$ is bounded and $\mathcal{B}_j : 2^{X_j} \rightarrow [0, 1]$. If the mapping $\mathcal{B} : 2^X \rightarrow [0, 1]$ is defined as follows:

$\mathcal{B}(A) = \bigwedge_{j\in J} \{\mathcal{B}_j\left(A_j\right) | f_j(A) = A_j\}$.

Then $\mathcal{B}$ is the product fuzzifying linear bornology on X.

Definition 2.6. (Jin and Yan [14]) Let $(X, \mathcal{B})$ be a fuzzifying bornological linear space. Then a unary predicate $T \in \mathcal{F}(\Sigma)$ called separation is defined as follows:

$(X, \mathcal{B}) \in T := ((\forall M \in Svec(X))((M \in \mathcal{B}) \rightarrow_L (M = \{\theta\}))$,

where the notation $Svec(X), \Sigma$ denote all linear subspaces of X and the family of all fuzzifying bornological linear spaces, respectively.

Moreover, the degree to which $(X, \mathcal{B})$ is separated is

$$[T(X,\mathcal{B})] = \bigwedge_{\substack{M\neq\{\theta\}\\ M\in Svec(X)}} \{1-\mathcal{B}(M)\}.$$

Definition 2.7. (Jin and Yan [14]) Let $(X,\mathcal{B})$ be a fuzzifying bornological linear space and let $S=\{x_n\}$ be a sequence in X. The degree to which S is convergent to x bornologically is

$$[x_n \overset{M}{\to} x] = \bigvee_{\substack{A\in Bal(X)\\ \lambda_n\to 0}} \{\mathcal{B}(A) : \forall n\in\mathbb{N}, x_n - x\in \lambda_n A\}.$$

Where $Bal(X)$ means the family of all balanced sets in X.

Theorem 2.8. (Jin and Yan [14]) Let $(X,\mathcal{B}_X)$ and $(Y,\mathcal{B}_Y)$ be fuzzifying bornological linear spaces and let $f:(X,\mathcal{B}_X)\to(Y,\mathcal{B}_Y)$ be a linear mapping. Then

$$\bigwedge_{S\in N(X)} \left([S \overset{M}{\to} x] \sqcap [Bd(f)]\right) \leq [f(S) \overset{M}{\to} f(x)],$$

where the operation $\sqcap$ is the Łukasiewicz t-norm, i.e.,$a\sqcap b=\max\{a+b-1,0\}$.

Theorem 2.9. (Jin and Yan [14]) Let $(X,\mathcal{B})$ be a fuzzifying bornological linear space. Then for all $S\in N(X)$ and $x,y\in X$, the following statement holds:

$$\left([S \overset{M}{\to} x] \wedge [S \overset{M}{\to} y]\right) \sqcap [T(X,\mathcal{B})] \leq [x=y].$$

Definition 2.10. (Jin and Yan [14]) Let $(X,\mathcal{B})$ be a fuzzifying bornological linear space. Then a unary predicate $BC\in\mathcal{F}(\Sigma)$ called bornologically closed is defined as follows:

$$A\in BC \overset{\Delta}{=} (\forall\{x_n\}\subseteq A)((x_n \overset{M}{\to} x) \to_L (x\in A)).$$

Moreover, the degree to which A is bornologically closed is

$$[BC(A)] = \bigwedge_{\substack{\{x_n\}\subseteq A\\ x\notin A}} \bigwedge_{\substack{B\in Bal(X)\\ \lambda_n\to 0}} \{1-\mathcal{B}(B) : \forall n\in\mathbb{N}, x_n - x\in\lambda_n B\}.$$

3. Fuzzifying completeness in fuzzifying bornological linear spaces

Definition 3.1. Let $(X,\mathcal{B})$ be a fuzzifying bornological linear space and $S\in N(X)$. Then the degree to which S is a Mackey-Cauchy sequence in $(X,\mathcal{B})$ is defined by

$$[CN(S)] = \bigvee_{A\in Bal(X),\lambda_{n,m}\to 0} \{\mathcal{B}(A) : \forall n,m\in\mathbb{N}, S(n)-S(m)\in\lambda_{n,m}A\}.$$

Example 3.2. Let $(\mathbb{R},|\cdot|)$ be an ordinary real line and N be a fuzzy set on $\mathbb{R}\times[0,+\infty)$ defining by

$$N(x,t) = \begin{cases} 0, & t=0;\\ \frac{1}{2}, & 0<t\leq|x|;\\ 1, & t>|x|. \end{cases}$$

Then $(\mathbb{R},N,\wedge)$ is a KM fuzzy normed space. For each $A\subseteq\mathbb{R}$, put $[Bd(A)]=\bigvee_{t>0}\bigwedge_{x\in A} N(x,t)$. Clearly, for every bounded set A of $(\mathbb{R},|\cdot|)$, $Bd(A)=1$, and each unbounded set A_1 of $(\mathbb{R},|\cdot|)$, $Bd(A_1)=\frac{1}{2}$. Moreover, it can deduce the conclusion which $(\mathbb{R},Bd(\cdot))$ is a fuzzifying bornological linear space. In fact, it needs to prove that $Bd(\cdot):2^{\mathbb{R}}\to[0,1]$ satisfies the conditions

(B1)–(B6) in Theorem 2.4. (B1) is clear, we only prove (B2)–(B6).

(B2) Let $[Bd(A)]<a$ for $A\subseteq B$. Then for all $t>0$ there exists $x\in A$ such that $N(x,t)<a$. Since $x\in A\subseteq B$, we obtain $[Bd(B)]<a$ easily. Thus, we get $[Bd(A)]\geq[Bd(B)]$.

(B3) Let $[Bd(A)]*[Bd(B)]>a$. Then $[Bd(A)]\wedge[Bd(B)]\geq[Bd(A)]*[Bd(B)]>a$ and there exist $t_1,t_2>0$ such that $N(x,t_1)\wedge N(y,t_2)>a$ for all $x\in A$ and $y\in B$. Setting $t=t_1\vee t_2$, it is clear that

$[Bd\,(A \cup B)] \geq a$ and $[Bd\,(A \cup B)] \geq [Bd\,(A)] * [Bd\,(B)]$.

(B4) By the inequality $N(x + y, t_1 + t_2) \geq N(x, t_1) \wedge N(y, t_2)$, taking the same method used in (B3), we may easily obtain $[Bd\,(A + B)] \geq [Bd\,(A)] \wedge [Bd\,(B)]$.

(B5) Let $[Bd\,(A)] > a$. Then there exists $t > 0$ such that $N(x, t) > a$ for all $x \in A$. If $\lambda = 0$, it is clear that $[Bd\,(\lambda A)] = 1$. Suppose that $\lambda \neq 0$, we have $N(\lambda x, t_1) = N(x, \frac{t_1}{|\lambda|})$. Setting $t_1 = |\lambda|\, t$, we obtain $N(\lambda x, |\lambda| t) > a$ for all $\lambda x \in \lambda A$. Hence, $[Bd\,(\lambda A)] \geq a$. Then $[Bd\,(\lambda A)] \geq [Bd\,(A)]$, as desired.

(B6) Let $[Bd\,(A)] > a$. Then there exists $t > 0$ such that $N(x, t) > a$ for all $x \in A$. Thus, we have $bx \in \bigcup_{|\alpha| \leq 1} \alpha A$ with $|b| \leq 1$. If $b = 0$, we have $N(bx, t) = 1$ for all $t > 0$. Suppose that $0 < |b| \leq 1$, we obtain $N(bx, t_1) = N(x, \frac{t_1}{|b|}) \geq N(x, t_1)$. Setting $t_1 = t$, then it is obvious that $\left[Bd\left(\bigcup_{|\alpha| \leq 1} \alpha A\right)\right] \geq a$ and $\left[Bd\left(\bigcup_{|\alpha| \leq 1} \alpha A\right)\right] \geq [Bd\,(A)]$.

Let $S = \frac{1}{n} \in N(\mathbb{R})$, $A = \{x : |x| \leq 1\}$, then $S\,(n) - S\,(m) = \frac{1}{n} - \frac{1}{m} = \frac{m-n}{nm} \in \frac{m-n}{mn} A$. Denote $\lambda_{n,m} = \frac{m-n}{nm}$, it is clear $\lambda_{n,m} \to 0$. Thus $[CN(S)] = 1$.

If $S_1 = \{n^2\} \in N(\mathbb{R})$, it follows that $S_1(n) - S_1(m) = n^2 - m^2 = (n + m)\sqrt{(n - m)^3}\frac{1}{\sqrt{n-m}} = \frac{1}{\sqrt{n-m}} A_1$, here $A_1 = \{(n + m)\sqrt{(n - m)^3} : n, m \in \mathbb{N}\}$. Obviously, $BD(A_1) = \frac{1}{2}$. In addition, for each bounded set A in $(\mathbb{R}, |\cdot|)$, there must be no double sequence $\{\lambda_{n,m}\}$ that $\lambda_{n,m} \to 0$ such that $S_1(n) - S_1(m) \in \lambda_{n,m} A$. So, $[CN(S_1)] = \frac{1}{2}$.

Theorem 3.3. Let $(X, \mathcal{B}_X)$, $(Y, \mathcal{B}_Y)$ be fuzzifying bornological linear spaces, $S \in N(X)$ and $f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is a linear mapping. Then

$$[CN(S)] \sqcap [Bd(f)] \leq [CN(f(S))].$$

Proof. We need to prove that $[Bd(f)] + [CN(S)] - 1 \leq [CN(f(S))]$, i.e., $[Bd(f)] \leq 1 - [CN(S)] + [CN(f(S))]$. For any $t > 1 - [CN(S)] + [CN(f(S))]$, let $[CN(S)] = a$ and $[CN(f(S))] = b$, then $1 - a + b < t$. Put $\varepsilon \in (0, t - 1 + a - b)$, there exist a balanced set $A \subseteq X$ and a double sequence $\{\lambda_{n,m}\}$ with $\lim_{n,m\to\infty} \lambda_{n,m} = 0$ such that $\mathcal{B}_X\,(A) > a - \varepsilon$ with $S\,(n) - S\,(m) \in \lambda_{n,m} A$ for all $n, m \in \mathbb{N}$. Since f is linear, it follows that $f(S\,(n)) - f(S\,(m)) = f(S\,(n) - S\,(m)) \in \lambda_{n,m} f(A)$ and $f(A)$ is balanced in Y. Thus $\mathcal{B}_Y(f(A)) \leq [CN(f(S))] = b$. Hence

$$\begin{aligned}[Bd(f)] &= \bigwedge_{B \subseteq X} \min\{1, 1 - \mathcal{B}_X\,(B) + \mathcal{B}_Y\,(f(B))\} \leq 1 - \mathcal{B}_X\,(A) + \mathcal{B}_Y\,(f(A)) \\ &< 1 - a + \varepsilon + b < t.\end{aligned}$$

Which means $[Bd(f)] \leq 1 - [CN(S)] + [CN(f(S))]$, i.e., $[CN(S)] \sqcap [Bd(f)] \leq [CN(f(S))]$.

Corollary 3.4. Let $(X, \mathcal{B}_X)$, $(Y, \mathcal{B}_Y)$ be fuzzifying bornological linear spaces and $f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is a bounded linear mapping. Then $[CN(S)] \leq [CN(f(S))]$ for all $S \in N(X)$.

Theorem 3.5. Let $(X, \mathcal{B})$ be a fuzzifying bornological linear space, $S \in N(X)$ and H a subsequence of S. Then

$$[H \xrightarrow{M} x] \wedge [CN(S)] \leq [S \xrightarrow{M} x].$$

Proof. Let $[H \xrightarrow{M} x] \wedge [CN(S)] > t$, then there exist $A, B \in Bal(X)$ and $t_j, \lambda_{n,m} \to 0$ such that $H(j) - x \in t_j A$, $S\,(n) - S\,(m) \in \lambda_{n,m} B$, $\mathcal{B}(A) > t$ and $\mathcal{B}(B) > t$ for all $j, n, m \in \mathbb{N}$. Since H is a subsequence of S, then for each $n \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that $n < n_j$. Thus, we have

$$S(n) - x = S(n) - S(n_j) + S(n_j) - x = S(\alpha) - S(n_j) + H(j) - x$$
$$\in \lambda_{n,n_j} B + t_j A \subseteq \max\{t_j, \lambda_{n,n_j}\}(A + B).$$

Clearly, $\max\{t_j, \lambda_{n,n_j}\} \to 0$ whenever $n \to \infty$, and $\mathcal{B}(A + B) > t$. It follows that $[S \overset{M}{\to} x] > t$, i.e., $[H \overset{M}{\to} x] \wedge [CN(S)] \leq [S \overset{M}{\to} x]$.

Theorem 3.6. Let $(X, \mathcal{B})$ be a fuzzifying bornological linear space and $S \in N(X)$. Then $[S \overset{M}{\to} x] \leq [CN(S)]$.

Proof. Let $[S \overset{M}{\to} x] > t$, then there exist $A \in Bal(X)$ and $\lambda_n \to 0$ such that $S(n) - x \in \lambda_n A$ and $\mathcal{B}(A) > t$ for all $n \in \mathbb{N}$. Thus, for any $n, m \in \mathbb{N}$,

$$S(n) - S(m) = S(n) - x + x - S(m) \in \lambda_n A + \lambda_m A \subseteq \max\{\lambda_n, \lambda_m\}A.$$

It is easy to find $\max\{\lambda_n, \lambda_m\} \to 0$. So $[CN(S)] > t$. This completes the proof.

Definition 3.7. Let Σ be the family of all fuzzifying bornological linear spaces. A unary fuzzy predicate $Com \in \mathcal{F}(\Sigma)$ is called bornological completeness predicate if it satisfies the following condition:

$$Com(X) := (\forall S \in N(X))(S \in CN) \longrightarrow_L (\exists x \in X)(S \overset{M}{\to} x).$$

The degree to which $(X, \mathcal{B})$ is bornological complete is defined by

$$[Com(X, \mathcal{B})] = \bigwedge_{S \in N(X)} (1 - [CN(S)] + \bigvee_{x \in X} [S \overset{M}{\to} x]).$$

Theorem 3.8. Let $(X_j, \mathcal{B}_j)$ be a family of fuzzifying bornological linear spaces indexed by a non-empty set J and let $X = \prod_{j \in J} X_j$ be the product of the sets X_j. For every $j \in J$, $f_j : X \to X_j$ is a canonical projection and $\mathcal{B} = \prod_{j \in J} \mathcal{B}_j$. Then

$$\bigwedge_{j \in J} [Com(X_j, \mathcal{B}_j)] \leq [Com(X, \mathcal{B})].$$

Proof. Let $\bigwedge_{j \in J} [Com(X_j, \mathcal{B}_j)] > t$, then for all $j \in J$, $[Com(X_j, \mathcal{B}_j)] > t$. It needs to prove that $1 - [CN(S)] + \bigvee_{x \in X} [S \overset{M}{\to} x] \geq t$ for any $S \in N(X)$. In fact, for every $S \in N(X)$ and $j \in J$, $S_j = f_j(S) \in N(X_j)$, it follows that $1 - [CN(S_j)] + \bigvee_{x_j \in X_j} [S_j \overset{M}{\to} x_j] > t$.

Suppose that $1 - [CN(S)] < a$, $a \in (0, t)$, then there exists $A \in Bal(X)$ and $\lambda_{n,m} \to 0$ such that $\mathcal{B}(A) > 1 - a$ and $S(n) - S(m) \in \lambda_{n,m} A$ for all $n, m \in \mathbb{N}$. Since $\mathcal{B}(A) = \bigwedge_{j \in J} \{\mathcal{B}_j \left(A_j\right) | f_j(A) = A_j\} > 1 - a$, we can easily get $\mathcal{B}_j(A_j) > 1 - a$ for all $j \in J$ with $f_j(A) = A_j$. Hence, $S_j(n) - S_j(m) = f_j(S(n)) - f_j(S(m)) \in \lambda_{n,m} f_j(A) = \lambda_{n,m} A_j$. Thus we may obtain $1 - [CN(S_j)] \leq a$ for all $j \in J$. Moreover, $\bigvee_{x_j \in X_j} [S_j \overset{M}{\to} x_j] > t - a$, which follows that there exists $B_j \in Bal(X_j)$ and $\lambda_{n,j} \to 0$ such that $\mathcal{B}_j(B_j) > t - a$ and $S_j(n) - x_j \in \lambda_{n,j} B_j$ for all $n \in \mathbb{N}$. Let $B = \prod_{j \in J} B_j, \lambda_n = (\lambda_{n,j})$, it is obvious that $\mathcal{B}(B) \geq t - a$ and $\prod_{j \in J} (S_j(n) - S_j(x)) = S(n) - x \in \lambda_n B$ for $n \in \mathbb{N}$. It follows that $t - \bigvee_{x \in X} [S \overset{M}{\to} x] \leq a$. By the arbitrariness of a, we have $1 - [CN(S)] \geq t - \bigvee_{x\in X}[S \xrightarrow{M} x]$, i.e., $1 - [CN(S)] + - \bigvee_{x\in X}[S \xrightarrow{M} x] \geq t$. This means $[Com(X,\mathcal{B})] \geq t$. Therefore $\bigwedge_{j\in J}[Com(X_j,\mathcal{B}_j)] \leq [Com(X,\mathcal{B})]$.

Theorem 3.9. Let $(X,\mathcal{B})$ be a family of fuzzifying bornological linear space and $A \subseteq X$. Then

$$[T(X,\mathcal{B})] \sqcap [Com(A)] \leq [BC(A)].$$

Proof. Let $[BC(A)] = \bigwedge_{S\in N(A), x\notin A} \bigwedge_{B\in Bal(X),\lambda_n\to 0} \{1 - \mathcal{B}(B) : \forall n \in \mathbb{N}, S(n) - x \in \lambda_n B\} < t$. Then there exists $S \in N(A)$, $x \notin A$, $B \in Bal(X)$ and $\lambda_n \to 0$ such that $1 - \mathcal{B}(B) < t$ and $S(n) - x \in \lambda_n B$ for all $n \in \mathbb{N}$. Let $\bigvee_{x\in A}[S \xrightarrow{M} x] \neq 0$. If $\bigvee_{x\in A}[S \xrightarrow{M} x] = 0$, then

$$\begin{aligned}&[T(X,\mathcal{B})] + \bigwedge_{S\in N(A)} (1 - [CN(S)] + \bigvee_{x\in A}[S \xrightarrow{M} x]) - 1\\ &\leq \bigwedge_{S\in N(A)} (1 - [CN(S)]) \leq [BC(A)].\end{aligned}$$

From Theorem 2.9, we obtain

$$\begin{aligned}&[T(X,\mathcal{B})] \sqcap (\bigvee_{x\in A}[S \xrightarrow{M} x] \wedge \bigvee_{y\notin A}[S \xrightarrow{M} y])\\ &= ([T(X,\mathcal{B})] + \bigvee_{x\in A}[S \xrightarrow{M} x]) \wedge ([T(X,\mathcal{B})] + \bigvee_{y\notin A}[S \xrightarrow{M} y]) - 1\\ &\leq [x = y] = 0.\end{aligned}$$

Thus, $[T(X,\mathcal{B})] + \bigvee_{x\in A}[S \xrightarrow{M} x] \leq 1$ or $[T(X,\mathcal{B})] + \bigvee_{y\notin A}[S \xrightarrow{M} y] \leq 1$.

Case 1. $[T(X,\mathcal{B})] + \bigvee_{x\in A}[S \xrightarrow{M} x] \leq 1$ holds. Obviously.

Case 2. $[T(X,\mathcal{B})] + \bigvee_{y\notin A}[S \xrightarrow{M} y] \leq 1$ holds.

Then $[T(X,\mathcal{B})] \leq 1 - \bigvee_{y\notin A}[S \xrightarrow{M} y] < 1 - (1 - t) = t$, which means $[T(X,\mathcal{B})] \sqcap [Com(A)] < t$. Hence $[T(X,\mathcal{B})] \sqcap [Com(A)] \leq [BC(A)]$.

Theorem 3.10. Let $(X,\mathcal{B})$ be a family of fuzzifying bornological linear space and $A \subseteq X$. Then

$$[Com(X,\mathcal{B})] \sqcap [BC(A)] \leq [Com(A)].$$

Proof. Let $[Com(A)] < t$, then there exists $S \in N(A)$ such that $(1 - [CN(S)] + \bigvee_{x\in A}[S \xrightarrow{M} x]) < t$. It needs to prove that $[BC(A)] < 1 + t - [Com(X,\mathcal{B})]$. Let $[BC(A)] > a$, then for all $S \in N(A)$, $x \notin A$, $B \in Bal(X)$ and $\lambda_n \to 0$, we get $\mathcal{B}(B) < 1 - a$ and $S(n) - x \in \lambda_n B$ for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned}[Com(X,\mathcal{B})] &= \bigwedge_{S\in N(X)} 1 - [CN(S)] + \bigvee_{x\in A}[S \xrightarrow{M} x]\\ &\leq 1 - [CN(S)] + \bigvee_{x\in A}[S \xrightarrow{M} x] \vee \bigvee_{y\notin A}[S \xrightarrow{M} y]\\ &\leq 1 - [CN(S)] + \bigvee_{x\in A}[S \xrightarrow{M} x] \vee (1 - a)\end{aligned}$$

$$= (1 - [CN(S)] + \bigvee_{x \in A} [S \overset{M}{\rightarrow} x]) \vee (1 - [CN(S)] + 1 - a)$$
$$< t \vee (1 - a + t) = 1 - a + t.$$

Hence, we obtain $[BC(A)] \leq 1+t-[Com(X, \mathcal{B})]$. It deduces that $[Com(X, \mathcal{B})] \sqcap [BC(A)] \leq [Com(A)]$.

4. Fuzzifying compactness in fuzzifying bornological linear spaces

Definition 4.1. Let Σ be the family of all fuzzifying bornological linear spaces. A unary fuzzy predicate $Compact \in \mathcal{F}(\Sigma)$ is called bornological compactness predicate if it satisfies the following condition:

$$Compact(A) := (\forall S)(S \in N(A) \longrightarrow_L (\exists H)(\exists x \in A)(H \leq S) \wedge (H \overset{M}{\rightarrow} x)).$$

The degree to which A is bornological compact is defined by

$$[Compact(A)] = \bigwedge_{S \in N(A)} \bigvee_{H \leq S} \bigvee_{x \in A} [H \overset{M}{\rightarrow} x].$$

Moreover, let $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ be fuzzifying bornological linear spaces, the degree to which f is bornological compact is defined by

$$[Compact(f)] = \bigwedge_{A \subseteq X} \min\{1, 1 - [Compact(A)] + [Compact(f(A))]\}.$$

Example 4.2. Suppose that fuzzifying bornological linear space $(\mathbb{R}, Bd(\cdot))$ is defined as Example 3.2, take $A = [0, 1], A_1 = (0, 1), A_2 = [0, +\infty)$, it is easy to check that $[Compact(A)] = 1, [Compact(A_1)] = [Compact(A_2)] = \frac{1}{2}$.

Theorem 4.3. Let $(X, \mathcal{B})$ be a fuzzifying bornological linear space and $f : (X, \mathcal{B}_X) \rightarrow (Y, \mathcal{B}_Y)$ is a linear mapping. Then for every $A \subseteq X$,

$$[Compact(A)] \sqcap [Bd(f)] \leq [Compact(f(A))].$$

Proof. The proof is similar to that of Theorem 3.3 and it can be omitted.

Theorem 4.4. Let $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ be fuzzifying bornological linear spaces. Then $[Bd(f)] \leq [Compact(f)]$.

Proof. For each $[Bd(f)] > t > 0$ and for every $A \in 2^X$, if $1 - [Compact(A)] < a$, we have for all $S \in N(A)$, there exist $H \leq S$ and $x \in A$ such that $[H \overset{M}{\rightarrow} x] > 1 - a$, i.e., there exists $B \in Bal(X)$, $\lambda_{n_k} \rightarrow 0$ such that $H(n_k) - x \in \lambda_{n_k} B$ and $\mathcal{B}_X(B) > 1 - a$ for all $k \in \mathbb{N}$. Since $[Bd(f)] > t$, we have $1 - \mathcal{B}_X(B) + \mathcal{B}_Y(f(B)) > t$. That is $t - \mathcal{B}_Y(f(B)) < a$, which follows that $t - [Compact(f(A))] < a$. This means $1 - [Compact(A)] \geq t - [Compact(f(A))]$, which implies $[Compact(f)] \geq t$.

Theorem 4.5. Let $(X, \mathcal{B})$ be a family of fuzzifying bornological linear space and $A \subseteq X$. Then

$$[Compact(A)] \leq [Com(A)].$$

Proof. By Theorem 3.5, we have

$$[Com(A)] = \bigwedge_{S \in N(A)} (1 - [CN(S)] + \bigvee_{x \in A} [S \overset{M}{\rightarrow} x])$$
$$\geq \bigwedge_{S \in N(A)} (1 - [CN(S)] + \bigvee_{x \in A} \bigvee_{H \leq S} ([H \overset{M}{\rightarrow} x] \wedge [CN(S)])).$$

For any $t \in (0, [Compact(A)])$, since $[Compact(A)] = \bigwedge_{S \in N(A)} \bigvee_{H \leq S} \bigvee_{x \in A} [H \overset{M}{\to} x]$, then for all $S \in N(D)$, there exist $H \leq S, x \in A$ such that $t < [H \overset{M}{\to} x]$. Thus

$$1 - [CN(S)] + \bigvee_{x \in A} \bigvee_{H \leq S} ([H \overset{M}{\to} x] \wedge [CN(S)]$$

$$\geq 1 - [CN(S)] + ([H \overset{M}{\to} x] \wedge [CN(S)] > 1 - [CN(S)] + t \wedge [CN(S)] \geq t.$$

By the arbitrariness of t and $S \in N(A)$, it follows that $[Com(A)] \geq [Compact(A)]$. This completes the proof.

Theorem 4.6. Let $(X, \mathcal{B})$ be a fuzzifying bornological linear space and $A \subseteq X$. Then, $[Compact(A)] = t$ if and only if A is bornological compact in $(X, \mathcal{B}^r)$ for every $r < t$ and A is not bornological compact in $(X, \mathcal{B}^r)$ for every $r > t$, where the notation $\mathcal{B}^r$ is from [14, 26].

Proof. Necessity. Suppose $[Compact(A)] = t > r$, then for all $S \in N(A)$, there exist $H \leq S$ and $x \in A$ such that $[H \overset{M}{\to} x] > r$, i.e., there exist $B \in Bal(X)$ and $\lambda_{n_k} \to 0$ such that $\mathcal{B}(B) > r$ and $H(n_k) - x \in \lambda_{n_k} B$ for all $k \in \mathbb{N}$. Thus, $B \in \mathcal{B}^r$, which follows that A is bornological compact in $(X, \mathcal{B}^r)$ for every $r < t$.

Suppose that there exists $r_1 > t$ such that A is bornological compact in $(X, \mathcal{B}^{r_1})$, then for all $S \in N(A)$, there exist $H \leq S$ and $x \in A$ such that $H \overset{M}{\to} x$. Thus there exist $B \in Bal(X)$ with $B \in \mathcal{B}^{r_1}$ and $\lambda_{n_k} \to 0$ such that $H(n_k) - x \in \lambda_{n_k} B$ for all $k \in \mathbb{N}$. Thus $[Compact(A)] = t \geq r_1$, which makes a contradiction. So, A is not bornological compact in $(X, \mathcal{B}^r)$ for every $r > t$.

Sufficiency. If A is bornological compact in $(X, \mathcal{B}^r)$ for every $r < t$, then for all $S \in N(A)$, there exist $H \leq S$, $B \in Bal(X)$ with $B \in \mathcal{B}^r$, $\lambda_{n_k} \to 0$ and $x \in A$ such that $H(n_k) - x \in \lambda_{n_k} B$ for all $k \in \mathbb{N}$. Thus, it is easy check that $[Compact(A)] \geq r$. By the arbitrariness of r, we obtain $[Compact(A)] \geq t$. Suppose that $[Compact(A)] > t$, we can easily we can obtain a contradiction. Therefore, we have $[Compact(A)] \leq t$, which follows that $[Compact(A)] = t$.

Theorem 4.7. Let $(X_j, \mathcal{B}_j)$ be a family of fuzzifying bornological linear spaces indexed by a non-empty set J and let $X = \prod_{j \in J} X_j$ be the product of the sets X_j. For every $j \in J$, let $f_j : X \to X_j$ is a canonical projection and $\mathcal{B} = \prod_{j \in J} \mathcal{B}_j$. Then for all $A \subseteq X$

$$\bigwedge_{j \in J} [Compact(f_j(A))] \leq [Compact(A)].$$

Proof. Let $\bigwedge_{j \in J} [Compact(f_j(A))] > t$. Then for all $S \in N(A)$, $j \in J$, $f_j(S) \in N(A_j)$. Thus there exist $H_j \leq f_j(S)$ and $x_j \in f_j(A)$ such that $[H_j \overset{M}{\to} x_j] > t$. It follows that there exist $B_j \in Bal(X_j)$ and $\lambda_{n_k^j} \to 0$ such that $\mathcal{B}_j(B_j) > t$ with $H_j(n_k^j) - x_j \in \lambda_{n_k^j} B_j$ for all $n_k^j \in \mathbb{N}$. Let $B = \prod_{j \in J} B_j$ and $H = \prod_{j \in J} H_j$. It is obvious that $H \leq S$. Since $\mathcal{B}(B) = \bigwedge_{j \in J} \{\mathcal{B}_j \left(B_j\right) | f_j(B) = B_j\} \geq t$, which together with $H(k) - x = \prod_{j \in J} H_j(n_k^j) - x_j \in \prod_{j \in J} \lambda_{n_k^j} B_j = \lambda_k B$, we obtain $[Compact(A)] \geq t$. That implies $\bigwedge_{j \in J} [Compact(f_j(A)] \leq [Compact(A)]$.

Definition 4.8. Let $(X, \mathcal{B})$ be a fuzzifying bornological linear spaces. A unary fuzzy predicate $Precompact \in \mathcal{F}(2^X)$ is called bornological precompactness predicate if it satisfies the following condition:

$$Precompact(A) := (\forall S)(S \in N(A) \longrightarrow_L (\exists H \leq S)(H \in CN)).$$

The degree to which A is bornological precompact in $(X, \mathcal{B})$ is defined by

$$[Precompact(A)] = \bigwedge_{S \in N(A)} \bigvee_{H \leq S} [CN(H)].$$

Taking the same method in the proof of Theorem 3.3, the next Theorem holds clearly, here we omit its proof.

Theorem 4.9. Let $(X, \mathcal{B})$ be a fuzzifying bornological linear space and $A \subseteq X$. Then

$$[Preompact(A)] \sqcap [Bd(f)] \leq [Preompact(f(A))].$$

Theorem 4.10. Let $(X, \mathcal{B})$ be a fuzzifying bornological linear space. Then for every $A \subseteq X$

$$[Compact(A)] \leq [Preompact(A)].$$

Proof. By Theorem 3.6, it is trivial.

Theorem 4.11. Let $(X, \mathcal{B})$ be a fuzzifying bornological linear space and $A \subseteq X$. Then $[Precompact(A)] = t$ if and only if A is bornological precompact in $(X, \mathcal{B}^r)$ for every $r < t$ and A is not bornological precompact in $(X, \mathcal{B}^r)$ for every $r > t$.

Proof. Necessity. Suppose $[Precompact(A)] = t > r$, then for all $S \in N(A)$, there exist $H \leq S$ such that $[CN(H)] > r$, i.e., there exist $B \in Bal(X)$ and $\lambda_{n_k,m_l} \to 0$ such that $\mathcal{B}(B) > r$ and $H(n_k) - H(m_l) \in \lambda_{n_k,m_l} B$ for all $k, l \in \mathbb{N}$. Thus, $B \in \mathcal{B}^r$, which follows that A is bornological precompact in $(X, \mathcal{B}^r)$ for every $r < t$.

Suppose that there exists $r_1 > t$ such that A is bornological precompact in $(X, \mathcal{B}^{r_1})$, then there exist $B \in Bal(X)$ with $B \in \mathcal{B}^{r_1}$ and $\lambda_{n_k,n_l} \to 0$ such that $H(n_k) - H(n_l) \in \lambda_{n_k,n_l} B$. Thus $[Precompact(A)] \geq r_1 > t$, which makes a contradiction. So, A is not bornological precompact in $(X, \mathcal{B}^r)$ for every $r > t$.

Sufficiency. If A is bornological precompact in $(X, \mathcal{B}^r)$ for every $r < t$, then for all $S \in N(A)$, there exist $H \leq S$, $B \in Bal(X)$ with $B \in \mathcal{B}^r$, $\lambda_{n_k,n_l} \to 0$ such that $H(n_k) - H(n_l) \in \lambda_{n_k,n_l} B$ for all $k, l \in \mathbb{N}$. Thus, it is easy check that $[Precompact(A)] > r$. By the arbitrariness of r, we obtain $[Precompact(A)] \geq t$. Suppose that $[Precompact(A)] > t$, we can easily obtain a contradiction. Therefore, we have $[Precompact(A)] \leq t$, which follows that $[Precompact(A)] = t$.

Theorem 4.12. Let $(X_j, \mathcal{B}_j)$ be a family of fuzzifying bornological linear spaces indexed by a non-empty set J and let $X = \prod_{j \in J} X_j$ be the product of the sets X_j. For every $j \in J$, let $f_j : X \to X_j$ is a canonical projection and $\mathcal{B} = \prod_{j \in J} \mathcal{B}_j$. Then for all $A \subseteq X$

$$\bigwedge_{j \in J} [Precompact(A_j)] \leq [Precompact(A)].$$

Proof. Similar to Theorem 4.6, here we omit it.

Theorem 4.13. Let $(X, \mathcal{B})$ be a fuzzifying bornological linear space. Then for every $A \subseteq X$

$$[Compact(A)] = [Preompact(A)] \sqcap [Com(A)].$$

Proof. For every $t < [Preompact(A)] \sqcap [Com(A)] = \bigwedge_{S \in N(A)} \bigvee_{H \leq S} [CN(H)] + [Com(A)] - 1$, there exists $H \leq S$ such that $[CN(H)] > t + 1 - [Com(A)]$. Equivalently, $[Com(A)] > t + 1 - [CN(H)]$, specially, we have $1 - [CN(H)] + \bigvee_{x\in A}[H \overset{M}{\to} x] > t + 1 - [CN(H)]$. It implies $\bigvee_{x\in A}[H \overset{M}{\to} x] > t$. So $[Compact(A)] = \bigwedge_{S\in N(A)} \bigvee_{H\leq S} \bigvee_{x\in A}[H \overset{M}{\to} x] > t$. By the arbitrariness of t, $[Compact(A)] \geq [Preompact(A)] \sqcap [Com(A)]$.

On the other hand, let $[Compact(A)] > t$, then for all $S \in N(X)$, there exist $H \leq S$ and $x \in A$ such that $[H \overset{M}{\to} x] > t$. Setting $[CN(S)] = a$, then $[CN(H)] \geq a$. Since $[CN(H)] \geq [H \overset{M}{\to} x] > t$, we obtain $[CN(H)] \geq a \vee t$. Furthermore, from Theorem 3.5, it implies that $[S \overset{M}{\to} x] \geq t \wedge a$. Thus,

$$\begin{aligned}
&[Preompact(A)] \sqcap [Com(A)] \\
&= \bigwedge_{S\in N(X)} \bigvee_{H\leq S}[CN(H)] + \bigwedge_{S\in N(X)}(1 - [CN(S)] + \bigvee_{x\in A}[S \overset{M}{\to} x]) - 1 \\
&\geq a \vee t + 1 - a + a \wedge t - 1 \\
&= t,
\end{aligned}$$

which follows that $[Preompact(A)] \sqcap [Com(A)] \geq [Compact(A)]$. That deduces $[Compact(A)] = [Preompact(A)] \sqcap [Com(A)]$.

5. Conclusions

In the present paper, the notions of fuzzifying Cauchy sequences, fuzzifying completeness, fuzzifying compactness and precompactness are introduced and some interesting properties of them are studied. The relationships among fuzzifying completeness, separation axiom and fuzzifying bornological closed set are discussed. Particularly, it is proved that a subset is fuzzifying bornological compact if and only if it is fuzzifying bornological precompact and bornological complete.

A direction worthy of future work is to study the fuzzifying complete bornologies. Also, studying the characterisations of fuzzifying completely bornological spaces is of interest.

Acknowledgments

We acknowledge the support of National Natural Science Foundation of China under Grant No. 12071225 and Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD).

Conflict of interest

The authors declare that there is no conflict of interest in this paper.

References

1. M. Abel, A. Šostak, Towards the theory of L-bornological spaces, *Iran. J. Fuzzy Syst.*, **8** (2011), 19–28. https://dx.doi.org/10.22111/ijfs.2011.233

2. G. Beer, S. Naimpally, J. Rodrígues-Lopés, S-topologies and bounded convergences, *J. Math. Anal. Appl.*, **339** (2008), 542–552. https://doi.org/10.1016/j.jmaa.2007.07.010

3. G. Beer, S. Levi, Gap, excess and bornological convergence, *Set-Valued Anal.*, **16** (2008), 489–506. https://doi.org/10.1007/s11228-008-0086-8
4. G. Beer, S. Levi, Pseudometrizable bornological convergence is Attouch-Wets convergence, *J. Convex Anal.*, **15** (2008), 439–453.
5. G. Birkhoff, *Lattice theory*, American Mathematical Society, 1995.
6. A. Caserta, G. Di Maio, L. Holá, Arzelá's theorem and strong uniform convergence on bornologies, *J. Math. Anal. Appl.*, **371** (2010), 384–392. https://doi.org/10.1016/j.jmaa.2010.05.042
7. A. Caserta, G. Di Maio, Lj. D. R. Kočinac, Bornologies, selection principles and function spaces, *Topol. Appl.*, **159** (2012), 1847–1852. https://doi.org/10.1016/j.topol.2011.04.025
8. A. Di Concilio, C. Guadagni, Bornological convergences and local proximity spaces, *Topol. Appl.*, **173** (2014), 294–307. https://doi.org/10.1016/j.topol.2014.06.005
9. V. Gregori, J. Miñana, D. Miravet, Fuzzy partial metric spaces, *Int. J. Gen. Syst.*, **48** (2019), 260–279. https://doi.org/10.1080/03081079.2018.1552687
10. H. Hogle-Nled, *Bornology and functional analysis*, North-Holland Publishing Company, 1977.
11. S. T. Hu, Boundedness in a topological space, *J. Math. Pure. Appl.*, **28** (1949), 287–320.
12. S.T. Hu, *Introduction to general topology*, Holden-Day, 1966.
13. Z. Jin, C. Yan, Induced L-bornological vector spaces and L-Mackey convergence, *J. Intell. Fuzzy Syst.*, **40** (2021), 1277–1285. https://doi.org/10.3233/JIFS-201599
14. Z. Jin, C. Yan, Fuzzifying bornological linear spaces, *J. Intell. Fuzzy Syst.*, **42** (2022), 2347–2358. https://doi.org/10.3233/JIFS-211644
15. A. Lechicki, S. Levi, A. Spakowski, Bornological convergence, *J. Math. Anal. Appl.*, **297** (2004), 751–770. https://doi.org/10.1016/j.jmaa.2004.04.046
16. H. Y. Li, F. G. Shi, Degrees of fuzzy compactness in L-fuzzy topological spaces, *Fuzzy Set. Syst.*, **161** (2010), 988–1001. https://doi.org/10.1016/j.fss.2009.10.012
17. R. Meyer, Smooth group representations on bornological vector spaces, *B. Sci. Math.*, **128** (2004), 127–166. https://doi.org/10.1016/j.bulsci.2003.12.002
18. A. M. Meson, F. Vericat, A functional approach to a topological entropy in bornological linear spaces, *J. Dyn. Syst. Geom. The.*, **3** (2005) 45–54. https://doi.org/10.1080/1726037X.2005.10698487
19. S. Oscag, Bornologies and bitopological function spaces, *Filomat*, **27** (2013), 1345–1349. https://doi.org/10.2298/FIL1307345O
20. B. Pang, F. G. Shi, Degrees of compactness in (L, M)-fuzzy convergence spaces and its applications, *Fuzzy Set. Syst.*, **251** (2014), 1–22. https://doi.org/10.1016/j.fss.2014.05.002
21. J. Paseka, S. Solovyov, M. Stehlík, On the category of lattice-valued bornological vector spaces, *J. Math. Anal. Appl.*, **419** (2014), 138–155. https://doi.org/10.1016/j.jmaa.2014.04.033
22. J. Paseka, S. Solovyov, M. Stehlík, Lattice-valued bornological systems, *Fuzzy Set. Syst.*, **259** (2015), 68–88. https://doi.org/10.1016/j.fss.2014.09.006
23. B. Pang, F. Shi, Strong inclusion orders between L-subsets and its applications in L-convex spaces, *Quaest. Math.*, **41** (2018), 1021–1043. https://doi.org/10.2989/16073606.2018.1436613

24. D. Qiu, Fuzzifying topological linear spaces, *Fuzzy Set. Syst.*, **147** (2004), 249–272. https://doi.org/10.1016/j.fss.2003.10.024

25. A. Sostak, I. Uljane, *L*-valued bornologies on powersets, *Fuzzy Set. Syst.*, **294** (2016), 93–104. https://doi.org/10.1016/j.fss.2015.07.016

26. Y. Liu, D. Zhang, Lowen spaces, *J. Math. Anal. Appl.*, **241** (2000), 30–38. https://doi.org/10.1006/jmaa.1999.6586