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ON CUBIC FUZZY GROUPS AND CUBIC FUZZY NORMAL SUBGROUPS

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ABSTRACT:

In this paper, the notions of cubic fuzzy groups and cubic fuzzy normal subgroups are introduced, and some related properties are investigated. Some relations between internal and external cubic sets with cubic fuzzy groups and cubic fuzzy normal subgroups are investigated. It is proved that the (P-, R-) intersection and (P-, R-) union of cubic fuzzy groups are also cubic fuzzy group. Moreover, the (P-, R-) intersection and (P-, R-) union of cubic fuzzy normal subgroups are proved to be cubic fuzzy normal subgroups.

KEYWORDS: Fuzzy set, interval-valued fuzzy set, internal and external cubic set, cubic fuzzy group, cubic fuzzy normal subgroup (P-,R-) intersection and (P-,R-) union.

1. INTRODUCTION

In (1965) fuzzy sets [18] are initiated by L. A. Zadeh, after he introduced the fuzzy sets, many researches were conducted on generalizations of the notion of fuzzy set. In fuzzy set theory, the membership of an element to a fuzzy set is a single value between zero and one. There are many authors who have conducted their research in various fields for the fuzzy set. For instance, Molodtsov [5] in soft set, Atanassov [1] in Intuitionistic fuzzy sets, Zhang [20] in Bipolar valued fuzzy sets, and Mahmood [16] in Bipolar soft sets. In [19] Zadeh (1975) developed the concept of interval-valued fuzzy set as a generalization of fuzzy set. After that, many authors investigated the similar topic and gotten some meaningful conclusions. In [3] Gorzalczany (1987) proposed and described an approximation inference approach based on the concept of an interval-valued fuzzy set, which is an extension of the concept of a fuzzy set. This method enables the creation of a formal fuzzy representation for speech decision algorithms. After that, many authors conducted their research on a fuzzy set and interval-valued fuzzy set together, and this case is called a cubic set. Furthermore, in [17] cubic sets are initiated by Young B. j., Chang S. K., and K. O. Yang in (2012), by using a fuzzy set and an interval-valued fuzzy set they introduced a new notion which is called (Internal, External) cubic set, and several related properties are investigated. There are many authors who have conducted their research in different aspects of the cubic set. [4] Ahmad B. Kh., Muhammad Q., Saleem A., Muhammad N. and Muneeza in (2021) an application of picture cubic fuzzy (PCF) aggregation operators with the picture cubic fuzzy information are used to solve problem of suppler selection with contradictory and insufficient attributes, such that compromise is appropriate for conflict resolve. Also they review some basic concepts of picture fuzzy set, and picture fuzzy aggregation operators. Moreover, they defined the idea of picture cubic fuzzy set, and their operational laws are discoursed. Further, they are proposed a series of PCF aggregation operators, like as PCF weighted average, PCF order weighted average, PCF weighted geometric and PCF order weighted geometric operator and discoursed their scarce properties. An approach are given to solve a decision-making problem with the PCF information, and in [13] they maked offers of some basic concepts of fuzzy set, intuitionistic fuzzy set, Pythagorean fuzzy set and operational laws of Pythagorean fuzzy sets. Also they defined the concept of Pythagorean cubic fuzzy Hamacher averaging aggregation operators and Pythagorean cubic fuzzy Hamacher geometric aggregation operators. [14] Sanum A. and Saleem A. (2021) they used Dombi operational laws and Heronian mean operators, to develop a new concept of cubic fuzzy Heronian mean Dombi aggregation operators, they presented new rules for the cubic fuzzy numbers (CFNs) based on Dombi t-norm and t-conorm (DTT), to present a new cubic fuzzy Dombi Heronian mean operators for the aggregation of CFNs and introduced an innovative style to decision making. [7] M. Qiyas, S. Abdullah and Muneeza (2020) they discussed the basic knowledge about the fuzzy set, intuitionistic fuzzy set and linguistic cubic variables. Also they presented some operational laws for linguistic intuitionistic cubic hesitant variables and their score function. Further, they presented average and geometric aggregation operators of linguistic intuitionistic cubic hesitant variables, and after that in [6] they gave theoretical information of linguistic intuitionistic cubic fuzzy variables (LICFVs), and initiate several fundamental laws of LICFVs, on the beginning of these functions a simple technique for classification of LICFVs presented. Since in decision making problems an aggregation operator is an important mathematical tool, they introduced the aggregation ability for linguistic intuitionistic cubic fuzzy information and establish numerous aggregation operators.

The concept of fuzzy sets in Algebraic structures was first introduced by Rosenfeld [11] in (1971) and he defined the notion of fuzzy subgroups. The terms 'fuzzy normal subgroup' and 'fuzzy coset' introduced by Mukherjee and Prabir [10] in (1984). They also demonstrated that a fuzzy normal subgroup is an analog of the concept of a normal subgroup of a group. They demonstrated that each member of a family of level subgroups of a fuzzy normal subgroup of a group G is a normal subgroup of G in the conventional sense. There are other researches also added their contribution to the study in this field on different branches of algebra in different aspects. For instance, Das [2] in fuzzy

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groups and level subgroups, Mukesh [9] in fuzzy group and their types, and Saleem A. [12] in a new type of fuzzy normal subgroups and fuzzy cosets. Also, some more general concepts on bipolar fuzzy algebra have been studied. Such as, Tahir M. and Muhammad M. [15] on bipolar fuzzy subgroups, and Muhiuddin G. [8] in bipolar fuzzy implicative ideals of BCK-algebras. The objective of this paper is to introduce the notion called 'cubic fuzzy group' and 'cubic fuzzy normal subgroup' on the general cubic set, and study (P-,R-) intersection and (P-,R-) union of cubic fuzzy group and cubic fuzzy normal subgroup and several related properties.

2. PRELEMINARIES

In this section, we will state certain useful definitions, properties and some existing results for cubic sets that will be useful for our discussion in the next sections.

Definition 2.1: suppose X be a universal set. A cubic set [17] $\mathcal{A}$ defined over the universal set X is an ordered pair which is defined as follows:

$$\mathcal{A} = \{\langle x, \mu(x), \lambda(x)\rangle : \mathrm{x} \in X\}$$

Where μ is a function such that $\mu: X \to [I]$, where $I = [0,1]$ which is called an interval-valued fuzzy set (IVFS) [19] in X. Denote $[I]^X$the set of all IVFSs in X. Also a function $\lambda: X \to I$, is called a fuzzy set [18] in a set X, denote I^X the collection of all fuzzy sets in a set X. Clarity, we denote the component of cubic set $\mathrm{A} = \langle \mu, \lambda\rangle$, and for any $\mu \in [I]^X$ and $x \in X$, where $\mu(\mathrm{x}) = [\mu^-(x), \mu^+(x)]$ which is called the grade of membership of an element x to μ, where $\mu^-: X \to I$ and $\mu^+: X \to I$ are fuzzy sets in X which are called a lower fuzzy set and an upper fuzzy set in X, respectively. C^X denotes the set of all cubic sets in X.

A cubic set $\mathcal{A} = \langle \mu, \lambda\rangle$ in which $\mu(\mathrm{x}) = \mathbf{0}$ and $\lambda(\mathrm{x}) = 1$ (resp. $\mu(\mathrm{x}) = \mathbf{1}$ and $\lambda(\mathrm{x}) = 0$) for all $x \in X$ is denoted by $\ddot{0}$ (resp. $\ddot{1}$).

Also if $\mu(\mathrm{x}) = \mathbf{0}$ and $\lambda(\mathrm{x}) = 0$ (then a cubic set $\mathcal{A} = \langle \mu, \lambda\rangle$ is called a zero cubic set) (resp. $\mu(\mathrm{x}) = \mathbf{1}$ and $\lambda(\mathrm{x}) = 1$) (then a cubic set $\mathcal{A} = \langle \mu, \lambda\rangle$ is called a unit cubic set) for all $x \in X$ is denoted by $\hat{0}$ (resp. $\hat{1}$). And in general if $\mu(\mathrm{x}) = \boldsymbol{\alpha}$ and $\lambda(\mathrm{x}) = \alpha$ (then a cubic set $\mathcal{A} = \langle \mu, \lambda\rangle$ is called α-cubic set).

Definition 2.2: [19] For any two IVFSs μ, υ in $[I]^X$. We say that $\mu \leq \upsilon$ iff $\mu^-(x) \leq \upsilon^-(x), and\ \mu^+(x) \leq \upsilon^+(x)$ for all $x \in X$.

Also we say that $\mu = \upsilon$ iff $\mu^-(x) = \upsilon^-(x)$, and $\mu^+(x) = \upsilon^+(x)$ for all $x \in X$.

Definition 2.3: The complement of $\mathcal{A} = \{\langle \mathrm{x}, \mu(\mathrm{x}), \lambda(\mathrm{x})\rangle : \mathrm{x} \in \mathrm{X}\}$ also is defined to be the cubic set $\mathcal{A}^c = \{\langle \mathrm{x}, \mu^c(\mathrm{x}), \lambda^c(\mathrm{x})\rangle : \mathrm{x} \in \mathrm{X}\}$, where $\mu^c(\mathrm{x}) = [1 - \mu^+(x), 1 - \mu^-(x)]$ and $\lambda^c(\mathrm{x}) = \{1 - \lambda(\mathrm{x})\}$ for all $x \in X$.

Obviously,$(\mathcal{A}^c)^c = \mathcal{A}$, $\ddot{0}^c = \ddot{1}$, $\ddot{1}^c = \ddot{0}$, $\hat{0}^c = \hat{1}\ and\ \hat{1}^c = \hat{0}$.

Definition 2.4: Let $\lambda, \alpha \in I^X$, we define the following on fuzzy sets

(i) $\lambda \leq \alpha$ by $\lambda(x) \leq \alpha(x)$ for every $x \in X$.

(ii) $\lambda \wedge \alpha \in I^X$ by $(\lambda \wedge \alpha)(x) = \min\{\lambda(x), \alpha(x)\}$ for each $x \in X$.

(iii) $\lambda \vee \alpha \in I^X$ by $(\lambda \vee \alpha)(x) = \max\{\lambda(x), \alpha(x)\}$ for each $x \in X$.

The union and intersection of the family $(\lambda_i : i \in \Lambda)$ where Λ is the indexing set of fuzzy sets in X are defined as follows:

$$(\vee \lambda_i)(x) = \sup\{\lambda_i(x) : i \in \Lambda\}$$
$$(\wedge \lambda_i)(x) = \inf\{\lambda_i(x) : i \in \Lambda\}$$

For a family of interval-valued fuzzy sets (IVFSs) $\{\mu_i : i \in \Lambda\}$, then their union $\mu = \sqcup\, \mu_{i(x)}$ and intersection $\beta = \sqcap\, \mu_{i(x)}$ are defined by$y$

$$\mu^-(x) = max[\mu_i{}^-(x)],$$
$$\mu^+(x) = max[\mu_i{}^+(x)], \forall\, x \in X$$

$\mu = \sqcup\, \mu_{i(x)} = \sup\,(\mu_{i(x)} : i \in \Lambda)\ and$

$$\beta^-(x) = min[\mu_i{}^-(x)],\ \beta^+(x) = min[\mu_i{}^+(x)], \forall\, x \in X$$

$\beta = \sqcap\, \mu_{i(x)} = \inf\,(\mu_{i(x)} : i \in \Lambda)$ respectively.

Definition 2.5: [18], (Zadeh, 1965), a fuzzy point x_a, where $a \in (0,1]$ is a fuzzy set in X defined by $x_a(x) = a$ and $x_a(y) = 0$ if $x \neq y$, x is called the support of x_a and a it's value.

Definition 2.6: [18], (Zadeh, 1965), a fuzzy point x_a is said to be contained in a fuzzy set, or belong to λ and denoted by $x_a \in \lambda$, if and only if $\lambda(x) \geq a$.

Definition 2.7: [3], (Gorzalczany,1987) for $\mu \in [I]^X$, the set $\{x \in X; \mu^+(x) > 0\}$ is called the support of μ and is denoted by μ_x. An interval-valued fuzzy point M_x is said to belong to an interval-valued fuzzy set μ if $M^- \leq \mu^-(x)$ and $M^+ \leq \mu^+(x)$.

3. ON CUBIC FUZZY GROUPS AND THEIR PROPERTIES

In this section, we will define the notion and concepts of cubic fuzzy groups and discuss some of its properties by depending on the fuzzy groups [11]. First we recall the definition of cubic fuzzy point.

Definition 3.1: Let G be a group on the non-empty universal set X, a cubic fuzzy point is defined by $\{\langle q, \mu(q), \lambda(q)\rangle : q \in G\}$ of the cubic set $\mathcal{A}$ and is denoted by C_q.

Definition 3.2: suppose $(G,*)$ be a group. A cubic set $\mathcal{A} = \langle \mu, \lambda\rangle$ is called a cubic fuzzy group $C(G)$ which is generated by G under $*$, if the following conditions are true:

1. $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$,

   $\lambda(x * y) \geq \min\{\lambda(x), \lambda(y)\}, \forall\, x, y \in G$, and
2. $\mu(x^{-1}) = \mu(x), \lambda(x^{-1}) = \lambda(x)$.

We denote the set of cubic fuzzy groups by $\mathcal{C}(G)$.

Example 3.1: suppose $(G,*)$ be a group where $G = \{1, \omega, \omega^2\}$ and ω is the cubic root of unity with the binary operation defined as below:

Table 1: composition table of cube roots of unity

$*$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Let $\mathcal{A} = \langle \mu, \lambda\rangle$ be a cubic set in G defined as below:

Table 2: cubic set $\mathcal{A} = \langle \mu, \lambda\rangle$

X	μ	λ
1	$\langle[0.3, 0.9]\rangle$	$\langle 0.6\rangle$
ω	$\langle[0.3, 0.6]\rangle$	$\langle 0.4\rangle$
ω^2	$\langle[0.3, 0.6]\rangle$	$\langle 0.4\rangle$

From Table 1 and Table 2, we get that A is a cubic fuzzy group.

Example 3.2: Suppose G be the set of all non-zero real number$s$, where the binary operation $*$ is defined as:

$a * b = ab$ for all $a, b \in G = \mathbb{R} - \{0\}$ be a group.

$$\mu(x) = \begin{cases} \left[|\mathrm{x}|, \frac{1}{|\mathrm{x}|+1}\right] & |\mathrm{x}| < 0.62 \\ \left[\frac{1}{|\mathrm{x}|+1}, |\mathrm{x}|\right] & 0.62 \leq |\mathrm{x}| \leq 1 \\ \left[\frac{1}{|\mathrm{x}|+2}, \frac{1}{|\mathrm{x}|+1}\right] & |\mathrm{x}| > 1 \end{cases} \quad \text{and}$$

$$\lambda(x) = \begin{cases} |\mathrm{x}| & |\mathrm{x}| \leq 1 \\ \frac{1}{\mathrm{x}^2+2} & |\mathrm{x}| > 1 \end{cases}$$

Then the set $\mathcal{A} = \langle \mu, \lambda\rangle$ is not a cubic fuzzy group because if we take $x = 0.1$,

$$\mu\big((0.1^{)-1}\big) = \mu(10) = \left[\frac{1}{12}, \frac{1}{11}\right] \neq \mu(0.1) = [0.1, 0.02].$$

Proposition 3.1: Zero cubic set, unit cubic set and all α-cubic sets of a group G are trivial $C(G)s$ of $\mathcal{C}(G)$.

Definition 3.3: [17] Let X be non-empty set. A cubic set $\mathcal{A} = \langle\mu, \lambda\rangle$ in X is said to be an internal cubic set (ICS) if $\mu^-(x) \leq \lambda(x) \leq \mu^+(x)$ for all $x \in X$.

Definition 3.4: [17] Let X be non-empty set. A cubic set $\mathcal{A} = \langle\mu, \lambda\rangle$ in X is said to be an external cubic set (ECS) if $\lambda(x) \notin (\mu^-(x), \mu^+(x))$ for all $x \in X$.

Remark 3.1: Let X be non-empty set. A cubic fuzzy group $\mathcal{A} = \langle\mu, \lambda\rangle$ of $\mathcal{C}(G)$ is said to be both (ICS) and (ECS) if $\mu^-(x) \leq \lambda(x) \leq \mu^+(x)$ and $\lambda(x) \notin (\mu^-(x), \mu^+(x))$ for all $x \in X$.

Example 3.3: Let $(G,*)$ be a group where $G = \{1, \omega, \omega^2\}$ and $*$ is the usual multiplication, then the cubic set $\mathcal{A} = \langle\mu, \lambda\rangle$ is a cubic fuzzy group.

Table 3: cubic set $\mathcal{A} = \langle\mu, \lambda\rangle$

G	μ	λ
1	$\langle[0.3,0.9]\rangle$	$\langle 0.9\rangle$
ω	$\langle[0.3,0.6]\rangle$	$\langle 0.6\rangle$
ω^2	$\langle[0.3,0.6]\rangle$	$\langle 0.6\rangle$

Then from the table and by the definition we get that $\mathcal{A}$ which is both (ICS) and (ECS).

Theorem 3.1: If $\mathcal{A} = \{\langle x, \mu(x), \lambda(x)\rangle : x \in X\}$ in a non-empty set X is an internal cubic set (ICS), then $\lambda(x) \geq \min\{\mu^-(x), \mu^+(x)\}$ for all $x \in X$.

Proof: Suppose that $\mathcal{A}$ is an ICS. Then by Definition **3.3**, we get

$\mu^-(x) \leq \lambda(x) \leq \mu^+(x) \quad \forall\, x \in X$. Thus, it is clear that, $\lambda(x) \geq \min\{\mu^-(x), \mu^+(x)\}\ \forall\, x \in X$.

Theorem 3.2: Let $\mathcal{A} = \langle\mu, \lambda\rangle$ be a cubic fuzzy group in $\mathcal{C}(G)$. If $\mathcal{A}$ is an (ICS) and (ECS), then

$$(\forall\, x \in X)(\lambda(x) \leq U(\mu) + L(\mu))$$

Where $L(\mu) = \{\mu^-(x) : x \in X\}$ and $U(\mu) = \{\mu^+(x) : x \in X\}$.

Proof: Suppose that $\mathcal{A}$ is an (ICS) and (ECS). By Definition 3.3 and Definition 3.4, we get $\mu^-(x) \leq \lambda(x) \leq \mu^+(x)$ and $\lambda(x) \notin (\mu^-(x), \mu^+(x)) \quad \forall x \in X$. Thus, $\lambda(x) = \mu^-(x)$ or $\lambda(x) = \mu^+(x)$, and so $\lambda(x) \leq U(\mu) + L(\mu)$.

Theorem 3.3: Let $\mathcal{A} = \langle\mu, \lambda\rangle$ be a cubic set in X. If $\mathcal{A}$ is an ICS (resp. ECS), then $\mathcal{A}^c$ is an ICS (resp. ECS). See [17].

Theorem 3.4: Let $\mathcal{A} = \langle\mu, \lambda\rangle$ be a cubic set in X. If $\mathcal{A}$ is a cubic fuzzy group, then $\mathcal{A}^c$ is also a cubic fuzzy group.

Proof: Since $\mathcal{A} = \langle\mu, \lambda\rangle$ is a cubic fuzzy group in $\mathcal{C}(G)$, then we have:

$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}, \lambda(x * y) \geq \min\{\lambda(x), \lambda(y)\}$, $\forall\, x, y \in G$

And $\mu(x^{-1}) = \mu(x), \lambda(x^{-1}) = \lambda(x)$.

This implies that

$[1 - \mu^+(x * y), 1 - \mu^-(x * y)] \leq \max\{\mu(x), \mu(y)\}$, $(1 - \lambda(x * y)) \leq \max\{\lambda(x), \lambda(y)\}$

And $\mu^c(x^{-1}) = [1 - \mu^+(x^{-1}), 1 - \mu^-(x^{-1})]$,

$\mu^c(x) = [1 - \mu^+(x), 1 - \mu^-(x)]$ implies that $\mu^c(x^{-1}) = \mu^c(x)$

Also

$\lambda^c(x^{-1}) = (1 - \lambda(x^{-1}))$, $\lambda^c(x) = (1 - \lambda(x))$, implies $\lambda^c(x^{-1}) = \lambda^c(x)$, hence $\mathcal{A}^c = \{\langle x, \mu^c(x), 1 - \lambda(x)\rangle : x \in G\}$ is a cubic fuzzy group in $\mathcal{C}(G)$.

Definition 3.5: Let $\mathcal{A}_1 = \{\langle x, \mu_1(x), \lambda_1(x)\rangle : x \in G\}$ and $\mathcal{A}_2 = \{\langle x, \mu_2(x), \lambda_2(x)\rangle : x \in G\}$ be two cubic fuzzy groups. Then we have

1- $\mathcal{A}_1 = \mathcal{A}_2 \leftrightarrow \mu_1(x) = \mu_2(x)$ and $\lambda_1(x) = \lambda_2(x)$ (Equality).
2- $\mathcal{A}_1 \sqsubseteq_p \mathcal{A}_2 \leftrightarrow \mu_1(x) \sqsubseteq \mu_2(x)$ and $\lambda_1(x) \leq \lambda_2(x)$ (p-order).
3- $\mathcal{A}_1 \sqsubseteq_R \mathcal{A}_2 \leftrightarrow \mu_1(x) \sqsubseteq \mu_2(x)$ and $\lambda_1(x) \geq \lambda_2(x)$ (R-order).

Proposition 3.2: [17] For any $\mathcal{A}_i = \{\langle x, \mu_i(x), \lambda_i(x)\rangle : x \in G\}$ where $i \in \Lambda$, then

1- $\sqcup_P \mathcal{A}_i = \{\langle x, \sup(\mu_i(x)), \sup(\lambda_i(x))\rangle : x \in G, i \in \Lambda\}$ (P-union)
2- $\sqcap_P \mathcal{A}_i = \{\langle x, \inf(\mu_i(x)), \inf(\lambda_i(x))\rangle : x \in G, i \in \Lambda\}$ (P-intersection)
3- $\sqcup_R \mathcal{A}_i = \{\langle x, \sup(\mu_i(x)), \inf(\lambda_i(x))\rangle : x \in G, i \in \Lambda\}$ (R-union)
4- $\sqcap_R \mathcal{A}_i = \{\langle x, \inf(\mu_i(x)), \sup(\lambda_i(x))\rangle : x \in G, i \in \Lambda\}$ (R-intersection).

In [17], they have proved that (P-union, P-intersection) of internal cubic sets are also internal cubic sets and showed by examples (P-union, P-intersection) of external cubic sets need not be external cubic sets, and the (R-union, R-intersection) of internal (resp. external) cubic sets need not be an internal (resp. external) cubic set. Also they have showed that the conditions of two external cubic sets for the (P-union) (resp. P-intersection) to be an internal cubic set, and gave conditions of two external cubic sets for the (P-union) (resp. R-union, R-intersection) to be an external cubic set.

Definition 3.6: Let $C(G)$ be a cubic fuzzy group generated by G. A proper subset $p(G)$ is called a cubic fuzzy subgroup if $p(G)$ is a cubic fuzzy group.

Example 3.4: Let $C(G)$ be a cubic fuzzy group, where $G = Z_4 = \{0,1,2,3\}$ and the membership functions μ and λ are given in the below table,

Table 4: cubic set $\mathcal{A} = \langle\mu, \lambda\rangle$

Z_4	μ	λ
0	[0.2,0.5]	(0.6)
1	[0.2,0.3]	(0.4)
2	[0.2,0.3]	(0.4)
3	[0.2,0.3]	(0.4)

We see that $H = \{0,2\}$ is a proper subset of Z_4 which is a group under addition in fact it is a cubic fuzzy group so the membership function $\mathcal{A}_H = \{\langle x, \mu(x), \lambda(x)\rangle : x \in H\}$ which is a cubic fuzzy group (subgroup).

Definition 3.7: A cubic fuzzy set $\mathcal{A} = \langle\mu, \lambda\rangle$ of $\mathcal{C}(G)$ is called a cubic fuzzy subgroup if $\mu(x^{-1}) \geq \mu(x), \lambda(x^{-1}) \geq \lambda(x)$ for all $x \in G$, where x is not an identity element in G.

In **Example 3.4** we notice that if $x \neq e$, then

$\mu(x^{-1}) \geq \mu(x), \lambda(x^{-1}) \geq \lambda(x)$ for all $x \in G$.

Theorem 3.5: Let $\mathcal{A} = \langle\mu, \lambda\rangle$ be a cubic fuzzy subgroup of $\mathcal{C}(G)$, then $\mu(e) \geq \mu(x), \lambda(e) \geq \lambda(x)$ for all $x \in G$, where e is the identity element of G.

Proof: $\mu(e) = \mu(x * x^{-1}) \geq \min\{\mu(x), \mu(x^{-1})\} = \mu(x)$ implies that $\mu(e) \geq \mu(x)$

Also $\lambda(e) = \lambda(x * x^{-1}) \geq \min\{\lambda(x), \lambda(x^{-1})\} = \lambda(x)$ implies that $\lambda(e) \geq \lambda(x)\ \forall\, x \in G$.

Theorem 3.6: In a cubic fuzzy subgroup $\mathcal{A} = \langle\mu, \lambda\rangle$ of $\mathcal{C}(G)$. If $\mu(x * y^{-1}) = \mu(e)$ *and* $\lambda(x * y^{-1}) = \lambda(e)$, then, $\mu(x) = \mu(y)$ *and* $\lambda(x) = \lambda(y)$.

Proof: $\mu(x) = \mu((x * y^{-1}) * y) \geq \min\{\mu(e), \mu(y)\} = \mu(y)$

$\mu(y) = \mu((y * x^{-1}) * x) \geq \min\{\mu(e), \mu(x)\} = \mu(x)$ implies that $\mu(x) = \mu(y)$

$\lambda(x) = \lambda((x * y^{-1}) * y) \geq \min\{\lambda(e), \lambda(y)\} = \lambda(y)$,

$\lambda(y) = \lambda((y * x^{-1}) * x) \geq \min\{\lambda(e), \lambda(x)\} = \lambda(x)$

This implies that $\lambda(x) = \lambda(y)$ and hence proved.

Theorem 3.7: A cubic set $\mathcal{A} = \langle\mu, \lambda\rangle$ is a cubic fuzzy subgroup of $\mathcal{C}(G)$ if and only if $\mu(x * y^{-1}) \geq \min\{\mu(x), \mu(y)\}$ and $\lambda(x * y^{-1}) \geq \min\{\lambda(x), \lambda(y)\}, \forall\, x, y \in G$.

Proof: Suppose that $\mathcal{A}$ is a cubic fuzzy subgroup then by the condition, $\mu(y^{-1}) = \mu(y), \lambda(y^{-1}) = \lambda(y)$ and for all $x, y \in G$, we have;

$\mu(x * y^{-1}) \geq \min\{\mu(x), \mu(y^{-1})\} = \min\{\mu(x), \mu(y)\}$

This implies that $\mu(x*y^{-1}) \geq \min\{\mu(x),\mu(y)\}$.
Also $\lambda(x*y^{-1}) \geq \min\{\lambda(x),\lambda(y^{-1})\} = \min\{\lambda(x),\lambda(y)\}$
And this implies that $\lambda(x*y^{-1}) \geq \min\{\lambda(x),\lambda(y)\}$.
Conversely: Let $\mu(x*y^{-1}) \geq \min\{\mu(x),\mu(y)\}$, $\lambda(x*y^{-1}) \geq \min\{\lambda(x),\lambda(y)\}\ \forall\, x,y \in G$,
we have to show that $\mathcal{A}$ is a cubic fuzzy subgroup of $\mathcal{C}(G)$;
Let $y = x$ then we get,
$\mu(e) \geq \mu(x), \lambda(e) \geq \lambda(x)$ ... $(**)$ for all $x \in G$; hence
$\mu(y^{-1}) = \mu(e*y^{-1}) \geq \min\{\mu(e),\mu(y)\} = \mu(y)$,
$\lambda(y^{-1}) = \lambda(e*y^{-1}) \geq \min\{\lambda(e),\lambda(y)\} = \lambda(y)$
Now let x and y be any two elements of G, then
$\mu(x*y) = \mu(x*(y^{-1})^{-1}) \geq \min\{\mu(x),\mu(y^{-1})\} \geq \min\{\mu(x),\mu(y)\}$
$\mu(x*y) \geq \min\{\mu(x),\mu(y)\}$ (1)
$\lambda(x*y) = \lambda(x*(y^{-1})^{-1}) \geq \min\{\lambda(x),\lambda(y^{-1})\} \geq \min\{\lambda(x),\lambda(y)\}$
$\lambda(x*y) \geq \min\{\lambda(x),\lambda(y)\}$ (2)
Now to prove $\mu(x^{-1}) = \mu(x), \lambda(x^{-1}) = \lambda(x)$, since we have $\mu(x^{-1}) \geq \mu(x), \lambda(x^{-1}) \geq \lambda(x)$, then we shall show that $\mu(x^{-1}) \leq \mu(x), \lambda(x^{-1}) \leq \lambda(x)$, let x be any member of G and e be the identity element of G, then
$\mu(x) = \mu(e*(x^{-1})^{-1}) \geq \min\{\mu(e),\mu(x^{-1})\} = \mu(x^{-1})$
Implies $\mu(x) \geq \mu(x^{-1})$, and we get that;
$\mu(x^{-1}) = \mu(x)$. (3)
Also, $\lambda(x) = \lambda(e*(x^{-1})^{-1}) \geq \min\{\lambda(e),\lambda(x^{-1})\} = \lambda(x^{-1})$
Implies $\lambda(x) \geq \lambda(x^{-1})$, and we get that;
$\lambda(x^{-1}) = \lambda(x)$ (4)
Now from 1,2,3 and 4, and for all $x,y \in G$, we get $\mathcal{A}$ is a cubic fuzzy subgroup of $\mathcal{C}(G)$.

Theorem 3.8: Let G be a group in the universal set X, $\mathcal{A}$ be a cubic fuzzy subgroup of $\mathcal{C}(G)$ and y ∈ G. then
$\mu(x*y) = \mu(x),\ \lambda(x*y) = \lambda(x)\quad \forall\, x \in G$ if and only if $\mu(y) = \mu(e),\ \lambda(y) = \lambda(e)$.
Proof: Suppose that $\mu(x*y) = \mu(x),\ \lambda(x*y) = \lambda(x)$ ∀ x ∈ G. Now by choosing $x = e$, then we obtain that $\mu(y) = \mu(e),\ \lambda(y) = \lambda(e)$.
Conversely: Suppose that $\mu(y) = \mu, \lambda(y) = \lambda(e)$. Then by Theorem 3.5, we have $\mu(y) \geq \mu(x)\ \ \forall\, x \in G$.
Now, $\mu(x*y) \geq \min\{\mu(x),\mu(y)\}$. Therefore, we have

$$\mu(x*y) \geq \mu(x)\ \ \forall\, x \in G$$

But

$$\mu(x) = \mu(x*y*y^{-1}) \geq \min\{\mu(x*y),\mu(y^{-1})\} \geq \min\{\mu(x*y),\mu(y)\}.$$

But $\mu(y) \geq \mu(x)\ \ \forall\, x \in G$. So $\min\{\mu(x*y),\mu(y)\} = \mu(x*y)$. Therefore, we have

$$\mu(x) \geq \mu(x*y)\quad \forall\, x \in G$$

Hence,

$$\mu(x*y) = \mu(x)\qquad \forall\, x \in G.$$

By the same way, we get $\lambda(x*y) = \lambda(x)\ \ \forall\, x \in G$.

Proposition 3.3: Let G be a group on the non-empty universal set X, if $\mathcal{A}_1 = \{\langle x,\mu_1(x),\lambda_1(x)\rangle : x \in G\}$ and $\mathcal{A}_2 = \{\langle x,\mu_2(x),\lambda_2(x)\rangle : x \in G\}$ are two cubic fuzzy groups of $\mathcal{C}(G)$, then $\mathcal{A}_1 \sqcap_P \mathcal{A}_2$ is also a cubic fuzzy group of $\mathcal{C}(G)$.
Proof: Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are cubic fuzzy groups of $\mathcal{C}(G)$, we have
$(\mathcal{A}_1 \sqcap_P \mathcal{A}_2)(x*y) = \{x*y, \inf(\mu_1(x*y),\mu_2(x*y)), \inf(\lambda_1(x*y),\lambda_2(x*y)) : x,y \in G\}$
$= \{x*y, \inf[\mu_i(x*y)], \inf[\lambda_i(x*y)] : x,y \in G\}$
$\geq \{x*y, \inf[\min\{\mu_i(x),\mu_i(y)\}], \inf[\min\{\lambda_i(x),\lambda_i(y)\}] : x,y \in G\}$
$= \{x*y, \min\{\inf(\mu_i(x)), \inf(\mu_i(y))\}, \min\{\inf(\lambda_i(x)), \inf(\lambda_i(y))\} : x,y \in G\}$
$= \{x*y, \min\{(\sqcap \mu_i)(x), (\sqcap \mu_i)(y)\}, \min\{(\wedge \lambda_i)(x), (\wedge \lambda_i)(y)\} : x,y \in G\}$
This implies that;

$$\inf[\mu_i(x*y)] = (\sqcap \mu_i(x*y)) \geq \min\{(\sqcap \mu_i)(x), (\sqcap \mu_i)(y)\},$$
$$\inf[\lambda_i(x*y)] = (\wedge \lambda_i(x*y)) \geq \min\{(\wedge \lambda_i)(x), (\wedge \lambda_i)(y)\}.$$

Now
$(\mathcal{A}_1 \sqcap_P \mathcal{A}_2)(x^{-1}) = \{x^{-1}, \inf[\mu_i(x^{-1})], \inf[\lambda_i(x^{-1})] : x^{-1} \in G\}$
$= \{e*x^{-1}, \inf[\mu_i(e*x^{-1})], \inf[\lambda_i(e*x^{-1})] : e, x^{-1} \in G\}$
$\geq \{e*x^{-1}, \inf[\min\{\mu_i(e),\mu_i(x)\}], \inf[\min\{\lambda_i(e),\lambda_i(x)\}] : e, x^{-1} \in G\}$
$= \{e*x^{-1}, \min\{\inf(\mu_i(e)), \inf(\mu_i(x))\}, \min\{\inf(\lambda_i(e)), \inf(\lambda_i(x))\} : e, x^{-1} \in G\}$
$= \{e*x^{-1}, \min\{(\sqcap \mu_i)(e), (\sqcap \mu_i)(x)\}, \min\{(\wedge \lambda_i)(e), (\wedge \lambda_i)(x)\} : e, x^{-1} \in G\}$
Since $\mu_i(e) \geq \mu_i(x), \lambda_i(e) \geq \lambda_i(x)\ \ \forall\, x \in G$
$= \{e*x^{-1}, (\sqcap \mu_i)(x), (\wedge \lambda_i)(x) : e, x^{-1} \in G\}$
Implies that;
$\inf[\mu_i(x^{-1})] = (\sqcap \mu_i)(x^{-1}) \geq (\sqcap \mu_i)(x), \inf[\lambda_i(x^{-1})] = (\wedge \lambda_i)(x^{-1}) \geq (\wedge \lambda_i)(x)$ …………. (1)
And
$(\mathcal{A}_1 \sqcap_P \mathcal{A}_2)(x) = \{x, \inf[\mu_i(x)], \inf[\lambda_i(x)] : x \in G\}$, since $x = (x^{-1})^{-1}$ then we can write it as follows
$= \{e*(x^{-1})^{-1}, \inf[\mu_i(e*(x^{-1})^{-1})], \inf[\lambda_i(e*(x^{-1})^{-1})] : e, x \in G\}$
$\geq \{e*(x^{-1})^{-1}, \inf[\min\{\mu_i(e),\mu_i(x^{-1})\}], \inf[\min\{(\lambda_i)(e),(\lambda_i)(x^{-1})\}] : e, x \in G\}$
$= \{e*(x^{-1})^{-1}, \min\{\inf(\mu_i(e)), \inf(\mu_i(x^{-1}))\}, \min\{\inf((\lambda_i)(e)), \inf((\lambda_i)(x^{-1}))\} : e, x \in G\}$
$= \{e*(x^{-1})^{-1}, \min\{(\sqcap \mu_i)(e), (\sqcap \mu_i)(x^{-1})\}, \min\{(\wedge \lambda_i)(e), (\wedge \lambda_i)(x^{-1})\} : e, x \in G\}$
Since $\mu_i(e) \geq \mu_i(x), \lambda_i(e) \geq \lambda_i(x)\ \ \forall x \in G$
$= \{e*(x^{-1})^{-1}, (\sqcap \mu_i)(x^{-1}), (\wedge \lambda_i)(x^{-1}) : e, x \in G\}$
And this implies that;
$\inf[\mu_i(x)] = (\sqcap \mu_i)(x) \geq (\sqcap \mu_i)(x^{-1})$,
$\inf[\lambda_i(x)] = (\wedge \lambda_i)(x) \geq (\wedge \lambda_i)(x^{-1})$ ………. (2)
Hence from (1) and (2) we get;
$(\sqcap \mu_i)(x^{-1}) = (\sqcap \mu_i)(x), (\wedge \lambda_i)(x^{-1}) = (\wedge \lambda_i)(x)$
And this implies that $\mathcal{A}_1 \sqcap_P \mathcal{A}_2$ is also a cubic fuzzy group.

Proposition 3.4: Let G be a group on the non-empty universal set X, if $\mathcal{A}_1 = \{\langle x,\mu_1(x),\lambda_1(x)\rangle : x \in G\}$ and $\mathcal{A}_2 = \{\langle x,\mu_2(x),\lambda_2(x)\rangle : x \in G\}$ are two cubic fuzzy groups of $\mathcal{C}(G)$, then $\mathcal{A}_1 \sqcap_R \mathcal{A}_2$ is also a cubic fuzzy group of $\mathcal{C}(G)$.
Proof: Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are cubic fuzzy groups of $\mathcal{C}(G)$, we have
$(\mathcal{A}_1 \sqcap_R \mathcal{A}_2)(x*y) = \{x*y, \inf(\mu_1(x*y),\mu_2(x*y)), \sup(\lambda_1(x*y),\lambda_2(x*y)) : x,y \in G\}$
$= \{x*y, \inf[\mu_i(x*y)], \sup[\lambda_i(x*y)] : x,y \in G\}$
$\geq \{x*y, \inf[\min\{\mu_i(x),\mu_i(y)\}], \sup[\min\{\lambda_i(x),\lambda_i(y)\}] : x,y \in G\}$
$= \{x*y, \min\{\inf(\mu_i(x)), \inf(\mu_i(y))\}, \min\{\sup(\lambda_i(x)), \sup(\lambda_i(y))\} : x,y \in G\}$
$= \{x*y, \min\{(\sqcap \mu_i)(x), (\sqcap \mu_i)(y)\}, \min\{(\vee \lambda_i)(x), (\vee \lambda_i)(y)\} : x,y \in G\}$
This implies that;
$\inf[\mu_i(x*y)] = (\sqcap \mu_i(x*y)) \geq \min\{(\sqcap \mu_i)(x), (\sqcap \mu_i)(y)\}$,
$\mathrm{Sup}[\lambda_i(x*y)] = (\vee \lambda_i(x*y)) \geq \min\{(\vee \lambda_i)(x), (\vee \lambda_i)(y)\}$.
Now
$(\mathcal{A}_1 \sqcap_R \mathcal{A}_2)(x^{-1}) = \{x^{-1}, \inf[\mu_i(x^{-1})], \sup[\lambda_i(x^{-1})] : x^{-1} \in G\}$
$= \{e*x^{-1}, \inf[\mu_i(e*x^{-1})], \sup[\lambda_i(e*x^{-1})] : e, x^{-1} \in G\}$
$\geq \{e*x^{-1}, \inf[\min\{\mu_i(e),\mu_i(x)\}], \sup[\min\{\lambda_i(e),\lambda_i(x)\}] : e, x^{-1} \in G\}$
$= \{e*x^{-1}, \min\{\inf(\mu_i(e)), \inf(\mu_i(x))\}, \min\{\sup(\lambda_i(e)), \sup(\lambda_i(x))\} : e, x^{-1} \in G\}$
$= \{e*x^{-1}, \min\{(\sqcap \mu_i)(e), (\sqcap \mu_i)(x)\}, \min\{(\vee \lambda_i)(e), (\vee \lambda_i)(x)\} : e, x^{-1} \in G\}$
Since $\mu_i(e) \geq \mu_i(x), \lambda_i(e) \geq \lambda_i(x)\ \ \forall x \in G$
$= \{e*x^{-1}, (\sqcap \mu_i)(x), (\vee \lambda_i)(x) : e, x^{-1} \in G\}$

Implies that;

$\inf[\mu_i(x^{-1})] = (\sqcap \mu_i)(x^{-1}) \geq (\sqcap \mu_i)(x), \sup[\lambda_i(x^{-1})] = (\vee \lambda_i)(x^{-1}) \geq (\vee \lambda_i)(x)$ (1)

And

$(\mathcal{A}_1 \sqcap_R \mathcal{A}_2)(x) = \{x, \inf[\mu_i(x)], \sup[\lambda_i(x)] : x \in G\}$, since we have $x = (x^{-1})^{-1}$ then we can write it as follows

$= \{e * (x^{-1})^{-1}, \inf[\mu_i(e * (x^{-1})^{-1})], \sup[\lambda_i(e * (x^{-1})^{-1})] : e, x \in G\}$

$\geq \{e * (x^{-1})^{-1}, \inf[\min\{\mu_i(e), \mu_i(x^{-1})\}], \sup[\min\{(\lambda_i)(e), (\lambda_i)(x^{-1})\}] : e, x \in G\}$

$= \{e * (x^{-1})^{-1}, \min\{\inf(\mu_i(e)), \inf(\mu_i(x^{-1}))\}, \min\{\sup((\lambda_i)(e)), \sup((\lambda_i)(x^{-1})) : e, x \in G\}$

$= \{e * (x^{-1})^{-1}, \min\{(\sqcap \mu_i)(e), (\sqcap \mu_i)(x^{-1})\}, \min\{(\vee \lambda_i)(e), (\vee \lambda_i)(x^{-1})\}: e, x \in G\}$

Since $\mu_i(e) \geq \mu_i(x), \lambda_i(e) \geq \lambda_i(x) \ \forall x \in G$

$= \{e * (x^{-1})^{-1}, (\sqcap \mu_i)(x^{-1}), (\vee \lambda_i)(x^{-1}): e, x \in G\}$

And this implies that;

$\inf[\mu_i(x)] = (\sqcap \mu_i)(x) \geq (\sqcap \mu_i)(x^{-1}), \sup[\lambda_i(x)] = (\vee \lambda_i)(x) \geq (\vee \lambda_i)(x^{-1})$ (2)

Hence from (1) and (2) we get;

$(\sqcap \mu_i)(x^{-1}) = (\sqcap \mu_i)(x), (\vee \lambda_i)(x^{-1}) = (\vee \lambda_i)(x)$

And this implies that $\mathcal{A}_1 \sqcap_R \mathcal{A}_2$ is also a cubic fuzzy group.

Proposition 3.5: Let G be a group on the non-empty universal set X, if $\mathcal{A}_1 = \{\langle x, \mu_1(x), \lambda_1(x)\rangle : x \in G\}$ and $\mathcal{A}_2 = \{\langle x, \mu_2(x), \lambda_2(x)\rangle : x \in G\}$ are two cubic fuzzy groups of $\mathcal{C}(G)$, then $\mathcal{A}_1 \sqcup_P \mathcal{A}_2$ is also a cubic fuzzy group of $\mathcal{C}(G)$.

Proof: Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are cubic fuzzy groups of $\mathcal{C}(G)$, we have

$(\mathcal{A}_1 \sqcup_P \mathcal{A}_2)(x * y) = \{x * y, \sup(\mu_1(x * y), \mu_2(x * y)), \sup(\lambda_1(x * y), \lambda_2(x * y)) : x, y \in G\}$

$= \{x * y, \sup[\mu_i(x * y)], \sup[\lambda_i(x * y)] : x, y \in G\}$

$\geq \{x * y, \sup[\min\{\mu_i(x), \mu_i(y)\}], \sup[\min\{\lambda_i(x), \lambda_i(y)\}] : x, y \in G\}$

$= \{x * y, \min\{\sup(\mu_i(x)), \sup(\mu_i(y))\}, \min\{\sup(\lambda_i(x)), \sup(\lambda_i(y))\} : x, y \in G\}$

$= \{x * y, \min\{(\sqcup \mu_i)(x), (\sqcup \mu_i)(y)\}, \min\{(\vee \lambda_i)(x), (\vee \lambda_i)(y)\} : x, y \in G\}$

This implies that;

$\sup[\mu_i(x * y)] = (\sqcup \mu_i(x * y)) \geq \min\{(\sqcup \mu_i)(x), (\sqcup \mu_i)(y)\}$,

$\sup[\lambda_i(x * y)] = (\vee \lambda_i(x * y)) \geq \min\{(\vee \lambda_i)(x), (\vee \lambda_i)(y)\}$.

Now

$(\mathcal{A}_1 \sqcup_P \mathcal{A}_2)(x^{-1}) = \{h^{-1}, \sup[\mu_i(x^{-1})], \sup[\lambda_i(x^{-1})] : x^{-1} \in G\}$

$= \{e * x^{-1}, \sup[\mu_i(e * x^{-1})], \sup[\lambda_i(e * x^{-1})] : e, x^{-1} \in G\}$

$\geq \{e * x^{-1}, \sup[\min\{\mu_i(e), \mu_i(x)\}], \sup[\min\{\lambda_i(e), \lambda_i(x)\}] : e, x^{-1} \in G\}$

$= \{e * x^{-1}, \min\{\sup(\mu_i(e)), \sup(\mu_i(x))\}, \min\{\sup(\lambda_i(e)), \sup(\lambda_i(x))\}: e, x^{-1} \in G\}$

$= \{e * x^{-1}, \min\{(\sqcup \mu_i)(e), (\sqcup \mu_i)(x)\}, \min\{(\vee \lambda_i)(e), (\vee \lambda_i)(x)\}: e, x^{-1} \in G\}$

Since $\mu_i(e) \geq \mu_i(x), \lambda_i(e) \geq \lambda_i(x) \ \forall x \in G$

$= \{e * x^{-1}, (\sqcup \mu_i)(x), (\vee \lambda_i)(x): e, x^{-1} \in G\}$

Implies that;

$\sup[\mu_i(x^{-1})] = (\sqcup \mu_i)(x^{-1}) \geq (\sqcup \mu_i)(x), \sup[\lambda_i(x^{-1})] = (\vee \lambda_i)(x^{-1}) \geq (\vee \lambda_i)(x)$ (1)

And

$(\mathcal{A}_1 \sqcup_P \mathcal{A}_2)(x) = \{x, \sup[\mu_i(x)], \sup[\lambda_i(x)] : x \in G\}$, since we have $x = (x^{-1})^{-1}$ then we can write it as follows

$= \{e * (x^{-1})^{-1}, \sup[\mu_i(e * (x^{-1})^{-1})], \sup[\lambda_i(e * (x^{-1})^{-1})] : e, x \in G\}$

$\geq \{e * (x^{-1})^{-1}, \sup[\min\{\mu_i(e), \mu_i(x^{-1})\}], \sup[\min\{(\lambda_i)(e), (\lambda_i)(x^{-1})\}] : e, x \in G\}$

$= \{e * (x^{-1})^{-1}, \min\{\sup(\mu_i(e), \sup(\mu_i(x^{-1}))\}, \min\{\sup((\lambda_i)(e)), \sup((\lambda_i)(x^{-1}))\} : e, x \in G\}$

$= \{e * (x^{-1})^{-1}, \min\{(\sqcup \mu_i)(e), (\sqcup \mu_i)(x^{-1})\}, \min\{(\vee \lambda_i)(e), (\vee \lambda_i)(x^{-1})\}: e, x \in G\}$

Since $\mu_i(e) \geq \mu_i(x), \lambda_i(e) \geq \lambda_i(x) \ \forall x \in G$

$= \{e * (x^{-1})^{-1}, (\sqcup \mu_i)(x^{-1}), (\vee \lambda_i)(x^{-1}): e, x \in G\}$

And this implies that;

$\sup[\mu_i(x)] = (\sqcup \mu_i)(x) \geq (\sqcup \mu_i)(x^{-1}), \sup[\lambda_i(x)] = (\vee \lambda_i)(x) \geq (\vee \lambda_i)(x^{-1})$ (2)

Hence from (1) and (2) we get;

$(\sqcup \mu_i)(x^{-1}) = (\sqcup \mu_i)(x), (\vee \lambda_i)(x^{-1}) = (\vee \lambda_i)(x)$

And this implies that $\mathcal{A}_1 \sqcup_P \mathcal{A}_2$ is also a cubic fuzzy group.

Proposition 3.6: Let G be a group on the non-empty universal set X, if $\mathcal{A}_1 = \{\langle x, \mu_1(x), \lambda_1(x)\rangle : x \in G\}$ and $\mathcal{A}_2 = \{\langle x, \mu_2(x), \lambda_2(x)\rangle : x \in G\}$ are two cubic fuzzy groups of $\mathcal{C}(G)$, then $\mathcal{A}_1 \sqcup_R \mathcal{A}_2$ is also a cubic fuzzy group of $\mathcal{C}(G)$.

Proof: Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are cubic fuzzy groups of $\mathcal{C}(G)$, we have

$(\mathcal{A}_1 \sqcup_R \mathcal{A}_2)(x * y) = \{x * y, \sup(\mu_1(x * y), \mu_2(x * y)), \inf(\lambda_1(x * y), \lambda_2(x * y)) : x, y \in G\}$

$= \{x * y, \sup[\mu_i(x * y)], \inf[\lambda_i(x * y)] : x, y \in G\}$

$\geq \{x * y, \sup[\min\{\mu_i(x), \mu_i(y)\}], \inf[\min\{\lambda_i(x), \lambda_i(y)\}] : x, y \in G\}$

$= \{x * y, \min\{\sup(\mu_i(x)), \sup(\mu_i(y))\}, \min\{\inf(\lambda_i(x)), \inf(\lambda_i(y))\} : x, y \in G\}$

$= \{x * y, \min\{(\sqcup \mu_i)(x), (\sqcup \mu_i)(y)\}, \min\{(\wedge \lambda_i)(x), (\wedge \lambda_i)(y)\} : x, y \in G\}$

This implies that;

$\sup[\mu_i(x * y)] = (\sqcup \mu_i(x * y)) \geq \min\{(\sqcup \mu_i)(x), (\sqcup \mu_i)(y)\}$,

$\inf[\lambda_i(x * y)] = (\wedge \lambda_i(x * y)) \geq \min\{(\wedge \lambda_i)(x), (\wedge \lambda_i)(y)\}$.

Now

$(\mathcal{A}_1 \sqcup_R \mathcal{A}_2)(x^{-1}) = \{x^{-1}, \sup[\mu_i(x^{-1})], \inf[\lambda_i(x^{-1})] : x^{-1} \in G\}$

$= \{e * x^{-1}, \sup[\mu_i(e * x^{-1})], \inf[\lambda_i(e * x^{-1})] : e, x^{-1} \in G\}$

$\geq \{e * x^{-1}, \sup[\min\{\mu_i(e), \mu_i(x)\}], \inf[\min\{\lambda_i(e), \lambda_i(x)\}] : e, x^{-1} \in G\}$

$= \{e * x^{-1}, \min\{\sup(\mu_i(e)), \sup(\mu_i(x))\}, \min\{\inf(\lambda_i(e)), \inf(\lambda_i(x))\}: e, x^{-1} \in G\}$

$= \{e * x^{-1}, \min\{(\sqcup \mu_i)(e), (\sqcup \mu_i)(x)\}, \min\{(\wedge \lambda_i)(e), (\wedge \lambda_i)(x)\}: e, x^{-1} \in G\}$

Since $\mu_i(e) \geq \mu_i(x), \lambda_i(e) \geq \lambda_i(x) \ \forall x \in G$

$= \{e * x^{-1}, (\sqcup \mu_i)(x), (\wedge \lambda_i)(x): e, x^{-1} \in G\}$, implies that;

$\sup[\mu_i(x^{-1})] = (\sqcup \mu_i)(x^{-1}) \geq (\sqcup \mu_i)(x), \inf[\lambda_i(x^{-1})] = (\wedge \lambda_i)(x^{-1}) \geq (\wedge \lambda_i)(x)$ (1)

And

$(\mathcal{A}_1 \sqcup_R \mathcal{A}_2)(x) = \{x, \sup[\mu_i(x)], \inf[\lambda_i(x)] : x \in G\}$, since we have $x = (x^{-1})^{-1}$ then we can write it as follows

$= \{e * (x^{-1})^{-1}, \sup[\mu_i(e * (x^{-1})^{-1})], \inf[\lambda_i(e * (x^{-1})^{-1})] : e, x \in G\}$

$\geq \{e * (x^{-1})^{-1}, \sup[\min\{\mu_i(e), \mu_i(x^{-1})\}], \inf[\min\{(\lambda_i)(e), (\lambda_i)(x^{-1})\}] : e, x \in G\}$

$= \{e * (x^{-1})^{-1}, \min\{\sup(\mu_i(e)), \sup(\mu_i(x^{-1}))\}, \min\{\inf((\lambda_i)(e)), \inf((\lambda_i)(x^{-1}))\}: e, x \in G\}$

$= \{e * (x^{-1})^{-1}, \min\{(\sqcup \mu_i)(e), (\sqcup \mu_i)(x^{-1})\}, \min\{(\wedge \lambda_i)(e), (\wedge \lambda_i)(x^{-1})\}: e, x \in G\}$

Since $\mu_i(e) \geq \mu_i(x), \lambda_i(e) \geq \lambda_i(x) \ \forall x \in G$

$= \{e * (x^{-1})^{-1}, (\sqcup \mu_i)(x^{-1}), (\wedge \lambda_i)(x^{-1}): e, x \in G\}$

And this implies that;

$\sup[\mu_i(x)] = (\sqcup \mu_i)(x) \geq (\sqcup \mu_i)(x^{-1}), \inf[\lambda_i(x)] = (\wedge \lambda_i)(x) \geq (\wedge \lambda_i)(x^{-1})$ (2)

Hence from (1) and (2) we get;

$(\sqcup \mu_i)(x^{-1}) = (\sqcup \mu_i)(x), (\wedge \lambda_i)(x^{-1}) = (\wedge \lambda_i)(x)$

And this implies that $\mathcal{A}_1 \sqcup_R \mathcal{A}_2$ is also a cubic fuzzy group.

Definition 3.8: A cubic set $\mathcal{A}$ on the non-empty universal set X, is called a cubic fuzzy semi group if;
$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, $\lambda(x * y) \geq \min\{\lambda(x), \lambda(y)\}$, $\forall\, x, y \in G$.
Clearly, by the definition that every cubic fuzzy group is a cubic semi group but in general the converse is not true, for example, in Example 3.2 if we take $x = 0.1$, then $\mu\big((0.1)^{-1}\big) = \mu(10) = \left[\frac{1}{12}, \frac{1}{11}\right] \neq \mu(0.1) = [0.1, 0.02]$.
Remark 3.2: Suppose $\mathcal{A} = \langle\mu, \lambda\rangle$ be a cubic set if $\mathcal{A}$ is both internal-external cubic set, then $\mathcal{A}$ is a cubic fuzzy semi group.

4. CUBIC FUZZY NORMAL SUBGROUPS

In this section, we present and study the concept of cubic fuzzy normal subgroups as an extension of fuzzy normal subgroups given in [10]. Some properties of this concept are discussed by membership functions.

Definition 4.1: Let $\mathcal{A} \in \mathcal{C}(G)$. Then $\mathcal{A}$ is called a cubic fuzzy normal subgroup of $\mathcal{C}(G)$ if for all $x, y \in G$,
$$\mu(x * y) = \mu(y * x), \lambda(x * y) = \lambda(y * x).$$
The set of cubic fuzzy normal subgroups of $\mathcal{C}(G)$ is denoted by $\mathcal{C}_n(G)$.
In **Example 3.1** we see that a cubic fuzzy group $\mathcal{A} = \langle\mu, \lambda\rangle$ which is a cubic fuzzy normal subgroup because;
$\mu(x * y) = \mu(y * x), \lambda(x * y) = \lambda(y * x)\ \forall\, x, y \in G$.
Theorem 4.1: Let $\mathcal{A}$ be a cubic fuzzy subgroup of $\mathcal{C}(G)$, then $\mathcal{A}$ is a cubic fuzzy normal subgroup of $\mathcal{C}_n(G)$ if and only if $\mu(y^{-1} * x * y) = \mu(x), \lambda(y^{-1} * x * y) = \lambda(x)\ \ \forall\, x, y \in G$.
Proof: Suppose that $\mathcal{A}$ is a cubic fuzzy normal subgroup of $\mathcal{C}_n(G)$, then
$\mu(y^{-1} * x * y) = \mu\big((y^{-1}) * (x * y)\big) = \mu((x * y) * (y^{-1})) = \mu((x) * (y * y^{-1})) = \mu(x)$
$\lambda(y^{-1} * x * y) = \lambda((y^{-1}) * (x * y)) = \lambda((x * y) * (y^{-1})) = \lambda((x) * (y * y^{-1})) = \lambda(x)$
Conversely: let $\mu(y^{-1} * x * y) = \mu(x), \lambda(y^{-1} * x * y) = \lambda(x)\ \ \forall\, x, y \in G$, then we have to show that $\mathcal{A}$ is a cubic fuzzy normal subgroup of $\mathcal{C}_n(G)$,
$\mu(y * x) = \mu(x^{-1} * x * y * x) = \mu(x^{-1} * (x * y) * x) = \mu(x * y)$
$\lambda(y * x) = \lambda(x^{-1} * x * y * x) = \lambda(x^{-1} * (x * y) * x) = \lambda(x * y)$
This implies that $\mathcal{A}$ is a cubic fuzzy normal subgroup of -$\mathcal{C}_n(G)$.
Definition 4.2: Let $\mathcal{A}$ be a cubic fuzzy subgroup of $\mathcal{C}(G)$ which is called a cubic fuzzy normal subgroup of $\mathcal{C}_n(G)$ if $\mu(y^{-1} * x * y) \geq \min\{\mu(x), \mu(y)\}, \lambda(y^{-1} * x * y) \geq \min\{\lambda(x), \lambda(y)\}\ \ \forall\, x, y \in G$.
A cubic fuzzy group $\mathcal{A} = \langle\mu, \lambda\rangle$ in Example 3.1 is a cubic fuzzy normal subgroup because it satisfies the condition;
$\mu(y^{-1} * x * y) \geq \min\{\mu(x), \mu(y)\}, \lambda(y^{-1} * x * y) \geq \min\{\lambda(x), \lambda(y)\}\ \ \forall\, x, y \in G$.
Proposition 4.1: Let G be a group on the non-empty universal set X, if $\mathcal{A}_1 = \{\langle x, \mu_1(x), \lambda_1(x)\rangle : x \in G\}$ and $\mathcal{A}_2 = \{\langle x, \mu_2(x), \lambda_2(x)\rangle : x \in G\}$ are two cubic fuzzy normal subgroups of $\mathcal{C}_n(G)$, then $\mathcal{A}_1 \sqcap_P \mathcal{A}_2$ is also a cubic fuzzy normal subgroup of $\mathcal{C}_n(G)$.
Proof: Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are cubic fuzzy normal subgroups of $\mathcal{C}_n(G)$, we have
$(\mathcal{A}_1 \sqcap_P \mathcal{A}_2)(y^{-1} * x * y)$
$= \{y^{-1} * x * y, \inf[\mu_i(y^{-1} * x * y)],$
$\inf[\lambda_i(y^{-1} * x * y)] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, \inf[\mu_i((y^{-1}) * (x * y))],$
$\inf[\lambda_i((y^{-1}) * (x * y))] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, \inf[\mu_i((x * y) * (y^{-1}))],$
$\inf[\lambda_i((x * y) * (y^{-1}))] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, \inf[\mu_i(x)], \inf[\lambda_i(x)] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, (\sqcap \mu_i(x)), (\sqcap \lambda_i(x)) : x, y, y^{-1} \in G\}$
This implies that;
$\inf[\mu_i(y^{-1} * x * y)] = \big(\sqcap \mu_i(y^{-1} * x * y)\big) = (\sqcap \mu_i(x))$,
$\inf[\lambda_i(y^{-1} * x * y)] = \big(\sqcap \lambda_i(y^{-1} * x * y)\big) = (\sqcap \lambda_i(x))$.
By Theorem 4.1, we get that $\mathcal{A}_1 \sqcap_P \mathcal{A}_2$ is also a cubic fuzzy normal subgroup of $\mathcal{C}_n(G)$.
Proposition 4.2: Let G be a group on the non-empty universal set X, if $\mathcal{A}_1 = \{\langle x, \mu_1(x), \lambda_1(x)\rangle : x \in G\}$ and $\mathcal{A}_2 = \{\langle x, \mu_2(x), \lambda_2(x)\rangle : x \in G\}$ are two cubic fuzzy normal subgroups of $\mathcal{C}_n(G)$, then $\mathcal{A}_1 \sqcap_R \mathcal{A}_2$ is also a cubic fuzzy normal subgroup of $\mathcal{C}_n(G)$.
Proof: Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are cubic fuzzy normal subgroups of $\mathcal{C}_n(G)$, we have
$(\mathcal{A}_1 \sqcap_R \mathcal{A}_2)(y^{-1} * x * y)$
$= \{y^{-1} * x * y, \inf[\mu_i(y^{-1} * x * y)],$
$\sup[\lambda_i(y^{-1} * x * y)] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, \inf[\mu_i((y^{-1}) * (x * y))],$
$\sup[\lambda_i((y^{-1}) * (x * y))] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, \inf[\mu_i((x * y) * (y^{-1}))],$
$\sup[\lambda_i((x * y) * (y^{-1}))] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, \inf[\mu_i(x)], \sup[\lambda_i(x)] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, (\sqcap \mu_i(x)), (\sqcup \lambda_i(x)) : x, y, y^{-1} \in G\}$
This implies that;
$\inf[\mu_i(y^{-1} * x * y)] = \big(\sqcap \mu_i(y^{-1} * x * y)\big) = (\sqcap \mu_i(x))$,
$\sup[\lambda_i(y^{-1} * x * y)] = \big(\sqcup \lambda_i(y^{-1} * x * y)\big) = (\sqcup \lambda_i(x))$.
By Theorem 4.1, we get $\mathcal{A}_1 \sqcap_R \mathcal{A}_2$ is also a cubic fuzzy normal subgroup of $\mathcal{C}_n(G)$.
Proposition 4.3: Let G be a group on the non-empty universal set X, if $\mathcal{A}_1 = \{\langle x, \mu_1(x), \lambda_1(x)\rangle : x \in G\}$ and $\mathcal{A}_2 = \{\langle x, \mu_2(x), \lambda_2(x)\rangle : x \in G\}$ are two cubic fuzzy normal subgroups of $\mathcal{C}_n(G)$, then $\mathcal{A}_1 \sqcup_P \mathcal{A}_2$ is also a cubic fuzzy normal subgroup of $\mathcal{C}_n(G)$.
Proof: Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are cubic fuzzy normal subgroups of $\mathcal{C}_n(G)$, we have
$(\mathcal{A}_1 \sqcup_P \mathcal{A}_2)(y^{-1} * x * y)$
$= \{y^{-1} * x * y, \sup[\mu_i(y^{-1} * x * y)],$
$\sup[\lambda_i(y^{-1} * x * y)] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, \sup[\mu_i((y^{-1}) * (x * y))],$
$\sup[\lambda_i((y^{-1}) * (x * y))] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, \sup[\mu_i((x * y) * (y^{-1}))],$
$\sup[\lambda_i((x * y) * (y^{-1}))] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, \sup[\mu_i(x)], \sup[\lambda_i(x)] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, (\sqcup \mu_i(x)), (\sqcup \lambda_i(x)) : x, y, y^{-1} \in G\}$
This implies that;
$\sup[\mu_i(y^{-1} * x * y)] = \big(\sqcup \mu_i(y^{-1} * x * y)\big) = (\sqcup \mu_i(x))$,
$\sup[\lambda_i(y^{-1} * x * y)] = \big(\sqcup \lambda_i(y^{-1} * x * y)\big) = (\sqcup \lambda_i(x))$.
Hence, by Theorem 4.1, we get that $\mathcal{A}_1 \sqcup_P \mathcal{A}_2$ is also a cubic fuzzy normal subgroup of $\mathcal{C}_n(G)$.
Proposition 4.4: Let G be a group on the non-empty universal set X, if $\mathcal{A}_1 = \{\langle x, \mu_1(x), \lambda_1(x)\rangle : x \in G\}$ and $\mathcal{A}_2 = \{\langle x, \mu_2(x), \lambda_2(x)\rangle : x \in G\}$ are two cubic fuzzy normal subgroups of $\mathcal{C}_n(G)$, then $\mathcal{A}_1 \sqcup_R \mathcal{A}_2$ is also a cubic fuzzy normal subgroup of $\mathcal{C}_n(G)$.
Proof: Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are cubic fuzzy normal subgroups of $\mathcal{C}_n(G)$, we have
$(\mathcal{A}_1 \sqcup_R \mathcal{A}_2)(y^{-1} * x * y) = \{y^{-1} * x * y, \sup[\mu_i(y^{-1} * x * y)],$
$\inf[\lambda_i(y^{-1} * x * y)] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, \sup[\mu_i((y^{-1}) * (x * y))],$
$\inf[\lambda_i((y^{-1}) * (x * y))] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, \sup[\mu_i((x * y) * (y^{-1}))],$
$\inf[\lambda_i((x * y) * (y^{-1}))] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, \sup[\mu_i(x)], \inf[\lambda_i(x)] : x, y, y^{-1} \in G\}$
$= \{y^{-1} * x * y, (\sqcup \mu_i(x)), (\sqcap \lambda_i(x)) : x, y, y^{-1} \in G\}$
This implies that;

$\sup[\mu_i(y^{-1} * x * y)] = (\sqcup \mu_i(y^{-1} * x * y)) = (\sqcup \mu_i(\mathrm{x}))$,
$\inf[\lambda_i(y^{-1} * x * y)] = (\sqcap \lambda_i(y^{-1} * x * y)) = (\sqcap \lambda_i(\mathrm{x}))$.
By Theorem 4.1, we get that $\mathcal{A}_1 \sqcup_R \mathcal{A}_2$ is also a cubic fuzzy normal subgroup of $\mathcal{C}_n(G)$.

5. CONCLUSION

A new concept of a fuzzy group namely the cubic fuzzy group on cubic fuzzy set is introduced by incorporating the features of a fuzzy set and an interval-valued fuzzy set. Several properties and theorems of a cubic fuzzy group and a cubic fuzzy normal subgroup are investigated and proved. We proved that (P-, R-) intersection and (P-,R-) union of two cubic fuzzy groups are cubic fuzzy group. Finally, we presented the (P-, R-) intersection and (P-, R-) union of two cubic fuzzy normal subgroups and proved that they are also cubic fuzzy normal subgroup. Our future research is to define and apply some mappings on cubic fuzzy groups, cubic fuzzy rings and cubic fuzzy fields.

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