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Strategic Priority-Based Course Allocation*

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Abstract: We present the conditional acceptance mechanism for the course allocation problem. This mechanism combines features of the immediate acceptance and the student optimal stable mechanisms. It implements the set of stable allocations in both Nash and undominated Nash equilibria under substitutable preferences and priorities. We model the post-allocation adjustment mechanism using a repeated version of the mechanism. This repeated mechanism reduces the wastefulness of out-of-equilibrium play and implements the set of stable allocation in Subgame Perfect Nash equilibrium under slot-speci#c preferences and priorities. Both mechanisms are easily implementable, reduce the complexity of eliciting students' preferences, and mimic the features of the mechanisms currently in use.

Keywords: conditional acceptance, immediate acceptance, multi-unit assignment problem, stability.

JEL Classification Numbers: C71; C78; D71.

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1 Introduction

We consider the multi-unit assignment problem and focus on the course allocation problem in which course schedules are assigned to students based on student preferences and course priorities (see Sönmez and Ünver, 2010; Budish, 2011; Kojima, 2013). Our results can also be applied to problems such as the time scheduling problem and the assignment of landing slots (see Schummer and Vohra, 2013; Schummer and Abizada, 2017).

Two issues arise in the allocation of course schedules. First, the allocation must be fair in the sense that it must respect both student preferences and course priorities. Failing to satisfy this requirement generates two unintended consequences: the existence of student-course blocking pairs and empty seats. The former generates post-allocation appeals that need to be resolved. The latter is wasteful. Both problems are usually solved by either an administrative allocation or a post-allocation adjustment mechanism. Second, eliciting student preferences over course schedules is difficult (see Budish et al., 2017). Even if we focus on stable (which is fair) allocations, no stable and strategy-proof mechanism exists for multi-unit assignment problems.[1] Additionally, the combinatorial nature of the course allocation problem requires students to form preferences on a large set of course schedules.

To overcome the impossibility of implementing stable allocations in dominant strategies, we relax the equilibrium concept and focus on the implementation in Nash equilibrium (NE). Additionally, we want to design mechanisms that are natural and in which students provide as little information as possible to mitigate the complexity of the strategy space. Therefore, we study mechanisms used in practice.

We start analyzing the student optimal stable (SO) mechanism (Gale and Shapley, 1962), widely employed in many-to-one assignment problems (see Abdulkadiroğlu and Andersson, 2022; Roth and Peranson, 1999). However, the NE outcome of the SO mechanism produces unstable allocations as Nash equilibrium outcome (see Roth and Sotomayor, 1990; Haeringer and Klijn, 2009). We prove that the NE of the game induced by the SO mechanism are stable if and only if the priorities satisfy essential homogeneity. This is a demanding requirement since it is equivalent to require strategy-proofness (see Kojima, 2013). We thus explore the immediate acceptance (IA) mechanism, which generalizes to the multi-unit case the so-called "Boston mechanism" (see Abdulkadiroğlu and Sönmez, 2003) introduced in the school assignment problem. We show that the IA mechanism implements the set of stable allocation in Nash equilibria under slot-specific priorities (see Kominers and Sönmez, 2016).[2] Thus, our results extend to the multi-unit assignment problem previous implementation results by Alcalde (1996) and Ergin and Sönmez (2006) for the

[1] A similar result holds even if we focus on efficient and individually rational allocations (see Sönmez,1999).

[2] Slot-specific priorities model situations in which a subgroup of students is given priority for a portion of the seats that are otherwise assigned according to a given criterion. For example, slot-specific priorities allow the introduction of diversity in the classroom (see Dur et al., 2016, 2018 for applications to school choice). Slot-specific priorities also encompass approaches such as majority quotas as defined in Kojima (2012) and minority reserves introduced by Hafalir et al. (2013).

marriage and school admission problems, respectively. There are situations where a more general priority structure is necessary. This is the case, for example, if the optimal class size is smaller than the course capacity. Substitutable priorities are compatible with the existence of stable allocations in this setting. However, under substitutable priorities the NE of the IA mechanism can result in unstable allocations. To extend the preference and priority domains we introduce the conditional acceptance (CA) mechanism. The CA mechanism implements the set stable of allocations in Nash equilibrium and undominated Nash equilibrium under substitutable preferences and priorities. To the best of our knowledge, this is the first paper to consider substitutable priorities in the course allocation problem (see Marutani, 2018, for the use of substitutable priorities in the school choice problem).

The CA mechanism combines characteristics of IA and SO mechanisms. Like the IA mechanism, the CA mechanism assigns seats at courses to students who rank them first and then to those who rank them second, and so on. Similar to the SO mechanism, student can lose a (tentatively) assigned course. The CA mechanism allows students to express the intensity of their preferences. By ranking a course higher, a student increases her chances of being admitted. Therefore, students have incentives to strategize.

Notice that the different results produced by the SO, IA, and CA mechanisms come from whether each assignment is tentative and whether students can request another course after they lose one. In the SO mechanism, students are assigned tentatively and can apply for a different course if they lose one. In the case of the IA mechanism, allocations are definitive and students can apply for course schedules in different stages. In the CA mechanism, the students can lose an assigned course, and they obtain courses in, at most, one stage.

In the SO, IA, and CA mechanisms out-of-equilibrium play is wasteful and costly for students. In practice, this problem is dealt with through a post-allocation adjustment. Consider, for example the Supplemental Offer and Acceptance Program in the National Resident Matching Program or $NRMP$.[3] We show that the design of a post-allocation adjustment mechanism can affect the strategic properties of the main mechanism if it is not stable or if it allows students to drop courses (see Example 4). We propose to allocate the vacant seats using additional rounds of the CA mechanism until there are not more seats to be distributed, or no student who wants them.[4] We call this mechanism the extended conditional acceptance (ECA) mechanism. The ECA mechanism mitigates the cost of out-of-equilibrium play while preserving incentives and implements the set of stable allocation in $SPNE$ when preferences and priorities are slot-specific.

Finally, there are practical situations where students have simpler preferences. For example, in the first years of several B.A. programs, students must take compulsory courses divided into sections that are, usually, held simultaneously. Another example are graduate seminars, courses

[3]https://www.nrmp.org/residency-applicants/soap/ Accessed 31/08/2022

[4]The idea of allocating the remaining courses using the same mechanism is not new. For example, it was proposed by Coles et al. (2010) for the $NRMP$.

organized by local libraries, or elective courses at small community colleges and universities where there are not overlapping time slots. In all those cases, we can represent student preferences using slot-specific preferences.[5] When student preferences are slot-specific it is easy to adapt the CA, IA, and ECA mechanisms to elicit preferences over individual courses. In this case, it is possible to simplify the student strategy space while preserving the incentives of the mechanisms.

1.1 Applicability of our results

In this subsection, we present three mechanisms that share features with the CA and ECA mechanisms. Our first example is the course allocation mechanism at University of Pennsylvania (UPenn). At Upenn undergraduate courses are treated as separated objects. Students have an adjusting period called "advance registration."[6] In the advance registration period, students are encouraged to use the information available as: "Strategically selecting and prioritizing courses increases the likelihood of receiving a favorable schedule, but there is no guarantee that students will be enrolled in all of their requested courses."[7] When the advance registration period ends, preferences become final and courses are allocated.

In the UPenn's allocation process, the order matters. In fact, the importance of the first choices is clearly stated to the point that students are encouraged to submit "an alternate request, especially for their top two choices." The role of the alternate request is that "if the primary request is not available, the alternate is treated with just as high a priority (better than if listed as #2)." After the allocation, a post-allocation adjustment round allows students to drop courses and register new ones. Our results, in particular Example 4, show that the possibility of dropping courses in the post-allocation adjustment round prevents UPenn's mechanism implementing stable allocations.

A similar mechanism is used at Eötvös Loránd University in Hungary (see Rusznak, Biró, and Fleiner, 2021). In this case, students first submit a preregistration course schedule that helps the university to estimate demand for each course and give students information on their chances to be admitted. Later there is a formal registration process where students register different courses before the capacity limits are imposed. Finally, an allocation is selected.

At the Department of Political Science in Aarhus University in Denmark, enroll its master's degree students using a mechanism similar to ECA with two rounds where students submit an application with a course schedule. If they are not allocated a seat on one or more of their desired courses, they must register anew in the second registration period. Also, "(a)ll students who have

[5]In particular, responsive preferences are slot-specific. The assumption of responsive preferences is common in the literature on course allocation and is used, among others, in Budish and Cantillon (2012), Kojima (2013), Kojima and Ünver (2014), and Doğan and Klaus (2018).

[6]The "Course selection starts about two weeks after advance registration ends and lasts approximately two and a half weeks. During the Course Selection Period, students move in and out of courses by adding and dropping on Penn InTouch. https://www.college.upenn.edu/registration-tips. Accessed 31/08/2022.

[7]https://www.college.upenn.edu/registration-tips.

been registered in their selected courses are bound by their choice. This means that you will not be able to cancel or change your registration for elective courses once the deadline for registration has expired, and the elective course is a binding part of your study programme."[8] Aarhus's design evidences that in environments with almost complete information course allocation can be made by declaring just one course schedule.

1.2 Alternative approaches to course allocation

The allocation of course schedules to students has been widely analyzed both in theory and in practice. It is sometimes conducted by a first-come-first-serve mechanism. This method is problematic. The students' rush to be first can overload the system (see Aziz et al., 2019) and produce unfair allocations. Another method commonly used is a serial dictatorship (SD) based on a particular criterion such as a random draw or the student average grade (see, among others, Pàpai, 2002; Ehlers and Klaus, 2003). The SD is group-strategy-proof and efficient. Under priorities, a SD mechanism produces stable allocations if all students are acceptable and there is a unique priority order for all the courses. The SD mechanism results in allocations where students with high priority systematically obtain their favorite schedules in detriment of the students with low priority (see Budish and Cantillon, 2012). An advantage of the SD mechanism is that it minimizes the need for reassignment after the initial allocation. However, this property depends on the practical implementation of the rule.[9]

Course schedules can also be allocated using bidding mechanisms. In these mechanisms, each student allocates fake money among the courses she wishes to register. Sönmez and Ünver (2010), studies bidding mechanisms used in several business schools. They propose the Gale-Shapley Pareto-dominant market mechanism that assigns priorities to courses to break ties, based on students' bids, and execute the SO mechanism. This mechanism can dominate the bidding mechanism.

Budish (2011) proposes the use of pseudo-markets to allocate course schedules without priorities. It introduces the approximate competitive equilibrium from equal incomes ($A-CEEI$). The $A-CEEI$ is efficient and approximately strategy-proof in large markets. Unfortunately, unstable allocations can survive even in large markets, maintaining the tension between efficiency and fairness (see Budish and Cantillon, 2012). The $A-CEEI$ bounds absolute envy, but this weak fairness concept is compatible with the existence of multiple blocking pairs. In addition, the implementation of the $A-CEEI$ is complex and computationally demanding (see Budish et al., 2017). Kornbluth and Kushnir (2021) present the Budget-Adjusted Pseudo-Market mechanism.

[8]See https://studerende.au.dk/en/studies/subject-portals/political-science/teaching/registration-for-courses/registration-for-masters-courses. Accessed 02/09/2022

[9]The Universidad Carlos III de Madrid allocates course schedules using a SD based on the student average grade. More than 1200 students appeal academic course to change their schedule (personal communication with Raúl Blanco, head of the student office. School of Social Sciences and Law at Universidad Carlos III de Madrid).

This mechanism introduces priorities in the $A-CEEI$ and preserves its properties. It also prevents justified schedule envy, a weaker fairness concept that guarantees that each student does not prefer the schedule of a lower priority student to her own.

The paper is organized as follows. Section 2 introduces the model and notation. Section 3 presents our results with one-shot mechanism. Section 4 presents our proposal for post-allocation adjustment. Section 5 presents simplified versions of the mechanisms, and Section 6 concludes. The proofs are in the Appendix.

2 The Model

There is a finite set of courses C and a finite set of students S, with $C \cap S = \emptyset$. Each course c has priorities over subsets of students. Priorities are described by a choice function $Ch_c : 2^S \to 2^S$, where $Ch_c(S') \subseteq S'$ for all $S' \subseteq S$.[10] We assume that the choice function is **substitutable**. Formally, if $S' \subseteq S$, $s, s' \in S \setminus S'$ and $s \notin Ch_c(S' \cup \{s\})$, then $s \notin Ch_c(S' \cup \{s, s'\})$. In words, Ch_c is substitutable if, whenever the course c rejects a student from a given subset of students, it rejects her when more students become available. We also assume that Ch_c satisfies **irrelevance of rejected students**.[11] Formally, if $S' \subseteq S$ and $s \notin Ch_c(S' \cup \{s\})$, then $Ch_c(S' \cup \{s\}) = Ch_c(S')$. Rejected students do not affect courses' choices. If Ch_c is substitutable and satisfies the irrelevance of rejected students, then it is rationalizable by a linear order on 2^S, P_c, which is $Ch_c(S') = \max_{P_c} \{S'' \mid S'' \subseteq S'\} = Ch_c(S', P_c)$ for all $S' \subseteq S$ (see Alva, 2018). A priority profile is a list $Ch_C = (Ch_c)_{c \in C}$ or, equivalently, $P_C = (P_c)_{c \in C}$, where P_c rationalizes Ch_c for all $c \in C$. When there is not ambiguity about P_c, we write $Ch_c(S')$ instead of $Ch_c(S', P_c)$. A particular class of substitutable priorities is the class of **slot-specific priorities** introduced by Kominers and Sönmez (2016). Under slot-specific priorities, each course $c \in C$ has a finite set of slots, Σ_c, with generic element σ. Each slot σ has a priority order $\succ_\sigma$, which is a strict, complete, and transitive binary relation over $S \cup \{\emptyset\}$, where $\{\emptyset\}$ represents the possibility of maintaining the slot empty. The higher a student is ranked under $\succ_\sigma$, the stronger the claim that she has for slot σ in the course c. If $\emptyset \succ_\sigma s$, student s is not acceptable for slot σ. If $\emptyset \succ_\sigma s$, for all slots σ, student s is not acceptable to c. If $s \succ_\sigma \emptyset$ for some σ then student s is acceptable to c. We denote by $A_c(P_c)$ the set of acceptable students for c. The total **supply** of course c is $q_c = |\Sigma_c|$. Let us define q as the vector of supplies for the various courses Σ_c, $q = (q)_{c \in C}$. We assume that the slots are numbered according to a linear **order of precedence** $\rhd_c$. Given two slots $\sigma, \sigma' \in \Sigma_c$, $\sigma \rhd_c \sigma'$ means that slot σ is to be filled before the slot σ' whenever possible. For each course c, we assume that slots in Σ_c are ordered in such a way that $\sigma^1 \rhd_c \sigma^2 \rhd_c \cdots \rhd_c \sigma^{q_c}$. Let $S' \subseteq S$. The choice of school c from S', denoted by $Ch_c(S')$, is obtained as follows: slots at school c are filled one at a time following

[10]Given a set X, by 2^X we denote the set of the subsets of X.

[11]The condition has been previously studied as "irrelevance of rejected contracts" in Aygün and Sönmez (2013) for models of allocation with contracts and as "irrelevance of rejected items" in Alva (2018) for general choice models.

the order of precedence. The highest-priority acceptable student in S' under $\succ_{\sigma^1}$, for example, student s^1, is chosen for slot σ^1 of school c; the highest-priority acceptable student in $S' \setminus \{s^1\}$ under $\succ_{\sigma^1}$, for example, student s^2, is chosen for slot σ^2 of school c, and so on. The choice function Ch_c satisfies substitutability (see Kominers and Sönmez, 2016; Hatfield and Kominers, 2017; and Chambers and Yenmez, 2018) and irrelevance of rejected students. A slot-specific priority profile is a tuple $\left(q, \left(\Sigma_c, (\succ_\sigma)_{\sigma\in\Sigma_c}, \rhd_c\right)_{c\in C}\right)$.

Each student $s \in S$ has a strict preference relation P_s over the set of subsets of C, 2^C. For each $C' \subseteq C$ and each $s \in S$, we denote by $Ch_s(C')$ the **choice set** of student s, which is her favorite combination of courses among the ones belonging to C'. Formally, $Ch_s(C', P_s) = \max_{P_s}\{D \mid D \subseteq C'\}$. When there is not ambiguity about P_s, we write $Ch_s(S')$ instead of $Ch_s(S', P_s)$. A subset of courses $C' \subseteq C$ is not acceptable to student s when $\emptyset P_s C'$. We assume that the choice set induced by each P_s is **substitutable** as previously defined for priorities. Let $\mathcal{P}$ be the set of substitutable preferences on 2^C. A more restrictive condition is responsiveness. We say that P_s is **responsive** (see Roth, 1985), with **supply** q_s if, for each $C' \subseteq C$ and for all $c, c' \in C \setminus C'$, the following holds: (1) if $|C'| < q_s$, then $C' \cup \{c\} P_s C' \cup \{c'\}$ if and only if $\{c\} P_s \{c'\}$, (2) if $|C'| < q_s$, then $C' \cup \{c\} P_s C'$ if and only if $\{c\} P_s \emptyset$, and (3) if $|C'| > q_s$, then $\emptyset P_s C'$.

For each $S' \subseteq S$, set $P_{S'} = (P_s)_{s\in S'}$. For each $s \in S$, set $P_{-s} = P_{S\setminus\{s\}}$. Given a preference relation P on 2^C, the restriction of P to $C' \subseteq C$, denoted by $P_{|C'}$, is a preference that ranks all subsets in $2^{C'}$ as P does and ranks all other subsets of courses as not acceptable. Formally, $P_{|C'}$ is such that, for all $Q, T \subseteq C'$, $QP_{|C'}T$ if and only if QPT and for all $Q \nsubseteq C'$, $\emptyset P_{|C'} Q$.

An **allocation** is a function $\mu : C \cup S \to 2^C \cup 2^S$ such that, for each $s \in S$ and each $c \in C$, $\mu(s) \in 2^C$, $\mu(c) \in 2^S$ and $c \in \mu(s)$ if and only if $s \in \mu(c)$. The set of all allocations is denoted by $\mathcal{M}$. Allocation μ is **individually rational** for $x \in C \cup S$ if $Ch(\mu(x)) = \mu(x)$. Allocation μ is **blocked** by a pair $(c, s) \in C \times S$ if $s \notin \mu(c)$, $c \in Ch_s(\mu(s) \cup \{c\})$, and $s \in Ch_c(\mu(c) \cup \{s\})$. Finally, an allocation μ is **stable** for (S, C, P_S, Ch_C) if it is individually rational for all $x \in C \cup S$ and there exists no pair blocking it. If P_S and Ch_C are substitutable and Ch_C satisfies irrelevance of rejected students, then a stable allocation exists (see Echenique and Oviedo, 2006).

A **mechanism** is a function φ that associates an allocation to every preference profile for students, $P = (P_s)_{s\in S}$, $\varphi : \mathcal{P}^{|S|} \to \mathcal{M}$. A mechanism is **stable** if $\varphi(P)$ is a stable allocation for each P. A mechanism is **strategy-proof** if $\varphi(P) R_s \varphi(P'_s, P_{-s})$ for each P, $s \in S$, and P'_s, where R_s denotes the weak preferences associated to P_s. Given a priority profile Ch_C and a preference profile $P \in \mathcal{P}^{|S|}$, a mechanism φ induces a normal form game $\mathcal{G}(P) = \left(S, \mathcal{P}^{|S|}, \varphi, P\right)$, where S is the set of players, $\mathcal{P}^{|S|}$ is the cross-product of students' strategy spaces, φ is the outcome function, and P is the profile of student preferences. Let $\Phi : \mathcal{P}^{|S|} \rightrightarrows \mathcal{M}$ be a correspondence. We say that φ **implements** Φ in **Nash equilibrium** if, for each $P \in \mathcal{P}^{|S|}$, the set of Nash equilibria of $\mathcal{G}(P) = \left(S, \mathcal{P}^{|S|}, \varphi, P\right)$, $NE(P)$ coincides with $\Phi(P)$. We say that φ **implements** Φ in **undominated Nash equilibrium** (UNE) if, for each $P \in \mathcal{P}^{|S|}$, the set of undominated Nash equilibria of $\mathcal{G}(P) = \left(S, \mathcal{P}^{|S|}, \varphi, P\right)$, $UNE(P)$ coincides with $\Phi(P)$.

3 One-shot mechanisms

In this section, we characterize preference domains in which the SO, and IA mechanisms implement stable allocations in the course allocation problem. Then we present the CA mechanism that extends the preference and priority domains in which we can implement the set of stable allocations.

3.1 The student optimal stable mechanism

Given priorities, the student optimal stable allocation associate to each substitutable profile of preferences $P_S = (P_s)_{s \in S}$ is the stable allocation μ which is optimal for all students. This is the stable allocation such that $Ch_s(\mu(s) \cup \nu(s)) = \mu(s)$ for all $s \in S$ and all stable allocations ν. Given preferences P, we denote by $SO(P)$ the student optimal stable allocation. In the multi-unit assignment case, the student optimal stable mechanism is not strategy-proof. Furthermore, the SO mechanism yields unstable allocations as NE outcomes.

Example 1 *Let $S = \{s_1, s_2, s_3\}$ and let $C = \{c_1, c_2\}$. Let preferences and priorities be as follows:*
$P_{s_1} : \{c_1, c_2\}, \{c_2\}, \{c_1\}$; $P_{s_2} : \{c_1\} \{c_2\}$; $P_{c_1} : \{s_1\}, \{s_2\}$; $P_{c_2} : \{s_2\}, \{s_1\}$.

There exists a unique stable allocation in the market (C, S, P), μ in which $\mu(s_1) = \{c_1\}$, $\mu(s_2) = \{c_2\}$, $\mu(s_3) = \emptyset$.

Let $P'_{s_1} : \{c_2\}, \{c_1\}$; $P'_{s_2} : \{c_1\}, \{c_2\}$. Strategy profile $P' = \left(P'_{s_i}\right)_{i=1,2}$ is a NE of the game induced by the student optimal stable allocation yielding allocation ν, in which $\nu(s_1) = \{c_2\}$, $\nu(s_2) = \{c_1\}$, $\nu(s_3) = \emptyset$, which is unstable since it is blocked by (c_2, s_1).

In the Example 1, the NE outcome μ is Pareto optimal and Pareto dominates the student optimal stable allocation. This is not always the case, as we show in the Example 2.

Example 2 *Let $S = \{s_1, s_2, s_3, s_4\}$ and let $C = \{c_1, c_2, c_3, c_4\}$. Let preferences and priorities be as follows:*

$P_{s_1} : \{c_1, c_2\}, \{c_2\}, \{c_1\}$; $P_{s_2} : \{c_1\}, \{c_2\}$; $P_{s_3} : \{c_4\}, \{c_3\}$; $P_{s_4} : \{c_3\}, \{c_4\}$; $P_{c_1} : \{s_1\}, \{s_2\}$; $P_{c_2} : \{s_2\}, \{s_1\}$; $P_{c_3} : \{s_3\}, \{s_4\}$; $P_{c_4} : \{s_4\}, \{s_3\}$.

There are two stable allocations μ and ν in which: $\mu(s_1) = \{c_1\}$, $\mu(s_2) = \{c_2\}$, $\mu(s_3) = \{c_4\}$, $\mu(s_4) = \{c_3\}$, $\rho(s_1) = \{c_1\}$, $\rho(s_2) = \{c_2\}$, $\rho(s_3) = \{c_3\}$, $\rho(s_3) = \{c_3\}$, $\rho(s_4) = \{c_4\}$.

Let $P'_{s_1} : \{c_2\}, \{c_1\}$; $P'_{s_2} : \{c_1\}, \{c_2\}$; $P'_{s_3} : \{c_3\}$; $P'_{s_4} : \{c_4\}$. Strategy profile $P' = \left(P'_{s_i}\right)_{i=1,2,3}$ is a NE of the game induced by the student optimal stable allocation yielding allocation ν, in which $\nu(s_1) = \{c_2\}$, $\nu(s_2) = \{c_1\}$, $\nu(s_3) = \{c_3\}$, $\nu(s_4) = \{c_4\}$, which is unstable since it is blocked by (c_1, s_1). It is also not Pareto optimal since it is dominated by τ, in which $\tau(s_1) = \{c_2\}$, $\tau(s_2) = \{c_1\}$, $\tau(s_3) = \{c_3\}$, $\tau(s_4) = \{c_4\}$. In addition, it does not Pareto dominate stable allocation μ.

Notice that, in Examples 1 and 2 there is a cycle which helps sustain the NE yielding ν as an outcome: $\{s_1\} P_{c_1} \{s_2\} P_{c_2} \{s_1\}$. Due to this cycle if student s_1 ranked $\{c_1, c_2\}$ first, she would block the admission of student s_2 to course c_1. Student s_2, having lost course c_1 would block the admission of student s_1 to course c_2. Thus, this kind of deviation would not be profitable to s_1.

In general, in multi-unit assignment models, a cycle can be formed with the lowest ranked students that can be admitted to two courses. If the priorities satisfy essential homogeneity (see Kojima, 2013) this potential cycle is prevented.

Definition 1 *Priorities $(P_c)_{c \in C}$ satisfy essential homogeneity if there is no $c_1, c_2 \in C$ such that:*

- $\{s_1\} P_{c_1} \{s_2\}$ *and* $\{s_2\} P_{c_2} \{s_1\}$*;*
- *there exist* $S_{c_1}, S_{c_1} \subseteq S \setminus \{s_1, s_2\}$ *such that* $|S_{c_1}| = q_{c_1} - 1$*,* $|S_{c_2}| = q_{c_2} - 1$*,* $\{s\} P_{c_1} \{s_2\}$ *for each* $s \in S_{c_1}$*, and* $\{s\} P_{c_2} \{s_1\}$ *for each* $s \in S_{c_2}$*.*

Essential homogeneity allows variation in priorities on the top q_c students for course c. Those students are admitted to course c whenever they apply, so their relative ranking does affect the NE outcome.

Proposition 1 *The SO mechanism implements the set of stable allocations in NE if preferences are substitutable and priorities are responsive and essentially homogeneous.*

Kojima (2013) proves that under essential homogeneity the SO mechanism is strategy-proof if and only if the priorities of the courses are responsive and essentially homogeneous. It follows from Kojima (2013) Theorem 1 and our Proposition 1 that to require stability to NE outcomes of the SO mechanism is equivalent to impose strategy-proofness.

3.2 The immediate acceptance mechanism

The many-to-many version of the classic IA mechanism goes as follows. First, each student submits her preferences. In the first stage, the favorite set of acceptable courses of each student is considered. Among the students claiming a course, those with the highest priorities for any given course are assigned to it. At the end of this stage, all students assigned to at least one course and all assigned seats are removed. At the n^{th} stage only the n^{th} choice in the preference list of the remaining students is considered. We repeat the procedure until no more seats or students remains.

Let $P = (P_s)_{s \in S}$ be a preference profile. Let $s \in S$ and let r be an integer such that $1 \leq r \leq 2^{|S|}$, and let $C^r_{P_s}$ be the r^{th} ranked acceptable course schedule according to P_s, if one exists. Let $C^r_{P_s}$ be empty otherwise.

Given a priority profile $(P_c)_{c \in C}$ and a preference profile $(P_s)_{s \in S}$, the following procedure describes the **immediate acceptance mechanism.**

Step 1: For each course c, let S_c^1 be the set of students who selected c among their first choices. Formally, $S_c^1 = \left\{s \in S \mid c \in C_{P_s}^1\right\}$. Define $\mu^1(c) = Ch_c(S_c^1)$. Every student in $\mu^1(c)$ is definitively enrolled in course c. Every student $s \in \bigcup_{c \in C} \mu^1(c)$ and every student s such that $C_{P_s}^1 = \emptyset$ is removed. Set $T^1 = S$. Let T^2 be the set of remaining students.

Step r, r $r \geq 2$: Only the r^{th} choices, among acceptable courses, of the students in T^r are considered. For each course c let $S_c^r = \left\{s \in T^r \mid c \in C_{P_s}^r\right\}$ be the set of students in T^r who selected c among their r^{th} choices. Let $\mu^r(c) = \max_{P_c}\{\mu^{r-1}(c) \cup S' \mid S' \subseteq S_c^r\}$. Every student $s \in \bigcup_{c \in C} \mu^r(c)$ and every student s such that $C_{P_s}^r = \emptyset$ is removed. Let T^{r+1} be the set of remaining students.

The procedure stops when all students have been removed. Formally, it stops at $r^* = \min\{r \mid T^{r+1} = \emptyset\}$. Let $IA(P) = \mu^{r^*}$ be the outcome. Notice that a student never loses the seat at a course she has been assigned to at some step of the mechanism, but she can be moved to seats of different precedence along the mechanism if the priorities are slot-specific.

Under substitutable preferences, all stable allocations are Nash equilibrium outcomes of the IA mechanism. However, not all Nash equilibrium outcomes are stable allocations. This is because not all outcomes of the mechanism are individually rational for courses, as we show in the Example 3.

Example 3 *Let $C = \{c_1, c_2\}$ and $S = \{s_1, s_2, s_3, s_4\}$. Each student wants to enroll in exactly one course. The maximal number of students c_1 can enroll is three, but the ideal number is two. Let preferences and priorities be as follows: $P_{s_1} : \{c_2\}, \{c_1\}$; $P_{s_2} : \{c_1\}$; $P_{s_3} : \{c_1\}$; $P_{s_4} : \{c_2\}$; $P_{c_1} : \{s_1, s_3\}, \{s_1, s_2, s_3\}, \{s_2, s_3\}, \{s_1, s_2\}, \{s_1\}, \{s_3\}, \{s_2\}$; $P_{c_2} : \{s_4\}, \{s_1\}, \{s_2\}, \{s_3\}$.*
All priorities are substitutable. Truth telling results in allocation μ, where $\mu(c_1) = \{s_1, s_2, s_3\}$ and $\mu(c_2) = \{s_4\}$, which is not individually rational because $Ch_{c_1}(\mu(c_1)) \neq \mu(c_1)$. However, truth telling is a Nash equilibrium of the IA mechanism because any student but s_1 is assigned to her preferred course, and s_1 has no profitable deviations.

The instability of NE allocations under the IA mechanism comes from the fact that acceptances are definitive. In the Example 3, when s_1's application arrives, course c_1's priorities prescribe the rejection of the student's application, but the IA mechanism does not allow it.

When priorities are slot-specific, this situation is not a concern because all outcomes of the IA mechanism are individually rational for the courses. In Lemma 2 we show that each student can obtain any course schedule that can be an outcome of the mechanism, ceteris paribus, by ranking it in first place.

Lemma 1 *Let $P = (P_s)_{s \in S}$ be a preference profile for students and let $\mu = IA(P)$. For each $s \in S$ and $C' \subseteq \mu(s)$ $C' = IA\left(P_{s|C'}, P_{-s}\right)$.*

This result allows us to prove Theorem 1.

Theorem 1 *The IA mechanism implements the set of stable allocation in NE if preferences are substitutable and priorities are slot-specific.*

The equilibrium strategies defined in part (ii) of the proof of Theorem 1 are undominated. Thus, we obtain Corollary 1.

Corollary 1 *The IA mechanism implements the set of stable allocation in UNE if preferences are substitutable and priorities are slot-specific.*

3.3 The conditional acceptance mechanism

The results in Subsections 3.1 and 3.2 show that the IA mechanism implements the set of stable allocation in richer preference and priority domains than the SO mechanism. The conditional acceptance mechanism extends this implementation domain. It exploits the structure of the IA mechanism to provide students with incentives to acquire and exploit the information about courses priorities and students' preferences. In the CA mechanism, the message space for each student is the set of preference profiles on course schedules. In the first stage, only the schedule that each student presents as her best is considered. Among the students demanding a given course, the group with the highest priority is chosen. At the end of this stage, all students assigned to at least one course and those students not demanding any course are removed. At the r^{th} step of the mechanism, only the r^{th} choice in the preference list of the remaining students is considered. Each course considers the students already assigned to it and the new students claiming a seat and allocate seats to the subset with the highest priority. All students who have been assigned at least one course at this stage are removed, together with the students not demanding any course. The mechanism stops when all students have been removed.

Let $P = (P_s)_{s\in S}$ be a preference profile. Let $s \in S$ and let r be an integer such that $1 \leq r \leq 2^{|S|}$, and let $C^r_{P_s}$ be the r^{th} ranked acceptable course schedule according to P_s, if one exists. Let $C^r_{P_s}$ be empty otherwise.

Given a priority profile $(P_c)_{c\in C}$ and a preference profile for students $(P_s)_{s\in S}$, the following procedure describes the **conditional acceptance mechanism**.

Step 1: Only the top choices of the students among acceptable courses are considered. For each course c, let S^1_c be the set of students who selected c among their first choices. Formally, $S^1_c = \left\{s \in S \mid c \in C^1_{P_s}\right\}$. Define $\mu^1(c) = Ch_c(S^1_c)$. Every student in $\mu^1(c)$ is enrolled in course c. Every student $s \in \bigcup_{c\in C} \mu^1(c)$ and very student s such that $C^1_{P_s} = \emptyset$ is removed. Set $T^1 = S$. Let T^2 be the set of remaining students.

Step r, $r \geq 2$: Only the r^{th} choices of students in T^r are considered. For each course c, let $S^r_c = \mu^{r-1}(c) \cup \left\{s \in T^r \mid c \in C^r_{P_s}\right\}$ be the set of students enrolled at c at the end of stage r and of the remaining students ranking a set containing c in the r^{th} place. Let $\mu^r(c) = Ch_c(S^r_c)$.

Every student $s \in \bigcup_{c \in C} \mu^r(c)$ and every student s such that $C^r_{P_s} = \emptyset$ is removed. Let T^{r+1} be the set of remaining students.

The procedure stops when all students have been removed. Formally, it stops at $r^* = \min\{r \mid T^{r+1} = \emptyset\}$. Let $CA(P) = \mu^{r^*}$ be the outcome. Notice that the mechanism produces an outcome even when preferences are not substitutable.

The CA mechanism has characteristics of both the IA and SO mechanisms. Students are accepted by courses at most once like in the IA mechanism and courses can replace previously accepted students with new ones.

We can think of an alternative version of the CA mechanism where students are asked each step for a unique course schedule. In this alternative version, our results follow with minor adaptation on the proofs. Lemma 2 illustrates that in the CA mechanism a student can obtain any of the possible outcomes of the mechanism by ranking them in the first place.

Lemma 2 *Let $P = (P_s)_{s \in S}$ be a preference profile for students and let $\mu = CA(P)$. If the priorities are substitutable, for each $s \in S$ and $C' \subseteq \mu(s)$, $C' = CA\left(P_{s|C'}, P_{-s}\right)(s)$.*

An immediate implication of Lemma 2 is that each student can obtain her favorite outcome of the mechanism by listing a a single course schedule.

The possibility to adapt the CA mechanism for students to declare a course schedule every round and the fact, that students can submit just one course schedule shows that the CA mechanism simplifies the problem of preference elicitation.

We present our main result in Theorem 2.

Theorem 2 *The CA mechanism implements the set of stable allocations in NE if preferences and priorities are substitutable.*

In the CA mechanism, unstable allocations are ruled out by the strategic behavior of the students. From Lemma 2, it follows that if pair (c, s) blocks an outcome allocation μ, $P_{s|Ch_s(\mu(s)\cup\{c\})}$ is a profitable deviation for s.

Notice that the equilibrium strategies defined in part (ii) of the proof of Theorem 2 are undominated. Thus, we present the following result.

Corollary 2 *The CA mechanism implements the set of stable allocation in UNE if preferences and priorities are substitutable.*

4 The extended conditional acceptance mechanism

The incentives used in the CA mechanism to induce stable allocations have a cost in case of out-of-equilibrium play. This is because any student who loses a course at any stage of the game will not be able to register an alternative course. In general, the problem of students with incomplete course schedule is solved either by an administrative allocation or by a new allocation procedure in which they can complete their course schedule. We call this stage post-allocation adjustment. Usually, in post-allocation adjustments, students are allowed to drop courses and register new ones. Example 4 shows how the addition of a post-allocation adjustment can distort the implementation of stable allocations.

Example 4 *Let $S = \{s_1, s_2, s_3, s_4\}$ and let $C = \{c_1, c_2, c_3, c_4\}$. Students can apply for more than one course. Let preferences and priorities be as follows:*
$P_{s_1} : \{c_1\}, \{c_3\}, \{c_2\}, \{c_4\}$; $P_{s_2} : \{c_4\}, \{c_1\}, \{c_2\}$; $P_{s_3} : \{c_4\}, \{c_3\}, \{c_1\}, \{c_2\}$; $P_{s_4} : \{c_4\}$;
$P_{c_1} : \{s_4\}, \{s_3\}, \{s_2\}, \{s_1\}$; $P_{c_2} : \{s_4\}, \{s_3\}, \{s_2\}, \{s_1\}$; $P_{c_3} : \{s_1\}, \{s_2\}, \{s_3\}$; $P_{c_4} : \{s_1\}, \{s_4\}, \{s_2\}, \{s_3\}$.

There is a unique stable allocation in the market (C, S, P), μ in which $\mu(s_1) = \{c_3\}$, $\mu(s_2) = \{c_2\}$, $\mu(s_3) = \{c_1\}$, $\mu(s_4) = \{c_4\}$. Notice that the strategy profile in which each student declares her course under μ as her unique acceptable course is a NE of the one-shot game for each student both under SO and under CA mechanisms.

Now assume that seats are assigned employing the CA mechanism and there is a post-allocation adjustment where the students can drop courses and register new ones with empty seats. Those empty seats are assigned employing the CA mechanism as well. The game has a $SPNE$ that does not have μ as an outcome: students play $P'_{s_1} : \{c_4, c_3\}$; $P'_{s_2} : \{c_2\}$; $P'_{s_3} : \{c_1\}$; $P'_{s_4} : \{c_4\}$ in the first stage. This profile gives the allocation ν: $\nu(s_1) = \{c_4, c_3\}$, $\nu(s_2) = \{c_2\}$, $\nu(s_3) = \{c_1\}$; $\nu(s_4) = \{\emptyset\}$. In the second stage, all students play their best response. This results in student s_1 dropping course c_4, students s_2 and student s_3 ranking c_2 and c_1 first, respectively, and agent s_4 not ranking any course. The described strategy constitutes a $SPNE$ yielding ν' as an outcome, in which $\nu'(s_1) = \{c_3\}$, $\nu'(s_2) = \{c_2\}$, $\nu'(s_3) = \{c_1\}$; $\nu'(s_4) = \emptyset$, which is wasteful and thus unstable in (C, S, P). Matching ν' is an equilibrium outcome also if CA repeats multiple times and students are allowed to drop courses and register only to empty seats. It is indeed easy to construct a $SPNE$ in which students play as above in the first stage and in which student s_1 drops course c_4 only at the last stage, preventing s_4 to register any course.

If the post-allocation adjustment consists in allowing students to trade courses, there are two additional $SPNE$. The first equilibrium consists in students playing $P'_{s_1} : \{c_3, c_4\}$; $P'_{s_2} : \{c_2\}$; $P'_{s_3} : \{c_1\}$, $P'_{s_4} : \{c_4\}$ in the first stage. Then s_1 and s_3 trade c_4 for c_1 and s_1 drops c_3. The second equilibrium consists in playing $P'_{s_1} : \{c_4\}$; $P'_{s_2} : \{c_2\}$; $P'_{s_3} : \{c_1\}$; $P'_{s_4} : \{c_4\}$ in the first stage. Then s_1 and s_3 trade c_4 for c_1. In both cases the outcome is matching ν'', in which $\nu''(s_1) = \{c_1\}$, $\nu''(s_2) = \{c_2\}$, $\nu''(s_3) = \{c_4\}$; $\nu''(s_4) = \emptyset$ which is unstable.

We introduce the extended conditional acceptance mechanism, a repeated version of the CA mechanism. The ECA models the post-allocation adjustment by repeating the CA mechanism as many times as needed with the students that still have courses to register and the courses that remain with empty seats. The assignment made at each CA stage is definitive and students cannot drop any of the courses they have obtained.

Given a priority profile $(P_c)_{c\in C}$ and a preference profile for students $(P_s)_{s\in S}$, the following procedure describes the **extended conditional acceptance mechanism**.

Step 1: Students submit a preference profile $P^1 = (P_s^1)_{s\in S}$.

- Set $\mu^1(P^1) = CA(P^1)$. For each $s \in S$,
  set $\mathcal{P}_s^1 = \{P_s \mid c \in A_s(P_s^1) \Rightarrow c \notin A_s(P_s)\}$.

Step r+1: Students submit a preference profile $P^{r+1} = (P_s^{r+1})_{s\in S} \in \prod_{s\in S} \mathcal{P}_s^r$. Set $A_s^r\left((P_s^i)_{i=1}^r\right) = \bigcup_{i=1}^r A_s(P_s^i)$. Define priorities P_c^{r+1} on S as follows. For $S', S'' \subseteq S \setminus \mu^r(c)$, $S' P_c^{r+1} S''$ if and only if $\mu^r(c) \cup S' P_c \mu^r(c) \cup C''$. Define $CA^{r+1}(P^{r+1})$ as the result of the conditional acceptance mechanism with priorities $(P_c^{r+1})_{c\in C}$ under profile of preferences P^{r+1}.

- Set $\mu^{r+1}(P^{r+1}) = \mu^r \cup CA^r(P^{r+1})$. For each $s \in S$,
  set $\mathcal{P}_s^{r+1} = \left\{P_s \mid c \in A_s^r\left((P_s^i)_{i=1}^r\right) \Rightarrow c \notin A_s(P^i)\right\}$.

The procedure stops at the lowest r^* such that $\mathcal{P}_s^{r^*} = \mathcal{P}_s^{r^*+1}$ for all $s \in S$. Set $ECA = \mu^{r^*}$. The procedure ends in a finite time since $\mathcal{P}_s^{r+1} \subseteq \mathcal{P}_s^r$, thus $r^* \leq |S|\,|C| = R^*$. Given a profile of preferences, the ECA mechanism induces an extensive game of complete information (see, among others, Maschler et al. 2013). A $SPNE$ is a profile of strategies that induce a NE in each subgame. We focus on the pure strategy $SPNE$ of the games induced by the ECA mechanism. Notice that the extensive form game induced by the extended conditional acceptance mechanism, can be stopped after R^* stages. Let $P = \left(P_s^h\right)_{s\in S}^{h\in H}$ be a strategy profile for students in which H is the set of non-terminal histories of the game induced by the ECA mechanism, let $ECA(P)$ be the outcome allocation when students play profile P, and let $ECA_s(P) = ECA(P)(s)$ for all $s \in S$.

Lemma 3 *Let $P = \left(P_s^h\right)_{s\in S}^{h\in H}$ be a strategy profile for students and let $\mu = ECA(P)$. Let $s\in S$ and let $C' \subseteq ECA_s(P)$. Let h_s be the Stage 1 history in which student s moves. If the priorities are slot-specific, for each $s \in S$, $C' = ECA_s\left(\left(P'^h{}_s, P_{-s}^h\right)^{h\in H}\right)$, in which $P'^{h_s}{}_s$is responsive and ranks C' acceptable and at the top of the list and $P'^h{}_s = \emptyset$ if h follows h_s, $\bar{P}_s^h = P_s^h$ for all h not following h_s, and $\bar{P}_s^h = \emptyset$ for all h following h_s.*

Theorem 3 shows that the ECA mechanism implements the set of stable allocation when both preferences and priorities are slot-specific.[12]

Theorem 3 *The ECA mechanism implements the set of stable allocation in $SPNE$ if preferences and priorities are slot-specific.*

[12]The definition of slot-specific preferences is analogous to the definition of slot-specific priorities.

Thus, the ECA mechanism maintains the strategic properties of the CA mechanism. If a student employs a non-equilibrium strategy, the penalty will be softened by the participation to a post-allocation adjustment. The addition of a post-allocation adjustment does not distort the strategic properties of the CA mechanism if preferences and priorities are slot-specific.

5 Simpler environments

Both the CA and IA mechanisms implement stable allocation under substitutable and slot-specific priorities, respectively. However, the complexity of the strategy space might hinder their practical implementation (see Budish et al., 2017). We concentrate on slot-specific preferences. Analogously to slot-specific priorities, slot-specific preferences generalize the assumption of responsive preferences, which is standard in the course allocation literature (see Sönmez and Ünver 2010; Budish, and Cantillon, 2012; or Aziz, et al., 2019).

We introduce two mechanisms derived from the CA and IA mechanisms where students a list of courses ordered by preference. Let $s \in S$. Assume that message for student s is $m_s = \left(\Sigma_s, (\succ_\sigma)_{\sigma \in \Sigma_s}, \rhd_s\right)$. Let M_s be the set of messages for student s. Given message m_s let $P_s = P_s(m_s)$. For each $s \in S$, let $m_s \in M_s$ and let $P_s = P_s(m_s)$ be preferences that rationalize m_s (see Theorem 1 in Alva, 2018). Given a priority profile $(Ch_c)_{c \in C}$ and a profile of messages $(m_s)_{s \in S}$, the simplified the CA mechanism (SCA) is defined by the following outcome function $SCA\left((m_s)_{s \in S}\right) = CA\left(P_s(m_s)_{s \in S}\right)$. In words, in the SCA, students play the game induced by the corresponding mechanism with preferences that rationalize the message of each student.[13]

Proposition 2 *Assume that the preferences are slot-specific and priorities are substitutable. The SCA implements the set of stable allocations in Nash equilibrium.*

We can also define a simplified version of the IA mechanism as follows. Given a priority profile $(Ch_c)_{c \in C}$ and $(\geq_s, q_s)_{s \in S}$, the simplified IA (SIA) mechanism is defined by the following outcome function $SIA\left((m_s)_{s \in S}\right) = IA\left((P_s(m_s))_{s \in S}\right)$.

Proposition 3 *Assume that preferences and priorities are slot-specific. The SIA implements the set of stable allocations in Nash equilibrium.*

The proof of Proposition 3 is similar to the proof of Proposition 2 and thus omitted.

The version of ECA under slot-specific preferences or simplified ECA ($SECA$) is natural after defining the SCA and Proposition 4 is a direct consequence of Theorem 3.

Proposition 4 *Assume that the preferences and priorities are slot-specific. The $SECA$ mechanism implements the set of stable allocation in $SPNE$.*

[13] If the preferences can be assumed as responsive which is if $\succ_\sigma = \succ_{\sigma'}$ for all $\sigma, \sigma' \in \Sigma_s$, for all $s \in S$, the message can be simplified to $(q_s, >_s)$ in which q_s is the demand of courses of student s and $>_s$ is a ranking of individual course, in an obvious way.

6 Conclusions

In this paper, we present a mechanism to allocate courses to students based on of preferences and priorities, the conditional acceptance mechanism or CA. Under substitutable preferences and priorities, the CA mechanism implements the set of stable allocation in Nash equilibrium and in undominated Nash equilibrium. The mechanism produces allocations that are fair, and its practical implementation is not computationally demanding. The CA mechanism is based on the IA mechanism but allows courses to reject students tentatively accepted to preserve individual rationality. This makes our new mechanism a hybrid between the IA and SO mechanism. out-of-equilibrium play in the CA mechanism is wasteful. To reduce the cost that incentives impose we introduce the ECA mechanism. This mechanism is based on the repetition of the CA mechanism and implements the set of stable allocation in $SPNE$ under slot-specific preferences and priorities. Our findings suggest that the post-allocation adjustment mechanism should not allow students to drop courses.

We conclude our analysis studying the design possibilities in markets with less complex preferences. Overall, our results proof the feasibility to design a mechanism that motivates students to acquire information and use it strategically to overcome the intrinsic difficulties present in the course allocation problem. Our findings support the design features present in mechanisms used in practice and point to the potential weaknesses of others.

Appendix: Proofs

Proof of the results in Subsection 3.1

Consider substitutable profiles of preferences $P = (P_s)_{s \in S}$ and priorities $P_C = (P_c)_{c \in C}$. Given $D = (D_s)_{s \in S} \in 2^{C|S|}$, define $D_c = \bigcup_{h \in D_{s'}} \{s'\}$, for each $c \in C$. For each $s \in S$, define $F_s(D, P) = \{c \mid \exists h \in H, s \in C_c(D_c \cup \{s\})\}$. For each $D = (C_s)_{s \in S} \in 2^S$, let $b_s(D, P) = Ch_s(F_s(D, P), P_s)$ and let $R_s(D, P) = S \setminus F_s(D)$. Finally, define $B_s(D, P) = b_s(D, P) \cup R_s(D, P)$. Set $D_s^0 = \emptyset$ for all $s \in S$ and define $D_s^{t+1}(P) = B_s\left((D_s^t(P))_{s \in S}, P\right)$ for all $t \geq 1$. This sequence is a realization of the "cumulative offer algorithm". From Proposition 6 in Romero-Medina and Triossi (2022) there exists t, such that $D_s(P) = D_s^t = D_s^{t+1}$ for all $s \in S$ and $SO(P)(c) = C_c(D_s(P)_h)$ for all c.

Lemma 4 *Let $P_S = (P_s)_{s \in S}$ be a profile of substitutable preference, let $P_C = (P_c)_{c \in C}$ be a substitutable profile of priorities, and let $P = (P_S, P_C)$. Let $\mu = SO(P)$. If the priorities are substitutable, for each $s \in S$ and $C' \subseteq \mu(s)$, $C' = SO\left(P_{s|C'}, P_{-s}\right)(s)$.*

Proof. Both $P_{s|\mu(s)}$ and $P_{s|C'}$ are substitutable. First, notice that $\mu(s) = SO\left(P_{s|\mu(s)}, P_{-s}\right)(s)$. Let $\nu = SO\left(P_{s|C'}, P_{-s}\right)(s)$. Notice $C' \subseteq \nu(s)$. We complete the proof of the claim, showing by contradiction, that $\nu(s) \subseteq C'$. Assume there exists $c \in C' \setminus \nu(s)$.

Let $P' = \left(P_{s|C'}, P_{-s}\right)$. Let $\nu = SO\left(P_{s|C'}, P_{-s}\right)$. Let $D_s^t = D_s^{t+1}(P)$, let $F_s^t = F_s\left(\left(D_{s'}^t\right)_{s' \in S}, P\right)$ for all $t \geq 1$. Let $D'^t = D'^{t+1}_s(P')$, let $F'^t_c = F_s\left(\left(D_c^t\right), P'\right)$ for all $s \in S$ and all $t \geq 1$.

Since both $P_{s|\mu(s)}$ and $P_{s|C'}$ are substitutable, we have (see the proof of Proposition 6 in Romero-Medina and Triossi, 2022).

- $F_{s'}^{t+1} \subseteq F_{s'}^t$ and $F'^{t+1}_{s'} \subseteq F'^t_{s'}$ for all $t \geq 0$ and all $s' \in S$;
- $F_{s'}^t \subseteq F'^t_{s'}$ for all $t \geq 1$ and $s' \in S \setminus \{s\}$.

Thus, for t large enough, $\mu^t = \mu$ and $\nu^t = \nu$ and $Ch_{s'}\left(F_{s'}^t \cup F'^t_{s'}\right) = Ch_{s'}\left(F'^t_{s'}\right)$ for all $t \geq 1$ and $s' \in S \setminus \{s\}$. This implies $Ch_{s'}\left(\mu(s') \cup \nu(s')\right) = \nu(s')$. Let $c \in C' \subseteq \mu(s)$. If $c \in \nu(s') \setminus \mu(s)$ for some $s' \in S \setminus \{s\}$, then $c \in Ch_{s'}\left(\mu(s') \cup \{c\}\right)$. Since μ is stable $s' \notin C_c\left(\mu(c) \cup \{s'\}\right)$. Because $s' \in \nu(c)$ and because preferences are substitutable $s' \notin C_c\left(\mu(c) \cup \nu(c)\right)$. This implies $C_c\left(\mu(c) \cup \nu(c)\right) = \mu(c)$. Since $s \in \mu(c)$, $s \in Ch_c\left(\nu(c) \cup \{s\}\right)$. Since C' is individually rational for s, $c \in Ch_s\left(\nu(s) \cup \{c\}\right)$, thus pair (c, s) blocks ν, which yields a contradiction. ■

Proof of Proposition 1. The proof of the claim is in two parts. First, we prove that all NE outcomes are stable allocations and then we prove that any stable allocation is a NE outcome of the game induced by the SO mechanism.

(i) Let P^* be a NE of the game induced by the SO and let $\mu = SO(P^*)$ when the profile of preferences is $P = (P_s)_{s \in S}$ be the preference profile of the students. By definition, allocation μ is individually rational for each course. Next, we prove by contradiction that μ is individually rational for students. Assume $Ch_s(\mu(s)) \neq \mu(s)$ for some $s \in S$. Then, from Lemma 4 $P^*_{s|Ch_s(\mu(s))}$ is a profitable deviation for student s, which yields a contradiction.
Finally, we prove by contraction that there is no course-student pair blocking μ. Assume (c, s) blocks μ. It follows that P_s^* does not rank $Ch_s\left(\mu(s) \cup \{c\}\right)$ above $\mu(s)$. Now, consider the following preferences for student s: $P_s' = P_{|Ch_s(\mu(s) \cup \{c\})}$ and $P_s'' = P_{|\mu(s)}$. Let $P' = \left(P_s', P_{-s}^*\right)$ and let $P'' = \left(P_s', P_{-s}^*\right)$.

We first prove that $\mu = SO(P^*) = SO(P'')$. Notice that μ is individually rational in for P'' as well. Also, s is not part of any pair blocking μ in the market (C, S, P'') because $\mu(s)\, R_s'' C'$ for all $C' \in 2^C$. No other pair (c, s'), $s' \neq S$ blocks μ in (C, S, P'') because $P''_{-s} = P^*_{-s}$. It follows that μ is stable in the market (C, S, P''). We prove by contradiction $\mu = SO(P'')$. Let $\nu = SO(P'')$. Assume that $\nu(s')\, P''_{s'} \mu(s')$ for some s' and observe that $\nu(s) = \mu(s)$ because $\mu(s)\, R_s'' C'$ for all $C' \in 2^C$. Then, ν is not stable in the market (C, S, P^*). Since ν Pareto dominates for students μ, $\succ_C$ is essentially homogeneous, which yields a contradiction. Thus, $\mu = SO(P'')$. Since $SO(P^*)(s)\, R_s SO(P_s', P^*)(s) = SO(P')$, $SO(P')$ is stable in the market (C, S, P''), thus, μ Pareto dominates $SO(P')$. Since μ is not stable in the market (C, S, P''), because (c, s) blocks it, $\succ_C$ is essentially homogeneous, which yields a contradiction.

(ii) Let μ be a stable allocation. Consider the following strategy profile: $\left(P_{s|\mu(s)}\right)_{s \in S}$. The stability of μ implies that the strategy profile is a NE.

■

Proof of the results in Subsection 3.2

Proof of Lemma 1. Let $s \in C$ and let $C' \subseteq \mu(s)$. Let $c \in C'$, let $r(c)$ be the step of the IA mechanism at which c has been assigned to s, and let $r(c) = \min_{r \leq r^*} \{r \mid s \in \mu^r(c)\}$. Notice $r(c) = r(c')$ for all $c, c' \in C'$. Let σ be the seat to which s is assigned at stage $r(c)$. Thus, the student s is the highest priority student for seat σ among the ones in $\mu^r(c)$ who are not assigned to a seat preceding σ. Formally, for each $c \in C'$, $r \leq r(c)$, if $s' \in \mu^r(c)$ and $s' \succ_\sigma s$, there exists a seat $\sigma' \in \Sigma_c$, $\sigma' \rhd_c \sigma$ such that $s' \succ_{\sigma'} s$. Thus, $C' = IA\left(P_{s|C'}, P_{-s}\right)$. ■

Proof of Theorem 1. The proof of the claim is in two parts. First, we prove that all NE outcomes are stable allocations and then we prove that any stable allocation is a NE outcome of the game induced by the IA mechanism.

(i) Let P^* be a NE of $\left(S, \mathcal{P}^{|S|}, IA, P\right)$ and let $\mu = IA(P^*)$. As observed, μ is individually rational for each course. We prove by contradiction that μ is individually rational for students. Assume $Ch_s(\mu(s)) \neq \mu(s)$ for some $s \in S$. Let $P'_s = P_{s|Ch_s(\mu(s))}$, by Lemma 1: $IA\left(P'_s, P^*_{-s}\right)(s) = Ch_s(\mu(s))$. Thus, the deviation is profitable to s, which yields a contradiction. Assume that there exists a pair blocking μ, $(c, s) \in C \times S$. Let $P' = P_{s|Ch_s(\mu(s) \cup \{c\})}$. Because $s \in Ch_c(\mu(c) \cup \{s\})$, the deviation is profitable to s, which yields a contradiction. Because μ is individually rational and cannot be blocked by a pair, μ is stable.

(ii) Let μ be a stable allocation. For each s, let $P^*_s = P_{s|\mu(s)}$. Set $P^* = (P^*_s)_{s \in S}$. We have $IA(P^*) = \mu$. We prove by contradiction that P^* is a Nash equilibrium. Assume that $s \in S$ has a profitable deviation, P'_s, and let $\mu' = IA\left(P'_s, P^*_{-s}\right)$. Let $c \in Ch_s(\mu(s) \cup \mu'(s)) \setminus \mu(s)$. Because P_s is substitutable, $c \in Ch_s(\mu(s) \cup \{c\})$. Let $P''_s = P_{s|Ch_s(\mu(s) \cup \{c\})}$, then $IA\left(P''_s, P^*_{-s}\right)(s) = Ch_s(\mu(s) \cup \{c\})$. It follows that (c, s) blocks μ, which yields a contradiction.

■

Proof of the results in Subsection 3.3

Proof of Lemma 2. Let $s \in S$ and let $C' \subseteq \mu(s)$. Let $c \in C'$, let $r(c)$ be the step of the CA mechanism at which c has been assigned to s for the first time along the mechanism, and let $r(c) = \min_{r \leq r^*} \{r \mid s \in \mu^r(c)\}$. Notice that $r(c) = r(c')$ for all $c, c' \in \mu(s)$ and that $\mu^r(s) = \emptyset$ for all $r < r(c)$. The substitutability of Ch_c implies that $C^{r(c)}_{P_s} P_s C'$; otherwise, $s \in \mu^r(c)$ for some $r < r(c)$. For all $i \leq r(c)$, let $P^{r(c)}_s$ be a preference profile over 2^C such that $C^{r(c)}_{P^{r(c)}_s} = C'$, and for $j \neq r(c)$: $C^{r(c)}_{P^{r(c)}_s} = C^j_{P_s}$ if $C^j_{P_s} \neq C'$ and $C^j_{P^{r(c)}_s} = C^{r(c)}_{P_s}$ if $C^j_{P_s} = C'$. Notice that

$CA\left(P_s^{r(c)}, P_{-s}\right)(s) = C'$. For all i, $i < r(c)$, let P_s^i be a preference over 2^C such that $C^i_{P_s^i} = C'$, and for $j \neq i$: $C^j_{P_s^j} = C^{j+1}_{P_s}$ if $C^{j+1}_{P_s^{j+1}} \neq C'$ and $C^j_{P_s^j} = C^{j+1}_{P_s^{j+1}}$ if $C^{j+1}_{P_s^{j+1}} = C'$. Intuitively, each P_s^j lifts C' to place j in the preference of s without changing the ranking above the j^{th} place.

We prove by contradiction that $CA\left(P_s^{i-1}, P_{-s}\right)(s) = CA\left(P_s^i, P_{-s}\right)(s) = C'$ for all i, $1 \leq i < r(c)$. For every preference on 2^C, Q_s, let $\mu^j_{Q_s}$ be the outcome at the stage j of the CA mechanism when preferences are (Q_s, P_{-s}). Notice that $\mu^i_{P_s} = \mu^i_{P_s^j}$ for all i, j, $2 \leq i < j \leq r(c)$. Thus, to prove that $CA\left(P_s^{i-1}, P_{-s}\right)(s) = CA\left(P_s^i, P_{-s}\right)(s)$ for all $i < r(c)$, it suffices to show that $s \in Ch_c\left(\mu^{i-1}_{P_s}(c) \cup \left\{s \in S \mid c \in \bigcup_{s' \neq s} C^{i-1}_{P_{s'}}\right\} \cup \{s\}\right)$ for all i, $2 \leq i \leq r(c)$. By contradiction, assume that it is not the case, and let j be the maximum integer such that
$s \notin Ch_c\left(\mu^{j-1}_{P_s}(c) \cup \left\{s \in S \mid c \in \bigcup_{s' \neq s} C^{i-1}_{P_{s'}}\right\} \cup \{s\}\right)$ and
$s \in Ch_c\left(\mu^{j}_{P_s}(c) \cup \left\{s \in S \mid c \in \bigcup_{s' \neq s} C^{i-1}_{P_{s'}}\right\} \cup \{s\}\right)$. Because P_c is substitutable, $s \in Ch_c\left(\mu^j_{P_s}(c) \cup \{s\}\right)$.
The j^{th} step of the mechanism when preferences are (P_s^j, P_{-s}) yields $\mu^j_{P_s^j}(c)$ to course c. We have $s \notin Ch_c\left(\mu^{j-1}_{P_s}(c) \cup \left\{s \in S \mid c \in \bigcup_{s' \neq s} C^{i-1}_{P_{s'}}\right\} \cup \{s\}\right) = \mu^j_{P_s^j}(c)$. Irrelevance of rejected students implies that
$Ch_c\left(\mu^{j-1}_{P_s}(c) \cup \left\{s \in S \mid c \in \bigcup_{s' \neq s} C^{i-1}_{P_{s'}}\right\} \cup \{s\}\right) = Ch_c\left(\mu^j_{P_s}(c) \cup \{s\}\right) = \mu^j_{P_s}(c)$. In particular, $s \notin Ch_c\left(\mu^j_{P_s}(c) \cup \{s\}\right)$, which yields a contradiction. Thus $CA\left(P_s^1, P_{-s}\right)(s) = C'$. It follows that $CA\left(P_{s|C'}, P_{-s}\right)(s) = C'$, which concludes the proof. ■

Proof of Theorem 2. The proof of the claim is in two parts. First, we prove that all NE outcomes are stable allocations and then we prove that any stable allocation is a NE outcome of the game induced by the CA mechanism. Fix preferences $P = (P_s)_{s \in S}$.

(*i*) Let P^* be a NE of the game induced by the CA and let $\mu = CA(P^*)$. Allocation μ is individually rational for each course by definition. We prove by contradiction that μ is individually rational for students. Assume $Ch_s(\mu(s)) \neq \mu(s)$ for some $s \in S$. Let $P'_s = P_{s|Ch_s(\mu(s))}$. Because P_s is substitutable, P'_s is substitutable as well. By Lemma 2: $CA\left(P'_s, P^*_{-s}\right)(s) = Ch_s(\mu(s))$. Thus, the deviation is profitable to s, which yields a contradiction. Assume that there exists a pair blocking μ, $(c, s) \in C \times S$. Let $P' = P_{s|Ch_s(\mu(s) \cup \{c\})}$. Because $s \in Ch_c(\mu(c) \cup \{s\})$, the deviation is profitable to s, which yields a contradiction. It follows that allocation μ is individually rational and cannot be blocked by any course-student pair; thus, it is stable.

(*ii*) Let μ be a stable allocation. For each s, let $P^*_s = P_{s|\mu(s)}$. Set $P^* = (P^*_s)_{s \in S}$. We have $CA(P^*) = \mu$. We prove by contradiction that P^* is a Nash equilibrium. Assume that $s \in S$ has a profitable deviation, P'_s, and let $\mu' = CA\left(P'_s, P^*_{-s}\right)$. Let $c \in Ch_s(\mu(s) \cup \mu'(s)) \setminus \mu(s)$. Because P_s is substitutable, $c \in Ch_s(\mu(s) \cup \{c\})$. Let $P''_s = P_{s|Ch_s(\mu(s) \cup \{c\})}$, then $CA\left(P''_s, P^*_{-s}\right)(s) = Ch_s(\mu(s) \cup \{c\})$. It follows that (c, s) blocks μ, which yields a contradiction.

■

Proof of the results in Section 4

Proof of Lemma 3. Let $r \geq 1$. Consider a stage $r+1$ history, say h. Assume that in all histories following h student s does not receive any additional course. Along the path leading to h, let C^d be the set of course in C' that student s receives at stage $d \leq r+1$. Observe that $C' = \bigcup_{d=1}^{r+1} = C^d$ and $C^d \cap C^{d'} = \emptyset$ for $d \neq d'$. Let $P^d_{s'} = P^{h'}_{s'}$ for all stage d histories preceding h and for all $s' \in S$, $1 \leq d \leq r$, and let $P^d_{s'} = \emptyset$ for all $s' \in S$. Let $P^d = \left(P^d_{s'}\right)_{s' \in S}$ and let h^d be the stage d predecessor of h. Let μ^d be the allocations obtained at stage d along the path leading to h and let $\mu^0 = \emptyset$. We have $\mu^{r+1} = \mu^r \cup CA\left(P^r\right)$ and $\mu^r = \mu^{r-1} \cup CA\left(P^{r-1}\right)$. We have $C^r \subseteq CA\left(P^r\right)$. Let $\hat{P}_s$ be a responsive profile of preferences listing as acceptable only the courses in $CA\left(P^{r-1}\right) \cup C^r$ and ranking $CA\left(P^{r-1}\right) \cup C'$ as a most preferred subset of courses. Not all seats in the courses in $CA\left(P^{r-1}\right) \cup C^r$ had been assigned in the histories preceding h''. Also, they had not been ranked as acceptable by s in the histories preceding h''. It follows that $\hat{P}_s \in \mathcal{R}^{r-1}_{h''}$, where $\mathcal{R}^r_{h''}$ is the set of responsive preferences that does not list as acceptable courses ranked as acceptable in the histories preceding h^{r-1}. Consider following strategy for student s, $\left(\bar{P}^{h'}_s\right)^{h' \in H}$: $\bar{P}^{h^{r-1}}_s = \hat{P}_s$, $\bar{P}^{h'}_s = P^{h'}_s$ if $h' \neq h^{r-1}$ and h' is not a successor of h^{r-1}, $\bar{P}^{h'}_s = \emptyset$ if h' is a successor of h^{r-1}. Employing strategy $\left(\bar{P}^{h'}_s\right)^{h' \in H}_{s \in S}$, yields $\mu^r(s) \cup C^r$, from Theorem 2. Similarly, proceeding backwards for all $1 \leq d < r$ one stage at a time yields the claim. ■

Proof of Theorem 3. The proof of the claim is in two parts. First, we prove that all NE outcomes are stable allocations and then we prove that any stable allocation is a NE outcome of the game induced by the ECA mechanism. Let P be a profile of slot-specific preferences.

(i) Let P^* be a NE of the game induced by the ECA and let $\mu = ECA\left(P^*\right)$. Allocation μ is individually rational for each course by definition. We prove by contradiction that μ is individually rational for students. Assume $Ch_s\left(\mu\left(s\right)\right) \neq \mu\left(s\right)$ for some $s \in S$. Since $Ch_s\left(\mu\left(s\right)\right) \subseteq \mu\left(s\right)$, by Lemma 3 there exists a strategy profile which yields $Ch_s\left(\mu\left(s\right)\right)$ to student s. It follows that student s has a profitable deviation from the equilibrium strategy P^*, which yields a contradiction.

Next, we prove by contradiction that μ has no blocking pairs. Assume that there exists a pair blocking μ, $(c, s) \in C \times S$. Notice that student s has never ranked c as acceptable. Let r, be the first stage of the game, such that, along the equilibrium path, student s ranks as acceptable a course in $\mu\left(s\right) \setminus Ch_s\left(\mu\left(s\right) \cup \{c\}\right)$ and let μ^r be the initial allocation of that history, say $\bar{h}$. Then, consider, $\bar{P}_s = \left(\bar{P}^h_s\right)^{h \in H}$ in which $\bar{P}^{\bar{h}}_s = P_{|Ch_s(\mu(s) \cup \{c\}) \setminus \mu^r(s)}$, $\bar{P}^h_s = P^{*h}_s$ if $h \neq \bar{h}$ and h is not a successor of $\bar{h}$, $\bar{P}^{\bar{h}}_s = \emptyset$ if h is a successor of $\bar{h}$. We have $ECA_s\left(\bar{P}_s, P^*_{-s}\right) = Ch_s\left(\mu\left(s\right) \cup \{c\}\right)$, thus $\bar{P}^{\bar{h}}_s$ is a profitable deviation to s, which yields a contradiction.

(*ii*) Let μ be a stable allocation. Let h_1 be the initial history. For each s, let $P_s^{h_1 *} = P_{s|\mu(s)}$. Let h be any history belonging to stage $r+1$, in which $r \geq 1$. Let μ^h be the "provisional" allocation from which the corresponding subgame starts. Let $\left(P_c^h\right)_{c\in C} = \left(P_c^{r+1}\right)_{c\in C}$ be the "stage-priorities" at at stage $r+1$ for history h as in the definition of the ECA mechanism at stage. Let C_s^h be the set of courses to which student s has not applied to in the history preceding h. For all s in S, let P_s^h a strict order defined on 2^C such that, for all $c \in C$, $C', C'' \subseteq C$: (a) if $c \notin C_s^h \cup \{\emptyset\}$ and $c \in C'$, $\emptyset P_s^h C'$; (b) if $C, C'' \subseteq C_s^h \setminus \mu^h(s)$ and $C' \succ_s^h C''$ if and only and $\left(\mu^{hr}(s) \cup C'\right) P_s \left(\mu^{hr}(s) \cup C'\right)$. Since P_s is slot-specific, then P_s^h is slot-specific as well. Let μ^h be a stable allocation for market $\left(C, S, P^h\right)$ where $P^h = \left(P_s^h, P_c^h\right)_{s\in S, c\in C}$. From Theorem 2 it follows that $\left(P_s^h\right)_{s\in S}$ is a $SPNE$ of the game starting at h yielding $\nu^h \cup \mu^h$ as an outcome. The stability of μ implies that the described strategy is a $SPNE$ yielding μ as an outcome.

■

Proof of the results in Section 5

Proof of Proposition 2. The proof of the claim is in two parts. First, we prove that all NE outcomes are stable allocations and then we prove that any stable allocation is a NE outcome of the game induced by the SCA mechanism.

(*i*) Let $m^* = \left(\Sigma_s^*, (\succ_\sigma^*)_{\sigma\in\Sigma_s^*}, \rhd_s^*\right)_{s\in S}$ be a NE of the game induced by the SCA when student preferences are given by $(P_s)_{s\in S}$ and let $\mu = SCA(m^*)$. Allocation μ is individually rational for each course. We prove by contradiction that μ is individually rational for students. Assume that μ is not individually rational for student $s \in S$. Let $P_s' = C_s(\mu(s), P_s)$. Preferences P_s' are slot-specific as well. Let $m_s' = \left(\Sigma_s', (\succ_\sigma')_{\sigma\in\Sigma_s^*}, \rhd_s\right)_{s\in S}$ in which Σ_s' is the set of slots for student s under P_s', $\rhd_s$ is the order of precedence of the slot in Σ_s' according to P_s' and for all $\sigma \in \Sigma_s$, $\succ_\sigma$ is the order induced by P_s' for slot σ, for every $\sigma \in \Sigma_s$. By Lemma 2, m_s' is a profitable deviation for student s, which yields a contradiction. We next prove by contradiction that μ is not blocked by any pair. Assume that there exists a pair blocking μ, $(c, s) \in C \times S$. Let $m_s' = \left(\Sigma_s', (\succ_\sigma')_{\sigma\in\Sigma_s^*}, \rhd_s\right)_{s\in S}$ obtained as above from the restriction of P_s to the individual courses in $C_s(\mu(s) \cup \{c\}, P_s)$. Because $s \in Ch_c(\mu(s) \cup \{c\}, P_s)$, the deviation is profitable to s, which yields a contradiction.

(*ii*) Let μ be a stable allocation. For each s, Let $q = \max_{P_s'} \{|C'| \mid C' \subseteq C, C' P_s \emptyset\}$. Let $m_s = \left(\Sigma_s, (\succ_\sigma)_{\sigma\in\Sigma_s^*}, \rhd_s\right)_{s\in S}$ be derived from the restriction of P_s to the individual courses in $\mu(s)$. Notice that $(m_s)_{s\in S}$ yields μ as an outcome. We prove by contradiction that $(m_s)_{s\in S}$ is a Nash equilibrium. Assume that student s has a profitable deviation, (m_s'), and let μ' be the outcome of such a deviation. Let $c \in Ch_s(\mu(s) \cup \mu'(s), P_s(m_s)) \setminus \mu(s)$. Because $P_s(m_s)$ is slot-specific, $c \in Ch_s(\mu(s) \cup \{c\}, P_s(m_s))$. Let m''_s be derived from the restriction of P_s to the individual courses of $Ch_s(\mu(s) \cup \{c\}, P_s(m_s))$. Then, (m''_s) is a profitable deviation as well, yielding $Ch_s(\mu(s) \cup \{c\}, P_s)$. Thus, the pair (c, s) blocks allocation μ, which yields a contradiction.

■

References

[1] Alcalde, J., 1996. Implementation of Stable Solutions to Marriage Problems. J. Econ. Theory 69, 240–254. https://doi.org/10.1006/jeth.1996.0050

[2] Alva, S., 2018. WARP and combinatorial choice. J. Econ. Theory 173, 320–333. https://doi.org/10.1016/j.jet.2017.11.007

[3] Abdulkadiroğlu, A., Sönmez, T., 2003. School Choice: A Mechanism Design Approach. Amer. Econ. Rev. 93, 729–747. https://doi.org/10.1257/000282803322157061

[4] Abdulkadiroğlu, A., Andersson, T., 2020. School Choice, NBER Working Paper 29822. https://www.doi.org/10.3386/w29822

[5] Aygün, O., Sönmez, T., 2013. Allocation with Contracts: Comment. Amer. Econ. Rev. 103, 2050–2051. https://doi.org/10.1257/aer.103.5.2050

[6] Aziz, H., Biro, P., Lang, J., Lesca, J., Monnot, J., 2019. Efficient reallocation under additive and responsive preferences. Theoretical Computer Science, 790: 1-15. https://doi.org/10.1016/j.tcs.2019.05.011

[7] Budish, E., 2011. The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes. J. Polit. Economy 119, 1061–1103. https://doi.org/10.1086/664613

[8] Budish, E., Cachon, G., Kessler, J., and Othman, A., 2017. Course Match: A Large-Scale Implementation of Approximate Competitive Equilibrium from Equal Incomes for Combinatorial Allocation. Oper. Res. 65(2), 314–336. https://doi.org/10.1287/opre.2016.1544

[9] Budish, E., Cantillon, E., 2012. The Multi-Unit Assignment Problem: Theory and Evidence from Course Allocation at Harvard. Amer. Econ. Rev. 102(5), 2237–2271. https://doi.org/10.1257/aer.102.5.2237

[10] Chambers, C. P., Yenmez, M. B., 2018. On lexicographic choice. Econ. Lett. 171, 222–224, https://doi.org/10.1016/j.econlet.2018.08.006

[11] Coles, P. A., Featherstone, C. R., Hatfield, J. W., Kojima, F., Kominers, S. D., Niederle, M., Pathak, P. A., Roth, A. E., 2010. Comment on the NRMP's "Supplemental Offer and Acceptance Program" Proposed to Replace the Post-Match Scramble. Mimeo. Available at: https://stanford. edu/~alroth/papers/NRMP%20comment.pdf. Accessed 31/08/2022.

[12] Doğan, B., Klaus B., 2018. Object allocation via immediate-acceptance: Characterizations and an affirmative action application. J. Math. Econ 79, 140–156 https://doi.org/10.1016/j.jmateco.2018.04.001

[13] Dur, U., Kominers, S. D., Pathak, P. A., and Sönmez, T., 2020. Explicit vs. statistical targeting in affirmative action: Theory and evidence from Chicago's exam schools. J. Econ. Theory 187. https://doi.org/10.1016/j.jet.2020.104996

[14] Dur, U., Kominers, S. D., Pathak, P. A., Sönmez, T., 2018. Reserve Design: Unintended Consequences and The Demise of Boston's Walk Zones. J. Polit. Economy, 126, No. 6, 2457–2479. https://doi.org/10.1086/699974

[15] Echenique, F., and Oviedo, J., 2006. A theory of stability in many-to-many allocation markets. Theor. Econ. 1(2), 233–273.

[16] Ehlers, L., Klaus B., 2003. Coalitional Strategy-Proofness, Resource-Monotonicity, and Separability for Multiple Assignment Problems, Soc. Choice Welfare, 21(2), 265-280. https://doi.org/10.1007/s00355-003-0259-1

[17] Ergin, H., Sönmez, T., 2006. Games of School Choice under the Boston Mechanism. J. Public. Econ. 90, 215–237. https://doi.org/10.1016/j.jpubeco.2005.02.002

[18] Gale, D., Shapley, L. S., 1962. College Admissions and the Stability of Marriage. The American Mathematical Monthly, 69(1) 9–15. https://doi.org/10.2307/2312726

[19] Hafalir, I. E., Yenmez, M. B., Yildirim, M. A., 2013. Effective affirmative action in school choice. Theor. Econ. 8, 325–363. https://doi. org/10. 3982/TE1135

[20] Haeringer, G., Klijn, F., 2009, Constrained school choice, J. Econ Theory, Volume 144, 1921–1947. https://doi.org/10.1016/j.jet.2009.05.002

[21] Hatfield, J. W., Kominers, S. D., 2017. Hidden substitutes. Mimeo.

[22] Kojima, F., 2012. School choice: Impossibilities for affirmative action, Games Econ. Behav. 75, 685–693. https://doi.org/10.1016/j.geb.2012.03.003

[23] Kojima, F., 2013. Efficient Resource Allocation under Multi-unit Demand. Games Econ. Behav. 82, 1–14. https://doi.org/10.1016/j.geb.2013.06.005

[24] Kojima, F., Ünver, U., 2014. The "Boston" school-choice mechanism: an axiomatic approach. Econ. Theory. 55, 515–544. https://doi.org/10.1007/s00199-013-0769-8

[25] Kominers, S. D., Sönmez, T., 2016. Allocation with Slot-Specific Priorities: Theory. Theor. Econ. 11(2), 683–710. https://doi.org/10.3982/TE1839

[26] Kornbluth, D, and Kushnir, A. I., 2021. Undergraduate Course Allocation through Pseudo-Markets (June 8, 2021). Available at SSRN: https://dx.doi.org/10.2139/ssrn.3901146

[27] Marutani, K., 2018. Gaming the deferred acceptance when message spaces are restricted. Math. Soc. Sci. 93, 153–158. https://doi.org/10.1016/j.mathsocsci.2018.03.007

[28] Maschler, M., Solan, E., Zemir, S., 2013: Game Theory. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511794216

[29] Pàpai, S., 2002. Strategyproof and Nonbossy Multiple Assignments. J. Public Econ. Theory, 3, 257–271. https://doi.org/10.1111/1097-3923.00066

[30] Romero-Medina, A., and Triossi, M., 2022. Take-it-or-Leave-it Contracts in Many-to-Many Matching Markets. Econ. Theory. https://doi.org/10.1007/s00199-022-01417-5

[31] Roth, A. E., 1985. The College Admissions Problem is not Equivalent to the Marriage Problem. J. Econ. Theory 36, 277–288. https://doi.org/10.1016/0022-0531(85)90106-1

[32] Roth, A. E., Peranson, E., 1999. The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design. American Economic Review 89 748-780. https://doi.org/10.1257/aer.89.4.748

[33] Roth, A. E., Sotomayor, M., 1990. Two-Sided allocation: A Study in Game-Theoretic Modeling and Analysis. Cambridge, Cambridge University Press. https://doi.org/10.1017/CCOL052139015X

[34] Rusznak, A., Biró, P., Fleiner, R., 2021. Seat transfers in the course allocation mechanism of Eötvös Loránd University. 503-508. 2021 IEEE 15th International Symposium on Applied Computational Intelligence and Informatics (SACI). https://doi.org/10.1109/SACI51354.2021.9465548

[35] Schummer, J., Abizada, A., 2017. Incentives in landing slot problems. J. Econ. Theory 170, 29–55. https://doi.org/10.1016/j.jet.2017.04.003

[36] Schummer, J., Vohra, R. V., 2013. Assignment of Arrival Slots. Amer. Econ. J. : Microeconomics 5, 164–185. https://doi.org/10.1257/mic.5.2.164

[37] Sönmez, T., Ünver, U., 2010. Course Bidding at Business Schools. Int. Econ. Rev. 51(1), 99–123. https://doi.org/10.1111/j.1468-2354.2009.00572.x