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# DEBRIS CLOUD EVOLUTION: MATHEMATICAL MODELLING AND APPLICATION TO SATELLITE CONSTELLATION DESIGN†

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**Abstract**—Orbital break-ups produce a large number of fragments, which constitute an obvious hazard for other satellites in nearby orbits. Of these fragments, many are too small to be detected by ground-based facilities: this leads to the need for mathematical modelling as a tool for adequate risk analysis. In this paper an average spatial density model is presented. It is based on the Gauss analogy and, for unperturbed Keplerian orbits, it matches the asymptotic density model developed by other authors.

Risk analysis for satellite constellations is an interesting application of debris cloud evolution models: the survivability of a constellation as a whole following the break-up of one of its satellites is obviously of primary concern in the constellation design. Risk analysis is conducted over a number of traditional configurations in order to achieve an additional constraint on the design parameters. Results indicate the remarkable influence of the fragmentation point position along the orbit; moreover, the higher risk for low orbit and the advantage of placing more satellites on a limited number of planes are assessed.


## 1. INTRODUCTION

As a result of the overcrowding of the Earth orbital environment, collision risks due to man-made debris have recently become a primary concern for all space missions. A detailed knowledge of the orbiting objects, population characteristics is necessary in the selection of the orbital altitude for a mission. This is true for a one satellite mission, and even more for satellite constellations, when the fragmentation of a single platform could start a catastrophic collisional cascading, due to the continuous orbital crossing.

The forecasting of the possible behaviours has, as its first step, the modelling of the evolution of the cloud deriving from the break-up of one of the platforms. Four phases are usually considered in the debris cloud evolution[1]: a pulsating ellipsoid, a torus with a pinch point corresponding to the fragmentation point, a band caused by the differential rotation of the orbits under the effects of perturbations, and a final stage where everything can be considered as being reduced to a background debris population (or even removed from the orbital environment by atmospheric drag).

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Crowther previously investigated the problem, giving a solution devoted to assess the risk of collision since the break-up of one satellite[2]. Crowther's solution is mostly related to the first phase of the cloud evolution, as it takes into account the pulsating ellipsoid behaviour.

In this paper, a different approach, based on the average spatial density, with the fragmented mass spread along the orbit, is presented. In such a way, time is no longer directly taken into account, but some useful considerations could be retained about the importance of the fragmentation position, and the geometry of the problem will nevertheless be preserved.

While the model described here is devoted to the second phase analysis, it could be noted that, in the case of the considered constellations, the third phase band will be very extended in space. Actually, the extension of the band is limited in latitude by the inclination of the original orbit, which is quite high (this is no question for polar models, but Walker configurations also usually have inclination angles of about 50 degrees). As a consequence the fragment density during the third phase will be very low.

The mathematical model is described in the next two paragraphs, and some results for classical constellation configurations are presented and discussed.

## 2. COLLISION PROBABILITY

A widely used expression[3] to determine the collision probability (PC) as a satellite crosses a debris cloud is:

$$\frac{dPC}{dt} = \rho A V_{rel}, \quad (1)$$

where $\rho$ is the cloud density, $V_{rel}$ is the relative velocity of the satellite with respect to the cloud, and $A$ is the cross-sectional area of the satellite.

The collision probability can be evaluated along the whole orbit:

$$PC' = \frac{(\Delta PC)_{orbit}}{\text{orbital period}} = \frac{1}{T}\int_0^T \rho A V_{rel}\, dt. \quad (2)$$

In this paper we are interested in evaluating the survivability of a constellation after the fragmentation of one of its $n$ satellites; the overall collision probability can be expressed as:

$$PC'_{TOT} = \sum_{j=1}^{n} PC'_j. \quad (3)$$

In the evaluation of the relative velocity an average cloud velocity can be defined; in the case of an isotropic break-up it turns out to be the original satellite orbital velocity $V$. Then $V_{rel}$ can be expressed as:

$$V_{rel} = 2V \sin\left(\frac{\gamma}{2}\right), \quad (4)$$

where $\gamma$ is the angle between the orbital planes of the $j$th satellite and of the fragmented one Fig. 1.

A remarkable consequence of the above stated relations is that collision probabilities involving the satellites in the same plane as the fragmented one are not considered. However, relative velocities between objects sharing the same orbital plane (of the order of 100 m/s) are much smaller than those arising from the intersection of two different orbital planes (about 10 km/s for LEO constellations).

The only term left to be evaluated in eqn (3) is the cloud density. Usually the short term evolution of the cloud is analysed making use of the linearised equations of the relative motion[4–6]. However, as the cloud dimensions increase, the errors deriving from the linearisation process can no longer be neglected. In the following, we will discuss a different approach, involving the determination of the time averaged spatial density; such an approach enables us to obtain exact results for keplerian orbits.

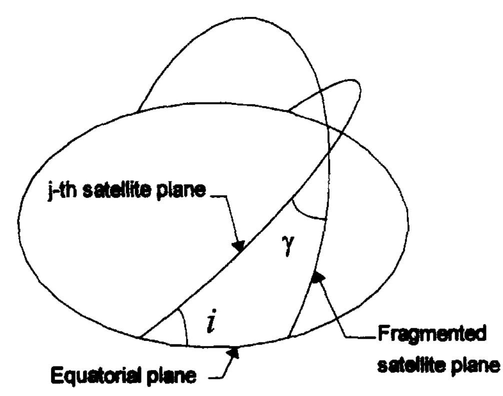


Fig. 1. Angles between orbital planes.

## 3. CLOUD DENSITY EVALUATION

The averaged density is defined as:

$$\bar{\rho} := \lim_{T\to\infty} \frac{1}{T}\int_0^T \rho(t)\, dt. \quad (5)$$

Its physical meaning is related to the number of objects that can be statistically expected to be found in a unitary volume at an arbitrarily chosen time.

In this paper we will carry out the calculations in detail for the 2D case, giving only the results for the 3D case. In this way, we can explain how to obtain the average density expression, without having to resort to cumbersome, if not really difficult, formulas (a complete derivation of both the 2D and 3D cases can be found in Ref. [7]).

The first step is the evaluation of the density for a single orbit:

$$\bar{\rho}_s(r, v) = \frac{1}{2\pi}\frac{\mu^2}{h^4}(1 - e^2)^{3/2}\hat{r}^2\, \delta[r - \hat{r}(\vec{h}, \vec{e}, v)], \quad (6)$$

where $r$ and $v$ are the radius and the anomaly from the fragmentation point, $\delta$ is the Dirac delta, $\vec{h}$ and $\vec{e}$ are the (specific) angular momentum and eccentricity vectors, and finally:

$$\hat{r} = \frac{h^2/\mu}{1 + e\cos\theta}. \quad (7)$$

Equation (6) is equivalent to considering the satellite mass as being distributed along its orbit and not concentrated (Gauss analogy).

The density of the cloud can be evaluated by integrating eqn (6) over all the orbits originated in the fragmentation (if not otherwise stated, the integration domain is considered to be the set of all the possible values of the independent variables):

$$\bar{\rho}(r, v) = \int \bar{\rho}_s(r, v, \vec{h}, \vec{e}) f(\vec{h}, \vec{e})\, d\vec{h}\, d\vec{e}. \quad (8)$$

In eqn (8) $f$ is a function describing the probability distribution for the orbits generated in the fragmentation event.

As a matter of fact, calculations are made easier if we consider as independent variables in the above integral the perturbed velocity due to the parent satellite disintegration.

Thus, we can write:

$$\bar{\rho}(r, v) = \frac{\mu^2}{2\pi}\int \frac{(1 - e^2(\vec{V}))^{3/2}}{h^4(\vec{V})}\hat{r}^2\delta(r - \hat{r})g(\vec{V})\, d\vec{V}, \quad (9)$$

where:

$$\vec{V} = [V_r, V_t]^T,$$

$$h = V_t r_0;$$

$$\beta_0 = \frac{h^2}{\mu r_0 - 1};$$

$$e^2 = \beta_0^2 + \left(\frac{V_r V_t r_0}{\mu}\right)^2.$$

Equation (9) can be simplified:

$$\bar{\rho}(r, v) = \frac{\mu^2}{2\pi}\int \frac{(1-e^2)^{3/2}}{h^4} r^2 \left|\frac{g(\vec{V})}{\partial \hat{r}/\partial V_r}\right|_{V_r = V_r^*} \mathrm{d}V_t, \quad (10)$$

where $V_r^*$ is the root of the equation $\hat{r} = r$, that is:

$$\frac{h^2}{\mu + \mu\beta_0 \cos v + V_r h \sin v} = r \quad (11)$$

solved for $V_r$.

Substitution yields:

$$\bar{\rho}(r, v) = \frac{\mu^2}{2\pi r_0^3 \sin^4 v} \cdot \int \frac{|\sin^2 v - \beta^2 - \beta_0^2 + 2\beta\beta_0 \cos v|^{3/2}}{v_t^3} \cdot g\left(\frac{\mu}{r_0 V_t \sin v}(\beta_0 \cos - \beta), V_t\right) \mathrm{d}V_t, \quad (12)$$

where:

$$\beta = \frac{r_0 V_t^2}{\mu} - 1.$$

This expression of the average density matches (apart from a $2\pi$ factor) the results obtained by Heard for asymptotic density[8]. Actually, if the spatial density $\rho$ approaches a constant value as time increases, it can be seen from eqn (5) that the average and asymptotic densities are the same.

Heard found a useful approximation for the integral (12), for an isotropic break-up, in which all the fragments are ejected with a $\Delta V$ much smaller than the orbital velocity, that is:

$$\alpha := \frac{\Delta \vec{V}}{\vec{V}_0} \ll 1.$$

In this case, the break-up velocity distribution can be expressed as:

$$g(\vec{V}) = \frac{1}{\pi}\delta[(V_r - V_{r_0})^2 + (V_t - V_{t_0})^2 - \Delta V^2];$$

$$\vec{V}_0 = \left[0, \sqrt{\frac{\mu}{r_0}}\right]^T.$$

Since we are interested in circular parent orbits, eqn (12) can be rewritten:

$$\bar{\rho}(r, v) = \frac{\mu^2}{2\pi_0^4 \sin^4 v} \cdot \int \frac{|\sin^2 v - \beta^2 - \beta_0^2 + 2\beta\beta_0 \cos v|^{3/2}}{V_t^3} \frac{\delta[\psi(V_t)]}{|\partial\psi/\partial V_t|} \mathrm{d}V_t \quad (13)$$

where:

$$\psi(V_t) := \left(\frac{\mu(\beta_0 \cos v - \beta)}{r_0 V_t \sin v}\right)^2 + \left(V_t - \sqrt{\frac{\mu}{r_0}}\right)^2 - \Delta V^2. \quad (14)$$

Equation (13) can be evaluated:

$$\bar{\rho}(r, v) = \frac{\mu^2}{2\pi^2 r_0^4 \sin^4 v} \cdot \sum_i \left.\frac{|\sin^2 v - \beta^2 - \beta_0^2 + 2\beta\beta_0 \cos v|^{3/2}}{V_t^3 |\partial\psi/\partial h|}\right|_{h = h_i^*}, \quad (15)$$

where $h_i^*$ are the roots of the equation

$$\psi(V_t) = 0.$$

The stated equation is a quartic, with two real roots. If $\alpha \ll 1$, it is possible to define:

$$\xi := \frac{r - r_0}{r_0}, \quad |\xi| \ll 1$$

$$\epsilon(\xi, \alpha) := \frac{V_t - V_{t_0}}{V_{t_0}}, \quad |\epsilon| \ll 1, \quad \epsilon(0, 0) = 0. \quad (16)$$

Using the relations (16) in (14) and (15), and retaining only the lower order terms:

$$\bar{\rho}(r, v) \cong \frac{1}{2\pi^2 r_0} \frac{1}{\sqrt{\alpha^2\sigma^2 - \xi^2}},$$

where:

$$\sigma^2 := \sin^2 v + 4(1 - \cos v)^2.$$

The results obtained so far can be extended to the 3D case[6]:

$$\bar{\rho}(r, \theta, \varphi) = \frac{\mu^2}{2\pi r_0^2} \frac{\cos^2\varphi |\sin\theta|}{(1 - \cos^2\theta\cos^2\varphi)^3} \cdot \int \frac{|1 - \cos^2\theta\cos^2\varphi - \beta^2 - \beta_0^2 + 2\beta\beta_0 \cos\theta\cos\varphi|^{3/2}}{V_t^2} \cdot g\left(\frac{\mu(\beta_0 \cos\theta\cos\varphi - \beta)\cos\varphi|\sin\theta|}{r_0 V_t (1 - \cos^2\theta\cos^2\varphi)}, \quad V_t, V_t \frac{\tan\varphi}{\sin\theta}\right) \mathrm{d}V_t \quad (17)$$

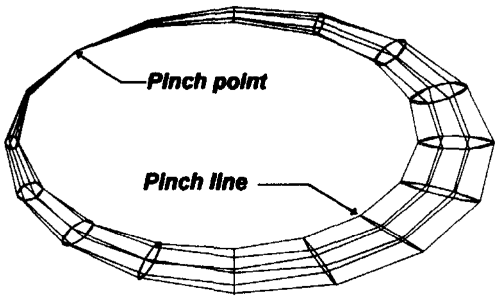


Fig. 2. Torus phase in debris cloud evolution.

and

$$\bar{\rho}(r,\theta,\varphi) \cong \frac{1}{4\pi^2\alpha|\sin\theta|}\frac{1}{\sqrt{\alpha^2\sigma^2-\chi^2}}, \quad (18)$$

where:

$$\chi^2 := \xi^2 + \varphi^2\frac{\sigma^2}{\sin^2\theta}. \quad (19)$$

Given an isotropic velocity perturbation distribution $\Gamma(\alpha)$, it is possible to write:

$$\bar{\rho} = 4\pi\int_0^\infty \bar{\rho}|_{\alpha=\hat{\alpha}}\hat{\alpha}^2\Gamma(\hat{\alpha})\,\mathrm{d}\hat{\alpha}. \quad (20)$$

The relations stated above are a useful tool for evaluating the cloud density, given the initial conditions at the break-up[9].

It is interesting to remark that the density distributions given above are singular at the original fragmentation point and on a line 180 degrees away (*pinch point* and *pinch line*) (Fig. 2).

#### 4. RESULTS FOR STANDARD CONSTELLATION MODELS

In satellite constellation design the collision risk evaluation should be a primary object of concern: more than simply considering the background debris level as in every space program, one has to take into account the risk for a catastrophic break-up of the entire constellation due to the fragmentation of a single platform, whose orbit will be continuously crossed by the other ones.

Figures 3–6 represent some results, obtained using the model previously described, of the interaction between the debris cloud, originated by a satellite belonging to a constellation, and the remaining platforms.

Since our model deals with an averaged density, the results have a meaning in space variables only, because we lost the explicit dependence on time during the averaging process. For this reason our attention will be focused on the position of the break-up event along the orbit. More precisely, the collision probability as a function of the anomaly of the fragmentation point will be analysed.

##### 4.1. *Polar constellation results*

Two models (i.e. the most popular ones, actually considered in practical design) are considered in the following. The first one is the polar model (Fig. 7 and Table 1), introduced by Lüders [10] and developed by Rider [11,12] and Adams [12]: the satellites are equally distributed among P planes, with the interplane angle always equal to $\pi/P$ in the general case (arbitrarily phased). Some improvement in the coverage properties could be obtained by controlling the relative motion of the satellites belonging to different planes (phased constellation case, with the inter-plane angle depending on the nature of the interfacing satellites, if co- or counter-rotating). From the debris collision point of view, there is no difference between these two cases, as the variations in the relative velocities, coming from eqn (4) and due to the slight shift of $\gamma$ values, are not actually relevant to the problem.

As we can expect, results show that the risk of a collision with objects coming from the fragmentation has a strong peak (actually a singular point) if the explosion of a satellite takes place near the pole (Fig. 3). The reason for the singularity is that, in the unperturbed orbit approximation, all the satellites, as well as all the produced fragments, will pass through this point. For other locations, risk is definitely at a lower level.

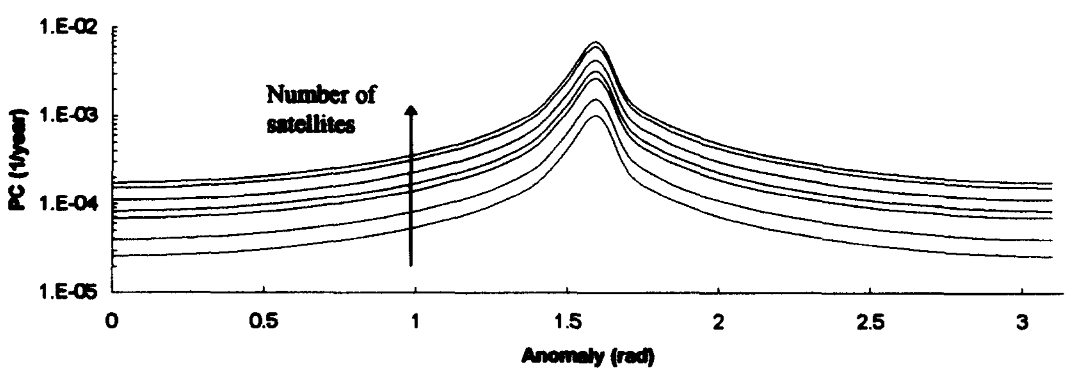


Fig. 3. Results for polar constellations—collision probability vs fragmentation event position along the orbit.

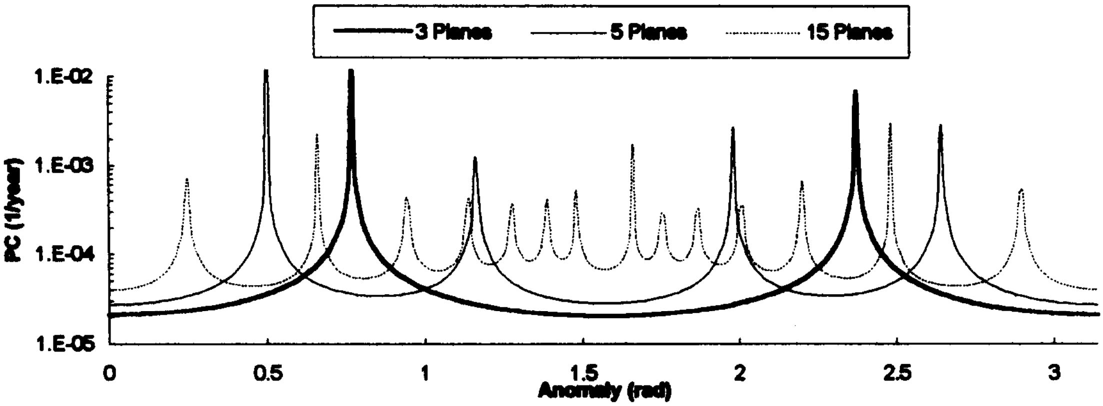


Fig. 4. Results for Walker constellations.

Moreover, the collisional risks are higher for low altitude constellations: the increased danger comes from two different causes. The first one is a reduced orbital length, making the density higher for an equal fragmented mass; the second one is the higher satellite number needed to cover the globe from limited altitudes, increasing the number of potential colliders.

### 4.2. *Walker constellation results*

A little more interesting is the Walker $\delta$ model case (Fig. 8), where the T satellites are placed in the most uniform way, equally distributed among P planes, which are equally inclined and spaced along the equator, the configuration being identified by the triple code T/P/F (where F gives the position–phase of a reference satellite). $\delta$ (i.e. the inclination parameter) values are classically in the order of 55° in the case of global coverage, and this important example will be considered in the following.

The plot of the risk distribution versus the fragmentation event position shows a certain number of peaks (Fig. 4), related to the intersection between the orbital planes. Distances between the peaks are not equal, as the lengths of the arcs are actually different (arc AB is shorter than BC in Fig. 8). Even if we consider that the peaks are singular points, and a certain attention must be devoted to manage them, a certain effect of broadening in the central area, where peaks are closer, is easy to be identified.

Some general remarks could be made on the basis of the previous result. Still keeping in mind that the model adopted is only valid until the perturbation effect is not so strong as to destroy the torus, we can positively conclude that the fragmentation event position is quite important: fragmentations in the equatorial zone are less dangerous than those happening at higher latitudes.

It could be useful to compare the risk of collision resulting from the fragmentation of one of the satellites of the constellation to that deriving from the pre-existing background. For example, data about the debris background taken from [13] could be compared with our results.

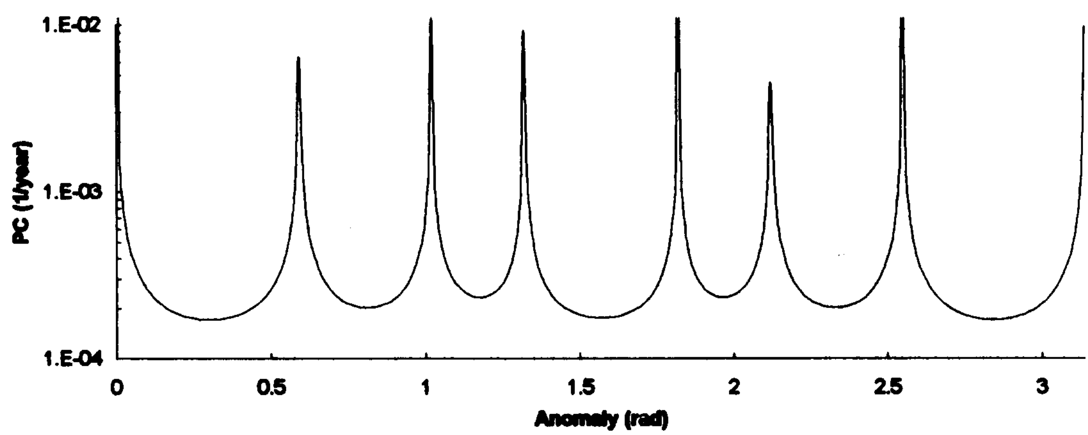


Fig. 5. Globalstar results.

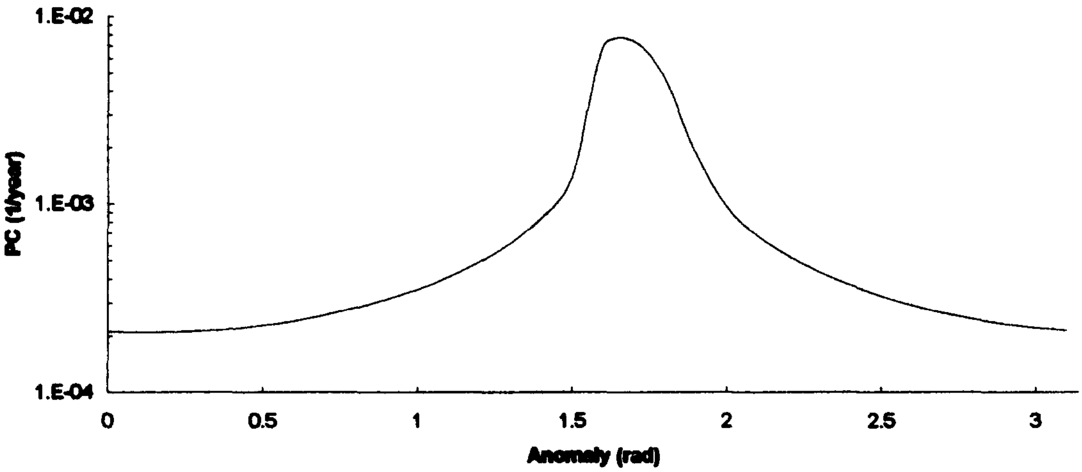


Fig. 6. Iridium results.

Concerning the effective design of the constellations, a basic question is the trade-off between the number of the satellites placed on each plane and the number of the planes. Actually, it comes from considerations about the launch effort, due to the advantages coming from clustered launches, and in certain cases about the ground station architecture, that it is better to have a reduced number of planes. Now, also from debris considerations we can conclude that many satellites on a small number of planes should be better than the opposite (see Fig. 4, where data from Walker constellations with equal satellites number but different P and S = T/P are presented).

### 4.3. *Actual constellations results*

Finally the previously described cloud density model is applied to actual constellation designs, in both the cases of Globalstar and Iridium. Obviously the results match the previously presented ones (Iridium is quite polar, with $i = 86.4°$ while Globalstar is similar to the Walker model). Globalstar distributes 48 satellites into 8 planes, with an altitude of 1389 km and an inclination of 52°. About Iridium, 66 satellites in 6 planes, $h$ = 780 km, $i = 86.4°$. Results are plotted in Figs 5 and 6.

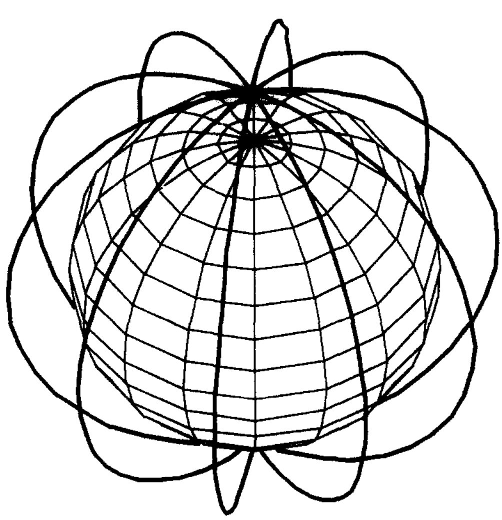

Fig. 7. Polar constellation model.

## 5. CONCLUSIONS

The evaluation of the collisional risk, which is an important parameter for all space missions, acquires a special significance in satellite constellation design, as a result of the possibility of a catastrophic cascading.

In the case of fragmentation of one of the constellation satellites, the model presented in this paper can be used to assess the survivability of the constellation as a whole, marking the importance of the position of the fragmentation event along the orbital path.

However, we are aware of some points in the model which need further development. First of all the model does not take into account the satellites placed in the same plane as the fragmented one. Actually this feature lowers the number of possible colliders, but

Table 1. Polar constellaton data

| Number of satellites | Number of planes | Orbit radius (km) |
|---|---|---|
| 21 | 3 | 8174 |
| 27 | 3 | 7838 |
| 32 | 4 | 7473 |
| 36 | 4 | 7346 |
| 44 | 4 | 7195 |
| 50 | 5 | 7051 |
| 55 | 5 | 6990 |

Table 2. Walker constellation data

| Number of satellites | Number of planes | Orbit radius (km) |
|---|---|---|
| 15 | 3 | 8596 |
| 15 | 5 | 8679 |
| 15 | 15 | 8679 |

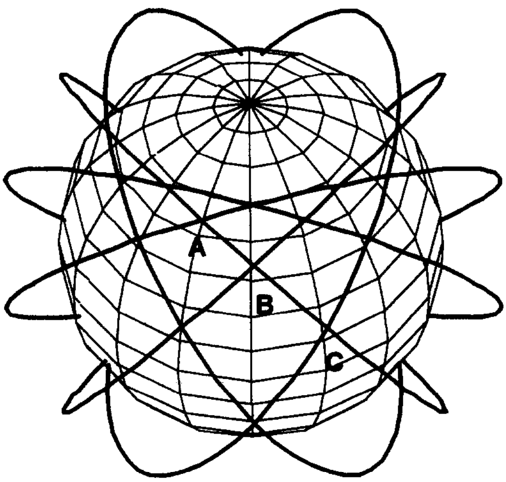


Fig. 8. Walker constellation model.

the cloud behaviour suggests that the differences in the relative velocities, so important in collision phenomena, are strictly limited between satellites (or objects coming from their fragmentation) belonging to the same orbital plane.

The main weakness of the model can possibly be given by the singularities in the density distribution (pinch point and pinch line), which is the reason for the peaks in the collision probabilities plots. These singularities present problems for both numerical and theoretical reasons. The numerical problem of integrating eqn (2) near the singularities is not really difficult to overcome (from an analytical point of view, the singularities can be integrated), but we believe that that expression for the collision probability can no longer be used for high values of $\rho$.

Further developments could finally be aimed at dealing with non-circular orbits: this will allow the analysis of different constellations models (as the Draim's one).

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