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ASYMPTOTIC ABELIAN COMPLEXITIES OF CERTAIN MORPHIC BINARY WORDS

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ABSTRACT

We study asymptotic Abelian complexities of morphic binary words. We complete the classification of upper Abelian complexities of pure morphic binary words initiated recently by F. Blanchet-Sadri, N. Rampersad, and N. Fox. We also study a class of morphic binary words having different asymptotic factor complexities despite having the same asymptotic Abelian complexity.

Keywords: morphic words, Abelian complexity, factor complexity

1. Introduction

The study of complexity measures of infinite words is a well-motivated and actively studied research area. The *factor complexity function*

$$\mathcal{P}_w : \mathbb{N} \to \mathbb{N}$$

of an infinite word $w \in \Sigma^{\mathbb{N}}$ counts, for each $n \in \mathbb{N}$, the number of distinct factors of w of length n. The notion is a fundamental one in combinatorics of infinite words. This can be seen, for instance, from the theorem of M. Morse and G.A. Hedlund [14], which characterises *ultimately periodic* words as exactly the words admitting $\mathcal{P}(n_0) \leq n_0$ for some $n_0 \in \mathbb{N}$. For surveys on factor complexity we refer the reader to [3, 4].

Inspired by the notion of factor complexity, other complexity measures have been developed. One such measure is the *Abelian complexity* of infinite words, the topic of this note. For other related complexity measures, see for instance [10, 17, 20]. Two finite words u, v are said to be *Abelian equivalent*, denoted by $u \sim v$, if, for each letter a, the word u contains equally many a's as the word v. Note that the Abelian equivalence is an equivalence relation on words. The *Abelian complexity function*

$$\mathcal{P}_w^{\mathrm{ab}} : \mathbb{N} \to \mathbb{N}$$

of an infinite word w then counts, for each n, the number of distinct Abelian equivalence classes of length n occurring in the word w. (The subscript is omitted when w

is clear from context.) E. M. Coven and G. A. Hedlund [5] characterise *purely periodic words* to be exactly the words w for which

$$\mathcal{P}_w^{\mathrm{ab}}(n_0) = 1$$

for some $n_0 \geq 1$. This creates the starting point of the study of the Abelian complexity function. While the notion has been around for a while, the study was formally initiated only recently by G. Richomme, K. Saari, and L. Q. Zamboni in [19].

The subject of this paper is the asymptotic Abelian complexity of *morphic binary words*. This has been motivated by the classification of asymptotic factor complexities of *pure morphic words* initiated by A. Ehrenfeucht, K. P. Lee, and G. Rozenberg ([8]) and completed by J.-J. Pansiot ([15], see also [3, 4]). It is thus natural to turn to other complexity classifications of such an important class of words. One such classification result by B. Adamczewski [1] classifies the upper bound growth of the *balance function* (see Definition 2) of primitive pure morphic words over arbitrary alphabets. The classification of the asymptotic Abelian complexities for pure morphic words was initiated in [2]. In that paper, the upper bound growth of the Abelian complexities of primitive binary words are classified (using the equivalence of the balance function and the Abelian complexity function in the binary case). They also classify the Abelian complexities of a large family of words fixed by non-primitive morphisms.

In this paper, we complete the classification of the limit superior Abelian complexities in the case of pure morphic binary words. The words studied here admit fluctuating Abelian complexity, that is, the limit inferior and limit superior Abelian complexities are of different order. This is in contrast to other words fixed by non-primitive binary morphisms. These words are uniformly recurrent, enabling the use of the notion of *derivated words* (see Definition 10) of uniformly recurrent words. We associate the limit superior Abelian complexity of a word to the limit superior balance function of one of its derivated words. We then apply the above mentioned result of [1] dealing with the balance function of primitive pure morphic words.

We also study the Abelian complexities of a class of morphic binary words which are not pure morphic. In particular, we focus on the relation between factor complexity and Abelian complexity. We define a sequence of morphic binary words having (pairwise) the same asymptotic Abelian complexities (up to a constant) despite having different asymptotic factor complexities.

The paper is organized as follows. In Section 2, we introduce basic notation and concepts. In Section 3, we gather results from the literature and state the main result: the classification of the upper bound growth of the Abelian complexities of pure morphic binary words. In Section 4, we prove the remaining cases from the classification theorem. Finally, in Section 5, we extend our interest to morphic, but not pure morphic, binary words having Abelian complexity of order $\Theta(n^r)$ and factor complexity of order $\Theta(n^s)$ for some $r, s \in \mathbb{Q}$, $r < 1 < s$. We construct, for any $r \in \mathbb{Q}$ with $0 < r < 1$ a sequence of morphic binary words $(y_s)_{s\geq 1}$ having

$$\mathcal{P}_{y_{s+1}}(n) = o(\mathcal{P}_{y_s}(n)) \quad \text{and} \quad \mathcal{P}_{y_s}^{\mathrm{ab}}(n) = \Theta(n^r)$$

for all $s \geq 1$.

2. Preliminaries and Notation

In this section, we introduce the notation used in the paper as well as recall relevant notions and results from the literature.

An *alphabet* Σ is a non-empty set of symbols called *letters*. In this paper, alphabets are assumed to be finite, unless explicitly otherwise stated. A finite or infinite sequence of letters over the alphabet Σ is called a *word*. The *empty word* is denoted by ε. The set of finite words over Σ is denoted by Σ^*, the set of non-empty finite words by Σ^+, and the set of infinite words by $\Sigma^{\mathbb{N}}$. More generally, for a set of words (or *language*) $S \subseteq \Sigma^*$, S^* denotes the language of finite sequences of elements of S interpreted as words over Σ. The sets S^+ and $S^{\mathbb{N}}$ are defined analogously. For $u \in \Sigma^+$, we let u^* and u^+ denote the sets $\{u\}^*$ and $\{u\}^+$, respectively. The infinite word u^ω denotes the singleton element of $\{u\}^{\mathbb{N}}$. When talking about *the binary alphabet*, we mean the alphabet $\mathbb{B} = \{a, b\}$. For a word $w \in \Sigma^*$, the *length* $|w|$ of w is the number of letters occurring in w. The set of words of length n over Σ is denoted by Σ^n.

A word $u \in \Sigma^*$ is a *factor* of $w \in \Sigma^*$ if there exist $p, q \in \Sigma^*$ such that $w = puq$. For a non-empty word u, we let $|w|_u$ denote the number of occurrences of u in w as a factor. The set of factors of w is denoted by $F(w)$. We let $F_n(w)$ denote the set

$$F(w) \cap \Sigma^n.$$

For w as above, if $p = \varepsilon$ (resp., $q = \varepsilon$) then u is called a *prefix* (resp., *suffix*) of w. Further, if $q \neq \varepsilon$ (resp., $p \neq \varepsilon$) then u is called *proper prefix* (resp., *proper suffix*). For $w = pq$, we define $p^{-1}w = q$. Similarly, we define $wq^{-1} = p$. The set of prefixes (resp., suffixes) of w is denoted by $\mathrm{pref}(w)$ (resp., $\mathrm{suff}(w)$) and the length k prefix (resp., suffix) of w, $|w| \geq k$, is denoted by $\mathrm{pref}_k(w)$ (resp., $\mathrm{suff}_k(w)$).

For an infinite word $x \in \Sigma^{\mathbb{N}}$, we define factors, prefixes, and left quotients analogously and we use the same notation as for finite words. An infinite word $y \in \Sigma^{\mathbb{N}}$ such that $x = uy$, for some $u \in \Sigma^*$, is called a *tail* of x. We call x *ultimately periodic* if there exist $u \in \Sigma^*, v \in \Sigma^+$ such that $x = uv^\omega$. If, in the above, $u = \varepsilon$ then x is called *purely periodic*. If no such u and v exist, then x is called *aperiodic*. The word x is called *recurrent* if each nonempty factor $u \in F(x)$ occurs infinitely many times in x. Moreover, x is called *uniformly recurrent* if, for each factor $u \in F(x)$, there exists an $N \in \mathbb{N}$ depending on u such that u occurs in each factor of x of length N. Further, x is called *linearly recurrent* if, for each $u \in F(x)$, there exists $K \in \mathbb{N}$ such that u occurs in each factor of x of length $K|u|$. We refer the reader to [4, 12] for more on basic notions in combinatorics on words.

Let $x \in \Sigma^* \cup \Sigma^{\mathbb{N}}$ and suppose u is a non-empty factor of x. The set of *complete first returns to u in x*, denoted by $\Re_u(x)$, is defined as

$$\Re_u(x) = \{\ v \in F(x) \mid u \in \mathrm{pref}(v), u \in \mathrm{suff}(v), \text{ and } |v|_u = 2\ \}.$$

We make use of the following result later on.

Proposition 1 [6, part of Proposition 2.6.]. *Let $p_1, \ldots, p_n \in \Re_u(x)u^{-1}$. Then*

$$|p_1 \cdots p_n u|_u = n + 1 \quad \textit{and} \quad u \in \mathrm{pref}(p_i \cdots p_n u)$$

for all $i = 1, \ldots, n$.

Consequently, the set

$$\Re_u(x)u^{-1}$$

is a code. *That is, if* $p_1 \cdots p_n = q_1 \cdots q_m$ *for some* $p_i, q_i \in \Re_u(x)u^{-1}$ *then* $m = n$ *and* $p_i = q_i$ *for all* $i = 1, \ldots, n$.

We refer the reader to [6, 21] for more on the notion of first return words.

A mapping

$$\varphi : \Delta^* \to \Sigma^*$$

between two alphabets Δ and Σ is called a *morphism* if

$$\varphi(uv) = \varphi(u)\varphi(v)$$

for all $u, v \in \Delta^*$. The notion of a morphism extends naturally to infinite words, and we will not make a distinction between the two. We say that φ is *uniform* if, for all letters $a, b \in \Sigma$,

$$|\varphi(a)| = |\varphi(b)|.$$

Throughout the text, when speaking of *binary morphisms*, we specifically mean morphisms

$$\mathbb{B}^* \to \mathbb{B}^*.$$

For an ordering of

$$\Sigma = \{a_1, a_2, \ldots, a_{|\Sigma|}\}$$

and a morphism

$$\varphi : \Sigma^* \to \Sigma^*,$$

the *incidence matrix* A_φ of φ is defined as

$$A_\varphi[i, j] = |\varphi(a_j)|_{a_i}.$$

In other words, the j-th entry of the i-th row equals the number of occurrences of a_i in $\varphi(a_j)$. For a morphism

$$\varphi : \Sigma^* \to \Sigma^*,$$

we have

$$A_{\varphi^n} = A_\varphi^n \quad \text{for all } n \in \mathbb{N}.$$

The morphism φ is called *primitive* if there exists $n_0 \in \mathbb{N}$ such that $A_\varphi^{n_0}$ contains only positive entries. In the case of the binary alphabet $\mathbb{B}$, we fix $a_1 = a$, $a_2 = b$ so that, given a binary morphism φ, A_φ is of the form

$$A_\varphi = \begin{pmatrix} |\varphi(a)|_a & |\varphi(b)|_a \\ |\varphi(a)|_b & |\varphi(b)|_b \end{pmatrix}.$$

Let φ be a morphism satisfying $\varphi(a) = ah$ for some $a \in \Sigma$ and a word $h \in \Sigma^+$ such that

$$\lim_{n\to\infty} |\varphi^n(h)| = \infty.$$

Then the word

$$\varphi^\omega(a) = \lim_{n\to\infty} \varphi^n(a)$$

exists and is a fixed point of φ. A word $x \in \Sigma^\mathbb{N}$ is called *pure morphic* if there exist a letter $a \in \Sigma$ and a morphism φ such that $x = \varphi^\omega(a)$. Further, w is called *primitive pure morphic*, if such a primitive φ exists. A word is said to be *morphic* if it is a morphic image of a pure morphic word. In other words, $y \in \Sigma^\mathbb{N}$ is morphic if there exist a pure morphic word $x \in \Delta^\mathbb{N}$ and a morphism $\gamma : \Delta \to \Sigma^*$ such that $y = \gamma(x)$.

We recall the Bachmann-Landau notation for asymptotic comparison of functions. Let $f, g : \mathbb{N} \to \mathbb{R}$ be functions with f non-negative and g positive. We write

- $f(n) = \mathcal{O}(g(n))$ if there exist $n_0 \in \mathbb{N}$ and $C > 0$ such that $f(n) \le Cg(n)$ for all $n \ge n_0$;
- $f(n) = \Omega(g(n))$ if there exist $n_0 \in \mathbb{N}$ and $C > 0$ such that $f(n) \ge Cg(n)$ for all $n \ge n_0$;
- $f(n) = \Theta(g(n))$ if both $f(n) = \mathcal{O}(g(n))$ and $f(n) = \Omega(g(n))$;
- $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

We also make brief use of the following notation in Theorem 3:

- $f(n) = \hat{\Omega}(g(n))$ if $\limsup_{n\to\infty} \frac{f(n)}{g(n)} > 0$ and
- $f(n) = (\mathcal{O} \cap \hat{\Omega})(g(n))$ if both $f(n) = \mathcal{O}(g(n))$ and $f(n) = \hat{\Omega}(g(n))$.

In general, the Abelian complexity function $\mathcal{P}^{\text{ab}}$ can be strongly fluctuating (see, e. g., [13, 11]), so, for our needs, it is more meaningful to study the asymptotic behavior of the Abelian complexity function. To this end, we define the *upper* (resp., *lower*) *Abelian complexity functions*, $\mathcal{U}_x^{\text{ab}}$ (resp., $\mathcal{L}_x^{\text{ab}}$), of a word $x \in \Sigma^\mathbb{N}$ as

$$\mathcal{U}_x^{\text{ab}}(n) = \max\{\, \mathcal{P}_x^{\text{ab}}(m) \mid 0 \le m \le n \,\} \text{ (resp., } \mathcal{L}_x^{\text{ab}}(n) = \min\{\, \mathcal{P}_x^{\text{ab}}(m) \mid m \ge n \,\}).$$

The asymptotic growth rates of these functions indicates how large the fluctuation of the Abelian complexity of x can be.

3. Background

We recall some related results from the literature. We define a complexity function closely related to the Abelian complexity. For this we need the following notation. For an infinite word $w \in \Sigma^{\mathbb{N}}$ and a letter $a \in \Sigma$, we define

$$\max_{w,a}(n) = \max \{ \, |u|_a \mid u \in F_n(w) \, \} .$$

The function $\min_{w,a} : \mathbb{N} \to \mathbb{N}$ is defined analogously.

Definition 2. Let $u \in \Sigma^{\mathbb{N}}$. The *balance function* B_u of u is defined as

$$B_u(n) = \max \{ \, \max_{u,a}(n) - \min_{u,a}(n) \mid a \in \Sigma \, \} .$$

It is straightforward to verify that, for $x \in \mathbb{B}^{\mathbb{N}}$,

$$\mathcal{P}_x^{\mathrm{ab}}(n) = B_x(n) + 1 \quad \text{for all } n \in \mathbb{N}.$$

The following result of B. Adamczewski is the first and deep starting point of the classification of the Abelian complexities of morphic words. The result classifies the asymptotic growth of the balance function of primitive pure morphic words. The asymptotic behaviour of $\mathcal{U}_x^{\mathrm{ab}}$ for binary words x can be extracted from the above, as was done in [2]. We state the theorem here since we shall make use of it in our later considerations. Before we do so, however, we recall some basic notions of linear algebra. When talking about eigenvalues of a morphism φ, we mean eigenvalues of A_φ. The multiplicity of the eigenvalue λ in the minimal polynomial of A_φ is denoted by α_λ. We let $\theta_1, \theta_2, \ldots, \theta_n$ be the distinct eigenvalues of φ ordered in such a way that $|\theta_i| \geq |\theta_{i+1}|$ and if $|\theta_i| = |\theta_{i+1}|$ then $\alpha_{\theta_i} \geq \alpha_{\theta_{i+1}}$. For a primitive φ, the Perron-Frobenius theorem (see, e.g., [18]) implies that $\theta_1 \in \mathbb{R}$, $\theta_1 > 1$, $\theta_1 > |\theta_2|$, and $\alpha_{\theta_1} = 1$. The eigenvalue θ_1 is called the *Perron*-eigenvalue of φ. We also make use of the eigenvalue θ_2, which can be seen as the second most significant eigenvalue of φ.

In the following, we let $\alpha_2 = \alpha_{\theta_2} - 1$.

Theorem 3 [1] (as formulated in [2]). *Let x be a fixed point of a primitive morphism φ. Then the following hold:*

(I) *If $|\theta_2| < 1$, then $B_x(n) = (\mathcal{O} \cap \hat{\Omega})(1)$.*

(II) *If $|\theta_2| > 1$, then $B_x(n) = (\mathcal{O} \cap \hat{\Omega})((\log n)^{\alpha_2} n^{\log_{\theta_1} |\theta_2|})$.*

(III) *If $|\theta_2| = 1$ and θ_2 is not a root of unity, then $B_x(n) = (\mathcal{O} \cap \hat{\Omega})((\log n)^{\alpha_2+1})$.*

(IV) *If $|\theta_2| = 1$ and θ_2 is a root of unity, then either*
- $B_x(n) = (\mathcal{O} \cap \hat{\Omega})((\log n)^{\alpha_2+1})$, *or*
- $B_x(n) = (\mathcal{O} \cap \hat{\Omega})((\log n)^{\alpha_2})$,

according to whether a certain constant $A_{\varphi,x}$ equals zero or not, respectively.

We refer the interested reader to [1] for more on computing the constant $A_{\varphi,u}$.

From the above, we immediately have that $\mathcal{U}_x^{\mathrm{ab}}(n)$, for a primitive pure morphic binary word x, is of order $\Theta(1)$, $\Theta(\log n)$, or $\Theta(n^{\log_{\theta_1} \theta_2})$ (since $\alpha_2 = 0$). In [2],

F. Blanchet-Sadri, N. Fox, and N. Rampersad go on to study Abelian complexities of fixed points of non-primitive binary morphisms. Before stating their result, we recall a straightforward characterization of such morphisms.

Proposition 4. *Let φ be a non-primitive binary morphism which admits an infinite fixed point $y = \varphi^\omega(a)$. Then either $\varphi(a) \in aa^+$ and $\varphi(b) \in \mathbb{B}^*$, or φ is of the form*

$$\varphi(a) \in a\Sigma^* b\Sigma^* \text{ and } \varphi(b) \in b^*, \tag{1}$$

where, if $\varphi(b) = \varepsilon$, then $|\varphi(a)|_a \geq 2$. Further, y is ultimately periodic if and only if $\varphi(a) \in aa^+$ or φ is of the form (1) and satisfies one of the following conditions:

- *$\varphi(a) \in ab^+$,*
- *$\varphi(b) = \varepsilon$, or*
- *$\varphi(a) = (ab^r)^s a$ and $\varphi(b) = b$ for some $r, s \geq 1$.*

Theorem 5 [2]. *Let φ be a non-primitive binary morphism as in (1) with $\varphi(b) = b^k$ for some $k \geq 1$. Suppose further that φ admits an aperiodic infinite fixed point $y = \varphi^\omega(a)$. Then the following holds:*

(I) *If $k = 1$ and $\varphi(a)$ ends with b, then $\mathcal{P}_y^{ab}(n) = \Theta(n)$.*

(II) *If $k \geq 2$, then*
- *$\mathcal{P}_y^{ab}(n) = \Theta(n)$ if $|\varphi(a)|_a > k$,*
- *$\mathcal{P}_y^{ab}(n) = \Theta(n/\log n)$ if $|\varphi(a)|_a = k$, and*
- *$\mathcal{P}_y^{ab}(n) = \Theta\left(n^{\log_k |\varphi(a)|_a}\right)$ if $|\varphi(a)|_a < k$.*

It is straightforward to check that the words fixed by non-primitive morphisms whose asymptotic (upper) Abelian complexities are not yet classified are as in (1), where $k = 1$ and $\varphi(a)$ ends with a. More precisely, φ is of the form

$$\varphi(a) = ab^{k_1}ab^{k_2}\cdots ab^{k_s}a, \quad \varphi(b) = b, \tag{2}$$

where $k_i \geq 0$ for all $i = 1, \ldots, s$ and there exist i, j such that $k_i < k_j$. Our aim is to complete the classification by proving the following in Section 4:

Theorem 6. *Let φ be as in (2) and $y = \varphi^\omega(a)$. Then*

$$\mathcal{U}_y^{ab}(n) = \Theta(\log n) \quad \text{and} \quad \mathcal{L}_y^{ab}(n) = \Theta(1).$$

In particular, morphisms of the form (2) are the only non-primitive binary morphisms whose fixed points have upper and lower Abelian complexities of different orders of growth.

4. The Proof of Theorem 6

In this section, we prove Theorem 6. We shall first consider the lower Abelian complexities of words fixed by morphisms of the form (2). After this, we focus on the upper Abelian complexity. We achieve this by finding a connection between the upper Abelian complexities and the balance functions of some *derivated words* of these words (see Subsection 4.1).

We first fix the notation for the remainder of this section. We let φ be a morphism as in (2) and we let $\mathbf{Y} = \varphi^\omega(a)$. We also let k_m (resp., k_M) denote the minimal (resp., maximal) of the exponents k_i, $i = 1, \ldots, s$ in (2).

We start with some elementary properties of the word $\mathbf{Y}$.

Lemma 7. *The word* $\mathbf{Y}$ *has the following properties.*

(i) *The set* $\Re_a(\mathbf{Y})$ *equals* $\Re_a(\varphi(a)) = \{ ab^{k_i}a \mid i = 1, \ldots, s \}$.

(ii) *For any fixed* $m \in \mathbb{N}$*, we have* $\mathbf{Y} \in \{ \varphi^m(a)b^{k_i} \mid i = 1, \ldots, s \}^\omega$.

(iii) *The word* $\mathbf{Y}$ *is linearly recurrent (so, in particular, uniformly recurrent).*

Proof. (i) Suppose this is not the case, $ab^ra \in F(\mathbf{Y}) \setminus F(\varphi(a))$ for some $r \in \mathbb{N}$. Suppose that $ab^ra \in \varphi(w)$, where $w \in F(\varphi^t(a))$, $t \geq 1$, t is the least such integer, and w is the shortest such factor of $\mathbf{Y}$. Now $w \notin \mathbb{B}$, and since $|ab^ra|_a = 2$, we have $w = ab^sa$ for some $s \geq 0$. Now

$$ab^ra \in F(\varphi(a)b^s\varphi(a)).$$

Since $ab^ra \notin \Re_a(\varphi(a))$, it follows that $s = r$, that is, $ab^ra \in F(\varphi^{t-1}(a))$, a contradiction.

(ii) The claim is true for $m = 0$ by the previous item. Suppose then that the claim is true for some $m \geq 0$;

$$\mathbf{Y} = \prod_{i=1}^{\infty} \varphi^m(a)b^{r_i}, \quad r_i \in \{k_1, \ldots, k_s\} \text{ for all } i \geq 1.$$

But then

$$\mathbf{Y} = \varphi(\mathbf{Y}) = \prod_{i=1}^{\infty} \varphi^{m+1}(a)b^{r_i}.$$

(iii) Let $u_m = \varphi^m(a)$ for each $m \geq 0$. It is straightforward to conclude that u_{m+1} contains each factor of length $|u_m|$ of $\mathbf{Y}$ for each $m \in \mathbb{N}$. Further, by Item (ii), any factor of length $2|u_{m+1}| + k_M$ contains u_m as a factor. The claim follows since $|u_{m+1}| \leq |u_1||u_m|$. □

Remark 8. Observe that

$$\lim_{n\to\infty} \frac{\min_{\mathbf{Y},c}(n)}{n} > 0$$

for both $c \in \mathbb{B}$. This is immediate by Item (ii) (case $m = 1$) in the above lemma together with $|\varphi(a)|_a, |\varphi(a)|_b > 0$. Observe that the limit always exists as the sequence

$$(\min_{\mathbf{Y},c}(n))_{n\geq 0}$$

is *subadditive*. In fact, since $\mathbf{Y}$ is linearly recurrent, we have

$$\lim_{n\to\infty} \min_{v\in F_n(\mathbf{Y})} \frac{|v|_u}{n} = \lim_{n\to\infty} \max_{v\in F_n(\mathbf{Y})} \frac{|v|_u}{n}$$

for any $u \in F(\mathbf{Y})$ ([7, Theorem 15] and [9, Proposition 7.2.10]). For us however, the first observation above is enough.

We are ready to show that the lower Abelian complexity of $\mathbf{Y}$ is bounded.

Lemma 9. *Let φ and $\mathbf{Y}$ be as above. Then $\mathcal{L}^{ab}_{\mathbf{Y}}(n) = \Theta(1)$.*

Proof. Let $u_m = \varphi^m(a)$ for each $m \in \mathbb{N}$. We claim that $\mathcal{P}^{\mathrm{ab}}_{\mathbf{Y}}(|u_m|)$ is bounded by a constant depending only on φ. Let now $v \in F(\mathbf{Y})$ have length $|u_m|$. By Item (ii) in the above lemma, it follows that v is a factor of $u_m b^{k_i} u_m$ for some $i \in \{1, \ldots, s\}$. In other words, we have $v = qb^r p$, where p (resp., q) is a (possibly empty) prefix (resp., suffix) of u_m and $r \leq k_M$. On the other hand, we have $u_m = puq$, for some $u \in F_r(\mathbf{Y})$. We thus conclude

$$|u_m|_b = |p|_b + |u|_b + |q|_b \leq |p|_b + r + |q|_b = |v|_b \leq |u_m|_b + r \leq |u_m|_b + k_M.$$

It follows that $\mathcal{P}^{\mathrm{ab}}_{\mathbf{Y}}(|u_m|) \leq k_M + 1$ for all $m \in \mathbb{N}$. The claim follows. □

The rest of this section is devoted to the upper Abelian complexity of $\mathbf{Y}$. We develop the tools needed in the following.

4.1. On Derivated Words of Uniformly Recurrent Words

We recall the definition of a *derivated word* of a uniformly recurrent word from [6]. We then study the derivated words of uniformly recurrent pure morphic words, and remark a slight generalization of a result from [6].

To this end, let $x \in \Sigma^{\mathbb{N}}$ be uniformly recurrent and let $p \in \mathrm{pref}(x)$ be non-empty. We recall the following property used in [6] implicitly. Let $y \in \Sigma^{\mathbb{N}}$ and assume that y admits a factorisation

$$y = \prod_{i=0}^{\infty} q_i,$$

where $q_i \in \Re_p(x)p^{-1}$ for all $i \geq 0$. Then this factorisation is unique. Indeed, the claim follows straightforwardly from the observation that

$$p \in \mathrm{pref}(\prod_{i=n}^{\infty} q_i)$$

for each $n \geq 0$ and p occurs nowhere else in y. To see this, we apply Proposition 1 to see that p is a prefix of $q_n \cdots q_{n+|u|}p$ (and thus a prefix of $q_n \cdots q_{n+|u|}$ since $|q_i| \geq 1$ for each $i \geq 0$) for each $n \geq 0$. Furthermore, if p occurs somewhere else in y, then $|q_n p|_p \geq 3$ for some $n \geq 0$, which contradicts the assumption $q_n \in \Re_p(x)p^{-1}$.

Definition 10. Let $x \in \Sigma^{\mathbb{N}}$ be uniformly recurrent and $p \in \mathrm{pref}(x)$ be non-empty. By the above discussion, we may write uniquely

$$x = \prod_{i=0}^{\infty} q_i,$$

where $q_i \in \Re_p(x)p^{-1}$ for each $i \in \mathbb{N}$. Let $\Delta_{p,x}$ be an alphabet with $|\Delta_{p,x}| = |\Re_p(x)|$, and let

$$\pi_{p,x} : \Delta_{p,x} \to \Re_p(x)p^{-1}$$

be a bijection. The *derivated word of* x *with respect to* p, denoted by $D_p(x)$, is defined as

$$D_p(x) = \prod_{i=0}^{\infty} \pi_{p,x}^{-1}(q_i) \in \Delta^{\omega}.$$

In the following, we order the elements of

$$\Re_p(x) = \{p_1, \ldots, p_d\}$$

in the order they occur for the first time in x. We then set

$$\Delta_{p,x} = \{\delta_1, \ldots, \delta_d\}$$

and fix $\pi_{p,x}$ by $\pi_{p,x}(\delta_i) = p_i$, $i = 1, \ldots, d$. We often omit the subscripts from $\Delta_{p,x}$ and $\pi_{p,x}$ whenever the word x and prefix p are clear from context.

Note that $\pi_{p,x}$ can be interpreted as a morphism

$$\pi_{p,x} : \Delta_{p,x}^* \to \Sigma^*,$$

whence $x = \pi_{p,x}(D_p(x))$. Note also that, since x is uniformly recurrent, then so is $D_p(x)$. The following result is a minor generalisation of [6, Proposition 5.1]. The proof is essentially the same, as suggested by Jarkko Peltomäki (personal communication).

Proposition 11. *Let* $\rho : \Sigma^* \to \Sigma^*$ *be a morphism admitting a uniformly recurrent fixed point* $x = \rho^{\omega}(a)$. *Let* p *be a non-empty prefix of* x. *Then* $D_p(x)$ *is primitive pure morphic.*

The ingredients of the proof of the above result are essential to our later considerations. In particular, we recall the construction of the primitive morphism

$$\mu : \Delta_{p,x} \to \Delta_{p,x}^*$$

satisfying $\mu^{\omega}(\delta_1) = D_p(x)$. To this end, it can be proved that $\rho\pi(\delta)$ is a return to p in x for any $\delta \in \Delta_{p,x}$. In other words, by Proposition 1, we may uniquely write

$$\rho\pi(\delta) = q_1 \cdots q_n,$$

where $q_i \in \Re_p(x)p^{-1}$ for each $i = 1, \ldots, n$. Finally, we define

$$\mu(\delta) = \pi^{-1}\rho\pi(\delta) = \pi^{-1}(q_1)\cdots\pi^{-1}(q_n)$$

for each $\delta \in \Delta_{p,x}$. We clarify the above construction by an example.

Example 12. Let

$$\varphi(a) = aabab^2a \quad \text{and} \quad \varphi(b) = b$$

so that φ is of the form (2). Let $y = \varphi^\omega(a)$. By Lemma 7,

$$\Re_a(y) = \{aa, aba, ab^2a\}$$

so we set $\Delta_{a,y} = \{\delta_1, \delta_2, \delta_3\}$. Now π is defined by $\pi(\delta_i) = ab^{i-1}$ for each $i = 1, 2, 3$. The primitive morphism μ is now defined by

$$\mu(\delta_i) = \pi^{-1}\varphi\pi(\delta_i) = \pi^{-1}(aabab^2ab^{i-1}) = \delta_1\delta_2\delta_3\delta_i$$

for each $i = 1, 2, 3$. The incidence matrix A_μ of μ is thus

$$A_\mu = \begin{pmatrix} |\mu(\delta_1)|_{\delta_1} & |\mu(\delta_2)|_{\delta_1} & |\mu(\delta_3)|_{\delta_1} \\ |\mu(\delta_1)|_{\delta_2} & |\mu(\delta_2)|_{\delta_2} & |\mu(\delta_3)|_{\delta_2} \\ |\mu(\delta_1)|_{\delta_3} & |\mu(\delta_2)|_{\delta_3} & |\mu(\delta_3)|_{\delta_3} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Note that, for example, the word fixed by the morphism $a \mapsto ab^2abab^3a$, $b \mapsto b$, has the same derivated word $D_a(y)$.

We note that in the above example, the obtained morphism μ is uniform with length $4 = |\varphi(a)|_a$. This is no coincidence when the morphism φ is of the form (2). Indeed, consider the construction of μ for our word $\mathbf{Y}$ and prefix a. We have

$$\Delta = \Delta_{a,\mathbf{Y}} = \{\delta_1, \ldots, \delta_d\}.$$

Now $\Re_a(\mathbf{Y})a^{-1} \subseteq ab^*$ so we may define $\pi = \pi_{a,\mathbf{Y}}$ by $\pi(\delta_i) = ab^{r_i} \in \Re_a(\varphi(a))a^{-1}$, for each $i = 1, \ldots, d$. By the definition of μ in the above construction, we obtain

$$\mu(\delta_i) = \pi^{-1}\varphi\pi(\delta_i) = \pi^{-1}(\varphi(a)b^{r_i}) = \pi^{-1}(\varphi(a)a^{-1}ab^{r_i}) = p\delta_i, \tag{3}$$

where $p = \pi^{-1}(\varphi(a)a^{-1})$. The morphism μ is thus uniform with length $|\varphi(a)|_a$.

Proposition 13. *We have* $B_{D_a(\mathbf{Y})}(n) = \mathcal{O}(\log n)$.

Proof. We aim to show that μ has eigenvalues $|\varphi(a)|$ and 1, both with multiplicities 1 (as roots of the minimal polynomial of A_μ). The claim then follows by the fourth point of Theorem 3. Indeed, the incidence matrix A_μ is of the form

$$A_\mu = \left(\Psi(p)^T \mid \Psi(p)^T \mid \cdots \mid \Psi(p)^T\right) + \mathbf{I}_{d\times d} = \mathbf{A} + \mathbf{I}_{d\times d},$$

where $\Psi(p)$ is the *Parikh vector* of p in (3), $\mathbf{I}_{d\times d}$ is the $d \times d$ *identity matrix*, and $\mathbf{A}$ is a $d \times d$ matrix, where each column is the same vector $\Psi(p)^T$.

Now let λ be an eigenvalue of A_μ. This implies that

$$0 = \det(A_\mu - \lambda \mathbf{I}_{d\times d}) = \det(\mathbf{A} + \mathbf{I}_{d\times d} - \lambda \mathbf{I}_{d\times d}) = \det(\mathbf{A} - (\lambda - 1)\mathbf{I}_{d\times d}).$$

It is straightforward to check that the only eigenvalues of $\mathbf{A}$ are

$$\sum_{i=1}^{d} |p|_{\delta_i} = |p| \quad \text{and} \quad 0$$

from which it follows that the eigenvalues of A_μ are $|\varphi(a)|_a$ and 1. We now claim that the minimal polynomial of A_μ is

$$x^2 - (|p| + 2)x + |p| + 1.$$

Indeed, it is straightforward to check that $\mathbf{A}^2 = |p|\mathbf{A}$, from which it follows that

$$A_\mu^2 - (|p| + 2)A_\mu + (|p| + 1)\mathbf{I}_{d\times d} = \mathbf{0}.$$

Now the minimal polynomial of A_μ is of degree 2 implying that the eigenvalue 1 has multiplicity 1, as was to be shown. □

4.2. The Upper Abelian Complexity

We are now ready to prove the upper Abelian complexity of $\mathbf{Y}$ to be of order

$$\mathcal{U}^{\text{ab}}_{\mathbf{Y}}(n) = \Theta(\log n).$$

For this, we bound the upper Abelian complexity of $\mathbf{Y}$ in terms of asymptotic balance function of $D_a(\mathbf{Y})$. We then show that for infinitely many m, we have

$$\mathcal{P}^{\text{ab}}_{\mathbf{Y}}(m) = \Theta(\log m).$$

Lemma 14. *We have $\mathcal{U}^{ab}_{\mathbf{Y}}(n) = \mathcal{O}(\log n)$.*

Proof. Let $D_a(\mathbf{Y})$, μ and π be as above and let $n \in \mathbb{N}$ with $\mathcal{P}^{ab}(n) > 2$. There exists a factor $u_M \in F_n(\mathbf{Y})$ such that $|u_M|_a = \max_{\mathbf{Y},a}(n)$ and u_M begins with a. Indeed, if $|v|_a = \max_{\mathbf{Y},a}(n)$ with $v \in b^r a\Sigma^*$, then, by considering a factor $vw \in F(\mathbf{Y})$ with $|w| = r$, we have

$$|\operatorname{suff}_n(vw)|_a = |v|_a + |w|_a \quad \text{and} \quad \operatorname{suff}_n(vw) \in a\Sigma^*.$$

Similarly, there exists a factor $u_m \in F_n(\mathbf{Y})$ such that $|u_m|$ begins with a and

$$|u_m|_a \leq \min_{\mathbf{Y},a}(n) + 1.$$

Observe that now $|u_M|_a > |u_m|_a$ by the choice of n.

As u_m begins with a, we may write $u_m \in \text{pref}(\pi(x))$ for some $x \in F_{|u_m|_a}(D_a(\mathbf{Y}))$, whence

$$|u_m|_a = |\pi(x)|_a \quad \text{and} \quad |u_m|_b \le |\pi(x)|_b \le |u_m|_b + k_M.$$

Similarly, we may write $u_M = \pi(z)v$, where $z \in F_{|u_m|_a}(D_a(\mathbf{Y}))$ and v begins with a. We now have $|u_M|_a - |u_m|_a = |v|_a$ so that

$$\mathcal{P}^{\text{ab}}_{\mathbf{Y}}(n) \le |v|_a + 2.$$

We claim that $|v| = \mathcal{O}(\log n)$ to conclude the proof. As

$$|\pi(z)| + |v| = |u_M| \le |\pi(x)|$$

and

$$|\pi(z)|_a = |\pi(x)|_a,$$

we have $|v| \le |\pi(x)| - |\pi(z)| = |\pi(x)|_b - |\pi(z)|_b$. Moreover, $|x| = |z|$ so, by Proposition 13,

$$|\pi(x)|_b - |\pi(z)|_b = \sum_{i=1}^{d} r_i(|x|_{a_i} - |z|_{a_i}) \le dk_M B_{D_a(\mathbf{Y})}(|x|) = \mathcal{O}(\log |x|).$$

Finally, $|x| = |u_m|_a \le n$ and thus $|v| = \mathcal{O}(\log n)$. The claim follows. □

We now proceed to show that

$$\mathcal{U}^{\text{ab}}_{\mathbf{Y}}(n) = \Omega(\log n).$$

To this end, let $\varphi(a) = gauah$ for some $g, u, h \in \Sigma^*$. We shall now construct a sequence of factors of $\mathbf{Y}$ defined recursively by

$$u_0^{g,h} = a \quad \text{and} \quad u_{n+1}^{g,h} = g^{-1}\varphi(u_n^{g,h})h^{-1} \text{ for } n \ge 0.$$

Note that the sequence is well-defined, as

$$\varphi(u_n) \in ga\mathbb{B}^*ah$$

for each $n \ge 0$. We now make some observations of the words in the sequence. In the following we let $\alpha = |\varphi(a)|_a$ and $\beta = |\varphi(a)|_b$ for ease of notation.

Lemma 15. *For all $n \in \mathbb{N}$,*

- $|u_n^{g,h}|_a = \left(1 - \frac{|gh|_a}{\alpha - 1}\right)\alpha^n + \frac{|gh|_a}{\alpha - 1},$
- $|u_n^{g,h}|_b = \frac{\beta}{\alpha - 1}\left(1 - \frac{|gh|_a}{\alpha - 1}\right)(\alpha^n - 1) + \left(\frac{\beta}{\alpha - 1}|gh|_a - |gh|_b\right)n.$

Proof. Define

$$\widehat{\Psi}_{g,h}(v) = (|v|_a, |v|_b, -|gh|_a, -|gh|_b)^T \in \mathbb{Z}^4$$

for all $v \in \Sigma^*$. Consider the following 4×4 matrix (in block form)

$$\widehat{A}_\varphi = \begin{pmatrix} A_\varphi & \mathbf{I} \\ \mathbf{0} & \mathbf{I} \end{pmatrix},$$

where $\mathbf{I} = \mathbf{I}_{2\times 2}$ and $\mathbf{0} = \mathbf{0}_{2\times 2}$ are the 2×2 identity matrix and zero matrix, respectively. It is readily verified that, for any $v \in a\Sigma^* a$, we have

$$\widehat{A}_\varphi \widehat{\Psi}_{g,h}(v) = \widehat{\Psi}_{g,h}(g^{-1}\varphi(v)h^{-1}).$$

This implies that $\widehat{A}_\varphi^n \widehat{\Psi}_{g,h}(a) = \widehat{\Psi}_{g,h}(u_n^{g,h})$ for all $n \in \mathbb{N}$. We then have, for all $n \in \mathbb{N}$,

$$\widehat{A}_\varphi^n = \begin{pmatrix} A_\varphi^n & \sum_{i=0}^{n-1} A_\varphi^i \\ \mathbf{0} & \mathbf{I} \end{pmatrix},$$

where

$$A_\varphi^n = \begin{pmatrix} \alpha^n & 0 \\ \beta\frac{\alpha^n-1}{\alpha-1} & 1 \end{pmatrix} \quad \text{and} \quad \sum_{i=0}^{n-1} A_\varphi^i = \begin{pmatrix} \frac{\alpha^n-1}{\alpha-1} & 0 \\ \beta\frac{\alpha^n-1-n(\alpha-1)}{(\alpha-1)^2} & n \end{pmatrix},$$

by straightforward induction. We finally have, for all $n \in \mathbb{N}$,

$$\widehat{A}_\varphi^n \widehat{\Psi}_{g,h}(a)[1,2] = \begin{pmatrix} \alpha^n - \frac{\alpha^n-1}{\alpha-1}|gh|_a \\ \beta\frac{\alpha^n-1}{\alpha-1} - \beta\frac{\alpha^n-1-n(\alpha-1)}{(\alpha-1)^2}|gh|_a - n|gh|_b \end{pmatrix}.$$

Rearranging the terms gives our claim. □

Recall that there exist $g_m, h_m, g_M, h_M \in \Sigma^*$ such that

$$\varphi(a) = g_m ab^{k_m} ah_m = g_M ab^{k_M} ah_M,$$

where k_m and k_M are as fixed in the beginning of this section. Let then

$$(u_n) = (u_n^{g_m,h_m}) \quad \text{and} \quad (v_n) = (u_n^{g_M,h_M})$$

be sequences constructed as above. Note that

$$|g_m h_m|_a = |g_M h_M|_a = \alpha - 2$$

and

$$|g_m h_m|_b - |g_M h_M|_b = k_M - k_m$$

whence, by the above lemma,

$$|v_n| - |u_n| = |v_n|_b - |u_n|_b = -n|g_M h_M|_b + n|g_m h_m|_b = n(k_M - k_m) \tag{4}$$

for all $n \in \mathbb{N}$. We are now in the position to complete the proof of Theorem 6.

Proof of Theorem 6. By Lemma 9 and Lemma 14, it is enough to show that

$$\mathcal{U}^{\mathrm{ab}}_{\mathbf{Y}}(n) = \Omega(\log n).$$

Let $(u_n)_n$ and $(v_n)_n$ be the sequences as discussed above. Let then $u_n f_n \in F(\mathbf{Y})$, so that $|u_n f_n| = |v_n|$, that is,

$$|f_n| = n(k_M - k_m).$$

Now, by Remark 8, for all large enough n there exists $\gamma < 1$ such that $|f_n|_b \le \gamma |f_n|$. From (4) we obtain

$$\mathcal{P}^{\mathrm{ab}}_{\mathbf{Y}}(|v_n|) \ge |v_n|_b - |u_n f_n|_b \ge (1-\gamma)(k_M - k_m)n.$$

Further, from the above lemma, we have

$$|v_n| = |v_n|_a + |v_n|_b = \frac{|\varphi(a)| - 1}{(\alpha - 1)^2}\alpha^n + \mathcal{O}(n) = \Theta(\alpha^n).$$

We thus conclude that $\mathcal{P}^{\mathrm{ab}}_{\mathbf{Y}}(|v_n|) = \Omega(\log |v_n|)$, whence $\mathcal{U}^{\mathrm{ab}}_{\mathbf{Y}}(n) = \Omega(\log n)$. □

We have shown that aperiodic words fixed by a morphism of the form (2) have Abelian complexity which fluctuates between constant and logarithmic growth. This completes the classification of the Abelian complexities of pure morphic words fixed by non-primitive binary morphisms. Further, the classification of upper Abelian complexities of pure morphic binary words is completed.

5. On Families of Words Having Asymptotically the Same Abelian Complexity

In this section, we extend our analysis to morphic binary words. In particular, we are interested in morphic words having Abelian complexity of the order $\Theta(n^{p/q})$, $p < q$.

We note that, in the case of pure morphic binary words, one can achieve such Abelian complexity with both primitive and non-primitive morphisms. Primitive morphisms having adjacency matrix, e. g., of the form

$$\begin{pmatrix} 2^q - 1 & 2^q - 2^p - 1 \\ 1 & 2^p + 1 \end{pmatrix}, \quad p < q,$$

yield words with $\mathcal{U}^{\mathrm{ab}}(n) = \Theta\left(n^{p/q}\right)$ (Item (ii) of Theorem 3). On the other hand, non-primitive morphisms having adjacency matrix of the form, e. g.,

$$\begin{pmatrix} 2^p & 0 \\ s & 2^q \end{pmatrix}, \; p < q, \; s \ge 1,$$

yield fixed points having $\mathcal{P}^{\text{ab}}(n) = \Theta\left(n^{p/q}\right)$ (Item (II) of Theorem 5, third point). What is worth noting is that both of the above types of words give $\Theta(n)$ factor complexity:

Lemma 16. *Let $y = \varphi^{\omega}(a)$ for some binary morphism φ. If*

$$\mathcal{P}_y^{ab}(n) = \Theta(n^r)$$

for some $r \in \mathbb{Q}$, $0 < r < 1$, then

$$\mathcal{P}_y(n) = \Theta(n).$$

Proof. Note that y is necessarily aperiodic. We show that the morphism φ is *everywhere-growing* and *quasi-uniform* (for definitions see [15, 16], or [3, Definitions 4.7.35 and 4.7.39]), that is, there exists $\beta > 1$ such that $|\varphi^n(a)|, |\varphi^n(b)| = \Theta(\beta^n)$ as n tends to infinity. The claim follows, as aperiodic fixed points of everywhere-growing quasi-uniform morphisms have linear factor complexity by Pansiot's result [15].

It is a simple exercise to show that a primitive morphism φ is everywhere-growing and quasi-uniform. If φ is non-primitive, then, by Theorems 5 and 6, $|\varphi(a)|_a < |\varphi(b)|_b$. It is simple to see that in this case the parameter β above equals $|\varphi(b)|$, and $|\varphi(b)| > 1$. □

The main result of this section is the following.

Theorem 17. *For each pair $p, q \in \mathbb{N}$, $p < q$, there exists a sequence of morphic binary words $(y_s)_{s\in\mathbb{N}}$ satisfying*

$$\mathcal{P}_{y_s}^{ab}(n) = \Theta(n^{p/q}) \quad \textit{and} \quad \mathcal{P}_{y_{s+1}}(n) = o\left(\mathcal{P}_{y_s}(n)\right).$$

There thus exists a family of morphic binary words having the same asymptotic Abelian complexity while the asymptotic factor complexities are different. We shall construct such sequences for each pair $p, q \in \mathbb{N}$.

5.1. The Construction and Initial Properties

We first fix the notation of the remainder of the section. For convenience, we use the infinite alphabet

$$\Sigma_{\mathbb{N}} = \{\ a_i \mid i \geq 0\ \}$$

indexed by the natural numbers. Let also

$$\Sigma_s = \{a_0, \ldots, a_s\}$$

and

$$\Gamma_s = \{\ a_i \mid i \geq s\ \}.$$

Define then the morphism

$$\gamma : \Sigma_{\mathbb{N}} \to \Sigma_{\mathbb{N}}^*$$

by

$$\gamma(a_0) = a_0 \text{ and}$$
$$\gamma(a_r) = a_r a_{r-1} \text{ for } r \geq 1.$$

Further, for each $s \in \mathbb{N}$, we define the morphism

$$\sigma_s : \Sigma_{\mathbb{N}} \to \mathbb{B}^*$$

by

$$\sigma_s(a_r) = \begin{cases} b & \text{if } a_r \in \Gamma_s \text{ and} \\ a & \text{otherwise.} \end{cases}$$

Now the infinite fixed point $\gamma^\omega(a_r)$ exists for each $r \geq 1$. For the remainder of this section we set, for each $r \geq 1$,

$$\mathbf{X}_r = \gamma^\omega(a_r).$$

Further, we let $\mathbf{X}_{s,r}$ denote the morphic word $\sigma_s(\mathbf{X}_r)$ for all $s \geq 0$, $r \geq 1$.

Example 18. We illustrate the words defined above. (Here we identify a_i with i for $i = 0, 1, 2, 3$.)

$$\begin{aligned} \mathbf{X}_3 &= 32212110211010021101001000211010010001000002\cdots \\ \mathbf{X}_{1,3} &= bbbbbbbabbbabaabbbabaabaaabbbabaabaaabaaaab\cdots \\ \mathbf{X}_{2,3} &= bbbabaaabaaaaaabaaaaaaaaaabaaaaaaaaaaaaaab\cdots . \end{aligned}$$

We recall that words similar to the words $\mathbf{X}_{s,s+1}$ have been studied previously [16] (see also [3, Subsection 4.7.1]) as examples of morphic, but not pure morphic, words.

The aim is to prove the following proposition.

Proposition 19. *Let $r, s \in \mathbb{N}$ with $1 \leq s < r$, and let $x = \mathbf{X}_{s,r}$. Then*
(I) $\mathcal{P}_x(n) = \Theta\left(n^{1+1/s}\right)$ *and*
(II) $\mathcal{P}_x^{ab}(n) = \Theta\left(n^{1-s/r}\right)$.

Let us first see how Theorem 17 follows from the above proposition.

Proof of Theorem 17. Let us fix $p, q \in \mathbb{N}$, $1 \leq p < q$. For each $s \geq 1$, let

$$y_s = \mathbf{X}_{s(q-p),sq}.$$

By Proposition 19, y_s has

$$\mathcal{P}_{y_s}(n) = \Theta\left(n^{1+1/s(q-p)}\right) \text{ and}$$
$$\mathcal{P}_{y_s}^{\text{ab}}(n) = \Theta\left(n^{1-s(q-p)/sq}\right) = \Theta\left(n^{p/q}\right).$$

It is clear that $\mathcal{P}_{y_{s+1}}(n) = o\left(\mathcal{P}_{y_s}(n)\right)$ as claimed. □

Before proving the above proposition, we shall list some properties of γ and its fixed points $\mathbf{X}_r$, $r \geq 1$. For this, we let

$$\Lambda_{m,l,r} = \prod_{i=m}^{l} \gamma^i(a_{r-1})$$

for all $m, r \in \mathbb{N}$ and $l \in \mathbb{N} \cup \{\infty\}$, with $r \geq 1$ and $l \geq m$. For technical reasons, we also allow $l = m - 1$, and we set $\Lambda_{m,m-1,r} = \varepsilon$.

Lemma 20. *The following properties hold for all $r \geq 1$.*

(I) $\mathbf{X}_r \in a_r \Sigma_{r-1}^{\omega}$. *In particular,* $\mathbf{X}_1 = a_1 a_0^{\omega}$ *and* $\mathbf{X}_{r,r} = ba^{\omega}$.

(II) *For all $n, m \in \mathbb{N}$, $n \geq m \geq 0$, we have*

$$\gamma^n(a_r) = a_r \Lambda_{0,n-1,r} = \gamma^m(a_r)\Lambda_{m,n-1,r} \quad \textit{and} \quad \mathbf{X}_r = \gamma^n(a_r)\Lambda_{n,\infty,r}.$$

(III) $F(\mathbf{X}_t) \subseteq F(\mathbf{X}_r)$ *and* $F(\mathbf{X}_{s,t}) \subseteq F(\mathbf{X}_{s,r})$ *for all $t, r, s \in \mathbb{N}$, $1 \leq t \leq r$.*

Proof. Item (I) is clear by the definition of γ and Item (II) is easily shown by induction. Item (III) is immediate by Item (II). □

To simplify notation, we define, for all $r \in \mathbb{N}$, the functions $p_r, p_{s,r} : \mathbb{N} \to \mathbb{N}$ by

$$p_r(n) = |\gamma^n(a_r)| \text{ and}$$
$$p_{s,r}(n) = |\gamma^n(a_r)|_{a_s}.$$

Thus,

$$p_r(n) = \sum_{i=0}^{r} p_{i,r}(n).$$

Lemma 21. *Let $r, s \in \mathbb{N}$ with $r \geq s \geq 0$ and $r > 0$. Then,*

(I) $p_{s,r}(n) = \binom{n}{r-s} = \frac{1}{(r-s)!} n^{r-s} + \mathcal{O}(n^{r-s-1})$ *and*

(II) $p_r(n) = \sum_{i=0}^{r} \binom{n}{i} = \frac{1}{r!} n^r + \mathcal{O}(n^{r-1})$.

Proof. For each $s \in \Sigma_r$, let $\mathbf{I}_s$ denote the $(r+1) \times (r+1)$ matrix having the entry $a_{ij} = 1$ if $j = i + s$ and $a_{ij} = 0$ otherwise. It is easy to check that $\mathbf{I}_1^t = \mathbf{I}_t$ for each $t = 1, \ldots, r$, that $\mathbf{I}_1^t = \mathbf{0}$ for $t \geq r$, and that $\mathbf{I}_0$ is the identity matrix.

Consider then the adjacency matrix $A_{\gamma,r}$ of γ restricted to the alphabet Σ_r (the top-left entry being $|\gamma(a_0)|_{a_0}$ while the bottom-right entry being $|\gamma(a_r)|_{a_r}$). We have $A_{\gamma,r} = \mathbf{I}_0 + \mathbf{I}_1$ so that

$$A_{\gamma,r}^n = \sum_{i=0}^{r} \binom{n}{i} \mathbf{I}_1^i = \sum_{i=0}^{r} \binom{n}{i} \mathbf{I}_i.$$

The rightmost column contains the entries $|\gamma^n(a_r)|_{a_i}$ for $i = 0, \ldots, r$, whence

$$p_{s,r}(n) = A^n_{\gamma,r}[s,r] = \binom{n}{r-s} = \frac{1}{(r-s)!}n^{r-s} + \mathcal{O}(n^{r-s-1}).$$

Finally,

$$p_r(n) = \sum_{i=0}^{r} \binom{n}{i} = \frac{1}{r!}n^r + \mathcal{O}(n^{r-1}).$$

The claims follow. □

5.2. Analysing the Factor Complexity

We shall first analyse the factor complexity of $\mathbf{X}_{s,r}$ for any pair $1 \leq s < r$. Our aim is to prove Proposition 19, Part (I). We start with a technical lemma.

Lemma 22. *Let $0 \leq s \leq r$. Then $\sigma_s\left(\gamma^n(a_r)\right)$ ends with*

$$ba^{p_s(n-r+s)-1}$$

for all $n > r - s$.

Proof. Let s be fixed. We shall prove the claim by induction on r. The base case $r = s$ is trivial, as

$$\sigma_s(\gamma^n(a_s)) = ba^{p_s(n)-1}$$

for all $n \geq 1$. Suppose the claim is true for r and consider the case of $r+1$. Let $n > r+1-s$. By Item (II) of Lemma 20, $\gamma^n(a_{r+1})$ ends with $\gamma^{n-1}(a_r)$. As $n-1 > r-s$, the induction hypothesis asserts that $\sigma_s(\gamma^{n-1}(a_r))$ ends with

$$ba^{p_s(n-(r+1)+s)-1}.$$

We have thus completed the induction step. □

We are in the position to analyse the factor complexity.

Proof of Proposition 19, Item (I). Let $s \geq 1$ be fixed. We prove, by induction on r, that $\mathbf{X}_{s,r}$ has the claimed factor complexity. The base case $r = s+1$ is a result in [16] (see also [3, Proposition 4.7.2]). Suppose then that the claim is true for the case r and consider the word $\mathbf{X}_{s,r+1}$. Let us fix n and estimate the size of

$$F_n(\mathbf{X}_{s,r+1}) \setminus F_n(\mathbf{X}_{s,r}).$$

Factorize $\mathbf{X}_{s,r+1}$ into three parts

$$\mathbf{X}_{s,r+1} = \sigma_s\left(a_{r+1}\Lambda_{0,k_1,r+1}\right) \cdot \sigma_s\left(\Lambda_{k_1+1,k_2,r+1}\right) \cdot \sigma_s\left(\Lambda_{k_2+1,\infty,r+1}\right),$$

where k_1 is minimal in the sense that $p_r(k_1) \geq n$ and k_2 is minimal in the sense that $\sigma_s(\gamma^{k_2}(a_r))$ ends with at least n a's. By Lemma 21,

$$p_t(x) = \frac{1}{t!}x^t + \mathcal{O}(x^{t-1})$$

for each $t \in \mathbb{N}$, so that $k_1 = \Theta(n^{1/r})$ and, by the above lemma, $k_2 = \Theta(n^{1/s})$.

Consider first the prefix. We first note that, by Lemma 20,

$$a_{r+1}\Lambda_{0,k_1,r+1} = \gamma^{k_1+1}(a_{r+1}).$$

Trivially,

$$|F_n\left(\sigma_s\left(a_{r+1}\Lambda_{0,k_1,r+1}\right)\right)| \leq |\gamma^{k_1+1}(a_{r+1})|$$

and we obtain, by Lemma 21, the rough upper bound

$$|\gamma^{k_1+1}(a_{r+1})| = \tfrac{1}{(r+1)!}k_1^{r+1} + \mathcal{O}(k_1^r) = \mathcal{O}\left(n^{1+1/r}\right).$$

Consider then the factors occurring in $\sigma_s(\Lambda_{k_1+1,k_2,r+1})$. Now any factor occurring in $\sigma_s(\gamma^i(a_r))$ occurs already in $\mathbf{X}_{s,r}$. By the choice of k_1, it suffices to consider factors that are of the form $\sigma_s(u_1u_2)$, where

$$\begin{aligned} &u_1 \in \mathrm{suff}(\gamma^i(a_r)) \text{ and} \\ &u_2 \in \mathrm{pref}(\gamma^{i+1}(a_r)) \end{aligned}$$

for some i satisfying $k_1 \leq i < k_2$. For each such i, there are at most $n-1$ choices of u_1 and u_2, and we obtain the upper bound

$$\sum_{i=k_1}^{k_2} n = n\mathcal{O}(n^{1/s}) = \mathcal{O}(n^{1+1/s}).$$

Finally, the factors occurring in the infinite tail have already been counted previously, either as factors of $\mathbf{X}_{s,r}$, or as a prefix of $\mathbf{X}_{s,r}$ preceded by a block of a's. We conclude, by the induction hypothesis,

$$\mathcal{P}_{\mathbf{X}_{s,r+1}}(n) = \mathcal{P}_{\mathbf{X}_{s,r}}(n) + \mathcal{O}(n^{1+1/s}) + \mathcal{O}(n^{1+1/r}) = \Theta\left(n^{1+1/s}\right).$$

□

5.3. Analysing the Abelian Complexity

We shall secondly analyse the Abelian complexity of $\mathbf{X}_{s,r}$ for $1 \leq s < r$. Our aim is to prove Proposition 19, Part (ii), the following lemma being crucial in doing so. In what follows, for $w \in \Sigma_{\mathbb{N}}^*$ and $s \in \mathbb{N}$, we let

$$|w|_{\Gamma_s} = \sum_{a\in\Gamma_s} |w|_a.$$

Lemma 23. *Let $1 \leq s \leq r$ and let $n, m \in \mathbb{N}$. Then*

$$|v|_{\Gamma_s} \leq |\operatorname{pref}_n(\Lambda_{m,\infty,r})|_{\Gamma_s}$$

for all $v \in F_n(\Lambda_{m,\infty,r})$. Further,

$$|\operatorname{pref}_n(\mathbf{X}_r)|_{\Gamma_s} = \max_{v \in F_n(\mathbf{X}_r)} |v|_{\Gamma_s}$$

for all $n \in \mathbb{N}$.

Proof. We prove these claims, for any fixed $s \geq 1$, by induction on r. Both of these are trivial for the base case $r = s$. Suppose the claims are true for some $r \geq s$, and consider the case of $r + 1$. Let n be fixed. We start by proving the following:

Claim 24. *If $v \in F_n(\Lambda_{m,\infty,r+1})$ is of the form*

$$v = e\Lambda_{m+1,l,r} f \tag{5}$$

for some $l, m \in \mathbb{N}$ with $l \geq m \geq 0$, $e \in \operatorname{suff}(\gamma^m(a_r))$, and $f \in \operatorname{pref}(\gamma^{l+1}(a_r))$, then

$$|v|_{\Gamma_s} \leq |\operatorname{pref}_n(\Lambda_{m,\infty,r+1})|_{\Gamma_s}.$$

Proof. Let

$$v \in F_n(\Lambda_{m,\infty,r+1})$$

be as in (5). Let

$$z \in \operatorname{pref}(\gamma^l(a_r) f)$$

so that

$$|\Lambda_{m,l-1,r+1} z| = |v|.$$

Thus, $\gamma^m(a_r) = ue$ for some $u \in \Sigma_r^*$ and

$$|z| = |\gamma^l(a_r)| + |f| - |u|.$$

Note that these notations are valid for the technical case $l = m$ also. The situation is illustrated in Figure 1.

Suppose first that $|z| \geq |\gamma^l(a_r)|$ whence $|u| \leq |f|$ and thus u is a prefix of f. In Figure 1, this corresponds to z ending at point 2). Let $v' = \operatorname{suff}_{|u|}(f)$. We have

$$|\Lambda_{m,l-1,r+1} z|_{\Gamma_s} - |v|_{\Gamma_s} = |u|_{\Gamma_s} - |v'|_{\Gamma_s} \geq 0$$

by applying the induction hypothesis to $u \in \operatorname{pref}(\mathbf{X}_r)$ and $v' \in F(\mathbf{X}_r)$.

Suppose then that $|z| < |\gamma^l(a_r)|$ whence $|f| < |u|$ and f is a proper prefix of u. In Figure 1, this corresponds to z ending at point 1). If $e = \varepsilon$ and $l = m$, then

$$z = v = f \in \operatorname{pref}(\gamma^m(a_r))$$

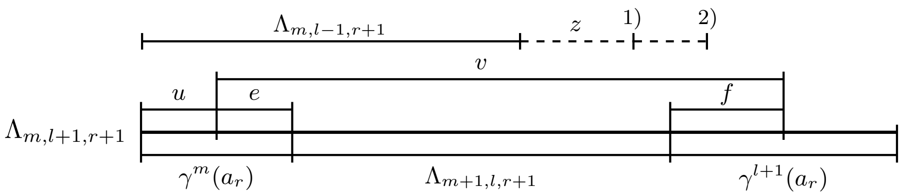

Figure 1: The words v and $\Lambda_{m,l-1,r+1}z$ in the proof of Claim 24.
Here z ends at point 1) if $|z| < |\gamma^l(a_r)|$, otherwise z ends at point 2).
If $l = m$, then $\Lambda_{m+1,l,r+1} = \Lambda_{m,l-1,r+1} = \varepsilon$, $v = ef$, and z is a prefix of uef.

and there is nothing to prove. Assume then that either $e \neq \varepsilon$ or $l > m$. Let

$$v' = \mathrm{suff}_{|u|-|f|}(\gamma^l(a_r)).$$

If $f = \varepsilon$, then we have

$$|\Lambda_{m,l-1,r+1}z|_{\Gamma_s} - |v|_{\Gamma_s} = |u|_{\Gamma_s} - |v'|_{\Gamma_s} \geq 0$$

by applying the induction hypothesis to $v' \in F(\mathbf{X}_r)$ and $u \in \mathrm{pref}(\mathbf{X}_r)$.

We are left with the case of f being a non-empty proper prefix of u. Write $u = fu'$ for some $u' \in \Sigma_r^+$, whence

$$|\Lambda_{m,l-1,r+1}z|_{\Gamma_s} - |v|_{\Gamma_s} = |u'|_{\Gamma_s} - |v'|_{\Gamma_s}.$$

Hence, to conclude the proof, it suffices to show that

$$|u'|_{\Gamma_s} \geq |v'|_{\Gamma_s}.$$

There exist $m_1 \in \mathbb{N}$, $0 \leq m_1 < m$, and words $g_1, g_2 \in \Sigma_r^*$ such that

$$f = \gamma^{m_1}(a_r)g_1 \quad \text{and} \quad \gamma^{m_1+1}(a_r) = \gamma^{m_1}(a_r)\gamma^{m_1}(a_{r-1}) = fg_2,$$

that is, $g_1g_2 = \gamma^{m_1}(a_{r-1})$. Now, by Item (ii) of Lemma 20,

$$\gamma^l(a_r) = \gamma^{m_1}(a_r)\Lambda_{m_1,l-1,r}.$$

We may thus write

$$g_1u' = \mathrm{pref}_{|g_1u'|}(\Lambda_{m_1,l-1,r}) \in F(\mathbf{X}_r).$$

Observe now that

$$\gamma^{l+1}(a_r) = \gamma^l(a_r)\gamma^l(a_{r-1}).$$

Since

$$v' \in \mathrm{suff}(\gamma^l(a_r)) \text{ and}$$
$$g_1 \in \mathrm{pref}(\gamma^{m_1}(a_{r-1})) \subseteq \mathrm{pref}(\gamma^l(a_{r-1})),$$

it follows that we may write

$$v'g_1 = e'\Lambda_{m_2,l-1,r}g_1 \in F(\Lambda_{m_1,\infty,r}),$$

where m_2 is minimal and $e' \in \operatorname{suff}(\gamma^{m_2-1}(a_r))$. Note that $m_2 > m_1$ since $e \neq \varepsilon$ or $l > m$. We apply the induction hypothesis on $v'g_1$ and g_1u to obtain

$$|v'g_1|_{\Gamma_s} \leq |g_1u'|_{\Gamma_s},$$

from which it follows that $|u'|_{\Gamma_s} \geq |v'|_{\Gamma_s}$. This concludes the proof of Claim 24. □□

From Claim 24, it follows that

$$|\operatorname{pref}_n(\Lambda_{m',\infty,r+1})|_{\Gamma_s} \leq |\operatorname{pref}_n(\Lambda_{m,\infty,r+1})|_{\Gamma_s}$$

for all $m' > m$. Indeed, since $\operatorname{pref}_n(\Lambda_{m',\infty,r+1})$ has a factorization of the form (5) (with m' in the role of $m+1$ and $e = \varepsilon$), we obtain

$$|\operatorname{pref}_n(\Lambda_{m',\infty,r+1})|_{\Gamma_s} \leq |\operatorname{pref}_n(\Lambda_{m'-1,\infty,r+1})|_{\Gamma_s} \leq \ldots \leq |\operatorname{pref}_n(\Lambda_{m,\infty,r+1})|_{\Gamma_s}.$$

Assume now that $v \in F_n(\Lambda_{m,\infty,r+1})$ has a factorization of the form $v = e\Lambda_{m'+1,l',r+1}f$ for some $l' \geq m' \geq m$, $e \in \operatorname{suff}(\gamma^{m'}(a_r))$, and $f \in \operatorname{pref}_n(\gamma^{l'+1}(a_r))$. By Claim 24 and the previous observation, we have

$$|v|_{\Gamma_s} \leq |\operatorname{pref}_n(\Lambda_{m',l',r+1}f)|_{\Gamma_s} \leq |\operatorname{pref}_n(\Lambda_{m,\infty,r+1})|_{\Gamma_s}.$$

If, on the other hand, $v \in F_n(\Lambda_{m,\infty,r+1})$ has no factorization of the form (5), then $v \in F(\mathbf{X}_r)$. By the induction hypothesis and the above observation, we have

$$|v|_{\Gamma_s} \leq |\operatorname{pref}_n(\mathbf{X}_r)|_{\Gamma_s} = |\operatorname{pref}_n(\Lambda_{m',\infty,r+1})|_{\Gamma_s} \leq |\operatorname{pref}_n(\Lambda_{m,\infty,r+1})|_{\Gamma_s},$$

where m' is minimal such that

$$|\gamma^{m'}(a_r)| \geq n.$$

We have proved that, for all $v \in F(\Lambda_{m,\infty,r+1})$,

$$|v|_{\Gamma_s} \leq |\operatorname{pref}_n(\Lambda_{m,\infty,r+1})|_{\Gamma_s},$$

that is, the first part of Lemma 23. It remains to prove that

$$|\operatorname{pref}_n(\mathbf{X}_{r+1})|_{\Gamma_s} = \max_{v \in F_n(\mathbf{X}_{r+1})} |v|_{\Gamma_s}.$$

But this is trivial since for all

$$v \in F_n(\mathbf{X}_{r+1}) \setminus \{\operatorname{pref}_n(\mathbf{X}_{r+1})\} = F_n(\Lambda_{0,\infty,r+1}),$$

we have

$$|v|_{\Gamma_s} \leq |\operatorname{pref}_n(\Lambda_{0,\infty,r+1})|_{\Gamma_s} = |\operatorname{pref}_n(a_{r+1}^{-1}\mathbf{X}_{r+1})|_{\Gamma_s} \leq |\operatorname{pref}_n(\mathbf{X}_{r+1})|_{\Gamma_s}.$$

We have thus completed the induction step, completing the proof of Lemma 23. □

Proof of Proposition 19, Item (ii). We now complete the proof by analysing the Abelian complexity of $\mathbf{X}_{s,r}$. Note that $\mathcal{P}^{\mathrm{ab}}_{\mathbf{X}_{s,r}}$ is monotonously increasing, since $\min_{\mathbf{X}_{s,r},b}(n) = 0$ for all $n \in \mathbb{N}$. By Lemmas 21 and 23, we have

$$\mathcal{P}^{\mathrm{ab}}_{\mathbf{X}_{r,s}}(p_r(k)) = |\gamma^k(a_r)|_{\Gamma_s} + 1 = \tfrac{1}{(r-s)!}k^{r-s} + \mathcal{O}(k^{r-s-1}).$$

In other words, we have

$$\mathcal{P}^{\mathrm{ab}}(n_k) = \Theta(n_k^{1-s/r})$$

for a sequence (n_k) of indices. Note also that there exists $\alpha \in \mathbb{R}$ such that $n_{k+1} \leq \alpha n_k$ for all large enough k. Let now $n \in \mathbb{N}$, such that $n_k < n \leq n_{k+1}$ for some large enough $k \in \mathbb{N}$. Now there exist $C_1, C_2 \in \mathbb{R}$ such that

$$\mathcal{P}^{\mathrm{ab}}(n) \leq \mathcal{P}^{\mathrm{ab}}(n_{k+1}) \leq C_1 n_{k+1}^{1-s/r} \leq C_1\alpha^{1-s/r}n^{1-s/r} \text{ and}$$
$$\mathcal{P}^{\mathrm{ab}}(n) \geq \mathcal{P}^{\mathrm{ab}}(n_k) \geq C_2 n_k^{1-s/r} \geq \tfrac{C_2}{\alpha^{1-s/r}}n^{1-s/r}.$$

Thus $\mathcal{P}^{\mathrm{ab}}(n) = \Theta(n^{1-s/r})$. □

6. Conclusions

We completed the classification of the asymptotic Abelian complexities of pure morphic binary words fixed by non-primitive morphisms. We note that the classification of lower Abelian complexities remains open for primitive pure morphic binary words. It is worth mentioning that the lower Abelian complexities of a large family of uniform binary morphisms is obtained in [2].

Classifying the Abelian complexities for primitive pure morphic words over larger alphabets remains totally open. The methods used here are specific to binary words and cannot be applied to larger alphabets directly. More precisely, the techniques rely on the equivalence of the balance function and the Abelian complexity of binary words. For larger alphabets, the link is not that clear.

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