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Research article

New fractional integral inequalities for preinvex functions involving Caputo-Fabrizio operator

Muhammad Tariq[1], **Hijaz Ahmad**[2,3], **Abdul Ghafoor Shaikh**[4], **Soubhagya Kumar Sahoo**[5], **Khaled Mohamed Khedher**[6,7] **and Tuan Nguyen Gia**[8,*]

[1] Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro 76062, Pakistan

[2] Istanbul Ticaret University, Information Technology Application and Research Center, Istanbul, Turkey

[3] Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39,00186 Roma, Italy

[4] Department of Basic Sciences and Related Studies, Quest NawabShah, Pakistan

[5] Department of Mathematics, Institute of Technical Education and Research, Siksha 'O' Anusandhan University, Bhubaneswar 751030, Odisha, India

[6] Department of Civil Engineering, College of Engineering, King Khalid University, Abha 61421, Saudi Arabia

[7] Department of acivil Engineering, High Institute of Technological Studies, Mrezgua University Campus, Nebeul 8000, Tunusia

[8] Department of Computing, University of Turku, 20500 Turku, Finland

* **Correspondence:** Email: tuan.nguyengia@utu.fi.

Abstract: It's undeniably true that fractional calculus has been the focus point for numerous researchers in recent couple of years. The writing of the Caputo-Fabrizio fractional operator has been on many demonstrating and real-life issues. The main objective of our article is to improve integral inequalities of Hermite-Hadamard and Pachpatte type incorporating the concept of preinvexity with the Caputo-Fabrizio fractional integral operator. To further enhance the recently presented notion, we establish a new fractional equality for differentiable preinvex functions. Then employing this as an auxiliary result, some refinements of the Hermite-Hadamard type inequality are presented. Also, some applications to special means of our main findings are presented.

Keywords: Caputo-Fabrizio fractional integral; preinvex functions; Hermite-Hadamard inequality; Hölder inequality; Hölder-İşcan inequality; Power-mean inequality
Mathematics Subject Classification: 26A51, 26A33, 26D07, 26D10, 26D15

1. Introduction

The term "convexity" is a subject of many mathematicians' research in the last century. This term has assumed a key part and has gotten exceptional consideration by numerous scientists in the improvement of different fields of pure and applied sciences. The theory of convexity portrays a crucial role in the field of financial mathematics, mathematical statistics, and functional analysis. Optimization of convex functions has many practical applications (circuit design, controller design, modeling, etc.). Due to a lot of uses and importance, the term "convexity" has become a rich factor of inspiration and mesmerizing field for scientists and mathematicians. We encourage the interested readers to see the references [1–7] for some discussion about convexity and its properties.

The term inequalities along with convexity property play an essential part in the present-day mathematical investigations. Both terminologies are closely related to each other. The term inequalities have a wide range of importance in mechanics, functional analysis, probability, numerical quadrature formulas, and statistical problems. In this manner, the hypothesis of inequalities might be viewed as an autonomous field of mathematical analysis. Interested readers can refer to [8–11].

Nowadays, the theory of inequality and fractional analysis have shown synchronous development. Fractional calculus has become a popular and promising research field in the past few decades in the diverse field of applied sciences. Some mathematicians have utilized newly introduced fractional derivatives and integrals with variant views and perspectives to be examined and solved by real-life problems in the various fields of applied sciences. Fractional calculus can be understood precisely by knowing some of the simple mathematical definitions like Gamma function, Beta function, Laplace transform, and Mittag-Leffler function. Probably, the first logical definition of a fractional derivative was given by Joseph Liouville and he published approximately nine papers on the fractional calculus between 1832 and 1837 and the last was in 1855. Probably the first application of fractional calculus was made by N. H. Abel during the year 1802–1829.

In this field, numerous mathematicians have concentrated on presenting new fractional operators and modeling that bring off real-world issues depending on their properties. The properties that make the various operators different from one another incorporate locality and singularity. The concept of Caputo owns several impressive characteristics and acknowledges traditional initial and boundary conditions to be incorporated in the problem formulation. Consequently, Caputo and Fabrizio in [12] studied a new fractional operator known as Caputo-Fabrizio fractional operator. The avocation behind introducing this new sort of derivative was to search for fractional derivatives with the nonsingular kernel and without the Gamma function. The feature of the said operator is exceptionally compelling in portraying heterogeneousness and frameworks with various scales with memory impacts, hence it is utilized in the investigation of many real-life modeling problems. Starting now and into the foreseeable future various experts have inspected and applied this new fractional operator for modelling of COVID-19 [13], modelling of Hepatitis-B epidemic [14], groundwater flow [15], and integro-differential equations [16–19]. Several scientists also worked on the generalized Atangana-Baleanu operator for fuzzy hybrid systems (see [20, 21]).

The rest of our article has the following organization. In Section 2, we briefly review some basic concepts and notions about preinvexity and fractional operators. We devote Section 3 to present new versions of Hermite-Hadamard type integral inequality and Pachpatte type integral inequalities with the aid of Caputo-Fabrizio fractional operator for preinvex function. Section 4 deals with the main findings, we establish an integral identity and employing this identity as the auxiliary result, some refinements of Hermite-Hadamard type inequality are discussed. In Section 5, we prove the usefulness of the main findings through applications to special means. Conclusion and future scopes are discussed in the last Section 6 .

2. Preliminaries

In this section, we recall some known concepts.

In the year 1994, Mititelu [22] investigated and explored the terminology of invex set, which is defined as

Definition 2.1. Let $\eta : \mathbb{X}\times\mathbb{X} \neq \emptyset \to \mathbb{R}$ be a real valued function, then $\mathbb{X}$ is said to be invex with respect to $\eta(.,.)$ if $\mathtt{g}_1 + \mathfrak{b}\eta(\mathtt{g}_2, \mathtt{g}_1) \in \mathbb{X}$, $\forall\ \mathtt{g}_1, \mathtt{g}_2 \in \mathbb{X}$ and $\mathfrak{b} \in [0, 1]$.

Note: The concept of invex set is more general than convex set. Means that every invex set is not convex but but the converse is true with the help of $\eta(\mathtt{g}_1, \mathtt{g}_2) = \mathtt{g}_1 - \mathtt{g}_2$ (see [22] and [23]).

In the year 1988, Weir and Mond [24] utilized the concept of invex set to investigate the concept of preinvexity.

Definition 2.2. [24] Let $\mathbb{X} \neq \emptyset \in \mathbb{R}$ be an invex set with respect to $\eta : \mathbb{X} \times \mathbb{X} \neq \emptyset \to \mathbb{R}$. Then the function $\Upsilon : \mathbb{X} \to \mathbb{R}$ is said to be preinvex with respect to η if

$$\Upsilon(\mathtt{g}_1 + \mathfrak{b}\eta(\mathtt{g}_2, \mathtt{g}_1)) \leq (1 - \mathfrak{b})\Upsilon(\mathtt{g}_1) + \mathfrak{b}\Upsilon(\mathtt{g}_2), \qquad \forall \mathtt{g}_1\,,\ \mathtt{g}_2 \in \mathbb{X}\,,\ \ \mathfrak{b} \in [0, 1]\,.$$

Note: The above function Υ is said to be preincave if and only if $-\psi$ is preinvex.

We can clearly see that every preinvex function is not convex but every convex function is preinvex by using the property of $\eta(\mathtt{g}_2, \mathtt{g}_1) = \mathtt{g}_2 - \mathtt{g}_1$ (see [25]). Many researchers proved that the concept of the preinvexity has interesting importance in the theory of optimization and mathematical programming.

In the year 2007, Noor [26] established a new version of the Hermite-Hadamard inequality for preinvex functions:

Theorem 2.1. *Let $\Upsilon : \mathbb{X} = [\mathtt{g}_1, \mathtt{g}_1 + \eta(\mathtt{g}_2, \mathtt{g}_1)] \to (0, \infty)$ be a preinvex function on the interval of real numbers $\mathbb{X}^\circ$ and $\mathtt{g}_1, \mathtt{g}_2 \in \mathbb{X}$ with $\mathtt{g}_1 < \mathtt{g}_1 + \eta(\mathtt{g}_2, \mathtt{g}_1)$. Then*

$$\Upsilon(\frac{2\mathtt{g}_1 + \eta(\mathtt{g}_2, \mathtt{g}_1)}{2}) \leq \frac{1}{\eta(\mathtt{g}_2, \mathtt{g}_1)} \int_{\mathtt{g}_1}^{\mathtt{g}_1+\eta(\mathtt{g}_2,\mathtt{g}_1)} \Upsilon(x)dx \leq \frac{\Upsilon(\mathtt{g}_1) + \Upsilon(\mathtt{g}_2)}{2}.$$

In the year 2011, Dragomir [27] examined the Hermite-Hadamard type inequality for differentiable preinvex function, which is stated as:

Theorem 2.2. *Suppose $\mathbb{X} \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$. Suppose $\Upsilon : \mathbb{X} \to \mathbb{R}$ is a differentiable function. If $|\Upsilon'|$ is preivex on $\mathbb{X}$ then, for every $\mathtt{g}_1, \mathtt{g}_2 \in \mathbb{A}$ with $\eta(\mathtt{g}_2, \mathtt{g}_1) \neq 0$. Then*

$$\left|\frac{\Upsilon(\mathtt{g}_1) + \Upsilon(\mathtt{g}_1 + \eta(\mathtt{g}_2, \mathtt{g}_1))}{2} - \frac{1}{\eta(\mathtt{g}_2, \mathtt{g}_1)} \int_{\mathtt{g}_1}^{\mathtt{g}_1+\eta(\mathtt{g}_2,\mathtt{g}_1)} \Upsilon(x)dx\right| \leq \frac{|\eta(\mathtt{g}_2, \mathtt{g}_1)|}{8} [\Upsilon(\mathtt{g}_1) + \Upsilon(\mathtt{g}_2)]\,.$$

Lemma 2.1. *[27] Let $\Upsilon : I^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0, \mathtt{g}_1, \mathtt{g}_2 \in I^0$ with $\mathtt{g}_1 < \mathtt{g}_1 + \eta(\mathtt{g}_2, \mathtt{g}_1)$ if $\Upsilon' \in L[\mathtt{g}_1, \mathtt{g}_1 + \eta(\mathtt{g}_2, \mathtt{g}_1)]$, then*

$$\begin{aligned}&-\frac{\Upsilon(\mathtt{g}_1) + \Upsilon(\mathtt{g}_1 + \eta(\mathtt{g}_2, \mathtt{g}_1))}{2} + \frac{1}{\eta(\mathtt{g}_2, \mathtt{g}_1)} \int_{\mathtt{g}_1}^{\mathtt{g}_1+\eta(\mathtt{g}_2,\mathtt{g}_1)} \Upsilon(x)dx \\ &\leq \frac{\eta(\mathtt{g}_2, \mathtt{g}_1)}{2} \int_0^1 (1 - 2\mathfrak{b})\Upsilon'(\mathtt{g}_1 + \mathfrak{b}\eta(\mathtt{g}_2, \mathtt{g}_1))d\mathfrak{b}.\end{aligned}$$

Later, several authors examined and collaborated their perspectives on the concept of preinvexity. We suggest interested readers to follow the published articles [28–31] for to know more about the concept of preinvexity.

Definition 2.3. [32–34] Let $\Upsilon \in H'(\mathsf{g}_1, \mathsf{g}_2), \mathsf{g}_1 < \mathsf{g}_2, \lambda \in [0, 1]$, then the fractional derivative and integral of Caputo-Fabrizio sense becomes

$$\left({}^{CFC}_{\mathsf{g}_1}D^{\lambda}\Upsilon\right)(\mathfrak{b}) = \frac{B(\lambda)}{(1-\lambda)}\int_{\mathsf{g}_1}^{\mathfrak{b}} \Upsilon'(x) e^{\frac{-\lambda(\mathfrak{b}-x)^{\lambda}}{1-\lambda}} dx,$$

$$\left({}^{CF}_{\mathsf{g}_1}I^{\lambda}\Upsilon\right)(\mathfrak{b}) = \frac{(1-\lambda)}{B(\lambda)}\Upsilon(\mathfrak{b}) + \frac{\lambda}{B(\lambda)}\int_{\mathsf{g}_1}^{\mathfrak{b}} \Upsilon(x) dx,$$

$$\left({}^{CFC}D^{\lambda}_{\mathsf{g}_2}\Upsilon\right)(\mathfrak{b}) = \frac{-B(\lambda)}{(1-\lambda)}\int_{\mathfrak{b}}^{\mathsf{g}_2} \Upsilon'(x) e^{\frac{-\lambda(x-\mathfrak{b})^{\lambda}}{1-\lambda}} dx,$$

and

$$\left({}^{CF}I^{\lambda}_{\mathsf{g}_2}\Upsilon\right)(\mathfrak{b}) = \frac{(1-\lambda)}{B(\lambda)}\Upsilon(\mathfrak{b}) + \frac{\lambda}{B(\lambda)}\int_{\mathfrak{b}}^{\mathsf{g}_2} \Upsilon(x) dx.$$

where $B(\lambda) > 0$ is a normalization function that satisfies $B(0) = B(1) = 1$.

In the year 2019, Imdat İşcan [35] provided the refinements of Hölder inequality called (Hölder-İşcan integral inequality), which is stated in the following theorem.

Theorem 2.3. *Let the real two functions namely* Υ_1 *and* Υ_2 *are defined on* $[\mathsf{g}_1, \mathsf{g}_2]$ *and* $|\Upsilon_1|^q, |\Upsilon_2|^q \in L[\mathsf{g}_1, \mathsf{g}_2]$ *for* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q}$, *then*

$$\begin{aligned}
&\int_{\mathsf{g}_1}^{\mathsf{g}_2} |\Upsilon_1(x)\Upsilon_2(x)|\, dx \\
&\le \frac{1}{\mathsf{g}_2 - \mathsf{g}_1}\Bigg[\left(\int_{\mathsf{g}_1}^{\mathsf{g}_2} (\mathsf{g}_2 - x)|\Upsilon_1(x)|^p dx\right)^{\frac{1}{p}}\left(\int_{\mathsf{g}_1}^{\mathsf{g}_2} (\mathsf{g}_2 - x)|\Upsilon_2(x)|^q dx\right)^{\frac{1}{q}} \\
&+ \left(\int_{\mathsf{g}_1}^{\mathsf{g}_2} (x - \mathsf{g}_1)|\Upsilon_1(x)|^p dx\right)^{\frac{1}{p}}\left(\int_{\mathsf{g}_1}^{\mathsf{g}_2} (x - \mathsf{g}_1)|\Upsilon_2(x)|^q dx\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

In the year 2019, another team of mathematicians namely M. Kadakal, I. İşcan and H. Kadakal [36] presented the refinements of power mean inequality(commonly called Improved power mean integral inequality), which is stated in the following theorem.

Theorem 2.4. *Let* Υ_1 *and* Υ_2 *be two real functions defined on* $[\mathsf{g}_1, \mathsf{g}_2]$ *and* $|\Upsilon_1|^q, |\Upsilon_2|^q \in L[\mathsf{g}_1, \mathsf{g}_2]$ *for* $p \ge 1$, *then*

$$\begin{aligned}
&\int_{\mathsf{g}_1}^{\mathsf{g}_2} |\Upsilon_1(x)\Upsilon_2(x)|\, dx \\
&\le \frac{1}{\mathsf{g}_2 - \mathsf{g}_1}\Bigg[\left(\int_{\mathsf{g}_1}^{\mathsf{g}_2} (\mathsf{g}_2 - x)|\Upsilon_1(x)| dx\right)^{1-\frac{1}{q}}\left(\int_{\mathsf{g}_1}^{\mathsf{g}_2} (\mathsf{g}_2 - x)|\Upsilon_2(x)|^q dx\right)^{\frac{1}{q}} \\
&+ \left(\int_{\mathsf{g}_1}^{\mathsf{g}_2} (x - \mathsf{g}_1)|\Upsilon_1(x)| dx\right)^{1-\frac{1}{q}}\left(\int_{\mathsf{g}_1}^{\mathsf{g}_2} (x - \mathsf{g}_1)|\Upsilon_2(x)|^q dx\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

Note: Throughout the paper we will use $B(\lambda)$ as a normalization function.

3. Hermite-Hadamard type inequality via fractional operator

Theorem 3.1. *Let* $\Upsilon : [\mathsf{g}_1, \mathsf{g}_1 + \eta(\mathsf{g}_2, \mathsf{g}_1)] \to (0, \infty)$ *be a preinvex function on* I^0 *and* $\Upsilon \in L[\mathsf{g}_1, \mathsf{g}_1 + \eta(\mathsf{g}_2, \mathsf{g}_1)]$. *If* $\lambda \in [0, 1]$, *then the following inequality holds:*

$$\Upsilon\left(\frac{2\mathsf{g}_1 + \eta(\mathsf{g}_2, \mathsf{g}_1)}{2}\right) \leq \frac{B(\lambda)}{\lambda\eta(\mathsf{g}_2, \mathsf{g}_1)}\left[\left({}^{CF}_{\mathsf{g}_1}I^{\lambda}\Upsilon\right)(k) + \left({}^{CF}I^{\lambda}_{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon\right)(k) - \frac{2(1-\lambda)}{B(\lambda)}\Upsilon(k)\right] \leq \frac{\Upsilon(\mathsf{g}_1) + \Upsilon(\mathsf{g}_2)}{2}, \tag{3.1}$$

where $k \in [\mathsf{g}_1, \mathsf{g}_1 + \eta(\mathsf{g}_2, \mathsf{g}_1)]$.

Proof. Since Υ is a preinvex function on $[\mathsf{g}_1, \mathsf{g}_1 + \eta(\mathsf{g}_2, \mathsf{g}_1)]$, we can write

$$2\Upsilon\left(\frac{2\mathsf{g}_1 + \eta(\mathsf{g}_2, \mathsf{g}_1)}{2}\right) \leq \frac{2}{\eta(\mathsf{g}_2, \mathsf{g}_1)}\int_{\mathsf{g}_1}^{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon(x)dx = \frac{2}{\eta(\mathsf{g}_2, \mathsf{g}_1)}\left(\int_{\mathsf{g}_1}^{k}\Upsilon(x)dx + \int_{k}^{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon(x)dx\right). \tag{3.2}$$

By multiplying both sides of (3.2) with $\frac{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}{2B(\lambda)}$ and adding $\frac{2(1-\lambda)}{B(\lambda)}\Upsilon(k)$ we have

$$\begin{aligned}
&\frac{2(1-\lambda)}{B(\lambda)}\Upsilon(k) + \frac{\lambda\eta(\mathsf{g}_2, \mathsf{g}_1)}{B(\lambda)}\Upsilon\left(\frac{2\mathsf{g}_1 + \eta(\mathsf{g}_2, \mathsf{g}_1)}{2}\right)\\
&\leq \frac{2(1-\lambda)}{B(\lambda)}\Upsilon(k) + \frac{\lambda}{B(\lambda)}\left(\int_{\mathsf{g}_1}^{k}\Upsilon(x)dx + \int_{k}^{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon(x)dx\right)\\
&= \left(\frac{(1-\lambda)}{B(\lambda)}\Upsilon(k) + \frac{\lambda}{B(\lambda)}\int_{\mathsf{g}_1}^{k}\Upsilon(x)dx\right) + \left(\frac{(1-\lambda)}{B(\lambda)}\Upsilon(k) + \frac{\lambda}{B(\lambda)}\int_{k}^{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon(x)dx\right)\\
&= \left({}^{CF}_{\mathsf{g}_1}I^{\lambda}\Upsilon\right)(k) + \left({}^{CF}I^{\lambda}_{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon\right)(k).
\end{aligned} \tag{3.3}$$

This completes the proof of the first inequality (3.1). For the proof of the second inequality, we use

$$\frac{2}{\eta(\mathsf{g}_2, \mathsf{g}_1)}\int_{\mathsf{g}_1}^{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon(x)dx \leq \Upsilon(\mathsf{g}_1) + \Upsilon(\mathsf{g}_2). \tag{3.4}$$

By making the same operation with (3.2) in (3.4), we have

$$\left({}^{CF}_{\mathsf{g}_1}I^{\lambda}\Upsilon\right)(k) + \left({}^{CF}I^{\lambda}_{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon\right)(k) \leq \frac{2(1-\lambda)}{B(\lambda)}\Upsilon(k) + \frac{\lambda\eta(\mathsf{g}_2, \mathsf{g}_1)}{2B(\lambda)}\left(\Upsilon(\mathsf{g}_1) + \Upsilon(\mathsf{g}_2)\right). \tag{3.5}$$

By recognising (3.5), the proof is completed. □

Corollary 3.1. *If we put* $\eta(\mathsf{g}_2, \mathsf{g}_1) = \mathsf{g}_2 - \mathsf{g}_1$ *in Theorem 3.1, we get Theorem 2 in [37].*

Theorem 3.2. *Let* $\Upsilon_1, \Upsilon_2 : [\mathsf{g}_1, \mathsf{g}_1 + \eta(\mathsf{g}_2, \mathsf{g}_1)] \to (0, \infty)$ *be a preinvex functions. If* $\Upsilon_1\Upsilon_2 \in L[\mathsf{g}_1, \mathsf{g}_1 + \eta(\mathsf{g}_2, \mathsf{g}_1)]$, *and* $k \in [\mathsf{g}_1, \mathsf{g}_1 + \eta(\mathsf{g}_2, \mathsf{g}_1)]$. *Then the following Caputo-Fabrizio fractional integral inequality holds:*

$$\frac{2B(\lambda)}{\lambda\eta(\mathsf{g}_2, \mathsf{g}_1)}\left[\left({}^{CF}_{\mathsf{g}_1}I^{\lambda}\Upsilon_1\Upsilon_2\right)(k) + \left({}^{CF}I^{\lambda}_{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon_1\Upsilon_2\right)(k) - \frac{2(1-\lambda)}{B(\lambda)}\Upsilon_1(k)\Upsilon_2(k)\right]$$

$$\leq \frac{2}{3}M(\mathrm{g}_1,\mathrm{g}_2)+\frac{1}{3}N(\mathrm{g}_1,\mathrm{g}_2),$$

where

$$M(\mathrm{g}_1,\mathrm{g}_2)=\Upsilon_1(\mathrm{g}_1)\Upsilon_2(\mathrm{g}_1)+\Upsilon_1(\mathrm{g}_2)\Upsilon_2(\mathrm{g}_2)$$

and

$$N(\mathrm{g}_1,\mathrm{g}_2)=\Upsilon_1(\mathrm{g}_1)\Upsilon_2(\mathrm{g}_2)+\Upsilon_1(\mathrm{g}_2)\Upsilon_2(\mathrm{g}_1).$$

Proof. Since Υ_1 and Υ_2 are preinvex function on $[\mathrm{g}_1,\mathrm{g}_1+\eta(\mathrm{g}_2,\mathrm{g}_1)]$, we have

$$\Upsilon_1(\mathrm{g}_1+\mathfrak{b}\eta(\mathrm{g}_2,\mathrm{g}_1))\leq(1-\mathfrak{b})\Upsilon_1(\mathrm{g}_1)+\mathfrak{b}\Upsilon_1(\mathrm{g}_2)$$

and

$$\Upsilon_2(\mathrm{g}_1+\mathfrak{b}\eta(\mathrm{g}_2,\mathrm{g}_1))\leq(1-\mathfrak{b})\Upsilon_2(\mathrm{g}_1)+\mathfrak{b}\Upsilon_2(\mathrm{g}_2).$$

Multiplying both the inequalities side by side, we have

$$\begin{aligned}&\Upsilon_1(\mathrm{g}_1+\mathfrak{b}\eta(\mathrm{g}_2,\mathrm{g}_1))\Upsilon_2(\mathrm{g}_1+\mathfrak{b}\eta(\mathrm{g}_2,\mathrm{g}_1))\\&\leq(1-\mathfrak{b})^2\Upsilon_1(\mathrm{g}_1)\Upsilon_2(\mathrm{g}_1)+\mathfrak{b}^2\Upsilon_1(\mathrm{g}_2)\Upsilon_2(\mathrm{g}_2)+\mathfrak{b}(1-\mathfrak{b})[\Upsilon_1(\mathrm{g}_1)\Upsilon_2(\mathrm{g}_2)+\Upsilon_1(\mathrm{g}_2)\Upsilon_2(\mathrm{g}_1)].\end{aligned}\tag{3.6}$$

Integrating (3.6) over [0, 1] and changing the variables, we obtain

$$\frac{2}{\eta(\mathrm{g}_2,\mathrm{g}_1)}\int_{\mathrm{g}_1}^{\mathrm{g}_1+\eta(\mathrm{g}_2,\mathrm{g}_1)}\Upsilon_1(x)\Upsilon_2(x)dx\leq\frac{2}{3}[\Upsilon_1(\mathrm{g}_1)\Upsilon_2(\mathrm{g}_1)+\Upsilon_1(\mathrm{g}_2)\Upsilon_2(\mathrm{g}_2)]+\frac{1}{3}[\Upsilon_1(\mathrm{g}_1)\Upsilon_2(\mathrm{g}_2)+\Upsilon_1(\mathrm{g}_2)\Upsilon_2(\mathrm{g}_1).$$

Which implies

$$\frac{2}{\eta(\mathrm{g}_2,\mathrm{g}_1)}\left[\int_{\mathrm{g}_1}^{k}\Upsilon_1(x)\Upsilon_2(x)dx+\int_{k}^{\mathrm{g}_1+\eta(\mathrm{g}_2,\mathrm{g}_1)}\Upsilon_1(x)\Upsilon_2(x)dx\right]\leq\frac{2}{3}M(\mathrm{g}_1,\mathrm{g}_2)+\frac{1}{3}N(\mathrm{g}_1,\mathrm{g}_2).$$

By multiplying both side with $\frac{\lambda\eta(\mathrm{g}_2,\mathrm{g}_1)}{2B(\lambda)}$ and adding $\frac{2(1-\lambda)}{B(\lambda)}\Upsilon_1(k)\Upsilon_2(k)$ we have

$$\begin{aligned}&\frac{\lambda}{B(\lambda)}\left[\int_{\mathrm{g}_1}^{k}\Upsilon_1(x)\Upsilon_2(x)dx+\int_{k}^{\mathrm{g}_1+\eta(\mathrm{g}_2,\mathrm{g}_1)}\Upsilon_1(x)\Upsilon_2(x)dx\right]+\frac{2(1-\lambda)}{B(\lambda)}\Upsilon_1(k)\Upsilon_2(k)\\&\leq\frac{\lambda\eta(\mathrm{g}_2,\mathrm{g}_1)}{2B(\lambda)}\left[\frac{2}{3}M(\mathrm{g}_1,\mathrm{g}_2)+\frac{1}{3}N(\mathrm{g}_1,\mathrm{g}_2)\right]+\frac{2(1-\lambda)}{B(\lambda)}\Upsilon_1(k)\Upsilon_2(k).\end{aligned}$$

Thus,

$$\left({}_{\mathrm{g}_1}^{CF}I^{\lambda}\Upsilon_1\Upsilon_2\right)(k)+\left({}^{CF}I^{\lambda}_{\mathrm{g}_1+\eta(\mathrm{g}_2,\mathrm{g}_1)}\Upsilon_1\Upsilon_2\right)(k)\leq\frac{\lambda\eta(\mathrm{g}_2,\mathrm{g}_1)}{2B(\lambda)}\left[\frac{2}{3}M(\mathrm{g}_1,\mathrm{g}_2)+\frac{1}{3}N(\mathrm{g}_1,\mathrm{g}_2)\right]+\frac{2(1-\lambda)}{B(\lambda)}\Upsilon_1(k)\Upsilon_2(k).$$

The proof gets completed after some rearrangements. □

Corollary 3.2. *If we put $\eta(\mathrm{g}_2,\mathrm{g}_1)=\mathrm{g}_2-\mathrm{g}_1$ in Theorem 3.2, we get the inequality in Theorem 3 in [37].*

Theorem 3.3. *Let a function $\Upsilon_1,\Upsilon_2:[\mathrm{g}_1,\mathrm{g}_1+\eta(\mathrm{g}_2,\mathrm{g}_1)]\to(0,\infty)$ be a preinvex function . If $\Upsilon_1\Upsilon_2\in L[\mathrm{g}_1,\mathrm{g}_1+\eta(\mathrm{g}_2,\mathrm{g}_1)]$, the set of integral function, then*

$$2\Upsilon\left(\frac{2\mathrm{g}_1+\eta(\mathrm{g}_2,\mathrm{g}_1)}{2}\right)\Upsilon_2\left(\frac{2\mathrm{g}_1+\eta(\mathrm{g}_2,\mathrm{g}_1)}{2}\right)$$

$$-\frac{B(\lambda)}{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}\left[\left({}^{CF}_{\mathsf{g}_1}I^{\lambda}\Upsilon_1\Upsilon_2\right)(k)+\left({}^{CF}I^{\lambda}_{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon_1\Upsilon_2\right)(k)-\frac{2(1-\lambda)}{B(\lambda)}\Upsilon_1(k)\Upsilon_2(k)\right]$$
$$\leq\frac{1}{6}M(\mathsf{g}_1,\mathsf{g}_2)+\frac{1}{3}N(\mathsf{g}_1,\mathsf{g}_2),\tag{3.7}$$

where $M(\mathsf{g}_1,\mathsf{g}_2), N(\mathsf{g}_1,\mathsf{g}_2)$ *are given in Theorem 3.2 and* $k\in[\mathsf{g}_1,\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)]$.

Proof. Since Υ_1 and Υ_2 are preinvex function on $[\mathsf{g}_1,\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)]$ for $\flat=1/2$, we have

$$\Upsilon_1\left(\frac{2\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\right)=\frac{\Upsilon_1(\mathsf{g}_1+\flat\eta(\mathsf{g}_2,\mathsf{g}_1))+\Upsilon_1(\mathsf{g}_1+(1-\flat)\eta(\mathsf{g}_2,\mathsf{g}_1))}{2},\quad\forall\mathsf{g}_1,\mathsf{g}_2\in I,\flat\in[0,1]$$

and

$$\Upsilon_2\left(\frac{2\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\right)=\frac{\Upsilon_2(\mathsf{g}_1+\flat\eta(\mathsf{g}_2,\mathsf{g}_1))+\Upsilon_2(\mathsf{g}_1+(1-\flat)\eta(\mathsf{g}_2,\mathsf{g}_1))}{2},\quad\forall\mathsf{g}_1,\mathsf{g}_2\in I,\flat\in[0,1].$$

Multiplying the above inequalities side by side, one has

$$\Upsilon\left(\frac{2\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\right)\Upsilon_2\left(\frac{2\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\right)$$
$$\leq\frac{1}{4}\Big[\Upsilon_1(\mathsf{g}_1+\flat\eta(\mathsf{g}_2,\mathsf{g}_1))\Upsilon_2(\mathsf{g}_1+\flat\eta(\mathsf{g}_2,\mathsf{g}_1))+\Upsilon_1(\mathsf{g}_1+(1-\flat)\eta(\mathsf{g}_2,\mathsf{g}_1))\Upsilon_2(\mathsf{g}_1+(1-\flat)\eta(\mathsf{g}_2,\mathsf{g}_1))$$
$$+\Upsilon_1(\mathsf{g}_1+\flat\eta(\mathsf{g}_2,\mathsf{g}_1))\Upsilon_2(\mathsf{g}_1+(1-\flat)\eta(\mathsf{g}_2,\mathsf{g}_1))+\Upsilon_1(\mathsf{g}_1+(1-\flat)\eta(\mathsf{g}_2,\mathsf{g}_1))\Upsilon_2(\mathsf{g}_1+\flat\eta(\mathsf{g}_2,\mathsf{g}_1))\Big]$$
$$\leq\frac{1}{4}\Big[\Upsilon_1(\mathsf{g}_1+\flat\eta(\mathsf{g}_2,\mathsf{g}_1))\Upsilon_2(\mathsf{g}_1+\flat\eta(\mathsf{g}_2,\mathsf{g}_1))+\Upsilon_1(\mathsf{g}_1+(1-\flat)\eta(\mathsf{g}_2,\mathsf{g}_1))\Upsilon_2(\mathsf{g}_1+(1-\flat)\eta(\mathsf{g}_2,\mathsf{g}_1))$$
$$+2\Big\{\flat(1-\flat)[\Upsilon_1(\mathsf{g}_1)\Upsilon_2(\mathsf{g}_1)+\Upsilon_1(\mathsf{g}_2)\Upsilon_2(\mathsf{g}_2)]+(1-\flat)^2\Upsilon_1(\mathsf{g}_1)\Upsilon_2(\mathsf{g}_2)+\flat^2\Upsilon_1(\mathsf{g}_2)\Upsilon_2(\mathsf{g}_1)\Big\}\Big].\tag{3.8}$$

Integrating the inequality (3.8) over $[0,1]$ and changing the variables, we have

$$\Upsilon_1\left(\frac{2\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\right)\Upsilon_2\left(\frac{2\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\right)$$
$$\leq\frac{1}{4}\left[\frac{2}{\eta(\mathsf{g}_2,\mathsf{g}_1)}\int_{\mathsf{g}_1}^{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon_1(x)\Upsilon_2(x)dx+\frac{1}{3}[\Upsilon_1(\mathsf{g}_1)\Upsilon_2(\mathsf{g}_1)+\Upsilon_1(\mathsf{g}_2)\Upsilon_2(\mathsf{g}_2)]\right.$$
$$\left.+\frac{2}{3}[\Upsilon_1(\mathsf{g}_1)\Upsilon_2(\mathsf{g}_2)+\Upsilon_1(\mathsf{g}_2)\Upsilon_2(\mathsf{g}_1)]\right].$$

Thus,

$$4\Upsilon_1\left(\frac{2\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\right)\Upsilon_2\left(\frac{2\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\right)$$
$$\leq\frac{2}{\eta(\mathsf{g}_2,\mathsf{g}_1)}\int_{\mathsf{g}_1}^{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon_1(x)\Upsilon_2(x)+\frac{1}{3}M(\mathsf{g}_1,\mathsf{g}_2)+\frac{2}{3}N(\mathsf{g}_1,\mathsf{g}_2).$$

By multiplying both sides with $\frac{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}{2B(\lambda)}$ and subtracting $\frac{2(1-\lambda)}{B(\lambda)}\Upsilon_1(k)\Upsilon_2(k)$ we have

$$\frac{2\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}{B(\lambda)}\Upsilon_1\left(\frac{2\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\right)\Upsilon_2\left(\frac{2\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\right)$$

$$-\frac{\lambda}{B(\lambda)}\left[\int_{\mathsf{g}_1}^{k}\Upsilon_1(x)\Upsilon_2(x)dx+\int_{k}^{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon_1(x)\Upsilon_2(x)dx\right]$$
$$-\frac{2(1-\lambda)}{B(\lambda)}\Upsilon_1(k)\Upsilon_2(k)\leq\frac{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}{2B(\lambda)}\left[\frac{1}{3}M(\mathsf{g}_1,\mathsf{g}_2)+\frac{2}{3}N(\mathsf{g}_1,\mathsf{g}_2)\right]-\frac{2(1-\lambda)}{B(\lambda)}\Upsilon_1(k)\Upsilon_2(k).$$

Cnsequently, we arrive at

$$\frac{2\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}{B(\lambda)}\Upsilon_1\left(\frac{2\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\right)\Upsilon_2\left(\frac{2\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\right)$$
$$-\left[\left({}^{CF}_{\mathsf{g}_1}I^{\lambda}\Upsilon_1\Upsilon_2\right)(k)+\left({}^{CF}I^{\lambda}_{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon_1\Upsilon_2\right)(k)\right]$$
$$\leq\frac{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}{2B(\lambda)}\left[\frac{1}{3}M(\mathsf{g}_1,\mathsf{g}_2)+\frac{2}{3}N(\mathsf{g}_1,\mathsf{g}_2)\right]-\frac{2(1-\lambda)}{B(\lambda)}\Upsilon_1(k)\Upsilon_2(k).$$

Multiplying both sides of the above inequality by $\frac{B(\lambda)}{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}$, we get the required inequality (3.7). □

Corollary 3.3. *If we put* $\eta(\mathsf{g}_2,\mathsf{g}_1)=\mathsf{g}_2-\mathsf{g}_1$ *in Theorem 3.3, we get the inequality in Theorem 4 in [37].*

4. Further consequences related to Caputo-Fabrizio fractional operator

Lemma 4.1. *Let* $\Upsilon : I=[\mathsf{g}_1,\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)]\to(0,\infty)$ *be a differentiable mapping on* I^0, $\mathsf{g}_1,\mathsf{g}_2\in I^0$ *with* $\mathsf{g}_1<\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)$ *if* $\Upsilon'\in L[\mathsf{g}_1,\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)]$, *then the following equality holds:*

$$\frac{\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\int_0^1(1-2\mathfrak{b})\Upsilon'(\mathsf{g}_1+\mathfrak{b}\eta(\mathsf{g}_2,\mathsf{g}_1))d\mathfrak{b}+\frac{2(1-\lambda)}{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon(k)$$
$$=-\frac{\Upsilon(\mathsf{g}_1)+\Upsilon(\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1))}{2}+\frac{B(\lambda)}{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}\left[\left({}^{CF}_{\mathsf{g}_1}I^{\lambda}\Upsilon\right)(k)+\left({}^{CF}I^{\lambda}_{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon\right)(k)\right],$$

where $k\in[\mathsf{g}_1,\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)]$.

Proof. It is easy to see that

$$\int_0^1(1-2\mathfrak{b})\Upsilon'(\mathsf{g}_1+\mathfrak{b}\eta(\mathsf{g}_2,\mathsf{g}_1))d\mathfrak{b}$$
$$=-\frac{\Upsilon(\mathsf{g}_1)+\Upsilon(\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1))}{2}+\frac{2}{(\eta(\mathsf{g}_2,\mathsf{g}_1))^2}\left(\int_{\mathsf{g}_1}^{k}\Upsilon(x)dx+\int_{k}^{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon(x)dx\right).$$

By multiplying both sides with $\frac{\lambda(\eta(\mathsf{g}_2,\mathsf{g}_1))^2}{2B(\lambda)}$ and adding $\frac{2(1-\lambda)}{B(\lambda)}\Upsilon(k)$ we have

$$\frac{\lambda(\eta(\mathsf{g}_2,\mathsf{g}_1))^2}{2B(\lambda)}\int_0^1(1-2\mathfrak{b})\Upsilon'(\mathsf{g}_1+\mathfrak{b}\eta(\mathsf{g}_2,\mathsf{g}_1))d\mathfrak{b}+\frac{2(1-\lambda)}{B(\lambda)}\Upsilon(k)$$
$$=-\frac{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}{B(\lambda)}\frac{\Upsilon(\mathsf{g}_1)+\Upsilon(\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1))}{2}+\left(\frac{(1-\lambda)}{B(\lambda)}\Upsilon(k)+\frac{\lambda}{B(\lambda)}\int_{\mathsf{g}_1}^{k}\Upsilon(x)dx\right)$$
$$+\left(\frac{(1-\lambda)}{B(\lambda)}\Upsilon(k)+\frac{\lambda}{B(\lambda)}\int_{k}^{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon(x)dx\right)$$
$$=-\frac{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}{B(\lambda)}\frac{\Upsilon(\mathsf{g}_1)+\Upsilon(\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1))}{2}+\left[\left({}^{CF}_{\mathsf{g}_1}I^{\lambda}\Upsilon\right)(k)+\left({}^{CF}I^{\lambda}_{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon\right)(k)\right].$$

This completes the proof. □

Corollary 4.1. *If we put* $\eta(\mathsf{g}_2,\mathsf{g}_1)=\mathsf{g}_2-\mathsf{g}_1$ *in Lemma 4.1, we get the equality in Lemma 2.1 in [37].*

Theorem 4.1. *Let* $\Upsilon: I=[\mathsf{g}_1,\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)]\rightarrow(0,\infty)$ *be a differentiable mapping on* I^0 *and* $|\Upsilon'|$ *be a preinvex on* $[\mathsf{g}_1,\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)]$ *if* $\Upsilon'\in L[\mathsf{g}_1,\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)]$, *where* $\mathsf{g}_1,\mathsf{g}_2\in I$ *with* $\mathsf{g}_1<\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)$. *Then, the following inequalities holds:*

$$\begin{aligned}&\left|-\frac{\Upsilon(\mathsf{g}_1)+\Upsilon(\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1))}{2}-\frac{2(1-\lambda)}{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon(k)+\frac{B(\lambda)}{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}\left[\left({}^{CF}_{\mathsf{g}_1}I^{\lambda}\Upsilon\right)(k)+\left({}^{CF}I^{\lambda}_{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon\right)(k)\right]\right|\\&\leq\frac{\eta(\mathsf{g}_2,\mathsf{g}_1)\left(|\Upsilon'(\mathsf{g}_1)|+|\Upsilon'(\mathsf{g}_2)|\right)}{8},\end{aligned}$$

where $k\in[\mathsf{g}_1,\mathsf{g}_1+\eta(b,\mathsf{g}_1)]$.

Proof. Applying lemma 4.1, properties of modulus and $|\Upsilon'|^q$ as a preinvex function, we have

$$\begin{aligned}&\left|-\frac{\Upsilon(\mathsf{g}_1)+\Upsilon(\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1))}{2}-\frac{2(1-\lambda)}{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon(k)+\frac{B(\lambda)}{\lambda\eta(b,\mathsf{g}_1)}\left[\left({}^{CF}_{\mathsf{g}_1}I^{\lambda}\Upsilon\right)(k)+\left({}^{CF}I^{\lambda}_{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon\right)(k)\right]\right|\\&\leq\frac{\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\int_0^1|1-2\mathfrak{b}||\Upsilon'(\mathsf{g}_1+\mathfrak{b}\eta(\mathsf{g}_2,\mathsf{g}_1))|d\mathfrak{b}\\&\leq\frac{\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\int_0^1|1-2\mathfrak{b}|\left((1-\mathfrak{b})|\Upsilon'(\mathsf{g}_1)|+\mathfrak{b}|\Upsilon'(\mathsf{g}_2)|\right)d\mathfrak{b}\\&=\frac{\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\Bigg(\int_0^{1/2}(1-2\mathfrak{b})\left((1-\mathfrak{b})|\Upsilon'(\mathsf{g}_1)|+\mathfrak{b}|\Upsilon'(\mathsf{g}_2)|\right)d\mathfrak{b}\\&+\int_{1/2}^1(1-2\mathfrak{b})\left((1-\mathfrak{b})|\Upsilon'(\mathsf{g}_1)|+\mathfrak{b}|\Upsilon'(\mathsf{g}_2)|\right)d\mathfrak{b}\Bigg)\\&=\frac{\eta(\mathsf{g}_2,\mathsf{g}_1)\left(|\Upsilon'(\mathsf{g}_1)|+|\Upsilon'(\mathsf{g}_2)|\right)}{8}.\end{aligned}$$

So the proof is completed. □

Corollary 4.2. *If we put* $\eta(\mathsf{g}_2,\mathsf{g}_1)=\mathsf{g}_2-\mathsf{g}_1$ *in Theorem 4.1, we get the inequality in Theorem 5 in [37].*

Theorem 4.2. *Let* $\Upsilon: I=[\mathsf{g}_1,\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)]\rightarrow(0,\infty)$ *be a differentiable mapping on I and* $|\Upsilon'|^q$ *be a preinvex on* $[\mathsf{g}_1,\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)]$ *,where* $p>1,\frac{1}{p}+\frac{1}{q}=1,\mathsf{g}_1,\mathsf{g}_2\in I$. *If* $\Upsilon'\in L[\mathsf{g}_1,\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)]$, *with* $\mathsf{g}_1<\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)$ *and* $\lambda\in[0,1]$, *the following inequalities holds*

$$\begin{aligned}&\left|-\frac{\Upsilon(\mathsf{g}_1)+\Upsilon(\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1))}{2}+\frac{2(1-\lambda)}{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon(k)+\frac{B(\lambda)}{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}\left[\left({}^{CF}_{\mathsf{g}_1}I^{\lambda}\Upsilon\right)(k)+\left({}^{CF}I^{\lambda}_{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon\right)(k)\right]\right|\\&\leq\frac{\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{|\Upsilon'(\mathsf{g}_1)|^q+|\Upsilon'(\mathsf{g}_2)|^q}{2}\right)^{1/q},\end{aligned}$$

where $k\in[\mathsf{g}_1,\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)]$.

Proof. By using Lemma 4.1, the Hölder inequality and preinvexity of $|\Upsilon'|^q$, we get

$$\begin{aligned}&\left|-\frac{\Upsilon(\mathsf{g}_1)+\Upsilon(\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1))}{2}-\frac{2(1-\lambda)}{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon(k)+\frac{B(\lambda)}{\lambda\eta(\mathsf{g}_2,\mathsf{g}_1)}\left[\left({}^{CF}_{\mathsf{g}_1}I^{\lambda}\Upsilon\right)(k)+\left({}^{CF}I^{\lambda}_{\mathsf{g}_1+\eta(\mathsf{g}_2,\mathsf{g}_1)}\Upsilon\right)(k)\right]\right|\\&\leq\frac{\eta(\mathsf{g}_2,\mathsf{g}_1)}{2}\int_0^1|1-2\mathfrak{b}||\Upsilon'(\mathsf{g}_1+\mathfrak{b}\eta(\mathsf{g}_2,\mathsf{g}_1))|d\mathfrak{b}\end{aligned}$$

$$\leq \frac{\eta(g_2,g_1)}{2}\left(\int_0^1 |1-2\mathfrak{b}|^p d\mathfrak{b}\right)^{1/p}\left(\int_0^1 |\Upsilon'(g_1+\mathfrak{b}\eta(g_2,g_1))|^q d\mathfrak{b}\right)^{1/q}$$
$$\leq \frac{\eta(g_2,g_1)}{2}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{|\Upsilon'(g_1)|^q+|\Upsilon'(g_2)|^q}{2}\right)^{1/q}.$$

So, we have the desired result. □

Corollary 4.3. *If we put $\eta(g_2,g_1)=g_2-g_1$ in Theorem 4.2 we get the inequality in Theorem 6 in [37].*

Theorem 4.3. *Let $\Upsilon : I=[g_1,g_1+\eta(g_2,g_1)]\to(0,\infty]$ be differentiable function on I° and $g_1,g_2\in I^\circ$ with $g_1<g_1+\eta(g_2,g_1)$, $q\geq 1$, and assuming that $\Upsilon'\in L[g_1,g_1+\eta(g_2,g_1)]$. If $|\Upsilon'|^q$ is a preinvex function on interval $[g_1,g_1+\eta(g_2,g_1)]$, then following inequality holds for $\mathfrak{b}\in[0,1]$,*

$$\left|-\frac{\Upsilon(g_1)+\Upsilon(g_1+\eta(g_2,g_1))}{2}-\frac{2(1-\lambda)}{\lambda\eta(g_2,g_1)}\Upsilon(k)+\frac{B(\lambda)}{\lambda\eta(g_2,a)}\left[{}^{CF}_{g_1}I^\lambda\Upsilon(k)+{}^{CF}I^\lambda_{g_1+\eta(g_2,g_1)}\Upsilon(k)\right]\right|$$
$$\leq \frac{\eta(g_2,g_1)}{4}\left(\frac{|\Upsilon'(g_1)|^q+|\Upsilon'(g_2)|^q}{2}\right)^{\frac{1}{q}},$$

where $k\in[g_1,g_1+\eta(g_2,g_1)]$.

Proof. Applying lemma 4.1, properties of modulus, power mean inequality and $|\Upsilon'|^q$ as a preinvex function, we have

$$\left|-\frac{\Upsilon(g_1)+\Upsilon(g_1+\eta(g_2,g_1))}{2}-\frac{2(1-\lambda)}{\lambda\eta(g_2,g_1)}\Upsilon(k)+\frac{B(\lambda)}{\lambda\eta(g_2,g_1)}\left[{}^{CF}_{g_1}I^\lambda\Upsilon(k)+{}^{CF}I^\lambda_{g_1+\eta(g_2,g_1)}\Upsilon(k)\right]\right|$$
$$\leq \frac{\eta(g_2,g_1)}{2}\int_0^1 |1-2\mathfrak{b}|\left|\Upsilon'(g_1+\mathfrak{b}\eta(g_2,g_1))\right| d\mathfrak{b}$$
$$\leq \frac{\eta(g_2,g_1)}{2}\left(\int_0^1 |1-2\mathfrak{b}|d\mathfrak{b}\right)^{1-\frac{1}{q}}\left(|1-2\mathfrak{b}|\left|\Upsilon'(g_1+\mathfrak{b}\eta(g_2,g_1))\right|^q d\mathfrak{b}\right)^{\frac{1}{q}}$$
$$\leq \frac{\eta(g_2,g_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2\mathfrak{b}|\left(|\Upsilon'(g_1)|^q[1-\mathfrak{b}]+|\Upsilon'(g_2)|^q\mathfrak{b}\right)d\mathfrak{b}\right)^{\frac{1}{q}}$$
$$\leq \frac{\eta(g_2,g_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(|\Upsilon'(g_1)|^q\int_0^1 |1-2\mathfrak{b}|[1-\mathfrak{b}]d\mathfrak{b}+|\Upsilon'(g_2)|^q\int_0^1 |1-2\mathfrak{b}|\mathfrak{b}d\mathfrak{b}\right)^{\frac{1}{q}}$$
$$\leq \frac{\eta(g_2,g_1)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{|\Upsilon'(g_1)|^q+|\Upsilon'(g_2)|^q}{4}\right)^{\frac{1}{q}}.$$

Further simplifications lead us to the desired proof. □

Remark 4.1. If we put $\eta(g_2,g_1)=g_2-g_1$ in the above theorem, then we get

$$\left|\frac{\Upsilon(g_1)+\Upsilon(g_2}{2}+\frac{2(1-\lambda)}{\lambda\eta(g_2,g_1)}\Upsilon(k)-\frac{B(\lambda)}{\lambda\eta(g_2,a)}\left[{}^{CF}_{g_1}I^\lambda\Upsilon(k)+{}^{CF}I^\lambda_{g_2}\Upsilon(k)\right]\right|$$
$$\leq \frac{(g_2-g_1)}{4}\left(\frac{|\Upsilon'(g_1)|^q+|\Upsilon'(g_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 4.4. *Let $\Upsilon : I = [\mathrm{g}_1, \mathrm{g}_1 + \eta(\mathrm{g}_2, \mathrm{g}_1)] \to (0, \infty]$ be differential function on I° and $\mathrm{g}_1, \mathrm{g}_2 \in I^\circ$ with $\mathrm{g}_1 < \mathrm{g}_1 + \eta(\mathrm{g}_2, \mathrm{g}_1)$, $q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and assume that $\Upsilon' \in L[\mathrm{g}_1, \mathrm{g}_1 + \eta(\mathrm{g}_2, \mathrm{g}_1)]$. If $|\Upsilon'|^q$ is a preinvex function on interval $[\mathrm{g}_1, \mathrm{g}_1 + \eta(\mathrm{g}_2, \mathrm{g}_1)]$, then following inequality holds for $\flat \in [0, 1]$,*

$$\left|-\frac{\Upsilon(\mathrm{g}_1) + \Upsilon(\mathrm{g}_1 + \eta(\mathrm{g}_2, \mathrm{g}_1))}{2} - \frac{2(1-\lambda)}{\lambda\eta(\mathrm{g}_2, \mathrm{g}_1)}\Upsilon(k) + \frac{B(\lambda)}{\lambda\eta(\mathrm{g}_2, \mathrm{g}_1)}\left[{}^{CF}_{\mathrm{g}_1}I^{\lambda}\Upsilon(k) + {}^{CF}I^{\lambda}_{\mathrm{g}_1+\eta(\mathrm{g}_2,\mathrm{g}_1)}\Upsilon(k)\right]\right|$$
$$\leq \frac{\eta(\mathrm{g}_2, \mathrm{g}_1)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{2|\Upsilon'(\mathrm{g}_1)|^q + |\Upsilon'(\mathrm{g}_2)|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{|\Upsilon'(\mathrm{g}_1)|^q + 2|\Upsilon'(\mathrm{g}_2)|^q}{3}\right)^{\frac{1}{q}}\right],$$

where $k \in [\mathrm{g}_1, \mathrm{g}_1 + \eta(\mathrm{g}_2, \mathrm{g}_1)]$.

Proof. Applying lemma 4.1, properties of modulus, Hölder İşcan inequality and preinvexity of $|\Upsilon'|^q$, we have

$$\left|-\frac{\Upsilon(\mathrm{g}_1) + \Upsilon(\mathrm{g}_1 + \eta(\mathrm{g}_2, \mathrm{g}_1))}{2} - \frac{2(1-\lambda)}{\lambda\eta(\mathrm{g}_2, \mathrm{g}_1)}\Upsilon(k) + \frac{B(\lambda)}{\lambda\eta(\mathrm{g}_2, \mathrm{g}_1)}\left[{}^{CF}_{\mathrm{g}_1}I^{\lambda}\Upsilon(k) + {}^{CF}I^{\lambda}_{\mathrm{g}_1+\eta(\mathrm{g}_2,\mathrm{g}_1)}\Upsilon(k)\right]\right|$$
$$\leq \frac{\eta(\mathrm{g}_2, \mathrm{g}_1)}{2}\int_0^1 |1 - 2\flat||\Upsilon'(\mathrm{g}_1 + \flat\eta(\mathrm{g}_2, \mathrm{g}_1))|$$
$$\leq \frac{\eta(\mathrm{g}_2, \mathrm{g}_1)}{2}\left(\int_0^1 (1-\flat)|1-2\flat|^p d\flat\right)^{\frac{1}{p}}\left(\int_0^1 (1-\flat)|\Upsilon'(\mathrm{g}_1 + \flat\eta(\mathrm{g}_2, \mathrm{g}_1))|^q d\flat\right)$$
$$+ \frac{\eta(\mathrm{g}_2, \mathrm{g}_1)}{2}\left(\int_0^1 \flat|1-2\flat|^p d\flat\right)^{\frac{1}{p}}\left(\int_0^1 \flat|\Upsilon'(\mathrm{g}_2 + \flat\eta(\mathrm{g}_2, \mathrm{g}_1))|^q d\flat\right)$$
$$\leq \frac{\eta(\mathrm{g}_2, \mathrm{g}_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(|\Upsilon'(\mathrm{g}_1)|^q\int_0^1 (1-\flat)(1-\flat)d\flat + |\Upsilon'(\mathrm{g}_2)|^q\int_0^1 (1-\flat)\flat d\flat\right)^{\frac{1}{q}}$$
$$+ \frac{\eta(\mathrm{g}_2, \mathrm{g}_1)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(|\Upsilon'(\mathrm{g}_1)|^q\int_0^1 \flat(1-\flat)d\flat + |\Upsilon'(\mathrm{g}_2)|^q\int_0^1 \flat^2 d\flat\right)^{\frac{1}{q}}$$
$$\leq \frac{\eta(\mathrm{g}_2, \mathrm{g}_1)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{2|\Upsilon'(\mathrm{g}_1)|^q + |\Upsilon'(\mathrm{g}_2)|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{|\Upsilon'(\mathrm{g}_1)|^q + 2|\Upsilon'(\mathrm{g}_2)|^q}{3}\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

Remark 4.2. If we put $\eta(\mathrm{g}_2, \mathrm{g}_1) = \mathrm{g}_2 - \mathrm{g}_1$ in the above theorem, then we get

$$\left|\frac{\Upsilon(\mathrm{g}_1) + \Upsilon(\mathrm{g}_2)}{2} + \frac{2(1-\lambda)}{\lambda(\mathrm{g}_2 - \mathrm{g}_1)}\Upsilon(k) - \frac{B(\lambda)}{\lambda(\mathrm{g}_2 - \mathrm{g}_1)}\left[{}^{CF}_{\mathrm{g}_1}I^{\lambda}\Upsilon(k) + {}^{CF}I^{\lambda}_{\mathrm{g}_2}\Upsilon(k)\right]\right|$$
$$\leq \frac{(\mathrm{g}_2 - \mathrm{g}_1)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{2|\Upsilon'(\mathrm{g}_1)|^q + |\Upsilon'(\mathrm{g}_2)|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{|\Upsilon'(\mathrm{g}_1)|^q + 2|\Upsilon'(\mathrm{g}_2)|^q}{3}\right)^{\frac{1}{q}}\right].$$

Theorem 4.5. *Let $\Upsilon : I = [\mathrm{g}_1, \mathrm{g}_1 + \eta(\mathrm{g}_2, a)] \to (0, \infty]$ be differential function on I° and $\mathrm{g}_1, \mathrm{g}_2 \in I^\circ$ with $\mathrm{g}_1 < \mathrm{g}_1 + \eta(\mathrm{g}_2, \mathrm{g}_1)$, $q \geq 1$, and assume that $\Upsilon' \in L[\mathrm{g}_1, \mathrm{g}_1 + \eta(\mathrm{g}_2, \mathrm{g}_1)]$. If $|\Upsilon'|^q$ is a preinvex function on interval $[\mathrm{g}_1, \mathrm{g}_1 + \eta(\mathrm{g}_2, \mathrm{g}_1)]$, then following inequality holds for $\flat \in [0, 1]$,*

$$\left|-\frac{\Upsilon(\mathrm{g}_1) + \Upsilon(\mathrm{g}_1 + \eta(\mathrm{g}_2, \mathrm{g}_1))}{2} - \frac{2(1-\lambda)}{\lambda\eta(\mathrm{g}_2, \mathrm{g}_1)}\Upsilon(k) + \frac{B(\lambda)}{\lambda\eta(\mathrm{g}_2, a)}\left[{}^{CF}_{\mathrm{g}_1}I^{\lambda}\Upsilon(k) + {}^{CF}I^{\lambda}_{\mathrm{g}_1+\eta(\mathrm{g}_2,\mathrm{g}_1)}\Upsilon(k)\right]\right|$$

$$\leq \frac{\eta(g_2,g_1)}{8}\left[\left(\frac{3|\Upsilon'(g_1)|^q+|\Upsilon'(g_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\Upsilon'(g_1)|^q+3|\Upsilon'(g_2)|^q}{4}\right)^{\frac{1}{q}}\right],$$

where $k \in [g_1, g_1+\eta(g_2,g_1)]$.

Proof. Applying Lemma 4.1, properties of modulus, improved power mean inequality and $|\Upsilon'|^q$ is a preinvex function, we have

$$\begin{aligned}
&\left|-\frac{\Upsilon(g_1)+\Upsilon(g_1+\eta(g_2,g_1))}{2}-\frac{2(1-\lambda)}{\lambda\eta(g_2,g_1)}\Upsilon(k)+\frac{B(\lambda)}{\lambda\eta(g_2,g_1)}\left[{}^{CF}_{g_1}I^{\lambda}\Upsilon(k)+{}^{CF}I^{\lambda}_{g_1+\eta(g_2,g_1)}\Upsilon(k)\right]\right|\\
&\leq \frac{\eta(g_2,g_1)}{2}\int_0^1|1-2b||\Upsilon'(g_1+b\eta(g_2,g_1))|\\
&\leq \frac{\eta(g_2,g_1)}{2}\left(\int_0^1(1-b)|1-2b|db\right)^{1-\frac{1}{q}}\left(\int_0^1(1-b)|1-2b||\Upsilon'(g_1+b\eta(g_2,g_1))|^q db\right)^{\frac{1}{q}}\\
&+\frac{\eta(g_2,g_1)}{2}\left(\int_0^1 b|1-2b|db\right)^{1-\frac{1}{q}}\left(\int_0^1 b|1-2b||\Upsilon'(g_1+b\eta(g_2,g_1))|^q db\right)^{\frac{1}{q}}\\
&\leq \frac{\eta(g_2,g_1)}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(|\Upsilon'(g_1)|^q\int_0^1(1-b)^2|1-2b|db+|\Upsilon'(g_2)|^q\int_0^1 b(1-b)|1-2b|db\right)^{\frac{1}{q}}\\
&+\frac{\eta(g_2,g_1)}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(|\Upsilon'(g_1)|^q\int_0^1 b(1-b)|1-2b|db+|\Upsilon'(g_2)|^q\int_0^1 b^2|1-2b|db\right)^{\frac{1}{q}}\\
&\leq \frac{\eta(g_2,g_1)}{8}\left[\left(\frac{3|\Upsilon'(g_1)|^q+|\Upsilon'(g_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\Upsilon'(g_1)|^q+3|\Upsilon'(g_2)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Remark 4.3. If we put $\eta(g_2,g_1)=g_2-g_1$ in the above theorem, then we get

$$\begin{aligned}
&\left|\frac{\Upsilon(g_1)+\Upsilon(g_2)}{2}+\frac{2(1-\lambda)}{\lambda(g_2-g_1)}\Upsilon(k)-\frac{B(\lambda)}{\lambda(g_2-g_1)}\left[{}^{CF}_{g_1}I^{\lambda}\Upsilon(k)+{}^{CF}I^{\lambda}_{g_2}\Upsilon(k)\right]\right|\\
&\leq \frac{(g_2-g_1)}{8}\left[\left(\frac{3|\Upsilon'(g_1)|^q+|\Upsilon'(g_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\Upsilon'(g_1)|^q+3|\Upsilon'(g_2)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

5. Applications

In this section, we examine and attain some applications regarding the above results.

(1) The arithmetic mean

$$A=A(g_1,g_2)=\frac{g_1+g_2}{2},\quad g_1,g_2\in R.$$

(2) The generalized logarithmic mean

$$L=L_r^r(g_1,g_2)=\frac{{g_2}^{r+1}-{g_1}^{r+1}}{(r+1)(g_2-g_1)},\quad r\in R\setminus\{-1,0\},\quad g_1,g_2\in R,\quad g_1\neq g_2.$$

Now using the results in Section 4, we present our results to attain some inequalities related to special means.

In all the results to follow we have taken $B(\lambda) = B(1) = 1$

Proposition 5.1. *Let* $g_1, g_1 + \eta(g_2, g_1) \in R^+, g_1 < g_1 + \eta(g_2, g_1)$*, then*

$$\left| - A({g_1}^2, (g_1 + \eta(g_2, g_1))^2) + L_2^2(g_1, g_1 + \eta(g_2, g_1)) \right| \leq \frac{\eta(g_2, g_1)}{4}[|g_1| + |g_2|].$$

Proof. If we prefer $\Upsilon(z) = z^2$ with $\lambda = 1$ in Theorem 4.1, then we have the desired result. □

Corollary 5.1. *If we set* $\eta(g_2, g_1) = g_2 - g_1$ *in Proposition 5.1 we get the inequality in Proposition 1 in [37].*

Proposition 5.2. *Let* $g_1, g_1 + \eta(g_2, g_1) \in R^+, g_1 < g_1 + \eta(g_2, g_1)$*, then*

$$\left| - A(e^{g_1}, {g_1}^{(g_1 + \eta(g_2, g_1))}) + L(e^{g_1}, e^{(g_1 + \eta(g_2, g_1))}) \right| \leq \frac{\eta(g_2, g_1)}{8}(e^{g_1} + e^{g_2}).$$

Proof. If we prefer $\Upsilon(z) = e^z$ with $\lambda = 1$ and $B(\lambda) = B(1) = 1$ in Theorem 4.1, then we have the desired result. □

Corollary 5.2. *If we set* $\eta(g_2, g_1) = g_2 - g_1$ *in Proposition 5.2 we get the inequality in Proposition 2 in [37].*

Proposition 5.3. *Let* $g_1, g_1 + \eta(g_2, g_1) \in R^+, g_1 < g_1 + \eta(g_2, g_1)$*, then*

$$\left| - A({g_1}^n, (g_1 + \eta(g_2, g_1))^n) + L_n^n(g_1, g_1 + \eta(g_2, g_1)) \right| \leq \frac{n\eta(g_2, g_1)}{8} \left[|{g_1}^{n-1}| + |{g_2}^{n-1}| \right].$$

Proof. If we prefer $\Upsilon(z) = z^n$ with $\lambda = 1$ and $B(\lambda) = B(1) = 1$ in Theorem 4.1, then we have the desired result. □

Corollary 5.3. *If we put* $\eta(g_2, g_1) = g_2 - g_1$ *in Proposition 5.3 we get the inequality in Proposition 3 in [37].*

6. Conclusions

Due to the potential applications fractional calculus has, the literature on fractional integral inequalities has become a rich source of attraction for many researchers in various fields. Refinements and estimations attained via preinvex functions produce better and sharper bounds when compared to convex functions. Finally, the innovative concept of Caputo-Fabrizio operator for preinvex function has a wide range of potential applications and importance in the direction of applied sciences. In this work, we investigated and explored a new version of Hermite-Hadamard type inequality involving a fractional integral operator in Caputo-Fabrizio sense. As a result, a new Kernel is attained and a new theorem valid for preinvex function is investigated for fractional-order integrals. To add more beauty to the paper, we attained the refinements of Hermite-Hadamard inequality with the help of Hölder, Hölder-İscan, power mean and improved power-mean inequality. One can observe that Theorem 4.2 provides better results when compared to Theorem 4.4. Similarly Theorem 4.5 provides better results when compared to Theorem 4.3. Finally, some applications of our main findings are provided. Our findings are the refinements and generalizations of the existing results that stimulate futuristic research.

Conflict of interest

The authors declare no conflicts of interest.

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