Locating-dominating codes in cycles

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Abstract

The smallest cardinality of an r-locating-dominating code in a cycle $\mathcal{C}_n$ of length n is denoted by $M_r^{LD}(\mathcal{C}_n)$. In this paper, we prove that for any $r \geq 5$ and $n \geq n_r$ when n_r is large enough ($n_r = \mathcal{O}(r^3)$) we have $n/3 \leq M_r^{LD}(\mathcal{C}_n) \leq n/3 + 1$ if $n \equiv 3 \pmod 6$ and $M_r^{LD}(\mathcal{C}_n) = \lceil n/3 \rceil$ otherwise. Moreover, we determine the exact values of $M_3^{LD}(\mathcal{C}_n)$ and $M_4^{LD}(\mathcal{C}_n)$ for all n.

Keywords: Locating-dominating code; optimal code; domination; graph; cycle

1 Introduction

Let $G = (V, E)$ be a simple connected and undirected graph with V as the set of vertices and E as the set of edges. Let u and v be vertices in V. If u and v are adjacent to each other, then the edge between u and v is denoted by uv. The *distance* $d(u, v)$ is the number of edges in any shortest path between u and v. Let r be a positive integer. We say that u *r-covers* v if the distance $d(u, v)$ is at most r. The *ball of radius r centered at u* is defined as

$$B_r(u) = \{x \in V \mid d(u, x) \leq r\}.$$

A non-empty subset of V is called a *code*, and its elements are called *codewords*. Let $C \subseteq V$ be a code and u be a vertex in V. An *I-set* (or an *identifying*

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set) of the vertex u with respect to the code C is defined as

$$I_r(C;u) = I_r(u) = B_r(u) \cap C.$$

Definition 1.1. Let r be a positive integer. A code $C \subseteq V$ is said to be *r-locating-dominating* in G if for all distinct vertices $u, v \in V \setminus C$ the set $I_r(C;u)$ is non-empty and

$$I_r(C;u) \neq I_r(C;v).$$

Let X and Y be subsets of V. The *symmetric difference* of X and Y is defined as $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. We say that the vertices u and v are *r-separated* by a code $C \subseteq V$ or by a codeword of $C \subseteq V$ if the symmetric difference $I_r(C;u) \triangle I_r(C;v)$ is non-empty. The definition of r-locating-dominating codes can now be reformulated as follows: $C \subseteq V$ is an r-locating-dominating code in G if and only if for all $u, v \in V \setminus C$ $(u \neq v)$ the vertex u is r-covered by a codeword of C and

$$I_r(C;u) \triangle I_r(C;v) \neq \emptyset.$$

The smallest cardinality of an r-locating-dominating code in a finite graph G is denoted by $M_r^{LD}(G)$. Notice that there always exists an r-locating dominating code in G. An r-locating-dominating code attaining the smallest cardinality is called *optimal*. In [3], it is shown that the problem of determining $M_r^{LD}(G)$ is NP-hard.

Locating-dominating codes are also known as locating-dominating *sets* in the literature. The concept of locating-dominating codes was first introduced by Slater in [13, 15, 16] and later generalized by Carson in [2]. The locating-dominating codes have been since studied in various papers such as [5], [6], [7], [8], [9], [10], [14], [17] and [18]. For other papers on the subject, we refer to the Web site [12]. Moreover, location-domination in cycles have been examined in [1], [4] and [16].

Let n be a positive integer such that $n \geq 3$. A cycle $\mathcal{C}_n = (V_n, E_n)$ is a graph such that the set of vertices is defined as $V_n = \{v_i \mid i \in \mathbb{Z}_n\}$ and the set of edges is defined as

$$E_n = \{v_i v_{i+1} \mid i = 0, 1, \ldots, n-2\} \cup \{v_{n-1} v_0\}.$$

Throughout the paper, we assume that the indices of $v_i \in V_n$ are calculated modulo n. Hence, the set of edges can be written as $E_n = \{v_i v_{i+1} \mid i \in \mathbb{Z}_n\}$. For the rest of the paper, we also assume that n and r are positive integers such that $n \geq 3$.

In [16], it is shown that $M_1^{LD}(\mathcal{C}_n) = \lceil 2n/5 \rceil$. For radius $r \geq 2$, Bertrand *et al.* [1] provide the lower bound

$$M_r^{LD}(\mathcal{C}_n) \geq \left\lceil \frac{n}{3} \right\rceil. \tag{1}$$

The exact values of $M_2^{LD}(\mathcal{C}_n)$ are determined in [4]. In particular, it is shown that for $n > 6$ if $n \equiv 3 \pmod 6$, then $M_2^{LD}(\mathcal{C}_n) = n/3 + 1$, else $M_2^{LD}(\mathcal{C}_n) = \lceil n/3 \rceil$. In Section 5, we determine the exact values of $M_3^{LD}(\mathcal{C}_n)$ and $M_4^{LD}(\mathcal{C}_n)$. In Section 4, we prove that for any $r \geq 5$ and $n \geq n_r$ when n_r is large enough ($n_r = \mathcal{O}(r^3)$) we have constructions showing $M_r^{LD}(\mathcal{C}_n) \leq n/3 + 1$ if $n \equiv 3 \pmod 6$ and $M_r^{LD}(\mathcal{C}_n) \leq \lceil n/3 \rceil$ otherwise. The latter constructions are optimal by the lower bound (1). Using the evidence provided in Sections 3 and 5, we conjecture that also the constructions in the case $n \equiv 3 \pmod 6$ are optimal.

In what follows, we begin in Section 2 by introducing some basic results concerning r-locating-dominating codes in cycles. Then, in Section 3, we proceed by considering r-locating-dominating codes in cycles $\mathcal{C}_n$ with small n (for a given r). In Section 4, we present constructions for r-locating-dominating codes in cycles for general r and, in Section 5, we consider r-locating-dominating codes in cycles when $2 \leq r \leq 4$.

2 Basics

We first present some useful observations concerning r-locating-dominating codes in cycles. For this, we need the concept of C-consecutive vertices introduced in [1]. Let i and j be positive integers. We say that (v_i, v_j) is a pair of *C-consecutive vertices* in $\mathcal{C}_n$ if $v_i, v_j \in V_n \setminus C$ and $v_k \in C$ for all $k = i+1, i+2, \ldots, j-1$ or for all $k = j+1, j+2, \ldots, i-1$. The following lemma is previously presented in [1, Remark 4].

Lemma 2.1 ([1])**.** *If $C \subseteq V_n$ is a code in $\mathcal{C}_n$, then each codeword of C can r-separate at most two pairs of C-consecutive vertices.*

Bertrand *et al.* in [1] also presented a useful characterization of r-locating-dominating codes in paths. The following lemma provides similar characterization in the case of cycles.

Lemma 2.2. *A code $C \subseteq V_n$ is r-locating-dominating in $\mathcal{C}_n$ if and only if*

(i) each vertex $u \in V_n \setminus C$ is r-covered by a codeword of C,

(ii) each pair (u, v) of C-consecutive vertices in $\mathcal{C}_n$ is r-separated by C and

(iii) there exists at most one vertex $u \in V_n \setminus C$ such that $I_r(u) = C$.

Proof. If C is an r-locating-dominating code in $\mathcal{C}_n$, then the conditions (i), (ii) and (iii) immediately follow. Assume then that $C \subseteq V_n$ is a code satisfying these three conditions. By the assumption, all the vertices of V_n are r-covered by a codeword of C. Let then u and v be two distinct vertices of V_n. If $I_r(u) = C$, then by the condition (iii), the vertices u and v are r-separated by a codeword.

Hence, we may assume that $I_r(u) \neq C$ and $I_r(v) \neq C$. If the intersection of $I_r(v)$ and $C \setminus I_r(u)$ is non-empty, then the vertices u and v are r-separated by a codeword of C. Otherwise, we have $I_r(v) \subseteq I_r(u)$. Then there exists a non-codeword $w \in V_n$ such that (u, w) is a pair of C-consecutive vertices and the symmetric difference $I_r(u) \triangle I_r(w)$ is a subset of $I_r(u) \triangle I_r(v)$. (Notice that if (u, v) is pair of C-consecutive vertices, then $v = w$.) Therefore, by the condition (ii), we have $I_r(u) \neq I_r(v)$. $\square$

In the previous characterization, the condition (iii) is necessary. Indeed, consider a code $\{v_0, v_2\}$ in $\mathcal{C}_6$ when $r = 2$. Clearly, the conditions (i) and (ii) now hold. However, the code is not 2-locating-dominating in $\mathcal{C}_6$ since $I_r(v_1) = I_r(v_4) = \{v_0, v_2\}$. Notice also that if $n \geq 4r + 2$ and the condition (i) holds, then there is no vertex $u \in V_n \setminus C$ such that $I_r(u) = C$.

The following lower bound is presented in [1, Theorem 13]. For Lemma 2.4, we include the proof of the lower bound here. In [1, Theorem 14], it is also shown that for each $r \geq 2$ there exist an infinite family of n such that $M_r^{LD}(\mathcal{C}_n) = \lceil n/3 \rceil$. In particular, it is shown that if r is even, $n > 6$ and $n \equiv 0 \pmod{3r}$ or if r is odd and $n \equiv 0 \pmod{3r+3}$, then the lower bound is attained.

Theorem 2.3 ([1]). *For all integers $n \geq 3$ and $r \geq 2$, we have*

$$M_r^{LD}(\mathcal{C}_n) \geq \left\lceil \frac{n}{3} \right\rceil .$$

Proof. Let C be an r-locating-dominating code in $\mathcal{C}_n$. By Lemma 2.1, each codeword of C can r-separate at most two pairs of C-consecutive vertices. On the other hand, by Lemma 2.2, each pair of C-consecutive vertices has to be r-separated by at least one codeword. Hence, we have $2|C| \geq n - |C|$. Thus, the claim immediately follows. □

The next lemma immediately follows from the previous proof.

Lemma 2.4. *Let n be divisible by three and $r \geq 2$. If C is an r-locating-dominating code in $\mathcal{C}_n$ with $n/3$ codewords, then*

(i) each codeword r-separates exactly two pairs of C-consecutive vertices and

(ii) each pair of C-consecutive vertices is r-separated by exactly one codeword of C.

For future consideration, we introduce the concept of C-block of codewords. Let t be a positive integer. Define $Q_t(i) = \{v_i, v_{i+1}, \ldots, v_{i+t-1}\}$ ($i \in \mathbb{Z}_n$). Let $C \subseteq V_n$ be a code. We say that $Q_t(i)$ is a *C-block (of codewords)* if the vertices $v_i, v_{i+1}, \ldots, v_{i+t-1} \in C$ and $v_{i-1}, v_{i+t} \notin C$. Moreover, if $Q_t(i)$ is a C-block of codewords, then the *length* of the C-block is t. Notice that if $Q_t(i)$ is a C-block, then (v_{i-1}, v_{i+t}) is a pair of C-consecutive vertices. Notice also that if $v_{i-1}, v_{i+1} \notin C$ and $v_i \in C$, then we say that $\{v_i\}$ is a C-block of length one.

Now we are ready to present the following two lemmas.

Lemma 2.5. *Let n be divisible by three and $r \geq 2$. If C is an r-locating-dominating code in $\mathcal{C}_n$ with $n/3$ codewords, then the length of any C-block of codewords is at most $r-1$.*

Proof. Let C be an r-locating-dominating code in $\mathcal{C}_n$ with $n/3$ codewords. Assume that there exists a C-block $Q_t(i)$ of length $t \geq r+1$. Then it is immediately clear that v_i (and v_{i+t-1}) r-separate at most one pair of C-consecutive vertices. This is a contradiction with Lemma 2.4 (i).

Assume then that $Q_r(i)$ is a C-block of length r. Since (v_{i-1}, v_{i+r}) is a pair of C-consecutive vertices, the symmetric difference $I_r(v_{i-1}) \triangle I_r(v_{i+r})$ contains exactly one codeword of C by Lemma 2.4 (ii). Therefore, without loss of generality, we may assume that $I_r(v_{i+r}) \setminus I_r(v_{i-1}) = \emptyset$. Since the pairs (v_j, v_{j+1}) of C-consecutive vertices, where $j = i+r, i+r+1, \ldots, i+2r-1$, are r-separated by exactly one codeword of C and $v_{j-r} \in I_r(v_j) \setminus I_r(v_{j+1})$, the vertices $v_{i+2r+1}, v_{i+2r+2}, \ldots, v_{i+3r} \notin C$. Hence, the set $I_r(v_{i+2r})$ is empty (a contradiction). Thus, the claim follows. □

Lemma 2.6. *Let n be divisible by three and $r \geq 2$. If C is an r-locating-dominating code in $\mathcal{C}_n$ with $n/3$ codewords, then the number of C-blocks of codewords is even.*

Proof. Let C be an r-locating-dominating code in $\mathcal{C}_n$ with $n/3$ codewords. Assume that $Q_t(i)$ is a C-block (for appropriate integers i and t). Hence, (v_{i-1}, v_{i+t}) is a pair of C-consecutive vertices. This pair is r-separated by a unique codeword. Assume that this codeword belongs to the C-block $Q_{t'}(i')$ (for some appropriate integers i' and t'). Now the pair $(v_{i'-1}, v_{i'+t'})$ of C-consecutive vertices is clearly r-separated by a unique codeword that belongs to the C-block $Q_t(i)$. Therefore, each C-block can be uniquely paired to another C-block. Thus, the number of C-blocks is even. $\square$

3 Cycles with a small number of vertices

In this section, we consider r-locating-dominating codes in $\mathcal{C}_n$ with small n (for a given r). The following easy theorem gives the exact values of $M_r^{LD}(\mathcal{C}_n)$ when $3 \leq n \leq 2r+1$.

Theorem 3.1. *Let n and r be positive integers such that $3 \leq n \leq 2r+1$ and $r \geq 2$. Then we have*

$$M_r^{LD}(\mathcal{C}_n) = n - 1.$$

Proof. Let C be an r-identifying code in $\mathcal{C}_n$. Assume that $|C| \leq n-2$. Then there exist $u, v \in V_n \setminus C$ such that $u \neq v$. Since $B_r(u) = B_r(v) = V_n$, we have $I_r(u) = C$ and $I_r(v) = C$. Therefore, $|C| \geq n-1$. On the other hand, $\{v_0, v_1, \ldots, v_{n-2}\}$ is an r-locating-dominating code in $\mathcal{C}_n$ with $n-1$ codewords. Thus, we have $M_r^{LD}(\mathcal{C}_n) = n-1$. $\square$

The following two theorems consider r-locating-dominating codes in the cycles $\mathcal{C}_{2r+2}$ and $\mathcal{C}_{2r+3}$.

Theorem 3.2. *Let $r \geq 2$. Then we have*

$$M_r^{LD}(\mathcal{C}_{2r+2}) = r + 1.$$

Proof. Let C be an r-locating-dominating code in $\mathcal{C}_n$ with $n = 2r+2$. For $v_i \in V_n \setminus C$, consider sets $B'_r(v_i) = V_n \setminus B_r(v_i) = \{v_{i+r+1}\}$. Since C is an r-locating-dominating code in $\mathcal{C}_{2r+2}$, the sets $B'_r(v_i) \cap C$ are unique for all $v_i \in V_n \setminus C$. Assume then that $|C| \leq r$. Since now $|V_n \setminus C| \geq r+2$, there exist (by the pigeonhole principle) vertices $v_i, v_j \in V_n \setminus C$ such that $v_i \neq v_j$ and $B'_r(v_i) \cap C = B'_r(v_j) \cap C$ (a contradiction). Thus, we have $|C| \geq r+1$.

By Lemma 2.2, it is straightforward to verify that $\{v_0, v_1, \ldots, v_r\}$ is an r-locating-dominating code in $\mathcal{C}_{2r+2}$. Therefore, we have $M_r^{LD}(\mathcal{C}_{2r+2}) = r+1$. $\square$

Theorem 3.3. *Let $r \geq 2$. Then we have*

$$M_r^{LD}(\mathcal{C}_{2r+3}) \geq \left\lceil \frac{2(2r+2)}{5} \right\rceil.$$

Proof. Let C be an r-locating-dominating code in $\mathcal{C}_n$ with $n = 2r + 3$. For $v_i \in V_n \setminus C$, consider again the sets $B'_r(v_i) = V_n \setminus B_r(v_i) = \{v_{i+r+1}, v_{i+r+2}\}$. Since C is an r-locating-dominating code in $\mathcal{C}_{2r+3}$, the sets $B'_r(v_i) \cap C$ are unique for all $v_i \in V_n \setminus C$. Hence, at most one of the sets $B'_r(v_i)$ can be empty and at most $|C|$ of them contains only one codeword of C. On the other hand, each codeword can belong to at most two sets $B'_r(v_i)$. Therefore, we have the inequality

$$|C| + 2(n - 2|C| - 1) \leq 2|C|.$$

Thus, the claim immediately follows. $\square$

Let $r = 5r' + 1$, where r' is a positive integer. Now, by the previous theorem, we have $M_r^{LD}(\mathcal{C}_{2r+3}) = M_r^{LD}(\mathcal{C}_{5(2r'+1)}) \geq 2(2r' + 1)$. Define then

$$C = \bigcup_{i=0}^{2r'} \{v_{5i}, v_{5i+1}\}.$$

It is straightforward to verify that C is an r-locating-dominating code in $\mathcal{C}_{2r+3}$ attaining the lower bound of Theorem 3.3. Thus, we have an infinite family of radii r for which $M_r^{LD}(\mathcal{C}_{2r+3}) = \lceil 2(2r + 2)/5 \rceil$.

Let us then determine the exact values of $M_r^{LD}(\mathcal{C}_{3r})$ and $M_r^{LD}(\mathcal{C}_{3r+3})$. The following theorem, which solves the exact values of $M_r^{LD}(\mathcal{C}_{3r})$ when r is even and $M_r^{LD}(\mathcal{C}_{3r+3})$ when r is odd, have previously been presented in [1].

Theorem 3.4 ([1])**.** *Let r be an integer such that $r \geq 3$.*

- *(i) If r is even, then $M_r^{LD}(\mathcal{C}_{3r}) = r$.*
- *(ii) If r is odd, then $M_r^{LD}(\mathcal{C}_{3r+3}) = r + 1$.*

The remaining cases are solved in the following theorem.

Theorem 3.5. *Let r be an integer such that $r \geq 3$.*

- *(i) If r is even, then $M_r^{LD}(\mathcal{C}_{3r+3}) = r + 2$.*
- *(ii) If r is odd, then $M_r^{LD}(\mathcal{C}_{3r}) = r + 1$.*

Proof. (i) Let $r \geq 3$ be an even integer. Assume that C is an r-locating-dominating code in $\mathcal{C}_{3r+3}$ with $r+1$ codewords. Let us first show that now each C-block of codewords is of length one. Assume to the contrary that $Q_t(i)$ is a C-block of codewords with $t \geq 2$ (for an appropriate integer i). Now (v_{i-1}, v_{i+t}) is a pair of C-consecutive vertices. The symmetric difference $B_r(v_{i-1}) \triangle B_r(v_{i+t}) = Q_{t+1}(i-r-1) \cup Q_{t+1}(i+r)$ contains at most one codeword, by Lemma 2.4 (ii). Without loss of generality, we may assume that $Q_{t+1}(i-r-1) \cap C$ is empty. Since the pairs $(v_{i-r+t-2}, v_{i-r+t-1})$ and $(v_{i-r+t-3}, v_{i-r+t-2})$ of C-consecutive vertices are r-separated, respectively, by the codewords v_{i+t-1} and v_{i+t-2}, the vertices $v_{i-2r+t-2}$ and $v_{i-2r+t-3}$ do not belong to C, by Lemma 2.4 (ii). By the considerations above, the symmetric difference $B_r(v_{i-2r+t-3}) \triangle B_r(v_{i-2r+t-2}) = \{v_{i-3r+t-3}, v_{i-r+t-2}\} = \{v_{i+t}, v_{i-r+t-2}\}$ does not contain codewords of C (a contradiction). Hence, each C-block is of length one.

By Lemma 2.6, we know that the number of C-blocks is even. Therefore, by the fact that each C-block is of length one, it immediately follows that the number of codewords in C is even. However, this contradicts the assumption that the number of vertices in C is equal to $r+1$. Thus, there does not exist an r-locating-dominating code in $\mathcal{C}_{3r+3}$ with $r+1$ codewords. Hence, we have $M_r^{LD}(\mathcal{C}_{3r+3}) \geq r+2$. On the other hand, it is straightforward to verify (using Lemma 2.2) that $\{v_0, v_1, \ldots, v_r, v_{2r+1}\}$ is an r-locating-dominating code in $\mathcal{C}_{3r+3}$ with $r+2$ codewords. Thus, we have $M_r^{LD}(\mathcal{C}_{3r+3}) = r+2$.

(ii) Let $r \geq 3$ be an odd integer. Assume that C is an r-locating-dominating code in $\mathcal{C}_{3r}$ with r codewords. Using similar ideas as in the case (i), it can be shown that each C-block is of length one. Then a contradiction again follows using Lemma 2.6. Thus, we have $M_r^{LD}(\mathcal{C}_{3r}) \geq r+1$. On the other hand, it is easy to verify that $\{v_0, v_1, \ldots, v_r\}$ is an r-locating-dominating code in $\mathcal{C}_{3r}$. Therefore, we have $M_r^{LD}(\mathcal{C}_{3r}) = r+1$. □

4 Cycles with a large number of vertices

Let r be an integer such that $r \geq 5$. In this section, we prove that for any $n \geq n_r$ when n_r is large enough we have constructions showing $M_r^{LD}(\mathcal{C}_n) \leq n/3+1$ if $n \equiv 3 \pmod 6$ and $M_r^{LD}(\mathcal{C}_n) \leq \lceil n/3 \rceil$ if $n \not\equiv 3 \pmod 6$. By Theorem 2.3, the latter constructions are optimal.

The path of length n is defined as $\mathcal{P}_n = (V_n, E'_n)$, where V_n is the same as in the case of cycles and $E'_n = \{v_i v_{i+1} \mid i = 0, 1, \ldots, n-2\}$. The following theorem provides a useful relation between the optimal r-locating-dominating codes in cycles and paths.

Theorem 4.1. *Let $n \geq 4r+2$. Then we have*

$$M_r^{LD}(\mathcal{C}_n) \leq M_r^{LD}(\mathcal{P}_{n+1}).$$

Proof. Let C be an r-locating-dominating code in $\mathcal{P}_{n+1}$. Assume first that $v_n \notin C$. Now each pair of C-consecutive vertices in $\mathcal{C}_n$ is r-separated by C, since each pair of C-consecutive vertices in $\mathcal{P}_{n+1}$ is r-separated by C. It is also easy to see that all the vertices of $\mathcal{C}_n$ are r-covered by a codeword of C and that there does not exist a vertex $u \in V_n \setminus C$ such that $B_r(u) = C$ (since $n \geq 4r+2$). Therefore, by Lemma 2.2, C is an r-locating-dominating code in $\mathcal{C}_n$.

If $v_0 \notin C$, then the proof is analogous to the previous case. Hence, assume that v_0 and v_n both belong to C. Let then $v_i, v_j, v_k \in V_n \setminus C$ be vertices such that $v_0, v_1, \ldots, v_{i-1} \in C$, $v_{j+1}, v_{j+2}, \ldots, v_n \in C$ and $v_{i+1}, v_{i+2}, \ldots, v_{k-1} \in C$. In other words, (v_j, v_i) and (v_i, v_k) are pairs of C-consecutive vertices. Consider then the code $C' = C \setminus \{v_n\}$ in $\mathcal{C}_n$. It is straightforward to verify that all the pairs except (v_i, v_j) of C'-consecutive vertices in $\mathcal{C}_n$ are r-separated by C'. Moreover, the symmetric difference of $B_r(v_j)$ and $B_r(v_k)$ contains a codeword of C'. Therefore, by Lemma 2.2, $C' \cup \{v_i\}$ is an r-locating dominating code in $\mathcal{C}_n$. Thus, in conclusion, we have $M_r^{LD}(\mathcal{C}_n) \leq M_r^{LD}(\mathcal{P}_{n+1})$. □

Assume that $r \geq 5$ and $n \geq 3r+2+6r((r-3)(2r+1)+r)$. In [5], it is shown that now $M_r^{LD}(\mathcal{P}_n) = \lceil (n+1)/3 \rceil$. Hence, if $n \equiv 1 \pmod 3$, then

$$\left\lceil \frac{n}{3} \right\rceil \leq M_r^{LD}(\mathcal{C}_n) \leq M_r^{LD}(\mathcal{P}_{n+1}) = \left\lceil \frac{n+2}{3} \right\rceil.$$

Therefore, $M_r^{LD}(\mathcal{C}_n) = \lceil n/3 \rceil$. If $n \equiv 3 \pmod 6$, we similarly obtain $n/3 \leq M_r^{LD}(\mathcal{C}_n) \leq M_r^{LD}(\mathcal{P}_{n+1}) = n/3+1$. We also conjecture that $M_r^{LD}(\mathcal{C}_n) = n/3+1$

(see Conjecture 5.4). In what follows, we give optimal constructions for the remaining cases when $n \equiv 0, 2$ or $5 \pmod 6$. For this, we first recall some preliminary definitions and results (previously presented in [5]).

Let i and s be non-negative integers. First, for $1 \leq i \leq r-2$, define

$$M_i(s) = \left(\bigcup_{\substack{j=0 \\ j \neq r-i-1}}^{r-1} \{v_{s+j}\} \right) \cup \{v_{s+2r-i}\}$$

and $M_i'(s) = M_i(s) \setminus \{v_{s+2r-i}\}$. Notice that $|M_i(s)| = r$. Furthermore, for $1 \leq i \leq r-3$, define

$$K_i(s) = M_i'(s) \cup \{v_{s+2r}, v_{s+3r-i}\} \cup \left(\bigcup_{\substack{j=3r+2 \\ j \neq 4r-i}}^{4r} \{v_{s+j}\} \right) \cup \{v_{s+5r-i}, v_{s+5r+2}\},$$

and $K_{r-2}(s) = M_{r-2}'(s) \cup \{v_{s+2r}, v_{s+2r+2}\}$. Notice that for $i = 1, 2, \ldots, r-3$, we have $|K_i(s)| = 2r+1$ and $|K_{r-2}(s)| = r+1$. Finally, define

$$L_2(s) = M_2(s) \cup \left(\bigcup_{\substack{j=3r+1 \\ j \neq 4r-1}}^{4r+1} \{v_{s+j}\} \right) \cup \{v_{s+6r}\}.$$

Notice that $|L_2(s)| = 2r+1$.

Denote by K_i and L_2 the patterns $\{v_s, v_{s+1}, \ldots, v_{s+\ell-1}\}$ where the codewords are determined by $K_i(s)$ and $L_2(s)$, respectively. The length ℓ of each pattern K_i and L_2 is equal to three times the number of codewords in the pattern. For example, the length of the pattern L_2 is equal to $6r+3$ (see the case (iii) below). The following lemma, which is a slightly reformulated version of [5], says for general $r \geq 5$ that the patterns K_i and L_2 can be concatenated to form r-locating dominating codes (because the beginning of each of them contains $M_i'(s)$).

Lemma 4.2 ([5])**.** *Let s be a non-negative integer and $r \geq 5$. Let C be a code in $\mathcal{C}_n$.*

- *(i) Let i be an integer such that $1 \leq i \leq r-3$. If $K_i(s) \cup M_{i+1}'(s+6r+3) \subseteq C$, then each pair (v_{j_1}, v_{j_2}) of C-consecutive vertices in $\mathcal{C}_n$ such that $s \leq j_1 \leq s+7r+2$ and $s \leq j_2 \leq s+7r+2$ is r-separated by a codeword of C.*
- *(ii) If $K_{r-2}(s) \cup M_1'(s+3r+3) \subseteq C$, then each pair (v_{j_1}, v_{j_2}) of C-consecutive vertices in $\mathcal{C}_n$ such that $s \leq j_1 \leq s+4r+2$ and $s \leq j_2 \leq s+4r+2$ is r-separated by a codeword of C.*
- *(iii) If $L_2(s) \cup M_1'(s+6r+3) \subseteq C$, then each pair (v_{j_1}, v_{j_2}) of C-consecutive vertices in $\mathcal{C}_n$ such that $s \leq j_1 \leq s+7r+2$ and $s \leq j_2 \leq s+7r+2$ is r-separated by a codeword of C.*

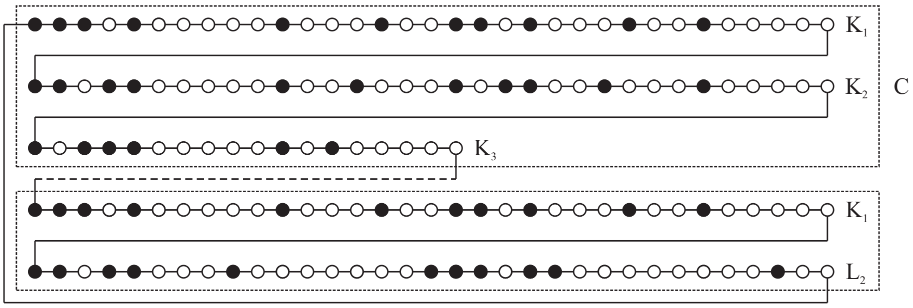

Figure 1: The r-locating-dominating code C_0 illustrated when $r = 5$. The pattern C, which obtained by concatenating the patterns K_1, K_2 and K_3, is repeated p times and the concatenation of K_1 and L_2 is repeated q times.

For the following constructions, we also define

$$C(s) = \bigcup_{i=0}^{r-3} K_{i+1}(s + i(6r + 3)).$$

Now we are ready to proceed with the remaining constructions of r-locating-dominating codes in cycles. These constructions are based on [5], although attention needs to be paid to details. First let $m = p((r-3)(6r+3)+3r+3)+q\cdot 2(6r+3)$, where p and q are non-negative integers. Define then

$$\begin{aligned} C_0 = & \bigcup_{j=0}^{p-1} C(j((r-3)(6r+3)+3r+3)) \\ & \cup \bigcup_{j=0}^{q-1} K_1(p((r-3)(6r+3)+3r+3)+2j(6r+3)) \\ & \cup \bigcup_{j=0}^{q-1} L_2(p((r-3)(6r+3)+3r+3)+(2j+1)(6r+3)). \end{aligned}$$

The code C_0 is illustrated in Figure 1. Notice that $M'_i(s) \subseteq K_i(s)$ and $M'_2(s) \subseteq L_2(s)$ for any s. Therefore, by Lemma 4.2, it is immediate that each pair (v_j, v_k) of C_0-consecutive vertices in $\mathcal{C}_m$ is r-separated by C_0. It is also obvious that all the vertices in $\mathcal{C}_m$ are r-covered by a codeword of C_0 and that there does not exist a vertex $u \in V_m \setminus C_0$ such that $I_r(u) = C_0$. Thus, by Lemma 2.2, it is easy to conclude that C_0 is an r-locating-dominating code in $\mathcal{C}_m$ with $m/3$ codewords.

Notice further that the greatest common divisor of $(r-3)(6r+3)+3r+3$ and $2(6r+3)$ is equal to 6. Hence, the greatest common divisor of $1/2\cdot((r-3)(2r+1)+r+1)$ and $2r+1$ is equal to 1. Thus, by [11, Theorem 8.3], if n' is an integer such that $n' \geq r((r-3)(2r+1)+r-1)$, then there exist non-negative integers p and q such that $n' = p/2\cdot((r-3)(2r+1)+r+1)+q(2r+1)$. Therefore, if n is an integer such that $n \geq 6r((r-3)(2r+1)+r-1)$ and $n \equiv 0 \pmod 6$, then there exist integers $p \geq 0$ and $q \geq 0$ such that $n = p((r-3)(6r+3)+3r+3)+q\cdot 2(6r+3)$.

Thus, if n is an integer such that $n \geq 6r((r-3)(2r+1)+r-1)$ and $n \equiv 0 \pmod 6$, then by the previous construction $M_r^{LD}(\mathcal{C}_n) \leq n/3$.

Let $m = 6r+2+p((r-3)(6r+3)+3r+3)+q \cdot 2(6r+3)$, where p and q are non-negative integers. Define

$$
\begin{aligned}
C_2 = K_{r-2}(r-1) &\cup \bigcup_{j=0}^{p-1} C(4r+2+j((r-3)(6r+3)+3r+3)) \\
&\cup \bigcup_{j=0}^{q-1} K_1(4r+2+p((r-3)(6r+3)+3r+3)+2j(6r+3)) \\
&\cup \bigcup_{j=0}^{q-1} L_2(4r+2+p((r-3)(6r+3)+3r+3)+(2j+1)(6r+3)) \\
&\cup M_1(4r+2+p((r-3)(6r+3)+3r+3)+2q(6r+3)).
\end{aligned}
$$

By Lemma 4.2, it is immediate that if (v_i, v_j) is a pair of C_2-consecutive vertices in $\mathcal{C}_m$ such that $r-1 \leq i \leq m-r-1$ and $r-1 \leq j \leq m-r-1$, then (v_i, v_j) is r-separated by C_2. Consider then the remaining pairs of C_2-consecutive vertices. For this, we first recall that $M_1(m-2r) = \{v_{-2r}, v_{-2r+1}, \ldots, v_{-r-3}, v_{-r-1}, v_{-1}\}$ and $K_{r-2}(r-1) = \{v_{r-1}, v_{r+1}, v_{r+2}, \ldots, v_{2r-2}, v_{3r-1}, v_{3r+1}\}$. Now it is easy to see that the pairs (v_{-r-2}, v_{-r}) and (v_{r-2}, v_r) are r-separated by the codeword v_{-1} and the pair (v_{-2}, v_0) is r-separated by the codeword v_{r-1}. Furthermore, for all $i = -r, -r+1, \ldots, -3$ and $j = 0, 1, \ldots, r-3$ the pairs (v_i, v_{i+1}) and (v_j, v_{j+1}) are r-separated by the codewords v_{i-r} and v_{j+1+r}, respectively. Thus, each pair of C_2-consecutive vertices in $\mathcal{C}_m$ is r-separated by C_2. Therefore, by Lemma 2.2, it is straightforward to verify that C_2 is an r-locating-dominating code in $\mathcal{C}_m$ with $\lceil m/3 \rceil$ codewords. Thus, as in the previous case, if n is an integer such that $n \geq 6r+2+6r((r-3)(2r+1)+r-1)$ and $n \equiv 2 \pmod 6$, then by the previous construction $M_r^{LD}(\mathcal{C}_n) \leq \lceil n/3 \rceil$.

Let $m = 12r+5+p((r-3)(6r+3)+3r+3)+q \cdot 2(6r+3)$, where p and q are non-negative integers. Define

$$
\begin{aligned}
C_5 = K_{r-2}(r) &\cup \bigcup_{j=0}^{p-1} C(4r+3+j((r-3)(6r+3)+3r+3)) \\
&\cup \bigcup_{j=0}^{q} K_1(4r+3+p((r-3)(6r+3)+3r+3)+2j(6r+3)) \\
&\cup \bigcup_{j=0}^{q-1} L_2(4r+3+p((r-3)(6r+3)+3r+3)+(2j+1)(6r+3)) \\
&\cup M_2(4r+3+p((r-3)(6r+3)+3r+3)+(2q+1)(6r+3)).
\end{aligned}
$$

Again, using Lemmas 2.2 and 4.2, it can be shown that C_5 is an r-locating-dominating code in $\mathcal{C}_m$ with $\lceil m/3 \rceil$ codewords. Thus, if n is an integer such that $n \geq 12r+5+6r((r-3)(2r+1)+r-1)$ and $n \equiv 5 \pmod 6$, then by the previous construction $M_r^{LD}(\mathcal{C}_n) \leq \lceil n/3 \rceil$.

Combining the previous results with the lower bound of Theorem 2.3, we immediately obtain the following theorem.

Theorem 4.3. *Let $r \geq 5$ and $n \geq 12r+5+6r((r-3)(2r+1)+r-1)$.*

(i) If $n \not\equiv 3 \pmod 6$*, then* $M_r^{LD}(\mathcal{C}_n) = \lceil n/3 \rceil$.

(ii) If $n \equiv 3 \pmod 6$*, then* $n/3 \leq M_r^{LD}(\mathcal{C}_n) \leq n/3 + 1$.

In the latter case of the previous theorem, we conjecture that actually $M_r^{LD}(\mathcal{C}_n) = n/3 + 1$ (see Conjecture 5.4).

5 On r-locating-dominating codes in cycles with $2 \leq r \leq 4$

In this section, we consider r-locating-dominating codes in $\mathcal{C}_n$ when $2 \leq r \leq 4$. The exact values of $M_2^{LD}(\mathcal{C}_n)$ are determined in [4]. In particular, it is shown that for $n > 6$ if $n \equiv 3 \pmod 6$, then $M_2^{LD}(\mathcal{C}_n) = n/3 + 1$, else $M_2^{LD}(\mathcal{C}_n) = \lceil n/3 \rceil$. In the following theorem, we provide an alternative (and shorter) proof for the lower bound in the case $n \equiv 3 \pmod 6$.

Theorem 5.1 ([4])**.** *Let* $n \equiv 3 \pmod 6$*. Then we have*

$$M_2^{LD}(\mathcal{C}_n) \geq n/3 + 1.$$

Proof. Let C be a 2-locating-dominating code in $\mathcal{C}_n$ with $n/3$ vertices. Now, by Lemma 2.5, each C-block is of length one. By Lemma 2.6, the number of C-blocks is even. Hence, by combining these two observations, the number of codewords of C is even. This contradicts with the fact that $|C| = n/3$ (an odd integer since $n \equiv 3 \pmod 6$). Thus, we have $M_2^{LD}(\mathcal{C}_n) \geq n/3 + 1$. □

With our new approach, a lower bound similar to the previous theorem can also be proved when $r = 3$ and $r = 4$. The following theorem shows the result for 3-locating-dominating codes.

Theorem 5.2. *Let* $n \equiv 3 \pmod 6$*. Then we have*

$$M_3^{LD}(\mathcal{C}_n) \geq n/3 + 1.$$

Proof. Let C be a 3-locating-dominating code in $\mathcal{C}_n$ with $n/3$ vertices. Notice that each C-block of codewords is now at most of length 2 (by Lemma 2.6). In what follows, we show that the number of C-blocks of length two is even.

Recall that according to Lemma 2.4 each pair of C-consecutive vertices is 3-separated by exactly one codeword of C. Assume then that $\{v_i, v_{i+1}\}$ is a C-block of length two. By the previous observation, the set $B_r(v_{i-1}) \triangle B_r(v_{i+2})$ contains exactly one codeword of C. Without loss of generality, we may assume that v_{i-4}, v_{i-3} and v_{i-2} do not belong to C. Then either v_{i+3} or v_{i+5} belongs to C. (Notice that if $v_{i+4} \in C$, then the pair (v_{i+3}, v_{i+5}) of C-consecutive vertices is 3-separated by at least two codewords.)

Assume first that $v_{i+5} \in C$. If now $v_{i+6} \notin C$, then the pair (v_{i+2}, v_{i+3}) of C-consecutive vertices is not r-separated by any codeword of C. Hence, $v_{i+6} \in C$ and further $v_{i+7} \notin C$. Therefore, $\{v_{i+5}, v_{i+6}\}$ is also a C-block of length two. Since the neighbourhoods of the C-blocks $\{v_i, v_{i+1}\}$ and $\{v_{i+5}, v_{i+6}\}$ are symmetrical to each other, these C-blocks of length two can be paired with each other.

Assume then that $v_{i+3} \in C$. Considering the pairs (v_{i+2}, v_{i+4}), (v_{i+4}, v_{i+5}) and (v_{i+6}, v_{i+7}), we obtain that v_{i+6}, v_{i+7}, v_{i+8} and v_{i+10} do not belong to

C. The pairs (v_{i+5}, v_{i+6}) and (v_{i+7}, v_{i+8}) of C-consecutive vertices imply that v_{i+9} and v_{i+11} belong to C. By the fact that now (v_{i+8}, v_{i+10}) is a pair of C-consecutive vertices, we know that either v_{i+12} or v_{i+13} is a codeword of C. If $v_{i+12} \in C$, then $\{v_{i+11}, v_{i+12}\}$ is a C-block and the neighbourhoods of the C-blocks $\{v_i, v_{i+1}\}$ and $\{v_{i+11}, v_{i+12}\}$ are symmetrical to each other. Therefore, these C-blocks of length two can be paired with each other. Assume then that $v_{i+13} \in C$. Consider then the symmetric difference $B_r(v_{i+10}) \triangle B_r(v_{i+12})$, where (v_{i+10}, v_{i+12}) is a pair of C-consecutive vertices. Now either v_{i+14} or v_{i+15} belongs to C. If $v_{i+14} \in C$, then the pair (v_{i+12}, v_{i+15}) of C-consecutive vertices is 3-separated by at least two codewords (a contradiction). Therefore, v_{i+15} belongs to C. Using similar arguments as above, we obtain that $v_{i+16}, v_{i+17}, v_{i+18}, v_{i+19}, v_{i+20}, v_{i+22} \notin C$ and $v_{i+21}, v_{i+23} \in C$. The situation is now analogous to the one in which we considered the pair (v_{i+8}, v_{i+10}) of C-consecutive vertices instead that here we have the pair (v_{i+20}, v_{i+22}).

The previous reasonings can now be repeated. However, since we are operating in a cycle, at some point the repetition has to end. Therefore, for some non-negative integer k we have that $\{v_i, v_{i+1}\}$ and $\{v_{i+11+12k}, v_{i+12+12k}\}$ are C-blocks with symmetrical neighbourhoods. Clearly, the sets $\{v_i, v_{i+1}\}$ and $\{v_{i+11+12k}, v_{i+12+12k}\}$ do not coincide. Thus, these C-blocks of length two can be paired with each other. In conclusion, each C-block of length two can be uniquely paired to another C-block of length two. Therefore, the number of C-blocks of length two is even.

By Lemma 2.6, the number of C-blocks is even. Hence, by the previous considerations, the number of C-blocks of length one is also even. Thus, the number of codewords of C is even. This contradicts with the fact that $|C| = n/3$ is odd. Therefore, we have $M_3^{LD}(\mathcal{C}_n) \geq n/3 + 1$. □

In the following theorem, a lower bound similar to the one in Theorems 5.1 and 5.2 is presented for 4-locating-dominating codes in cycles.

Theorem 5.3. *Let $n \equiv 3 \pmod 6$. Then we have*

$$M_4^{LD}(\mathcal{C}_n) \geq n/3 + 1.$$

Proof. Let C be a 4-locating-dominating code in $\mathcal{C}_n$ with $n/3$ vertices. As earlier, we start by showing that the number of C-blocks of length two is even.

Let $\{v_i, v_{i+1}\}$ be a C-block of length two. Without loss of generality, we can again assume that v_{i-5}, v_{i-4} and v_{i-3} do not belong to C. As in the previous proof, we can also conclude that v_{i+5} does not belong to C. Moreover, since v_{i-1} and v_{i+2} are 4-separated by C, either $v_{i+4} \in C$ or $v_{i+6} \in C$ by Lemma 2.4.

In what follows, we are going to classify C-blocks of length two into different types depending on their neighbourhood. If $\{v_{i+6}, v_{i+7}, v_{i+8}\}$ is a C-block of length three, then we say that C-block $\{v_i, v_{i+1}\}$ is of type A_1. If $\{v_{i+6}, v_{i+7}\}$ is a C-block of length two, then a contradiction follows since the pair (v_{i+3}, v_{i+4}) of C-consecutive vertices is not 4-separated by a codeword. Assume that $\{v_{i+6}\}$ is a C-block of length one. Then v_{i+4} does not belong to C. If $v_{i+3} \notin C$, then the C-block $\{v_i, v_{i+1}\}$ is said to be of type A_2. Assume further that $v_{i+3} \in C$. If now $v_{i-2} \notin C$, then we say that $\{v_i, v_{i+1}\}$ is of type A_3, else it is of type A_4.

If $\{v_{i+3}, v_{i+4}\}$ is a C-block of length two, then ($v_{i+6} \notin C$ and) $\{v_i, v_{i+1}\}$ is of type A_5. Assume now that $\{v_{i+4}\}$ is a C-block of length one. Then v_{i+6} does not belong to C. If $v_{i-2} \in C$, then the C-block $\{v_i, v_{i+1}\}$ is of type A_6, else it is of type A_7.

For each of the previous types A_i we also have a symmetrical pair A'_i which is considered as a reflection of the neighbourhood of type A_i (between the vertices v_i and v_{i+1}. For example, if $v_{i-4}, v_{i+4}, v_{i+5}, v_{i+6} \notin C$ and $\{v_{i-7}, v_{i-6}, v_{i-5}\}$ is a C-block of length three, then we say that C-block $\{v_i, v_{i+1}\}$ is of type A'_1. By the previous considerations, it is straightforward to verify that each C-block of length two is one of the types A_i or A'_i.

Assume that the C-block $\{v_i, v_{i+1}\}$ is of type A'_1. Then v_{i-2}, v_{i-1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5} and v_{i+6} do not belong to C. Considering the pairs (v_{i+2}, v_{i+3}), (v_{i+3}, v_{i+4}), (v_{i+4}, v_{i+5}) and (v_{i+5}, v_{i+6}) of C-consecutive vertices, we have that $v_{i+7}, v_{i+8} \in C$ and $v_{i+9}, v_{i+10} \notin C$. The pair (v_{i+9}, v_{i+10}) of C-consecutive vertices imply that $v_{i+14} \in C$. Therefore, considering the pair (v_{i+6}, v_{i+9}) of C-consecutive vertices, we obtain that the C-block $\{v_{i+7}, v_{i+8}\}$ of length two is either of type A_1 or A_7. The proof of the following symmetrical result is analogous: if the C-block $\{v_i, v_{i+1}\}$ is of type A_1, then the C-block $\{v_{i-7}, v_{i-6}\}$ of length two is either of type A'_1 or A'_7.

Assume that the C-block $\{v_i, v_{i+1}\}$ is of type A_2. Then the vertices v_{i+2}, v_{i+3}, v_{i+4} and v_{i+5} do not belong to C. Considering the pairs (v_{i+2}, v_{i+3}), (v_{i+3}, v_{i+4}) and (v_{i+4}, v_{i+5}) of C-consecutive vertices, we obtain that $v_{i+7}, v_{i+9} \notin C$ and $v_{i+8} \in C$. Since $v_{i+1} \in B_4(v_{i+5}) \triangle B_4(v_{i+7})$, then $v_{i+10}, v_{i+11} \notin C$. Considering the pair (v_{i+7}, v_{i+9}) of C-consecutive vertices, we know that either v_{i+12} or v_{i+13} belong to C. If $v_{i+13} \in C$, then it is straightforward to conclude (using similar arguments as before) that $\{v_{i+13}, v_{i+14}\}$ is a C-block of type A'_2. Otherwise, it can be seen that $v_{i+12}, v_{i+14} \in C$ and $v_{i+13}, v_{i+15}, v_{i+16}, v_{i+17} \notin C$. The situation is now analogous to the one in which we considered the pair (v_{i+7}, v_{i+9}) of C-consecutive vertices instead that here we have the pair (v_{i+13}, v_{i+15}). The previous reasonings can be repeated. However, since we are operating in a cycle, at some point the repetition has to end. Therefore, for some non-negative integer k we have that $\{v_i, v_{i+1}\}$ and $\{v_{i+13+6k}, v_{i+14+6k}\}$ are C-blocks of type A_2 and A'_2, respectively. The following symmetrical result also holds: if $\{v_i, v_{i+1}\}$ is a C-block of type A'_2, then for some non-negative integer k we have that $\{v_{i-13-6k}, v_{i-12-6k}\}$ is a C-block of type A_2.

In the following, we list the results of the previous two paragraphs and other analogous ones, which can be obtained using similar arguments:

- If $\{v_i, v_{i+1}\}$ is a C-block of type A'_1, then $\{v_{i+7}, v_{i+8}\}$ is a C-block either of type A_1 or A_7.
- If $\{v_i, v_{i+1}\}$ is a C-block of type A_2, then for some non-negative integer k we have that $\{v_{i+13+6k}, v_{i+14+6k}\}$ is a C-block of type A'_2.
- If $\{v_i, v_{i+1}\}$ is a C-block of type A'_3, then either $\{v_{i+11}, v_{i+12}\}$ is a C-block of type A'_6 or $\{v_{i+13}, v_{i+14}\}$ is a C-block of type A'_4.
- If $\{v_i, v_{i+1}\}$ is a C-block of type A_4, then $\{v_{i+13}, v_{i+14}\}$ is a C-block either of type A_3 or A_5.
- If $\{v_i, v_{i+1}\}$ is a C-block of type A'_5, then either $\{v_{i+11}, v_{i+12}\}$ is a C-block of type A'_6 or $\{v_{i+13}, v_{i+14}\}$ is a C-block of type A'_4.
- If $\{v_i, v_{i+1}\}$ is a C-block of type A_6, then $\{v_{i+11}, v_{i+12}\}$ is a C-block either of type A_3 or A_5.

- If $\{v_i, v_{i+1}\}$ is a C-block of type A'_7, then $\{v_{i+7}, v_{i+8}\}$ is a C-block either of type A_1 or A_7.

The obvious symmetrical results also hold. For example, if $\{v_i, v_{i+1}\}$ is a C-block of type A'_4, then $\{v_{i-13}, v_{i-12}\}$ is a C-block either of type A'_3 or A'_5.

The results listed above provide an approach to pair C-blocks of length two. The C-block $\{v_i, v_{i+1}\}$ depending on its type is paired with the C-block of length two suggested by the previous results. For example, the C-block $\{v_i, v_{i+1}\}$ of type A'_3 is paired with $\{v_{i+11}, v_{i+12}\}$ or $\{v_{i+13}, v_{i+14}\}$ depending on which one of these sets is a C-block. Using the results listed above, it is straightforward to verify that this way each C-block of length two is uniquely paired with another such one. Therefore, the number of C-blocks of length two is even.

By Lemma 2.6, the number of C-blocks is even. Hence, since the number of C-blocks of length two is even, the number of C-blocks that are of length one or three is also even. Thus, the number of codewords of C is even. This contradicts with the fact that $|C| = n/3$. Therefore, we have $M_4^{LD}(\mathcal{C}_n) \geq n/3 + 1$. $\square$

Theorems 3.5, 5.1, 5.2 and 5.3 suggest the following conjecture.

Conjecture 5.4. *Let n be a positive integer such that $n \equiv 3 \pmod 6$. Then for any r we have*

$$M_r^{LD}(\mathcal{C}_n) \geq n/3 + 1.$$

In what follows, we concentrate on constructing optimal r-locating-dominating codes in $\mathcal{C}_n$ when $3 \leq r \leq 4$. In order to do this, we first need to present some preliminary definitions and results.

Define an infinite path $\mathcal{P}_\infty = (V_\infty, E_\infty)$, where $V_\infty = \{v_i \mid i \in \mathbb{Z}\}$ and $E_\infty = \{v_i v_{i+1} \mid i \in \mathbb{Z}\}$. Define then

$$C = \{v_i \in V_\infty \mid i \equiv 0, 2 \bmod 6\}.$$

In [8], it is stated that if r is an integer such that $r \geq 2$ and $r \equiv 1, 2, 3$ or $4 \pmod 6$, then C is an r-locating-dominating code in $\mathcal{P}_\infty$. This result is rephrased in the case of cycles in the following lemma when $r = 3$ or $r = 4$.

Lemma 5.5. *Let n and k be integers such that*

$$D = \{v_k, v_{k+2}, v_{k+6}, v_{k+8}, v_{k+12}, v_{k+14}\} \subseteq V_n.$$

If a pair (v_i, v_j) of D-consecutive vertices in $\mathcal{C}_n$ is such that $k+5 \leq i \leq k+13$ and $k+5 \leq j \leq k+13$, then v_i and v_j are 3- and 4-separated by D. Moreover, for each vertex $v_i \in V_n \setminus D$ such that $k+6 \leq i \leq k+11$ we have $\emptyset \subsetneq I_3(D; v_i) \subsetneq D$ and $\emptyset \subsetneq I_4(D; v_i) \subsetneq D$.

Consider then 3-locating-dominating codes in $\mathcal{C}_n$. The exact values of $M_3^{LD}(\mathcal{C}_n)$ when $3 \leq n \leq 8$ are determined in Theorems 3.1 and 3.2. Let p be a non-negative integer. Define then

$$D(p) = \bigcup_{i=0}^{p} \{v_{6i}, v_{6i+2}\}.$$

It is straightforward to verify that $D(1)$ and $D(2)$ are 3-locating-dominating codes in $\mathcal{C}_9$, $\mathcal{C}_{10}$, $\mathcal{C}_{11}$, $\mathcal{C}_{12}$ and $\mathcal{C}_{15}$, $\mathcal{C}_{16}$, $\mathcal{C}_{17}$, $\mathcal{C}_{18}$, respectively. Therefore, by combining Lemmas 2.2 and 5.5, it can be concluded that $D(p)$ is a 3-locating-dominating code in $\mathcal{C}_{6p+3}$, $\mathcal{C}_{6p+4}$, $\mathcal{C}_{6p+5}$ and $\mathcal{C}_{6p+6}$ with $2(p+1)$ codewords when $p \geq 1$. Similarly, it can be shown that $D(p) \cup \{v_{6p+5}\}$ is a 3-locating-dominating code in $\mathcal{C}_{6p+8}$ with $2p+3$ codewords when $p \geq 1$. Furthermore, $D(p) \cup \{v_{6p+5}, v_{6p+8}, v_{6p+10}\}$ is a 3-locating-dominating code in $\mathcal{C}_{6p+13}$ with $2p+5$ codewords when $p \geq 0$. In conclusion, the constructions given above attain the lower bounds of Theorems 2.3 and 5.2. Thus, the exact values of $M_3^{LD}(\mathcal{C}_n)$ are determined for all positive integers n.

Consider now 4-locating-dominating codes in $\mathcal{C}_n$. By Theorems 3.1 and 3.2, the exact values of $M_4^{LD}(\mathcal{C}_n)$ are known when $3 \leq n \leq 10$. By Lemma 5.5, $D_1(p)$ is a 4-locating-dominating code in $\mathcal{C}_{6p+6}$ when $p \geq 2$. Using analogous arguments as above in the case $r = 3$, the following results can be shown:

- The code $D(p) \cup \{v_{6p+5}, v_{6p+7}, v_{6p+8}\}$ is 4-locating-dominating in $\mathcal{C}_{6p+13}$ with $2p+5$ codewords when $p \geq 0$.
- The code $D(p) \cup \{v_{6p+7}\}$ is 4-locating-dominating in $\mathcal{C}_{6p+8}$ with $2p+3$ codewords when $p \geq 1$.
- The code $D(p) \cup \{v_{6p+4}, v_{6p+7}, v_{6p+9}, v_{6p+10}\}$ is 4-locating-dominating in $\mathcal{C}_{6p+15}$ with $2p+6$ codewords when $p \geq 0$.
- The code $D(p) \cup \{v_{6p+4}, v_{6p+6}\}$ is 4-locating-dominating in $\mathcal{C}_{6p+10}$ with $2p+4$ codewords when $p \geq 1$.
- The code $D(p) \cup \{v_{6p+7}, v_{6p+8}, v_{6p+10}, v_{6p+15}, v_{6p+18}, v_{6p+21}\}$ is 4-locating-dominating in $\mathcal{C}_{6p+23}$ with $2p+8$ codewords when $p \geq 0$.

In conclusion, by Theorems 2.3 and 5.3, the exact values of $M_4^{LD}(\mathcal{C}_n)$ are determined for all n except 11, 12 or 17. The missing values can be easily determined since it is straightforward to verify that $\{v_0, v_1, v_3, v_4\}$, $\{v_0, v_2, v_4, v_6\}$ and $\{v_0, v_1, v_4, v_7, v_{10}, v_{11}\}$ are 4-locating-dominating codes in $\mathcal{C}_{11}$, $\mathcal{C}_{12}$ and $\mathcal{C}_{17}$, respectively, attaining the lower bound of Theorem 2.3.

The following theorem summarizes the previous considerations on 3- and 4-locating-dominating codes.

Theorem 5.6. *Let $n \geq 3$ and $3 \leq r \leq 4$. Then we have the following results:*

(i) $M_r^{LD}(\mathcal{C}_n) = n - 1$ *if* $3 \leq n \leq 2r + 1$.

(ii) $M_r^{LD}(\mathcal{C}_{2r+2}) = r + 1$.

(iii) $M_r^{LD}(\mathcal{C}_n) = n/3 + 1$ *if* $n > 2r + 2$ *and* $n \equiv 3 \pmod 6$.

(iv) $M_r^{LD}(\mathcal{C}_n) = \lceil n/3 \rceil$ *if* $n > 2r + 2$ *and* $n \not\equiv 3 \pmod 6$.

In finding the optimal families of r-locating-dominating codes in the cases $r = 3$ and $r = 4$, some computer searches were applied to obtain the initial codes. In what follows, we present a couple of approaches that were used to increase the efficiency of the search algorithms.

Consider each vertex of $\mathcal{C}_n$ in order. There are two possibilities: either the vertex is in the code, or it is not in the code. So there are 2^n cases to consider. We can reduce the number of cases we have to consider in two ways. First, we note that after we have decided whether or not vertex v_i is in the code, we can check vertices v_r through v_{i-r} to make sure they have distinct identifying sets. (Note that for paths, we could check vertices v_0 through v_{i-r}.) The second method for limiting the search uses a running count on the number of vertices in the code. By symmetry, we can assume that the number of codewords in the first half of the cycle is at most the number of codewords in the second half of the cycle. We also know that among any consecutive set of vertices v_i to v_j that contains (exactly) k codewords, the number of codewords must be at least $\log_2(j-i+1-2r-k)$. These two categories of checks were sufficient to reduce the running time of the algorithm to a manageable level.

6 Conclusions

Previously, the exact values of $M_1^{LD}(\mathcal{C}_n)$ and $M_2^{LD}(\mathcal{C}_n)$ have been determined in [16] and [4], respectively. In Section 5, we solved the exact values of $M_3^{LD}(\mathcal{C}_n)$ and $M_4^{LD}(\mathcal{C}_n)$ for any n. In Section 3, we determined the exact values of $M_r^{LD}(\mathcal{C}_n)$ when $3 \leq n \leq 2r+2$. Furthermore, in Section 4, it is shown that when n is large enough we have $n/3 \leq M_r^{LD}(\mathcal{C}_n) \leq n/3+1$ if $n \equiv 3 \pmod 6$ and $M_r^{LD}(\mathcal{C}_n) = \lceil n/3 \rceil$ otherwise. Moreover, we have Conjecture 5.4 stating that $M_r^{LD}(\mathcal{C}_n) \geq n/3+1$ when $n \equiv 3 \pmod 6$.

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