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Locally univalent approximations of analytic functions

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ABSTRACT

In the present paper, we introduce a measure of the non-univalency of an analytic function, and we use it in order to find the best approximation of analytic function by a locally univalent normalized analytic function.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \qquad z \in \Delta, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in Δ will be denoted by $\mathcal{U}$.

Following [4], for $\alpha \in \mathbb{R}$ we consider the class $\mathcal{G}(\alpha)$ consisting of locally univalent functions $f \in \mathcal{A}$ which satisfy the condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{\alpha}{2}, \qquad z \in \Delta. \tag{1.2}$$

It is easy to see that the identity function satisfies the above inequality for $\alpha > 0$, thus $\mathcal{G}(\alpha) \neq \varnothing$ if $\alpha > 0$, and we will make this assumption on α in the sequel. In [5], Ozaki introduced the class $\mathcal{G} \equiv \mathcal{G}(1)$ and proved that functions in $\mathcal{G}$ are univalent in Δ. In [11], Umezawa generalized Ozaki's result for a version

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of the class $\mathcal{G}$ (convex functions in one direction). It is also known that the functions in the class $\mathcal{G}(1)$ are starlike in Δ (see for example the particular case $\alpha = 1$ of (16) in [8], or [9,10]).

Since $\mathcal{G}(\alpha) \subset \mathcal{G}(\alpha')$ whenever $\alpha < \alpha'$, it readily follows that the class $\mathcal{G}(\alpha)$ is included in the class $\mathcal{S}$ of starlike functions in Δ whenever $\alpha \in (0,1]$, which in particular shows that $\mathcal{G}(\alpha)$ consists only of univalent functions for any $\alpha \in (0,1]$. In the present paper we will investigate the properties of the class $\mathcal{G}(\alpha)$ (and of a certain subclass $\mathcal{G}^*(\alpha)$ of it), and then we will determine the best approximation of an analytic function by functions in the class $\mathcal{G}(\alpha)$ in the sense of L^2 norm. The method is based on solving a certain semi-infinite quadratic problem, in the spirit of [6] and [7].

The structure of the paper is the following. In Section 2 we introduce the subclass $\mathcal{G}^*(\alpha) \subset \mathcal{G}(\alpha)$, defined by a certain inequality in terms of the Taylor coefficients of the function. Next, we investigate the connection between the class $\mathcal{G}(\alpha)$ (for various values of $\alpha > 0$) and the classical classes of starlike and convex functions (Theorem 2.1).

As indicated above, it is an open problem whether $\mathcal{G}(\alpha) \subset \mathcal{U}$ for $\alpha > 1$. In Theorem 2.2 we give a partial result for this problem, which shows that for $\alpha \in [1, 4.952)$ the radius of univalence of the class $\mathcal{G}(\alpha)$ is at least $1/\alpha$. The section concludes with a result (Proposition 2.1) which shows that for certain values of $\alpha \in (0,1]$ the class $\mathcal{G}^*(\alpha)$ interpolates between subclasses of starlike and convex functions, and that the result is sharp.

In order to investigate the problem of the best approximation of an analytic function by functions in the class $\mathcal{G}^*(\alpha)$ (in the sense of L^2 norm), in Section 3 we introduce and solve a semi-infinite quadratic programming problem, which may be of independent interest (Theorem 3.2). The paper concludes with Section 4, in which we apply the results of the previous section in order to settle the problem of the best approximation of an analytic function by functions in the class $\mathcal{G}^*(\alpha)$ (Theorem 4.1), and to present some numerical examples (Example 2.1).

2. Results on the classes $\mathcal{G}(\alpha)$ and $\mathcal{G}^*(\alpha)$

It can be easily seen that functions in $\mathcal{G}(\alpha)$ are not necessarily univalent in Δ if $\alpha > 1$, as shown by the following example.

Example 2.1. Consider the function $f : \Delta \to \mathbb{C}$ defined by $f(z) = \frac{1}{3}(z-1)^3 + \frac{1}{3}$, $z \in \Delta$. It is easy to see that the function f belongs to the class $\mathcal{A}$ and it is locally univalent. Since

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \operatorname{Re}\left(1 + \frac{2z}{z-1}\right) < 2 = 1 + \frac{2}{2}, \qquad z \in \Delta,$$

it follows that $f \in \mathcal{G}(2)$.

It is easy to see that $z_1 = 1 + 0.5e^{3\pi i/4}, z_2 = 1 + 0.5e^{3\pi i/4 + 2\pi i/3} \in \Delta$ and $f(z_1) = f(z_2)$, which shows that f is not univalent in Δ. It follows that for $\alpha \geq 2$ the class $\mathcal{G}(\alpha)$ does not consist entirely of univalent functions. It is an open question whether $\mathcal{G}(\alpha) \subset \mathcal{U}$ for $\alpha \in (1,2)$ (see Theorem 2.2 for a partial result on this problem).

We begin by investigating the connection between the class $\mathcal{G}(\alpha)$ (for various $\alpha > 0$) and the classical subclasses of (normalized) univalent functions consisting of starlike and convex functions, denoted by $\mathcal{S}$, respectively by $\mathcal{K}$.

It is known (see for example [2], p. 52) that if the Taylor coefficients of a normalized analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfy the inequality

$$\sum_{n=2}^{\infty} n |a_n| \leq 1, \tag{2.1}$$

then $f \in \mathcal{S}$, and if

$$\sum_{n=2}^{\infty} n^2 |a_n| \leq 1, \tag{2.2}$$

then $f \in \mathcal{K}$. We will denote by $\mathcal{S}^*$ and $\mathcal{K}^*$ the subclasses of $\mathcal{S}$, respectively $\mathcal{K}$, consisting of functions which satisfy (2.1), respectively (2.2) above.

Similarly, one can find sufficient conditions on the Taylor coefficients of the function which guarantee that it belongs to the class $\mathcal{G}(\alpha)$. One such sufficient condition is given by the following.

Lemma 2.1 *([4], Theorem 4). Suppose that $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, $z \in \Delta$, satisfies*

$$\sum_{n=2}^{\infty} n\left(2(n-1) - \alpha\right) |a_n| \leq \alpha, \tag{2.3}$$

for some $0 < \alpha \leq 1$. Then $f \in \mathcal{G}(\alpha)$.

As a particular example, consider the following.

Example 2.2. Let $f \in \mathcal{A}$ be given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{\alpha e^{i\theta}}{n^2(n-1)[2(n-1)-\alpha]} z^n, \qquad z \in \Delta,$$

for some $0 < \alpha \leq 1$ and $\theta \in \mathbb{R}$. The coefficient inequality (2.3) becomes

$$\sum_{n=2}^{\infty} n[2(n-1)-\alpha]|a_n| = \alpha \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \alpha$$

and by Lemma 2.1 it follows that $f \in \mathcal{G}(\alpha)$.

For $\alpha > 0$, we define $\mathcal{G}^*(\alpha)$ to be the class consisting of normalized analytic functions in the unit disk for which the corresponding Taylor coefficients satisfy the inequality (2.3) above. By Lemma 2.1 above it follows that $\mathcal{G}^*(\alpha) \subset \mathcal{G}(\alpha)$ whenever $\alpha \in (0,1]$, hence by the previous results it follows $\mathcal{G}^*(\alpha)$ consists only of univalent functions, for any $\alpha \in (0,1]$.

We will first we establish some connections between the classes $\mathcal{G}(\alpha)$ and $\mathcal{S}$, respectively between $\mathcal{G}(\alpha)$ and $\mathcal{K}$.

Theorem 2.1. *a) Consider $f \in \mathcal{A}$ locally univalent and let $F(z) = \int_0^z \frac{1}{f'(w)} dw$. Then $F \in \mathcal{A}$, and moreover $f \in \mathcal{K}$ iff $F \in \mathcal{G}(2)$.*

b) Consider $f \in \mathcal{A}$, $f(z) \neq 0$ for $z \in \Delta - \{0\}$, and let $F(z) = \int_0^z \frac{w}{f(w)} dw$. Then $F \in \mathcal{A}$, and moreover $f \in \mathcal{S}$ iff $F \in \mathcal{G}(2)$.

Proof. a) First note that since $f \in \mathcal{A}$ and f is locally univalent, F is an analytic function in Δ. It is immediate from the definition that $F(0) = 0$ and $F'(0) = \frac{1}{f'(0)} = 1$, so $F \in \mathcal{A}$, and F is also locally univalent.

Next, note that from the definition we have $F'(z) f'(z) = 1$ for all $z \in \Delta$. Differentiating we obtain $F''(z) f'(z) + F'(z) f''(z) = 0$ for $z \in \Delta$, and dividing by $F'(z) f'(z) = 1$ we conclude

$$\frac{F''(z)}{F'(z)} = -\frac{f''(z)}{f'(z)}, \qquad z \in \Delta. \tag{2.4}$$

Recalling the well-known characterization of convex functions $f \in \mathcal{A}$ (see for example [2], Theorem 2.2.3)

$$f \in \mathcal{K} \Longleftrightarrow \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \qquad z \in \Delta \tag{2.5}$$

and using the above, we obtain

$$\begin{aligned} F \in \mathcal{G}(2) &\Longleftrightarrow \operatorname{Re}\left(1 + \frac{zF''(z)}{F'(z)}\right) < 2, \qquad z \in \Delta \\ &\Longleftrightarrow \operatorname{Re}\left(1 - \frac{zf''(z)}{f'(z)}\right) < 2, \qquad z \in \Delta \\ &\Longleftrightarrow \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \qquad z \in \Delta \\ &\Longleftrightarrow f \in \mathcal{K}, \end{aligned}$$

concluding the proof of the first claim.

b) Under the given hypotheses on f, it is not difficult to check that $F \in \mathcal{A}$ and is locally univalent. From the definition of F we obtain

$$F'(z) f(z) = z, \qquad z \in \Delta$$

and differentiating again we have $F''(z) f(z) + F'(z) f'(z) = 1$, $z \in \Delta$. Dividing the last two relation we arrive at

$$z\frac{F''(z)}{F'(z)} + z\frac{f'(z)}{f(z)} = 1, \qquad z \in \Delta.$$

Using the above and the characterization of starlike functions (see for example [2], Theorem 2.2.2), we obtain

$$\begin{aligned} F \in \mathcal{G}(2) &\Longleftrightarrow \operatorname{Re}\left(1 + \frac{zF''(z)}{F'(z)}\right) < 2, \qquad z \in \Delta \\ &\Longleftrightarrow \operatorname{Re}\left(2 - z\frac{f'(z)}{f(z)}\right) < 2, \qquad z \in \Delta \\ &\Longleftrightarrow \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad z \in \Delta \\ &\Longleftrightarrow f \in \mathcal{S}, \end{aligned}$$

concluding the proof. □

Remark 2.1. Note that in part a) of the above lemma, the construction is symmetric in terms of f and F, i.e. we have the symmetric relation $f'(z) F'(z) = 1$ for all $z \in \Delta$. Using this observation, and interchanging the roles of f and F, the same proof shows that we also have the equivalence $F \in \mathcal{K}$ iff $f \in \mathcal{G}(2)$.

In [5], Ozaki proved that the functions in the class $\mathcal{G}(1)$ are univalent. However, this result does not hold in general for the class $\mathcal{G}(\alpha)$ if $\alpha > 1$, as Example 2.1 shows it. This raises the question about the radius of injectivity of the class $\mathcal{G}(\alpha)$ for $\alpha > 1$, with an answer provided by the following.

Theorem 2.2. *For $1 \le \alpha < 4.952$, the radius of injectivity of the class $\mathcal{G}(\alpha)$ is at least $\frac{1}{\alpha}$. That is, any function $f \in \mathcal{G}(\alpha)$ with $1 \le \alpha < 4.952$ is univalent in the disk $\left\{z \in \mathbb{C} : |z| < \frac{1}{\alpha}\right\}$.*

Proof. Recall Ahlfors's univalence criterion (see [1], or Theorem 3.3.2 in [2]): if $f \in \mathcal{A}$ and there exists a constant $c \in \mathbb{C}$ with $|c| \le 1$, $c \ne -1$, such that

$$\left|\frac{zf''(z)}{f'(z)} + c\frac{|z|^2}{1-|z|^2}\right| \le \frac{1}{1-|z|^2}, \qquad z \in \Delta, \tag{2.6}$$

then f is univalent in Δ.

Recall that if $p : \Delta \to \mathbb{C}$ has positive real part and $p(0) = 1$, then we have the estimate (see for example [2], p. 31)

$$\left|p(z) - \frac{1+|z|^2}{1-|z|^2}\right| \le \frac{2|z|}{1-|z|^2}, \qquad z \in \Delta. \tag{2.7}$$

If $f \in \mathcal{G}(\alpha)$, then $p(z) = 1 - \frac{2zf''(z)}{\alpha f'(z)}$ has positive real part and satisfies $p(0) = 1$, so the above estimate gives

$$\left|\frac{zf''(z)}{\alpha f'(z)} + \frac{|z|^2}{1-|z|^2}\right| \le \frac{|z|}{1-|z|^2}, \qquad z \in \Delta. \tag{2.8}$$

Consider $F(z) = \alpha f\left(\frac{z}{\alpha}\right)$, $z \in \Delta$. It is easy to see that $F \in \mathcal{A}$ and

$$\frac{1}{\alpha}\frac{f''\left(\frac{z}{\alpha}\right)}{f'\left(\frac{z}{\alpha}\right)} = \frac{F''(z)}{F'(z)}, \qquad z \in \Delta.$$

From (2.8) we obtain

$$\left|\frac{\frac{z}{\alpha}f''\left(\frac{z}{\alpha}\right)}{\alpha f'\left(\frac{z}{\alpha}\right)} + \frac{|z/\alpha|^2}{1-|z/\alpha|^2}\right| \le \frac{|z/\alpha|}{1-|z/\alpha|^2}, \qquad z \in \Delta,$$

or equivalent

$$\left|\frac{zF''(z)}{F'(z)} + \frac{\frac{1}{\alpha}|z|^2}{1-|z/\alpha|^2}\right| \le \frac{|z|}{1-|z/\alpha|^2}, \qquad z \in \Delta.$$

Using the above and the triangle inequality, we get

$$\begin{aligned}\left|\frac{zF''(z)}{F'(z)} + \frac{3-\alpha}{2}\cdot\frac{|z|^2}{1-|z|^2}\right| &\le \frac{|z|}{1-|z/\alpha|^2} + \left|-\frac{3-\alpha}{2}\cdot\frac{|z|^2}{1-|z|^2} + \frac{\frac{1}{\alpha}|z|^2}{1-|z/\alpha|^2}\right| \\ &= \frac{|z|}{1-|z/\alpha|^2} + |z|^2\,\frac{(\alpha-1)\left|3|z|^2+\alpha(2-\alpha)\right|}{2\alpha^2\left(1-|z|^2\right)\left(1-|z/\alpha|^2\right)} \\ &= \frac{1}{1-|z|^2}\,\frac{2\alpha^2|z|\left(1-|z|^2\right)+(\alpha-1)|z|^2\left|3|z|^2+\alpha(2-\alpha)\right|}{2\alpha^2\left(1-|z/\alpha|^2\right)} \\ &\le \frac{1}{1-|z|^2}.\end{aligned} \tag{2.9}$$

To justify the last inequality, we have left to show that for any $t \in [0,1)$ and $\alpha \in [1, 4.952]$ we have

$$\frac{2\alpha^2 t\left(1-t^2\right)+(\alpha-1)t^2\left|3t^2+\alpha(2-\alpha)\right|}{2\left(\alpha^2-t^2\right)} \leq 1,$$

or equivalent

$$-2\alpha^2 t^3 + 2t^2 + 2\alpha^2 t - 2\alpha^2 \leq (\alpha-1)t^2\left(3t^2+\alpha(2-\alpha)\right) \leq 2\alpha^2 t^3 - 2t^2 - 2\alpha^2 t + 2\alpha^2. \tag{2.10}$$

The right inequality above is equivalent to

$$(\alpha - t)\left(3(\alpha-1)t^3 + \alpha(\alpha-3)t^2 + 2(1-\alpha)t + 2\alpha\right) \geq 0,$$

which reduces to showing that $g(\alpha,t) = 3(\alpha-1)t^3 + \alpha(\alpha-3)t^2 + 2(1-\alpha)t + 2\alpha \geq 0$. We note that

$$\frac{\partial g}{\partial \alpha}(\alpha,t) = 3t^3 + (2\alpha-3)t^2 - 2t + 2 = (1+t)(3t^2-4t+2) + 2(\alpha-1)t^2 \geq 0,$$

hence $g(\alpha,t)$ is an increasing function of $\alpha \geq 1$ for any $t \in [0,1]$ arbitrarily fixed. It follows that $g(\alpha,t) \geq g(1,t) = 2 - 2t^2 \geq 0$, for any $\alpha \geq 1$ and $t \in [0,1]$, thus proving the right inequality in (2.10).

Proceeding similarly, the left inequality in (2.10) is equivalent to

$$(\alpha + t)\left(3(\alpha-1)t^3 + \alpha(3-\alpha)t^2 - 2(1+\alpha)t + 2\alpha\right) \geq 0,$$

and reduces to showing that $h(\alpha,t) = 3(\alpha-1)t^3 + \alpha(3-\alpha)t^2 - 2(1+\alpha)t + 2\alpha \geq 0$.

We note that $\frac{\partial^2 h}{\partial \alpha^2}(\alpha,t) = -2t^2 \leq 0$, hence $h(\alpha,t)$ is a concave function of $\alpha \geq 1$ for any $t \in [0,1]$ arbitrarily fixed. It follows that for $\alpha \in [1, 4.952]$ and $t \in [0,1]$ we have

$$h(\alpha,t) \geq \min\left\{h(1,t), h(4.952,t)\right\}. \tag{2.11}$$

We have $h(1,t) = 2(t-1)^2 \geq 0$ and $h(4.952,t) = \frac{2}{15625}(77375 - 93000t - 75518t^2 + 92625t^3)$. Simple calculus shows that the cubic defining $h(4.952,t)$ attains its minimum on the interval $[0,1]$ at $t_0 = 2(37759 + \sqrt{7886335831})/277875 \approx 0.9109$, and

$$h(4.952,t_0) = \frac{22414744979838922 - 252362746592\sqrt{7886335831}}{3619430419921875} \approx 0.00101 > 0.$$

Combining the above with (2.11) we obtain that $h(\alpha,t) \geq \min\{0, h(4.952,t_0)\} > 0$ for any $t \in [0,1]$ and $\alpha \in [1, 4.952]$, which concludes the proof of the claim (2.10).

Using Ahlfors's criterion (with $c = \frac{3-\alpha}{2} \in (-1,1]$ for $\alpha \in [1, 4.952]$), from (2.9) we deduce that $F(z)$ is univalent in the unit disk Δ. In turn, since $F(z) = \alpha f\left(\frac{z}{\alpha}\right)$, this shows that f is univalent in the disk $\left\{z \in \mathbb{C} : |z| < \frac{1}{\alpha}\right\}$, and therefore the radius of injectivity of the class $\mathcal{G}(\alpha)$ is at least $1/\alpha$, for any $\alpha \in [1, 4.952]$, thus concluding the proof. □

Remark 2.2. Note that in the particular case $\alpha = 1$, the above theorem shows that the radius of univalence of the class $\mathcal{G}(1)$ is at least (and therefore equal to) 1, i.e. the class $\mathcal{G}(1)$ consists entirely of univalent functions, and thus we obtain as a particular case of our theorem the result proved by Ozaki ([5]). We do not know whether the lower bound $\frac{1}{\alpha}$ in the previous theorem actually coincides with the radius of univalence of the class $\mathcal{G}(\alpha)$ for $\alpha \in (1, 4.952]$.

The following result shows that the subclass $\mathcal{G}^*(\alpha) \subset \mathcal{G}(\alpha)$ interpolates between the subclasses S^* and $\mathcal{K}^*$ of starlike, respectively convex functions.

Proposition 2.1. *$\mathcal{G}^*(\alpha) \subset \mathcal{S}^*$ for any $\alpha \in (0, 1]$, and $\mathcal{G}^*(\alpha) \subset \mathcal{K}^*$ for any $\alpha \in (0, \frac{2}{3}]$. Moreover, the result is sharp in the sense that $\mathcal{G}^*(\alpha) \not\subset \mathcal{S}^*$ for $\alpha > 1$ and $\mathcal{G}^*(\alpha) \not\subset \mathcal{K}^*$ for $\alpha > \frac{2}{3}$.*

Proof. Note that for $\alpha \in (0, 1]$ we have

$$n \leq n[2(n-1) - \alpha],$$

for any $n \geq 1 + \frac{1+\alpha}{2}$, in particular for $n \geq 2$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{G}^*(\alpha)$, we have

$$\sum_{n=2}^{\infty} n |a_n| \leq \sum_{n=2}^{\infty} n [2(n-1) - \alpha] |a_n| \leq \alpha \leq 1$$

and therefore $f \in \mathcal{S}^*$.

Similarly, if $\alpha \in (0, \frac{2}{3}]$ we have

$$n^2 \leq n(2(n-1) - \alpha),$$

for any $n \geq 2 + \alpha$, in particular for $n \geq 3$.

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{G}^*(\alpha)$, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n^2 |a_n| &\leq 4|a_2| + \sum_{n=3}^{\infty} n[2(n-1) - \alpha] |a_n| \\
&\leq 4|a_2| + \alpha - 2(2-\alpha)|a_2| \\
&\leq \alpha (1 + 2|a_2|) \\
&\leq \alpha \left(1 + \frac{\alpha}{2-\alpha}\right) \\
&= \frac{2\alpha}{2-\alpha} \\
&\leq 1,
\end{aligned}$$

which shows that $f(z) \in \mathcal{K}^*$ for $\alpha \in \left(0, \frac{2}{3}\right]$, concluding the first part of the proof.

From (2.3) it is not difficult to see that the class $\mathcal{G}^*(\alpha)$ is monotone increasing with respect to $\alpha > 0$. In order to prove the last assertion of the lemma it suffices therefore to consider the case $\alpha \in (1, 2)$.

Consider the function $f(z) = z + \beta z^2$ with $\beta > 0$, and note that $f(z) \in \mathcal{G}^*(\alpha)$ iff $\beta \leq \frac{\alpha}{2(2-\alpha)} \in \left(\frac{1}{2}, \infty\right)$ for $\alpha \in (1, 2)$.

For fixed $\alpha \in (1, 2)$, considering $\beta = \frac{\alpha}{2(2-\alpha)}$ we have $f(z) \in \mathcal{G}^*(\alpha)$, but $f(z) \notin \mathcal{S}^*$ since $\beta = \frac{\alpha}{2(2-\alpha)} > \frac{1}{2}$ for any $\alpha \in (1, 2)$. This shows that $\mathcal{G}^*(\alpha) \not\subset \mathcal{S}^*$ for any $\alpha > 1$.

Similarly, for any $\alpha \in \left(\frac{2}{3}, 2\right)$, considering $\beta = \frac{\alpha}{2(2-\alpha)}$ we have $f(z) \in \mathcal{G}^*(\alpha)$ but $f(z) \notin \mathcal{K}^*$ since $\beta = \frac{\alpha}{2(2-\alpha)} > \frac{1}{4}$ for any $\alpha \in \left(\frac{2}{3}, 2\right)$. This shows that $\mathcal{G}^*(\alpha) \not\subset \mathcal{K}^*$ for any $\alpha > 1$, concluding the proof. □

3. Approximation by functions in the subclass $\mathcal{G}^*(\alpha) \subset \mathcal{G}(\alpha)$

As a measure of (non)univalency of a function $f \in \mathcal{A}$, in [6,7] the authors considered

$$\operatorname{dist}(f,\mathcal{U}) = \inf_{g\in\mathcal{U}} \left(\iint\limits_{\Delta} |f(x+iy) - g(x+iy)|^2 dxdy \right)^{\frac{1}{2}}, \tag{3.1}$$

with similar definitions for the subclasses $\mathcal{K}$, $\mathcal{S}$, $\mathcal{K}^*$, and $\mathcal{S}^*$. In the same spirit, we consider the following.

Definition 3.1. For $f \in \mathcal{A}$ we define

$$\operatorname{dist}(f,\mathcal{G}(\alpha)) = \inf_{g\in\mathcal{G}(\alpha)} \left(\iint\limits_{\Delta} |f(x+iy) - g(x+iy)|^2 dxdy \right)^{\frac{1}{2}}, \tag{3.2}$$

with a similar definition for $\operatorname{dist}(f,\mathcal{G}^*(\alpha))$.

Although $\operatorname{dist}(\cdot,\mathcal{G}(\alpha))$ is not a norm in $\mathcal{A}$ (see Theorem 3.1), $dist(f,\mathcal{G}(\alpha))$ is a measure showing how "far" is the function f from the class $\mathcal{G}(\alpha)$, and the same is true for $dist(f,\mathcal{G}^*(\alpha))$.

We will use the following result from [6], which shows that the L^2 norm of an analytic function $f : \Delta \to \mathbb{C}$ can be expressed in term of the coefficients of the Taylor series of f, as follows.

Lemma 3.1. *([6]) If $f : \Delta \to \mathbb{C}$ is analytic in Δ and has series expansion $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$, $z \in \Delta$, then*

$$\iint\limits_{\Delta} |f(x+iy)|^2 dxdy = \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}. \tag{3.3}$$

The class $\mathcal{G}(\alpha)$ can be characterized in terms of $\operatorname{dist}(\cdot,\mathcal{G})$ as follows (a similar characterization can be given for the class $\mathcal{G}^*(\alpha)$).

Theorem 3.1. *For $f \in \mathcal{A}$, $\operatorname{dist}(f,\mathcal{G}(\alpha)) = 0$ if and only if $f \in \mathcal{G}(\alpha)$.*

Proof. The proof being similar to the proof of the characterization of the class $\mathcal{K}$ [7, Theorem 3] (with the inequality (2.5) which characterizes the class $\mathcal{K}$ replaced by the inequality (1.2) which characterizes the class $\mathcal{G}(\alpha)$), we will just briefly sketch the argument and refer the reader to [7].

If $\operatorname{dist}(f,\mathcal{G}(\alpha)) = 0$, we can find a sequence $(f_n)_{n\geq 1} \subset \mathcal{G}(\alpha)$ which converges to f uniformly on compact subsets of Δ. Since f_n are locally univalent (hence $f' \neq 0$ in Δ) and using Hurwitz's theorem, we conclude that f is also locally univalent in Δ. Since $f_n \in \mathcal{G}(\alpha)$, passing to the limit with $n \to \infty$ in the inequality (1.2) which characterizes this class, we conclude that f also satisfies this inequality, thus $f \in \mathcal{G}(\alpha)$. The converse implication being obvious, this concludes the proof. □

In order to find the best approximation of an analytic function $f \in \mathcal{A}$ in the subclass $\mathcal{G}^*(\alpha) \subset \mathcal{G}(\alpha)$, motivated by Definition 3.1, Lemma 3.1, and an argument embedded in the proof of Theorem 4.1, we were led to consider the problem of finding

$$\inf \sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1}, \tag{3.4}$$

where $(a_n)_{n\geq 2}$ is a given sequence of non-negative real numbers, and the infimum is taken over all non-negative sequences $(x_n)_{n\geq 2}$ of real numbers satisfying

$$\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]x_n \le 1. \tag{3.5}$$

We first note that the solution of the above problem is trivial if

$$\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]a_n \le 1$$

(the above infimum is 0, attained for $x_n = a_n$, $n \ge 2$). We will therefore consider the following additional hypothesis on the sequence $(a_n)_{n\ge 2}$

$$\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]a_n > 1. \tag{3.6}$$

The above problem is a particular case of a semi-infinite quadratic programming problem (see for example [3]), with corresponding Lagrangian given by

$$L = \sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1} + \mu \left(\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]x_n - 1 \right). \tag{3.7}$$

The solution of the quadratic problem (3.4)–(3.5) is given by the Karush–Kuhn–Tucker conditions (see [3] or [6,7], and assume for the moment that the same conditions can be used for an infinite instead of a finite number of variables, as detailed in Remark 3.1 below), which in this case become:

$$\frac{\partial L}{\partial x_n} = 2\frac{x_n - a_n}{n+1} + \mu \left(\frac{n}{\alpha}[2(n-1)-\alpha] \right) \ge 0, \qquad n \ge 2, \tag{3.8}$$

$$\frac{\partial L}{\partial \mu} = \sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]x_n - 1 \le 0, \tag{3.9}$$

$$x_n \frac{\partial L}{\partial x_n} = x_n \left[2\frac{x_n - a_n}{n+1} + \mu \left(\frac{n}{\alpha}[2(n-1)-\alpha] \right) \right] = 0, \qquad n \ge 2, \tag{3.10}$$

$$\mu \frac{\partial L}{\partial \mu} = \mu \left(\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]x_n - 1 \right) = 0, \tag{3.11}$$

$$x_n \ge 0, \qquad n \ge 2, \tag{3.12}$$

$$\mu \ge 0. \tag{3.13}$$

The equation (3.11) shows that either $\mu = 0$ or $\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]x_n = 1$. However, the hypothesis (3.6) shows that we cannot have $\mu = 0$. This is so for otherwise from (3.10) we obtain $x_n = 0$ or $x_n = a_n$, and since (3.8) shows that $x_n \ge a_n$, we conclude that $x_n = a_n$ for all $n \ge 2$. In turn, this shows that

$$\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]x_n = \sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]a_n > 1,$$

contradicting (3.9).

We have therefore $\mu > 0$, and we can rewrite the system (3.8)–(3.13) as follows:

$$2\frac{x_n - a_n}{n+1} + \mu \left(\frac{n}{\alpha}[2(n-1)-\alpha] \right) \ge 0, \qquad n \ge 2, \tag{3.14}$$

$$x_n \left[2\frac{x_n - a_n}{n+1} + \mu\left(\frac{n}{\alpha}[2(n-1)-\alpha]\right)\right] = 0, \qquad n \geq 2, \tag{3.15}$$

$$\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]x_n = 1, \tag{3.16}$$

$$x_n \geq 0, \qquad n \geq 2, \tag{3.17}$$

$$\mu > 0. \tag{3.18}$$

The equation (3.15) shows that either $x_n = 0$ or

$$x_n = a_n + \mu n(n+1)\left(\frac{\alpha - 2(n-1)}{2\alpha}\right)$$

and we will denote by $\mathcal{I}$ be the set of indices $n \geq 2$ for which the latter equality holds (therefore $x_n = 0$ for $n \in \mathcal{I}^c = \{2, 3, \ldots\} - \mathcal{I}$). Assuming the additional hypothesis $\alpha \in (0, 2)$, from (3.14) and (3.17) we obtain

$$\mu \geq \frac{2\alpha a_n}{n(n+1)[2(n-1)-\alpha]}, \qquad n \in \mathcal{I}^c, \tag{3.19}$$

respectively

$$\mu \leq \frac{2\alpha a_n}{n(n+1)[2(n-1)-\alpha]}, \qquad n \in \mathcal{I}. \tag{3.20}$$

Note that if we also impose the additional hypothesis

$$\lim_{n\to\infty} \frac{a_n}{n^3} = 0, \tag{3.21}$$

the last inequality cannot hold for infinitely many indices n (i.e. $\mathcal{I}$ must be finite). This is so for otherwise we can pass to the limit in (3.20) along a sequence of indices in $\mathcal{I}$ converging to ∞ and conclude $\mu \leq 0$, thus contradicting (3.18). From (3.16) we now obtain

$$\begin{aligned} 1 &= \sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]x_n \\ &= \sum_{n\in\mathcal{I}} \frac{n}{\alpha}[2(n-1)-\alpha]\left(a_n + \mu n(n+1)\left(\frac{\alpha-2(n-1)}{2\alpha}\right)\right) \\ &= \sum_{n\in\mathcal{I}} \frac{n}{\alpha}[2(n-1)-\alpha]a_n - \frac{\mu}{2\alpha^2}\sum_{n\in\mathcal{I}} n^2(n+1)[2(n-1)-\alpha]^2 \end{aligned}$$

and therefore

$$\mu = \frac{2\alpha^2\left(\sum_{n\in\mathcal{I}} \frac{n}{\alpha}[2(n-1)-\alpha]a_n - 1\right)}{\sum_{n\in\mathcal{I}} n^2(n+1)[2(n-1)-\alpha]^2} > 0. \tag{3.22}$$

In order to find the solution of the problem (3.4) – (3.5), it remains to find the set of indices $\mathcal{I}$ (the last equality gives then the value

$$x_n = a_n + \mu n(n+1)\left(\frac{\alpha-2(n-1)}{2\alpha}\right),$$

for $n \in \mathcal{I}$ and $x_n = 0$ for $n \in \mathcal{I}^c$). To do this, recall that μ given by (3.22) must satisfy (3.19) and (3.20).

The choice of the set $\mathcal{I}$ depends on whether the all the terms of the sequence $(a_n)_{n\geq 2}$ are positive or not, so we introduce the set of indices $\mathcal{P} = \{n \geq 2 : a_n > 0\}$, and distinguish the following cases.

Case 1: $\mathcal{P} = \{2, 3, \dots\}$.

The hypothesis (3.21) shows that

$$\left(\frac{2\alpha a_n}{n(n+1)[2(n-1)-\alpha]}\right)_{n\geq 2},$$

is a sequence of positive numbers converging to 0, so we can choose a permutation $(i_n)_{n\geq 2}$ of the indices in $\mathcal{P}$ such that

$$\alpha_n = \frac{2\alpha a_{i_n}}{i_n(i_n+1)[2(i_n-1)-\alpha]}, \qquad n \geq 2,$$

is a non-increasing sequence (to see this, note that each interval in the partition $[1,\infty) \cup \left(\cup_{m\geq 1}[\frac{1}{m+1}, \frac{1}{m})\right) = (0, \infty)$ contains only a finite number of terms of the original sequence).

Since

$$\sum_{n=2}^{\infty} \frac{i_n}{\alpha}[2(i_n-1)-\alpha]a_{i_n} = \sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]a_n > 1,$$

there exists an integer $n_0 \geq 2$ such that

$$\sum_{n=2}^{n_0} \frac{i_n}{\alpha}[2(i_n-1)-\alpha]a_{i_n} > 1$$

and assume that $n_0 \geq 2$ is the smallest index with this property. Setting

$$\mu_n = \frac{2\alpha^2\left(\sum_{m=2}^{n} \frac{i_m}{\alpha}[2(i_m-1)-\alpha]a_{i_m} - 1\right)}{\sum_{m=2}^{n} i_m^2(i_m+1)[2(i_m-1)-\alpha]^2}, \qquad n \geq 2,$$

first note that we must have $0 < \mu_{n_0} \leq \alpha_{n_0}$. This is so for if $n_0 = 2$, then

$$\mu_2 = \frac{2\alpha^2\left(\frac{i_2}{\alpha}[2(i_2-1)-\alpha]a_{i_2} - 1\right)}{i_2^2(i_2+1)[2(i_2-1)-\alpha]^2} \leq \frac{2\alpha a_{i_2}}{i_2(i_2+1)[2(i_2-1)-\alpha]} = \alpha_2,$$

so the claim holds in this case. If $n_0 > 2$, by the choice of n_0 we have

$$\mu_{n_0-1} = \frac{2\alpha^2\left(\sum_{n=2}^{n_0-1} \frac{i_n}{\alpha}[2(i_n-1)-\alpha]a_{i_n} - 1\right)}{\sum_{n=0}^{n_0-1} i_n^2(i_n+1)[2(i_n-1)-\alpha]^2} \leq 0 < \frac{2\alpha a_{i_{n_0}}}{i_{n_0}(i_{n_0}+1)[2(i_{n_0}-1)-\alpha]} = \alpha_{n_0}$$

and using the observation that $\frac{a}{b} \leq \frac{c}{d}$ with $b, d > 0$ implies $\frac{a+c}{b+d} \leq \frac{c}{d}$, we obtain

$$\mu_{n_0} = \frac{2\alpha^2\left(\sum_{n=2}^{n_0-1} \frac{i_n}{\alpha}[2(i_n-1)-\alpha]a_{i_n} - 1\right) + 2\alpha i_{n_0}[2(i_{n_0}-1]-\alpha]a_{i_{n_0}}}{\sum_{n=0}^{n_0-1} i_n^2(i_n+1)[2(i_n-1)-\alpha]^2 + i_{n_0}^2(i_{n_0}+1)[2(i_{n_0}-1)-\alpha]^2} \leq \alpha_{n_0},$$

concluding the proof of the claim.

We distinguish now the following subcases.

Case 1a): $\mu_{n_0} \geq \alpha_{n_0+1}$.

Since the sequence $(\alpha_n)_{n\geq 2}$ is non-increasing, we have

$$\mu_{n_0} \leq \alpha_{n_0} \leq \alpha_n, \qquad n \in \{2, 3, \dots, n_0\},$$

and

$$\mu_{n_0} \geq \alpha_{n_0+1} \geq \alpha_n, \qquad n \in \{n_0+1, n_0+2, ...\},$$

so we can chose $\mathcal{I} = \{i_2, i_3, \dots, i_{n_0}\}$, and thus $\mu = \mu_{n_0}$ satisfies (3.19)–(3.20), giving the solution in this case.

Case 1b): $\mu_{n_0} < \alpha_{n_0+1}$.

In this case, using again the above observation we have

$$\mu_{n_0} \leq \mu_{n_0+1} \leq \alpha_{n_0+1}$$

and either $\mu_{n_0+1} \geq \alpha_{n_0+2}$ or $\mu_{n_0+1} < \alpha_{n_0+2}$.

If $\mu_{n_0+1} \geq \alpha_{n_0+2}$, proceeding as in Case 1a) above, we can choose $\mathcal{I} = \{i_2, i_3, ..., i_{n_0+1}\}$, and thus $\mu = \mu_{n_0+1}$ satisfies (3.19)–(3.20), giving the solution in this case.

If $\mu_{n_0+1} < \alpha_{n_0+2}$, we obtain:

$$\mu_{n_0} \leq \mu_{n_0+1} \leq \mu_{n_0+2} \leq \alpha_{n_0+2}$$

and proceeding inductively, either

$$0 < \mu_{n_0} \leq \mu_{n_0+1} \leq \mu_{n_0+2} \leq ... \leq \mu_{n_0+k} < \alpha_{n_0+k}, \qquad k \geq 0, \tag{3.23}$$

or we can find an integer $k \geq 2$ for which

$$\alpha_{n_0+k+1} \leq \mu_{n_0+k} \leq \alpha_{n_0+k}. \tag{3.24}$$

Since by construction the sequence $(\alpha_n)_{n\geq 2}$ converges to 0, the inequalities in (3.23) cannot hold for every $k \geq 0$, and therefore the first possibility above is ruled out. It follows that we can always find an integer k for which (3.24) holds, and proceeding as in Case 1a) above we can choose $\mathcal{I} = \{i_2, i_3, ..., i_{n_0+k}\}$, and thus $\mu = \mu_{n_0+k}$ satisfies (3.19)–(3.20), giving the solution in this case.

Case 2: $\mathcal{P} = \{n \geq 2 : a_n > 0\} \subsetneq \{2, 3, \dots\}$.

We distinguish the following subcases.

Case 2a): the set $\mathcal{P}$ is infinite.

Since $a_n = 0$ for $n \in \{2, 3, \dots\} - \mathcal{P}$, we have

$$\sum_{n\in\mathcal{P}} \frac{n}{\alpha}[2(n-1) - \alpha]a_n = \sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1) - \alpha]a_n > 1$$

(additional hypothesis (3.6)). We can therefore apply the argument in Case 1 above to the sequence $(a_n)_{n\in\mathcal{P}}$ of positive numbers and obtain a solution (i.e. a choice of the set of indices $\mathcal{I} \subset \mathcal{P}$, as indicated above) of the problem

$$\inf \sum_{n\in\mathcal{P}} \frac{(x_n - a_n)^2}{n+1},$$

where the infimum is taken over all non-negative sequences $(x_n)_{n\in\mathcal{P}}$ with

$$\sum_{n\in\mathcal{P}} \frac{n}{\alpha}[2(n-1)-\alpha]x_n \leq 1.$$

It is not difficult to see that the solution of the above minimization problem is also a solution of the original minimization problem (3.4)–(3.5) (for $n \in \{2,3,\dots\} - \mathcal{P} \subset \mathcal{I}^c$ we have $x_n = a_n = 0$).

Case 2b): the set $\mathcal{P}$ is finite.

The hypothesis (3.6) shows that $\mathcal{P}$ cannot be empty, so $|\mathcal{P}| = p$ for some $p \geq 1$. If $(i_n)_{n=2,\dots,p+1}$ is a permutation of the indices in $\mathcal{P}$ such that

$$\alpha_n = \frac{2\alpha a_{i_n}}{i_n(i_n+1)[2(i_n-1)-\alpha]}, \qquad (n = 2,\dots,p+1),$$

is a non-increasing sequence, proceeding as in Case 1 above, either we can find an integer $k \geq 0$ such that the index set $\mathcal{I} = \{i_2,\dots,i_{n_0+k}\}$ gives the solution, or else

$$0 < \mu_{n_0} \leq \mu_{n_0+1} \leq \dots \leq \mu_{p+1} \leq \alpha_{p+1}.$$

In the latter case we can chose $\mathcal{I} = \{i_2, i_3, ..., i_{p+1}\}$ and note that $\mu = \mu_{p+1}$ satisfies the necessary conditions (3.19)–(3.20), so the index set $\mathcal{I}$ gives the solution of the minimization problem (3.4)–(3.5) in this last case.

The above analysis can be summarized in the following result.

Theorem 3.2. *If $0 < \alpha < 2$ and $(a_n)_{n\geq 2}$ is a sequence of non-negative real numbers satisfying*

$$\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]a_n > 1 \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{a_n}{n^3} = 0, \tag{3.25}$$

there exists an integer $N \geq 2$ such that the minimum of the quadratic problem (3.4)–(3.5) is given by

$$\sum_{n\in\mathcal{I}^c} \frac{a_n^2}{n+1} + \frac{\left(\sum_{n\in\mathcal{I}} n[2(n-1)-\alpha]a_n - \alpha\right)^2}{\sum_{n\in\mathcal{I}} n^2(n+1)[2(n-1)-\alpha]^2},$$

attained for the sequence $(x_n)_{n\geq 2}$ defined by

$$x_n = \begin{cases} a_n - \mu_N n(n+1)\frac{2(n-1)-\alpha}{2\alpha}, & n \in \mathcal{I}, \\ 0, & n \in \mathcal{I}^c, \end{cases}$$

where

$$\mu_N = \frac{2\alpha^2\left(\sum_{n\in\mathcal{I}} \frac{n}{\alpha}[2(n-1)-\alpha]a_n - 1\right)}{\sum_{n\in\mathcal{I}} n^2(n+1)[2(n-1)-\alpha]^2},$$

$\mathcal{I} = \{i_2, i_3, \dots, i_N\}$ and $(i_n)_{n=2,3,\dots,|\mathcal{P}|+1}$ is a is a permutation of the indices in $\mathcal{P} = \{n \geq 2 : a_n > 0\}$ such that

$$\alpha_n = \frac{2\alpha a_{i_n}}{i_n(i_n+1)[2(i_n-1)-\alpha]}, \qquad (n = 2,3,\dots,|\mathcal{P}|+1),$$

is a non-increasing sequence.

Moreover, we can take $N = \min\{n \geq 2 : \alpha_{n+1} \leq \mu_n \leq \alpha_n\}$, *where*

$$\mu_n = \frac{2\alpha^2\left(\sum_{m=2}^{n} \frac{i_m}{\alpha}[2(i_m-1)-\alpha]a_{i_m} - 1\right)}{\sum_{m=2}^{n} i_m^2(i_m+1)[2(i_m-1)-\alpha]^2}, \qquad (n = 2, 3, \dots, |P|+1).$$

Remark 3.1. To complete the proof of the above theorem, we have left to justify that we can use the Karush–Kuhn–Tucker conditions for the quadratic programming problem (3.4)–(3.5), with an infinite (instead of a finite) number of variables. The reasoning being similar to [7, Remark 3], we will just briefly outline it.

The idea is to observe that for any integer $m \geq 2$ we have

$$\inf \sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1} \geq \inf \sum_{n=2}^{m} \frac{(x_n - a_n)^2}{n+1}, \tag{3.26}$$

where both infima are taken over all non-negative sequences $(x_n)_{n\geq 2}$ of real numbers with $\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]a_n \leq 1$. Since $x_{m+1}, x_{m+2}, \dots$ do not appear in the objective function in the second infimum above, the second infimum is the same when taken over all finite truncated sequences $(x_n)_{n=2,\dots,m}$ with $\sum_{n=2}^{m} \frac{n}{\alpha}[2(n-1)-\alpha]a_n \leq 1$. Solving the Karush–Kuhn–Tucker conditions for this finite-dimensional problem (the calculations are identical as in the proof above) and using the notation of Theorem 3.2, it follows that for $m \geq \max\{i_n, \dots, i_N\}$ the second infimum in (3.26) is attained for the sequence $x_2, \dots, x_m$ given by

$$x_n = \begin{cases} a_n - \mu_N n(n+1)\frac{2(n-1)-\alpha}{2\alpha}, & n \in \mathcal{I}, \\ 0, & n \in \mathcal{I}_m^c = \{2, \dots, m\} - \mathcal{I}. \end{cases}$$

Combining with (3.26), we obtain

$$\inf \sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1} \geq \sum_{n\in\mathcal{I}_m^c} \frac{a_n^2}{n+1} + \frac{\left(\sum_{n\in I} n[2(n-1)-\alpha]a_n - \alpha\right)^2}{\sum_{n\in I} n^2(n+1)[2(n-1)-\alpha]^2}$$

and passing to the limit with $m \to \infty$ we obtain

$$\begin{aligned}\inf \sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1} &\geq \lim_{m\to\infty} \sum_{n\in\mathcal{I}_m^c} \frac{a_n^2}{n+1} + \frac{\left(\sum_{n\in\mathcal{I}} n[2(n-1)-\alpha]a_n - \alpha\right)^2}{\sum_{n\in\mathcal{I}} n^2(n+1)[2(n-1)-\alpha]^2} \\ &= \sum_{n\in\mathcal{I}^c} \frac{a_n^2}{n+1} + \frac{\left(\sum_{n\in\mathcal{I}} n[2(n-1)-\alpha]a_n - \alpha\right)^2}{\sum_{n\in\mathcal{I}} n^2(n+1)[2(n-1)-\alpha]^2},\end{aligned}$$

which is just the value of the objective function $\sum_{n=2}^{\infty} \frac{(x_n-a_n)^2}{n+1}$ for the sequence $(x_n)_{n\geq 2}$ defined in Theorem 3.2. It follows that the infimum of the quadratic problem (3.4)–(3.5) is attained for the sequence in the statement of Theorem 3.2, completing the argument used in the proof.

4. Applications

As an application of Theorem 3.2, we will determine the best approximation of a normed analytic function in the subclass $\mathcal{G}^*(\alpha)$, that is, we will find

$$\operatorname{dist}(f, \mathcal{G}^*(\alpha)) = \inf_{g\in\mathcal{G}^*(\alpha)} \left(\int_{\Delta} |f(x+iy) - g(x+iy)|^2 dxdy \right)^{\frac{1}{2}},$$

for a given function $f \in \mathcal{A}$, and we will determine the extremal function $g \in \mathcal{G}^*(\alpha)$ for which the minimum is attained. The result is the following.

Theorem 4.1. *Assume that* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ *and* $0 < \alpha < 2$. *If*

$$\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]|a_n| \le 1,$$

then $\operatorname{dist}(f, \mathcal{G}^*(\alpha)) = 0$ *(attained for* $g = f \in \mathcal{G}^*(\alpha) \subset \mathcal{G}(\alpha)$*) and if*

$$\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]|a_n| > 1 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{|a_n|}{n^3} = 0,$$

then we have

$$\operatorname{dist}(f, \mathcal{G}^*(\alpha)) = \left(\pi \sum_{n\in I^c} \frac{|a_n|^2}{n+1} + \pi \frac{\left(\sum_{n\in I} n[2(n-1)-\alpha]|a_n| - \alpha\right)^2}{\sum_{n\in I} n^2(n+1)[2(n-1)-\alpha]^2} \right)^{\frac{1}{2}}, \tag{4.1}$$

where $\mathcal{I} = \{i_2, ..., i_N\}$ *is given by* Theorem 3.2 *with* a_n *replaced by* $|a_n|$.

Moreover, the minimum value of $\operatorname{dist}(f, \mathcal{G}^*(\alpha))$ *above is attained for the function* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{G}^*(\alpha)$, *where*

$$b_n = \begin{cases} \left(|a_n| - \mu_N n(n+1)\frac{2(n-1)-\alpha}{2\alpha}\right) e^{i\arg a_n}, & n \in \mathcal{I}, \\ 0, & n \in \mathcal{I}^c, \end{cases} \tag{4.2}$$

where $\mu_N = 2\alpha^2 \frac{\sum_{n\in\mathcal{I}} \frac{n}{\alpha}[2(n-1)-\alpha]|a_n| - 1}{\sum_{n\in\mathcal{I}} n^2(n+1)[2(n-1)-\alpha]^2}$.

Proof. The claim is obvious in the first case, so assume that $\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]|a_n| > 1$. Using Lemma 3.1 and the triangle inequality we obtain

$$\begin{aligned} \operatorname{dist}(f, \mathcal{G}^*(\alpha)) &= \left(\pi \inf \sum_{n=2}^{\infty} \frac{|a_n - b_n|^2}{n+1}\right)^{\frac{1}{2}} \ge \left(\pi \inf \sum_{n=2}^{\infty} \frac{(|a_n| - |b_n|)^2}{n+1}\right)^{\frac{1}{2}} \\ &= \left(\pi \inf \sum_{n=2}^{\infty} \frac{(|a_n| - x_n|)^2}{n+1}\right)^{\frac{1}{2}}, \end{aligned}$$

where the second and the third infimum are taken over all sequences $(b_n)_{n\ge2}$ of complex numbers satisfying $\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]|b_n| \le 1$, and the last infimum is taken over all non-negative sequences $(x_n)_{n\ge2}$ of real numbers satisfying $\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]x_n \le 1$.

Applying Theorem 3.2 with $|a_n|$ instead of a_n, we obtain that the last infimum above is attained for the sequence $(x_n)_{n\ge2}$ given by

$$x_n = \begin{cases} |a_n| - \mu_N n(n+1)\frac{2(n-1)-\alpha}{2\alpha}, & n \in \mathcal{I}, \\ 0, & n \in \mathcal{I}^c. \end{cases}$$

Observing that the triangle inequality $|a_n - b_n| \ge ||a_n| - |b_n||$ becomes an equality if $\arg a_n = \arg b_n$, it follows that

$$\operatorname{dist}(f,\mathcal{G}^*(\alpha)) = \left(\pi \inf \sum_{n=1}^{\infty} \frac{|a_n - b_n|^2}{n+1}\right)^{\frac{1}{2}},$$

is attained for the sequence $(b_n)_{n\geq 2}$ of complex numbers with $b_n = x_n e^{i\arg a_n}$, $n \geq 2$ (note that if $a_n = 0$, from the proof of Theorem 3.2 we have $n \in \mathcal{I}^c$, so $x_n = 0$ and therefore $b_n = x_n e^{i\arg a_n} = 0$ is unambiguously defined).

Since $\mathrm{b}_n = 0$ for $n \in \mathcal{I}^c$ and $|b_n| = x_n \geq 0$ for $n \in \mathcal{I}$, we obtain

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{n}{\alpha}[2(n-1)-\alpha]|b_n| &= \sum_{n\in\mathcal{I}} \frac{n}{\alpha}[2(n-1)-\alpha]\left[|a_n| - \mu_N n(n+1)\frac{2(n-1)-\alpha}{2\alpha}\right] \\ &= \sum_{n\in\mathcal{I}} \frac{n}{\alpha}[2(n-1)-\alpha]|a_n| - \frac{1}{2\alpha^2}\mu_N \sum_{n\in\mathcal{I}} n^2(n+1)(2(n-1)-\alpha)^2 \\ &= 1,\end{aligned}$$

which shows that $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{G}^*(\alpha)$ and

$$\begin{aligned}\left(\int_{\Delta} |f(x+iy) - g(x+iy)|^2 dxdy\right)^{\frac{1}{2}} &= \left(\pi \sum_{n=2}^{\infty} \frac{|a_n - b_n|^2}{n+1}\right)^{\frac{1}{2}} \\ &= \left(\pi \sum_{n=2}^{\infty} \frac{(|a_n| - |b_n|)^2}{n+1}\right)^{\frac{1}{2}} \\ &= \left(\pi \sum_{n\in\mathcal{I}^c} \frac{|a_n|^2}{n+1} + \pi \frac{\left(\sum_{n\in\mathcal{I}} n[2(n-1)-\alpha]|a_n| - \alpha\right)^2}{\sum_{n\in\mathcal{I}} n^2(n+1)[2(n-1)-\alpha]^2}\right)^{\frac{1}{2}} \\ &= \operatorname{dist}(f,\mathcal{G}^*(\alpha)),\end{aligned}$$

as needed, concluding the proof. □

As applications of the previous theorem, we have the following.

Example 4.1. Consider the function $f_{\beta,\gamma} : \Delta \to \mathbb{C}$ defined by $f_{\beta,\gamma}(z) = z + \beta z^2 + \gamma z^3$, where $\beta, \gamma \in \mathbb{C}$. Applying Theorem 4.1 and Theorem 3.2 for an arbitrarily fixed $\alpha \in (0,2)$, we obtain the following.

If $2(2-\alpha)|\beta| + 3(4-\alpha)|\gamma| \leq \alpha$, then $f_{\beta,\gamma} \in \mathcal{G}^*(\alpha)$ and $\operatorname{dist}(f_{\beta,\gamma}, \mathcal{G}^*(\alpha)) = 0$.

Assuming now $2(2-\alpha)|\beta| + 3(4-\alpha)|\gamma| > \alpha$, we distinguish the following cases.

a) If $\gamma = 0$ (hence $\beta \neq 0$), in the notation of Theorem 3.2, we have $\mathcal{P} = \{2\}$, $i_2 = 2$, $N = 2$, $\mathcal{I} = \{i_2\} = \{2\}$, and $\operatorname{dist}(f_{\beta,0}, \mathcal{G}^*(\alpha)) = \sqrt{\frac{\pi}{3}}\left(|\beta| - \frac{\alpha}{2(2-\alpha)}\right)$ is attained for the function $g_{\alpha,\beta,0} \in \mathcal{G}^*(\alpha)$ defined by

$$g_{\alpha,\beta,0}(z) = z + \frac{\alpha}{2(2-\alpha)} e^{i\arg\beta} z^2, \qquad z \in \Delta.$$

Fig. 1 shows a comparison of the images of the unit disk under the function $f_{1,0}$ and of its various approximations $g_{\alpha,1,0}$. Note that by Proposition 2.1 we have that $g_{2/3,1,0} \in \mathcal{K}^*$ is a convex function, $g_{1,1,0} \in \mathcal{S}^*$ is a starlike function, and $g_{7/6,1,0}$ is in not univalent.

b) If $\beta = 0$ (hence $\gamma \neq 0$), we have $\mathcal{P} = \{3\}$, $i_2 = 3$, $N = 2$, $\mathcal{I} = \{i_2\} = \{3\}$, and $\operatorname{dist}(f_{0,\gamma}, \mathcal{G}^*(\alpha)) = \frac{\sqrt{\pi}}{2}\left(|\gamma| - \frac{\alpha}{3(4-\alpha)}\right)$ is attained for the function $g_{\alpha,0,\gamma} \in \mathcal{G}^*(\alpha)$ defined by

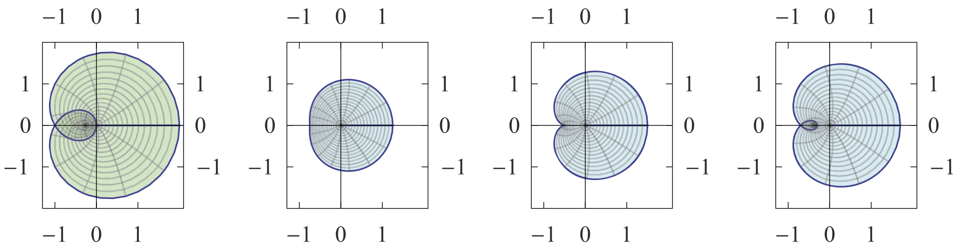

Fig. 1. In order from left to right: the image of the unit disk under $f_{1,0}$, $g_{2/3,1,0}$, $g_{1,1,0}$, and $g_{7/6,1,0}$.

$$g_{\alpha,0,\gamma}(z) = z + \frac{\alpha}{3(4-\alpha)} e^{i \arg \gamma} z^3, \qquad z \in \Delta.$$

c) If $\beta, \gamma \neq 0$, then $\mathcal{P} = \{2, 3\}$ and we distinguish the following subcases.

i) If $2(4-\alpha)|\beta| \geq (2-\alpha)|\gamma| + \frac{\alpha(4-\alpha)}{2-\alpha}$, then $i_2 = 2$ and $i_3 = 3$, $N = 2$, $\mathcal{I} = \{i_2\} = \{2\}$ and $\operatorname{dist}(f_{\beta,\gamma}, \mathcal{G}^*(\alpha)) = \sqrt{\frac{\pi}{3}\left(|\beta| - \frac{\alpha}{2(2-\alpha)}\right)^2 + \frac{\pi}{4}|\gamma|^2}$ is attained for the function $g_{\alpha,\beta,\gamma} \in \mathcal{G}^*(\alpha)$ defined by

$$g_{\alpha,\beta,\gamma}(z) = z + \frac{\alpha}{2(2-\alpha)} e^{i \arg \beta} z^2, \qquad z \in \Delta.$$

ii) If $(2-\alpha)|\gamma| + \frac{\alpha(4-\alpha)}{2-\alpha} > 2(4-\alpha)|\beta| \geq (2-\alpha)|\gamma|$, then $i_2 = 2$ and $i_3 = 3$, $N = 3$, $\mathcal{I} = \{i_2, i_3\} = \{2, 3\}$. The minimal distance $\operatorname{dist}(f_{\beta,\gamma}, \mathcal{G}^*(\alpha))$ and the extremal function $g_{\alpha,\beta,\gamma}(z) = z + b_2 z^2 + b_3 z^3 \in \mathcal{G}^*(\alpha)$ are given by (4.1), respectively by (4.2).

iii) If $(2-\alpha)|\gamma| \geq 2(4-\alpha)|\beta| + \frac{\alpha(2-\alpha)}{3(4-\alpha)}$, then $i_2 = 3$ and $i_3 = 2$, $N = 2$, $\mathcal{I} = \{i_2\} = \{3\}$ and $\operatorname{dist}(f_{\beta,\gamma}, \mathcal{G}^*(\alpha)) = \sqrt{\frac{\pi}{3}|\beta|^2 + \frac{\pi}{4}\left(|\gamma| - \frac{\alpha}{3(4-\alpha)}\right)^2}$ is attained for the function $g_{\alpha,\beta,\gamma} \in \mathcal{G}^*(\alpha)$ defined by

$$g_{\alpha,\beta,\gamma}(z) = z + \frac{\alpha}{3(4-\alpha)} e^{i \arg \gamma} z^3, \qquad z \in \Delta.$$

iv) If $2(4-\alpha)|\beta| + \frac{\alpha(2-\alpha)}{3(4-\alpha)} > (2-\alpha)|\gamma| > 2(4-\alpha)|\beta|$, $i_2 = 3$ and $i_3 = 2$, $N = 3$, $\mathcal{I} = \{i_2, i_3\} = \{2, 3\}$. The minimal distance $\operatorname{dist}(f_{\beta,\gamma}, \mathcal{G}^*(\alpha))$ and the extremal function $g_{\alpha,\beta,\gamma}(z) = z + b_2 z^2 + b_3 z^3 \in \mathcal{G}^*(\alpha)$ are given by (4.1), respectively by (4.2).

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