Doctopic: Theory of natural computing ARTICLE IN PRESS TCS:13093

Theoretical Computer Science ••• (••••) •••–•••

Contents lists available at ScienceDirect

Theoretical Computer Science

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On the domino problem of the Baumslag-Solitar groups

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ARTICLE INFO

Article history:
Received 31 March 2021
Received in revised form 27 August 2021
Accepted 1 September 2021
Available online xxxx

Keywords:
Domino problem
Tiling
Undecidability
Baumslag-Solitar group

ABSTRACT

In [1] we construct aperiodic tile sets on the Baumslag-Solitar groups $BS(m,n)$. Aperiodicity plays a central role in the undecidability of the classical domino problem on $\mathbb{Z}^2$, and analogously to this we state as a corollary of the main construction that the Domino problem is undecidable on all Baumslag-Solitar groups. In the present work we elaborate on the claim and provide a full proof of this fact. We also provide details of another result reported in [1]: there are tiles that tile the Baumslag-Solitar group $BS(m,n)$ but none of the valid tilings is recursive. The proofs are based on simulating piecewise affine functions by tiles on $BS(m,n)$.

1. Introduction

Classical Wang tilings are colorings of the two-dimensional Euclidean grid $\mathbb{Z}^2$ that respect some local constraints, usually given in terms of matching edge colorings of unit square tiles, known as Wang tiles. The domino problem (DP) is the decision problem to determine if given local constraints admit any valid colorings, i.e., whether a valid tiling of the infinite grid exists. Wang tiles and the domino problem were introduced by Hao Wang in order to study decision problems for formulas in the predicate calculus with a quantifier prefix $\forall\exists\forall$ [2].

The domino problem was proved undecidable by R. Berger [3]. To prove undecidability, one draws computation steps by Turing machines on tilings. With this idea, Wang tiles also provide computationally universal models for bio-computation and self-assembly. Winfree et al. [4] demonstrated a way to build DNA tiles that work as Wang tiles. Jonoska et al. [5] introduced a flexible tile model in three dimensions that can more readily self-assemble into non-planar structures. See also [6] for details and comparisons of the models.

In the present work we consider tilings on other regular structures. The concept of Wang tilings immediately generalizes to grids that are finitely gen- erated groups. In this setting, the domino problem is known to be decidable on virtually free groups, and it is even conjectured that virtually free groups are the only ones with decidable DP [7]. The problem enjoys nice inheritance properties: having a subgroup H, a quotient G/H with a finitely generated $H \trianglelefteq G$ or a translation-like action of H on G, all with undecidable DP for H, is enough for DP to be also undecidable on G. The decidabilty of DP is thus a commensurability invariant, and also a quasi-isometry invariant for finitely presented groups [8], hence a geometric property of the group. The conjecture mentioned above is true for polycyclic groups [9], groups $G_1 \times G_2$ with G_1 and G_2 infinite [10] and fundamental groups of oriented surfaces [11]. We refer to [12] for a recent survey.

In the conference paper [1] we study tilings of Baumslag-Solitar groups and exhibit examples of weakly aperiodic tile sets. Aperiodicity plays a central role in the undecidability of the domino problem on $\mathbb{Z}^2$, and analogously to this we state

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https://doi.org/10.1016/j.tcs.2021.09.002

Please cite this article as: N. Aubrun and J. Kari, On the domino problem of the Baumslag-Solitar groups, Theoretical Computer Science, https://doi.org/10.1016/j.tcs.2021.09.002

as a corollary of the main construction that the Domino problem is undecidable on all Baumslag-Solitar groups $BS(m,n)$. In the present work we elaborate on the claim and provide a full proof of this fact. Note that for $|m|>1$ and $|n|>1$ the result also follows directly from the fact that the group $BS(m,n)$ has $\mathbb{Z}^2$ as a subgroup [13] (see also Remark 1 below); it is namely easy to see that if a group has a decidable domino problem then also its finitely generated subgroups must have a decidable domino problem [12, Proposition 9.3.30]. However, the interesting cases $m=1, n>1$ are not covered by this reasoning and our method provides a unified proof that works for all parameter values. We believe this deserves a full explanation with the details that were not published in [1] due to page limit restrictions. We also provide details of another result reported in [1]: there are tiles that tile the Baumslag-Solitar group $BS(m,n)$ but none of the valid tilings is recursive.

The paper is organized as follows. In Section 2 we define central notions and notations that we use, including subshifts on groups and the domino problem on groups. We also define and discuss the Baumslag-Solitar groups $BS(m,n)$. Section 3 is the main body of the paper containing the details of our construction for the cases $m,n\geq 1$. In Section 4 we observe that the main proof actually provides a slightly stronger inseparability result that we can use to cover also the remaining cases of negative parameter values, as well as to prove the undecidability of the domino problem on a natural linear homomorphic image of $BS(m,n)$. The homomorphism erases the grid structure $\mathbb{Z}^2$ from $BS(m,n)$ so that the homomorphic image no longer has $\mathbb{Z}^2$ inside it. In Section 5 we show how the main construction also yields a tile set that admits a tiling of $BS(m,n)$ but does not admit any recursive tiling.

2. Background

2.1. Subshifts and aperiodicity

In the sequel A is a finite alphabet and $\mathbb{G}$ is a finitely generated group. Endowed with the product topology, the set $A^{\mathbb{G}}$ is compact and metrizable. Elements of $A^{\mathbb{G}}$ are called *configurations* that can be thought of as colorings of the group $\mathbb{G}$ with symbols from A. If $c\in A^{\mathbb{G}}$ is a configuration, we may simplify the notation $c(g)$ into c_g. The group $\mathbb{G}$ acts to the left on the set of configurations $A^{\mathbb{G}}$ through the shift action:

$$\mathfrak{S}:\begin{pmatrix} \mathbb{G}\times A^{\mathbb{G}} & \to & A^{\mathbb{G}} \\ (g,c) & \mapsto & \mathfrak{S}_g(c)\end{pmatrix}$$

where $\mathfrak{S}_g(c)$ is the configuration defined by $\left(\mathfrak{S}_g(c)\right)_h = c_{g^{-1}\cdot h}$ for every $h\in\mathbb{G}$. The dynamical system $\left(A^{\mathbb{G}},\mathfrak{S}\right)$ is called the *full shift*. A set of configurations $X\subset A^{\mathbb{G}}$ is a *subshift* if it is closed and shift invariant.

Let $c\in A^{\mathbb{G}}$ be a configuration. Its orbit is $orb_{\mathfrak{S}}(c)=\left\{\mathfrak{S}_g(c)\mid g\in\mathbb{G}\right\}$ and its stabilizer is $stab_{\mathfrak{S}}(c)=\left\{\ g\in\mathbb{G}\mid\mathfrak{S}_g(c)=c\right\}$. Note that the stabilizer is always a subgroup of $\mathbb{G}$.

A configuration $x\in A^{\mathbb{G}}$ is *periodic* if it has finite orbit $|orb_{\mathfrak{S}}(x)|<\infty$. A subshift $X\subset A^{\mathbb{G}}$ is *weakly aperiodic* if every configuration $x\in X$ has infinite orbit $|orb_{\mathfrak{S}}(x)|=\infty$. A subshift $X\subset A^{\mathbb{G}}$ is *strongly aperiodic* if every configuration $x\in X$ has trivial stabilizer $stab_{\mathfrak{S}}(x)=\{1_{\mathbb{G}}\}$.

2.2. Subshifts of finite type and the domino problem

A finite *pattern* is a coloring $p\in A^D$ of a finite set $D\subseteq\mathbb{G}$. A configuration $c\in A^{\mathbb{G}}$ *avoids* pattern p if $\mathfrak{S}_g(c)_{|D}\neq p$ for all $g\in\mathbb{G}$. For a set F of finite patterns we denote by X_F the set of configurations that avoid all $p\in F$. The set X_F is a subshift and, in fact, every subshift is equal to X_F for some F. We call elements in F *forbidden* patterns that define X_F. If F is a finite set of patterns then X_F is called a *subshift of finite type (SFT)*. Note that the same SFT may be defined by different sets of forbidden patterns, some of which may be infinite. But the subshift is of finite type as long as some finite F defines it.

The domino problem is the algorithmic question that asks whether for a given finite set F of finite patterns the subshift X_F is empty or not. The input instance F can be encoded using generators of the group: Since the group $\mathbb{G}$ is finitely generated, each group element has a representation as a word of generators and their inverses. A finite pattern is then represented as a finite set of assignments $g\mapsto a$ for $g\in\mathbb{G}$ and $a\in A$. Note that such an assignment can be inconsistent, i.e., two different words that represent the same group element may be assigned different symbols. This is allowed: no configuration contains such an inconsistent "pattern" so that having them in F does not affect the defined SFT. Now each finite set F of finite patterns has a finite representation that can be used as a coding of the input to the domino problem.

Typically we are interested to know whether the domino problem is decidable or undecidable. The decidability status of the domino problem depends on the group $\mathbb{G}$, but it does not depend on the choice of the finite generator set. It is a property of the group. The classical domino problem concerns the free commutative group $\mathbb{Z}^2$, and the problem is undecidable in this group [3]. In contrast, the domino problem is decidable on the infinite cyclic group $\mathbb{Z}$ and, more generally, on all finitely generated free groups. We recommend [12] for more details on the topic. In this work we consider the domino problem on the Baumslag-Solitar groups.

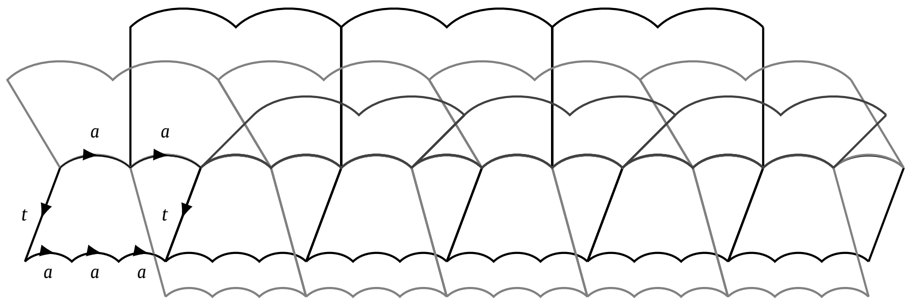

Fig. 1. A portion of the Cayley graph of the group $BS(2,3)$ with generating set $\{a,t\}$. Three sheets from the top merge and separate into two sheets to the bottom.

2.3. Baumslag-Solitar groups

Given two non-zero integers m and n, the Baumslag-Solitar group of order (m,n) is the two-generator and one-relator group with the presentation

$$BS(m,n) = \langle a, t \mid a^m t = t a^n \rangle.$$

Since $BS(-m,-n)$ is isomorphic to $BS(m,n)$, it is enough to consider groups with $m > 0$. For simplicity, we first also assume $n > 0$. The case $n < 0$ is covered in Corollary 19. Geometrically, the Cayley graph of $BS(m,n)$ with generating set $\{a,t\}$ is made of sheets that branch off each other along cosets of $\langle a \rangle$ with n-fold and m-fold branches up and down, respectively, so that the global structure of the sheets looks like a tree in which all nodes have degree $m+n$. Each of these sheets is quasi-isometric to the hyperbolic plane $\mathbb{H}^2$. See Fig. 1 for an illustration.

There is a homomorphism Φ from $BS(m,n)$ to the general linear group of invertible 2×2 matrices that maps the generators as follows:

$$\Phi(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \Phi(t) = \begin{pmatrix} \frac{m}{n} & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

For any $x, y \in \mathbb{R}$ we have

$$\begin{aligned} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \Phi(a) &= \begin{pmatrix} x & x+y \\ 0 & 1 \end{pmatrix} \text{ and} \\ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \Phi(t) &= \begin{pmatrix} \frac{m}{n}x & y \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{2}$$

It is now easy to verify that $\Phi(a)^m\Phi(t) = \Phi(t)\Phi(a)^n$ so that Φ is indeed well-defined on $BS(m,n)$. We also see that for any $g \in BS(m,n)$ there are numbers $\alpha(g) \in \mathbb{Q}$ and $\beta(g) \in \mathbb{Z}$ such that

$$\Phi(g) = \begin{pmatrix} \left(\frac{m}{n}\right)^{\beta(g)} & \alpha(g) \\ 0 & 1 \end{pmatrix}. \tag{3}$$

Clearly $\alpha(g)$ is uniquely determined by g, and so is $\beta(g)$ if $|m| \neq |n|$. In this case we can deduce from (2) that for any $g \in BS(m,n)$

$$\begin{cases} \beta(ga) &= \beta(g), \\ \beta(gt) &= \beta(g)+1, \\ \alpha(gt) &= \alpha(g), \\ \alpha(ga) &= \alpha(g) + \left(\frac{m}{n}\right)^{\beta(g)}. \end{cases} \tag{4}$$

Also when $|m| = |n|$ we can choose $\beta(g)$ in (3) such that (4) remains valid and $\beta(\varepsilon) = 0$ for the identity element ε of the group. The value $\beta(g)$ is the sum of the exponents of t in any word over the alphabet $\{a, t, a^{-1}, t^{-1}\}$ that represents g. We call $\beta(g)$ the *level* of g.

Remark 1. If $|m| > 1$ and $|n| > 1$ then the homomorphism Φ is not injective. For example, $b = tat^{-1}ata^{-1}t^{-1}a^{-1}$ is not the identity element of $BS(m,n)$ but it is in the kernel of Φ. In fact, b has infinite order so that $\langle b \rangle$ is isomorphic to $\mathbb{Z}$. Noting that $a^m b = b a^m$ and that the matrix $\Phi(a^m)$ has infinite order, we see that the subgroup of $BS(m,n)$ generated by b and a^m is isomorphic to $\mathbb{Z}^2$.

To reduce clutter we introduce the following function $\lambda : BS(m,n) \to \mathbb{Q}$ that merges α and β into a single useful quantity as follows:

$$\lambda(g) := \frac{1}{m}\left(\frac{m}{n}\right)^{-\beta(g)} \alpha(g).$$

The function satisfies the equalities

$$\begin{cases} \lambda(ga) &= \lambda(g) + \frac{1}{m}, \\ \lambda(gt) &= \frac{n}{m}\lambda(g) \end{cases} \tag{5}$$

that we can easily deduce from (4).

Remark 2. If $|m| \neq |n|$ then $\Phi(g)$ uniquely determines $\alpha(g)$ and $\beta(g)$. But $\Phi(g)$ uniquely determines $\lambda(g)$ in all cases, including $|m| = |n|$.

3. Domino problem on Baumslag-Solitar groups

In this section we fix $m, n \geq 1$ and prove that the domino problem is undecidable on the group $\mathbb{G} = BS(m,n)$. The proof idea is similar to the proof of the analogous result on the discrete hyperbolic plane in [14,15], which in turn is an adaptation of a former construction of a strongly aperiodic SFT on $\mathbb{Z}^2$ [16]. Modifications of these proofs are needed, for example, due to the overlapping sheets in the Cayley graphs of Baumslag-Solitar groups.

In the following subsections we introduce basic ingredients of the proof. First we define what is meant by tiles that compute affine functions and discuss horizontal rows of such tiles. In the next Subsection 3.2 we explicitly construct tiles to compute a given affine function f on a given input $\vec{x}$ at a given group element $g \in \mathbb{G}$, and show how to associate to any two-way infinite sequence of applications of affine maps a tiling of $\mathbb{G}$ with the corresponding tiles. In the Subsection 3.3 we review the immortality problem of affine maps from [14,15], and show how the constructions in the previous subsection yield a finite tile set that can be effectively constructed. The proof is finalized by showing that this tile set admits a tiling of $\mathbb{G}$ if and only if the corresponding system of affine maps has an immortal point.

3.1. Tiles that compute affine maps on Baumslag-Solitar groups

Our configurations on $\mathbb{G}$ are most conveniently represented as colorings of the edges of the Cayley graph of $\mathbb{G}$ with generators a and t. In a configuration $c \in A^{\mathbb{G}}$ each $g \in \mathbb{G}$ stores the colors of edges $g \to ga$ and $g \to gt$. The local tiling constraint is given in terms of a set of allowed tiles. A *tile* is a coloring of the edges

$$\begin{array}{lcl} \varepsilon & \longrightarrow a \longrightarrow a^2 \longrightarrow \cdots \longrightarrow & a^m \\ \downarrow & & \downarrow \\ t & \longrightarrow ta \longrightarrow ta^2 \longrightarrow \cdots \longrightarrow & ta^n = a^m t, \end{array}$$

i.e., the edges around the shape that is shown in Fig. 2(a) positioned at a group element g. In a valid tiling, for each group element g the pattern of this shape found at position g must be among given allowed tiles. It is then clear that the set of valid tilings is an SFT, defined by forbidding the non-allowed tiles.

The colors on the edges will be elements of $\mathbb{R}^2$. We say that a tile with colors $\vec{x}_1, \ldots, \vec{x}_m$ on the top edges, $\vec{y}_1, \ldots, \vec{y}_n$ on the bottom edges, and $\vec{\ell}$ and $\vec{r}$ on the left and on the right, as shown in Fig. 2(b), computes an affine function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ if

$$f\left(\frac{\vec{x}_1 + \ldots \vec{x}_m}{m}\right) + \vec{\ell} = \frac{\vec{y}_1 + \ldots \vec{y}_n}{n} + \vec{r}. \tag{6}$$

Such tile thus computes the image by f of the average of the elements on the top edges, and redistributes this image on the bottom edges. This is performed up to calculation errors, that are propagated to neighbors through the left and right edges of the tile.

Consider a finite horizontal segment of k tiles that all compute the same affine function f, and let $\vec{x}_1^{(i)}, \ldots, \vec{x}_m^{(i)}$, $\vec{y}_1^{(i)}, \ldots, \vec{y}_n^{(i)}$ and $\vec{\ell}^{(i)}, \vec{r}^{(i)}$ be the colors of the i'th tile, $1 \leq i \leq k$. See Fig. 3 for an illustration. Assuming consecutive tiles match in color we have $\vec{r}^{(i)} = \vec{\ell}^{(i+1)}$ for all i. Let us sum up the equations (6) for the k tiles, cancel the identical colors $\vec{r}^{(i)}$ and $\vec{\ell}^{(i+1)}$ from the two sides of the obtained equation, and divide the final equation by k. Using the fact that affine functions commute with the operation of taking averages we obtain that

$$f\left(\frac{\sum_{i=1}^{k}\sum_{j=1}^{m}\vec{x}_j^{(i)}}{km}\right) + \frac{\vec{\ell}^{(1)}}{k} = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n}\vec{y}_j^{(i)}}{kn} + \frac{\vec{r}^{(k)}}{k}, \tag{7}$$

Fig. 2. (a) The shape of a tile at position $g \in BS(m,n)$ and (b) the colors on the edges of a tile.

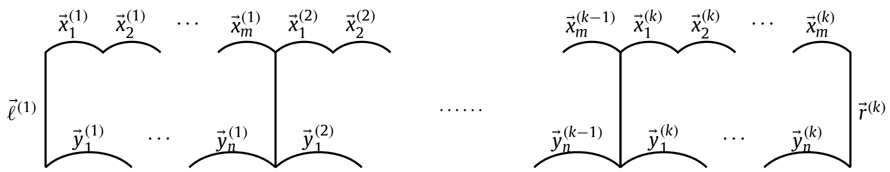

Fig. 3. A row of k matching tiles.

showing that the segment – viewed as a single tile with km top edges and kn bottom edges – also computes f with diminishing calculation errors as $k \longrightarrow \infty$.

Moving to infinite sequences, we say that a two-way infinite sequence $\vec{x}_i \in \mathbb{R}^2$ of vectors, $i \in \mathbb{Z}$, *codes* vector $\vec{x} \in \mathbb{R}^2$ if arbitrarily long segments of the sequence have averages arbitrarily close to $\vec{x}$. More precisely, for $j = 1, 2, \ldots$ there are segment boundaries $n_j, m_j \in \mathbb{Z}$ such that

$$\begin{array}{l} n_j < m_j, \\ \lim_{j\to\infty} (m_j - n_j) = \infty, \text{ and} \\ \lim_{j\to\infty} \frac{1}{m_j - n_j + 1} \sum_{i=n_j}^{m_j} \vec{x}_i = \vec{x}. \end{array} \tag{8}$$

Note that one sequence may code several different vectors. A simple compactness argument, however, shows that if $\{\vec{x}_i \mid i \in \mathbb{Z}\}$ is finite then the sequence codes at least one vector. Indeed, the averages over finite subsegments belong to a bounded subset of $\mathbb{R}^2$ and thus any sequence of averages has a converging subsequence.

Consider next an infinite horizontal segment of matching tiles that all compute the same affine function f. Assume that only finitely many different tiles are used. Let $\vec{x}_i$ and $\vec{y}_i$, for $i \in \mathbb{Z}$, be the sequence of labels on the top and the bottom edges in this segment, aligned so that the top and the bottom colors of the i'th tile are $\vec{x}_{im+1}, \ldots, \vec{x}_{im+m}$ and $\vec{y}_{in+1}, \ldots, \vec{y}_{in+n}$, respectively, and let $\vec{r}^{(i)} = \vec{\ell}^{(i+1)}$ be the color on the right edge of tile i and the left edge of tile $i+1$. Let $\vec{x}$ be a vector that is coded by the top sequence $\vec{x}_i$, and let $n_j, m_j \in \mathbb{Z}$ be the corresponding segment boundaries, $j = 1, 2, \ldots$, that satisfy (8). Note that decreasing n_j and increasing m_j by at most m positions does not change the fact that (8) holds. Thus we may assume that $n_j = ma_j + 1$ and $m_j = mb_j + m$ for some integers $a_j \le b_j$, meaning that $\vec{x}_i$ for $i = n_j, \ldots, m_j$ are the top colors of the tiles $a_j, \ldots, b_j$. By (7) the average of the bottom colors $\vec{y}_i$ in the corresponding segment $i = na_j + 1, \ldots, nb_j + n$ approaches $f(\vec{x})$ as $i \to \infty$. Note that the vectors $\vec{r}^{(i)} = \vec{\ell}^{(i+1)}$ are bounded as there are only finitely many different tiles used so that the error terms $\frac{\vec{\ell}^{(1)}}{k}$ and $\frac{\vec{r}^{(k)}}{k}$ in (7) approach zero as $k = b_j - a_j + 1 \longrightarrow \infty$.

We have thus shown the following.

Lemma 3. *Consider an infinite horizontal sequence of matching tiles that all compute the same affine function f and only contain finitely many different tiles, and let $\vec{x}$ be a vector that is coded by the sequence of top labels. Then $f(\vec{x})$ is coded by the sequence of bottom labels.*

Next we design a concrete implementation of tiles that compute a given affine map.

3.2. *Specific tiles to compute the affine map f on input $\vec{x}$ at $g \in \mathbb{G}$*

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an affine map $\vec{x} \mapsto M\vec{x} + \vec{b}$ defined by a 2×2 real matrix M and a vector $\vec{b} \in \mathbb{R}^2$. For every $\vec{x} \in \mathbb{R}^2$ and $g \in \mathbb{G}$ we define a tile $T_f(g, \vec{x})$ with the following edge colors and show that it computes f, where the names on the left refer to the notations of Fig. 2(b):

$$\begin{aligned}
\vec{x}_k &= \lfloor (m\lambda(g)+k)\,\vec{x}\rfloor - \lfloor (m\lambda(g)+(k-1))\,\vec{x}\rfloor \text{ for } k=1\ldots m\\
\vec{y}_k &= \lfloor (n\lambda(g)+k)\,f(\vec{x})\rfloor - \lfloor (n\lambda(g)+(k-1))\,f(\vec{x})\rfloor \text{ for } k=1\ldots n\\
\vec{\ell} &= \tfrac{1}{m} f\left(\lfloor m\lambda(g)\vec{x}\rfloor\right) - \tfrac{1}{n}\lfloor n\lambda(g) f(\vec{x})\rfloor + \lfloor \lambda(g) - \tfrac{1}{m}\rfloor \vec{b}\\
\vec{r} &= \tfrac{1}{m} f\left(\lfloor (m\lambda(g)+m)\,\vec{x}\rfloor\right) - \tfrac{1}{n}\lfloor (n\lambda(g)+n)\,f(\vec{x})\rfloor + \lfloor \lambda(g)+1-\tfrac{1}{m}\rfloor \vec{b}
\end{aligned} \tag{9}$$

where we apply the floor function on vectors coordinate-wise, meaning that $\lfloor (x,y)^T\rfloor = (\lfloor x\rfloor, \lfloor y\rfloor)^T$ for all $x, y \in \mathbb{R}$. We denote these colors also by $\vec{x}_k(f,g,\vec{x})$, $\vec{y}_k(f,g,\vec{x})$, $\vec{\ell}(f,g,\vec{x})$ and $\vec{r}(f,g,\vec{x})$, respectively, when we want to identify the tile $T_f(g,\vec{x})$ in question.

Remark 4. The group element g is only used in the tile through the function λ so that if $\lambda(g_1)=\lambda(g_2)$ then also $T_f(g_1,\vec{x}) = T_f(g_2,\vec{x})$.

Remark 5. The floor function in (9) discretizes the colors, and this is used in the Subsection 3.3 to guarantee that a tile set is finite. Observe that with the floor functions in (9) omitted one would have a tile with colors $\vec{x}_k = \vec{x}$ for all $k = 1\ldots m$, $\vec{y}_k = f(\vec{x})$ for all $k=1\ldots n$, and $\vec{\ell} = \vec{r} = \vec{0}$, i.e., a tile that does not depend on g.

A direct calculation gives the following.

Lemma 6. *The tile $T_f(g,\vec{x})$ computes f.*

Proof. By replacing every term $\vec{y}_k$ and $\vec{x}_k$ by its expression in (9), the two sums $\vec{x}_1+\cdots+\vec{x}_m$ and $\vec{y}_1+\cdots+\vec{y}_n$ telescope to

$$\begin{aligned}
\vec{x}_1+\cdots+\vec{x}_m &= \lfloor (m\lambda(g)+m)\,\vec{x}\rfloor - \lfloor m\lambda(g)\vec{x}\rfloor,\\
\vec{y}_1+\cdots+\vec{y}_n &= \lfloor (n\lambda(g)+n)\,f(\vec{x})\rfloor - \lfloor n\lambda(g) f(\vec{x})\rfloor.
\end{aligned}$$

Using the property $f(c\vec{y}-c\vec{z}) = cf(\vec{y}) - cf(\vec{z}) + \vec{b}$ of the affine function $f:\vec{x}\mapsto M\vec{x}+\vec{b}$ we obtain

$$f\left(\frac{\vec{x}_1+\cdots+\vec{x}_m}{m}\right) = \frac{1}{m} f\left(\lfloor (m\lambda(g)+m)\,\vec{x}\rfloor\right) - \frac{1}{m} f\left(\lfloor (m\lambda(g))\,\vec{x}\rfloor\right) + \vec{b}.$$

These calculations, together with the expressions of $\vec{\ell}$ and $\vec{r}$ in (9), give

$$f\left(\frac{\vec{x}_1+\cdots+\vec{x}_m}{m}\right) - \frac{\vec{y}_1+\cdots+\vec{y}_n}{n} + \vec{\ell} - \vec{r} = \vec{b} + \lfloor \lambda(g) - \frac{1}{m}\rfloor \vec{b} - \lfloor \lambda(g)+1-\frac{1}{m}\rfloor \vec{b} = \vec{0},$$

verifying (6). □

Consider any two-way infinite sequence of real vectors $\vec{x}_i \in \mathbb{R}^2$ such that for every $i\in\mathbb{Z}$ the condition $\vec{x}_{i+1} = f_i(\vec{x}_i)$ holds for some affine function f_i. Let us show that if one puts at each position $g\in\mathbb{G}$ the tile $T_{f_i}(g,\vec{x}_i)$ with $i=\beta(g)$ then one obtains a consistent coloring of the edges of the Cayley graph of $\mathbb{G}$.

Lemma 7. *The edges of the Cayley graph of $\mathbb{G}$ can be labeled so that the tile at each $g\in\mathbb{G}$ is $T_{f_i}(g,\vec{x}_i)$ where $i=\beta(g)$ is the level of g.*

Proof. For every $g\in\mathbb{G}$ denote $T(g) = T_{f_i}(g,\vec{x}_i)$ where $i=\beta(g)$, and label the edges $g\to ga$ and $g\to gt$ of the Cayley graph by the colors $\vec{x}_1$ and $\vec{\ell}$ of the tile $T(g)$. Let us prove that the tile created by this coloring at any $g\in\mathbb{G}$ is $T(g)$.

- For every $k=1\ldots m$ the color $\vec{x}_k$ of the tile $T(g)$ is the same as the color $\vec{x}_1$ of the tile $T(ga^{k-1})$, that is, the color of the edge $ga^{k-1}\to ga^k$: elements g and ga^{k-1} have the same level $i=\beta(ga^{k-1})=\beta(g)$ so that $T(g)=T_f(g,\vec{x})$ and $T(ga^{k-1}) = T_f(ga^{k-1},\vec{x})$ for the same $f=f_i$ and $\vec{x}=\vec{x}_i$. The claimed result $\vec{x}_k(f,g,\vec{x}) = \vec{x}_1(f,ga^{k-1},\vec{x})$ follows directly from (9) using $m\lambda(g)+k = m\lambda(ga^{k-1})+1$.
- The color $\vec{r}$ of the tile $T(g)$ is the same as the color $\vec{\ell}$ of the tile $T(ga^m)$, that is, the color of the edge $ga^m\to ga^m t$: again g and ga^m have the same level i so that $T(g)=T_f(g,\vec{x})$ and $T(ga^m)=T_f(ga^m,\vec{x})$ for the same $f=f_i$ and $\vec{x}=\vec{x}_i$. The result $\vec{r}(f,g,\vec{x}) = \vec{\ell}(f,ga^m,\vec{x})$ follows now from $\lambda(ga^m)=\lambda(g)+1$ because replacing $\lambda(g)$ by $\lambda(g)+1$ in the formula for $\vec{\ell}$ in (9) gives the formula of $\vec{r}$ in (9).

- For every $k = 1 \ldots n$ the color $\vec{y}_k$ of tile $T(g)$ is the same as the color $\vec{x}_1$ of the tile $T(gta^{k-1})$, that is, the color of the edge $gta^{k-1} \to gta^k$: Now $\beta(gta^{k-1}) = \beta(g)+1$ so that $T(g) = T_{f_i}(g, \vec{x}_i)$ and $T(ga^m) = T_{f_{i+1}}(ga^m, f_{i+1}(\vec{x}_i))$ for $i = \beta(g)$. From (5) we can compute $m\lambda(gta^{k-1}) + 1 = n\lambda(g) + k$ so that from (9) we get

$$\begin{aligned}\vec{y}_k(f_i, g, \vec{x}_i) &= \lfloor (n\lambda(g)+k)\, f_i(\vec{x}_i)\rfloor - \lfloor (n\lambda(g)+(k-1))\, f_i(\vec{x}_i)\rfloor \\ &= \lfloor \big(m\lambda(gta^{k-1})+1\big)\,\vec{x}_{i+1}\rfloor - \lfloor \big(m\lambda(gta^{k-1})\big)\,\vec{x}_{i+1}\rfloor \\ &= \vec{x}_1(f_{i+1}, gta^{k-1}, \vec{x}_{i+1}).\end{aligned}$$

We have seen that colors of the edges around the tile shape at position g form the tile $T_{f_i}(g, \vec{x}_i)$, as claimed. □

3.3. Reduction from the immortality problem for rational piecewise affine maps to the domino problem on $\mathbb{G}$

Now we have the necessary ingredients in place to proceed to a reduction from the immortality problem for rational piecewise affine maps on the plane. In this problem we are given a finite family, indexed by $i \in I$ for a finite index set I, of disjoint closed unit size squares $U_i = [n_i, n_i+1] \times [m_i, m_i+1]$ of integer corners, $n_i, m_i \in \mathbb{Z}$, and associated affine maps $f_i : U_i \longrightarrow \mathbb{R}^2$ with rational coefficients. These define a function $f : U \longrightarrow \mathbb{R}^2$ where $U = \cup_{i\in I} U_i$ and $f(\vec{x}) = f_i(\vec{x})$ for $\vec{x} \in U_i$. The function f can be iterated on an initial point $\vec{x} \in U$ for as long as the image stays inside U. We say that $\vec{x} \in U$ is an *immortal point* if $f^k(\vec{x}) \in U$ for all $k \in \mathbb{N}$, and it is mortal otherwise. The immortality problem for rational piecewise affine maps is the decision problem that takes the family $\{(U_i, f_i) \mid i \in I\}$ as its input and outputs `Yes` if f possesses an immortal point, and `No` otherwise. This problem is undecidable due to a reduction from the known undecidable immortality problem for Turing machines [17]. We refer to [15] for details of this reduction.

Theorem 8 *([14,15]). The immortality problem for rational piecewise affine maps is undecidable.*

Now we proceed with an effective construction of a tile set on the group $\mathbb{G} = BS(m,n)$ that admits a valid configuration if and only if f has an immortal point. For every $U_i = [n_i, n_i+1] \times [m_i, m_i+1]$ and associated affine f_i we show how to construct a finite tile set T_i that satisfies the following conditions:

(i) every tile in T_i computes f_i, i.e., it satisfies (6) for $f = f_i$, and
(ii) tiles $T_{f_i}(g, \vec{x})$ as defined in (9) are in T_i for all $\vec{x} \in U_i$ and all $g \in \mathbb{G}$.

Noting that $r - 1 < \lfloor r \rfloor \le r$ for all $r \in \mathbb{R}$ and that colors $\vec{x}_k$ in (9) are integers we see that tiles $T_{f_i}(g, \vec{x})$ for $\vec{x} \in U_i$ have their colors $\vec{x}_k$ from the four element set $\{n_i, n_i+1\} \times \{m_i, m_i+1\}$. Similarly, finite sets can be computed that contain colors $\vec{y}_k$, $\vec{\ell}$ and $\vec{r}$ of $T_{f_i}(g, \vec{x})$ for all $\vec{x} \in U_i$ and all $g \in \mathbb{G}$. Thus the tile set T_i can be constructed by taking all tiles that compute f_i and have colors in these finite sets. We also

(iii) assign the additional label $i \in I$ in the two vertical edges of the tiles in T_i,

guaranteeing this way that tiles in any positions g and ga^m compute the same affine map f_i since they share a vertical edge.

Our tile set T is the union of all T_i over $i \in I$. It is finite and can be effectively constructed for a given family $\{(U_i, f_i) \mid i \in I\}$ of rational affine maps. We next prove that T admits a tiling if and only if the family $\{(U_i, f_i) \mid i \in I\}$ has an immortal point. We split the two directions of this statement in two separate lemmas.

Lemma 9. *If the family $\{(U_i, f_i) \mid i \in I\}$ has an immortal point then T admits a tiling of the group $\mathbb{G} = BS(m,n)$.*

Proof. Suppose an immortal point exists. A simple compactness argument then provides a two-way infinite orbit $\ldots, \vec{x}_{-1}, \vec{x}_0, \vec{x}_1, \ldots$ of $f : U \longrightarrow \mathbb{R}^2$, where for each $i \in \mathbb{Z}$ there is $j(i) \in I$ such that $\vec{x}_i \in U_{j(i)}$ and $\vec{x}_{i+1} = f_{j(i)}(\vec{x}_i)$. By Lemma 7 the edges of the Cayley graph of $\mathbb{G}$ can be labeled so that the tile at each $g \in \mathbb{G}$ is $T_{f_{j(i)}}(g, \vec{x}_i)$ where $i = \beta(g)$. By the construction of T, and in particular the property (ii) above, the tile $T_{f_{j(i)}}(g, \vec{x}_i)$ is in $T_{j(i)} \subseteq T$, and is thus an allowed pattern. Moreover, tiles at g and ga^m that share a vertical edge have the same level $i = \beta(g) = \beta(ga^m)$ so that the additional labeling (iii) is consistent. □

Remark 10. The tiling of Lemma 9 has identical tiles in any positions $g_1, g_2 \in \mathbb{G}$ that satisfy $\alpha(g_1) = \alpha(g_2)$ and $\beta(g_1) = \beta(g_2)$. Indeed, from $\lambda(g_1) = \lambda(g_2)$ we have that $T_f(g_1, \vec{x}) = T_f(g_2, \vec{x})$ for all f and $\vec{x}$, and from $\beta(g_1) = \beta(g_2)$ we get that the same f and $\vec{x}$ are used in both positions g_1 and g_2. This fact is used to deduce a recursive inseparability result in Lemma 15.

Lemma 11. *If T admits a tiling of the group $\mathbb{G} = BS(m,n)$ then the family $\{(U_i, f_i) \mid i \in I\}$ has an immortal point.*

Proof. Fix a valid configuration, that is, a coloring of the edges of the Cayley graph such that the tiles at all $g \in \mathbb{G}$ belong to the set T of allowed tiles. For any $g \in \mathbb{G}$ and any $k \in \mathbb{Z}$ let $\vec{x}_k^{(g)}$ be the color of the edge $ga^k \to ga^{k+1}$. Consider any fixed $g \in \mathbb{G}$. The tiles at positions ga^{jm} for $j \in \mathbb{Z}$ form a two-way infinite horizontal matching sequence of tiles, whose top labels read the color sequence $\vec{x}_k^{(g)}$, $k \in \mathbb{Z}$, and whose bottom labels read the sequence $\vec{x}_k^{(gt)}$, $k \in \mathbb{Z}$. Due to condition (iii) all these tiles belong to the same set T_i, $i \in I$, so that they all compute the same affine function f_i, and $\vec{x}_k^{(g)} \in \{n_i, n_i+1\} \times \{m_i, m_i+1\}$ for all $k \in \mathbb{Z}$. By the latter condition, all averages of top labels belong to $U_i = [n_i, n_i+1] \times [m_i, m_i+1]$ so that also every $\vec{x} \in \mathbb{R}^2$ that the sequence $\vec{x}_k^{(g)}$ codes belongs to U_i. Moreover, by Lemma 3 the vector $f_i(\vec{x})$ is coded by the bottom sequence $\vec{x}_k^{(gt)}$, $k \in \mathbb{Z}$.

Because $g \in \mathbb{G}$ was arbitrary, we have the following property: any vector $\vec{x}$ coded by any horizontal sequence of colors in the Cayley graph belongs to one U_i set and its image $f_i(\vec{x})$ is also coded by a horizontal sequence of colors. An easy induction then shows that any such vector $\vec{x}$ is an immortal point of the family $\{(U_i, f_i) \mid i \in I\}$. As every horizontal sequence codes at least one vector we see that there are immortal points. □

Combining the two lemmas we have our first main result.

Theorem 12. *The domino problem on the Baumslag-Solitar group $BS(m,n)$ with $m,n \geq 1$ is undecidable.*

4. Extensions

Recall that an algorithm to solve the domino problem is inherited from a group to its subgroups:

Proposition 13 *([12], Proposition 9.3.30). Let H be a finitely generated subgroup of a finitely generated group G. If the domino problem is decidable on G then it is also decidable on H.*

As noted in Remark 1, the Baumslag-Solitar group $BS(m,n)$ contains $\mathbb{Z}^2$ as a subgroup when $m > 1$ and $n > 1$:

Proposition 14 *([13], Proposition 7.11). Group $\mathbb{Z}^2$ embeds in $BS(m,n)$ if and only if $|n|, |m| > 1$ or $|n| = |m| = 1$.*

As the domino problem is undecidable on $\mathbb{Z}^2$, these two propositions imply its undecidability on $BS(m,n)$ with $n, m > 1$. However, the case $m = 1$, $n > 1$ cannot be proved in this manner since $\mathbb{Z}^2$ does not embed in $BS(m,n)$. Our proof provides a unified way to see the undecidability of the domino problem on all Baumslag-Solitar groups, including the case $m = 1$, $n > 1$. The grid structure inside $BS(m,n)$ with $n, m > 1$ is not used in the proof, which is also evidenced by the discussion below leading to Corollaries 18 and 19.

Let us use the following terminology. Let $\gamma : G \longrightarrow S$ be a function from a finitely generated group G to some set S. We say that a configuration $c \in A^G$ is *γ-consistent* if it factors though γ in the sense that $c(g_1) = c(g_2)$ for any $g_1, g_2 \in G$ such that $\gamma(g_1) = \gamma(g_2)$.

Recall the functions α and β from Subsection 2.3. Let us denote (α, β) for the function $BS(m,n) \longrightarrow \mathbb{Q} \times \mathbb{Z}$ that maps $g \mapsto (\alpha(g), \beta(g))$. From $(\alpha(g), \beta(g))$ we can uniquely compute $\Phi(g)$ and $\lambda(g)$. As pointed out in Remark 10, the tiling reported in Lemma 9 is (α, β)-consistent. This means that Lemmas 9 and 11 imply a sharper recursive inseparability result than stated in Theorem 12:

Lemma 15. *Let $m, n \geq 1$. The following two families of SFTs on $BS(m,n)$ are recursively inseparable:*

(a) *empty SFT, and*
(b) *SFTs that contain an (α, β)-consistent configuration.*

We make the following general observation.

Lemma 16. *Let G and H be finitely generated groups and $\gamma : G \longrightarrow H$ a homomorphism. Suppose that in G it is undecidable whether a given SFT contains an γ-consistent configuration. Then the domino problem on H is undecidable.*

Proof. By Proposition 13 the undecidability of the domino problem on $\gamma(G)$ implies its undecidability on H. By considering $\gamma(G)$ in place of H we can thus assume in the following that γ is surjective.

Suppose the domino problem is decidable on H. Then we can decide as follows if a given SFT on G contains an γ-consistent configuration. Let F be the given finite set of forbidden patterns on G. First, we can eliminate from F patterns that are not γ-consistent: if $p \in F$ is such that $\gamma(g_1) = \gamma(g_2)$ and $p(g_1) \neq p(g_2)$ for some $g_1, g_2 \in G$ then p cannot appear in any γ-consistent configuration, and therefore removing p from F does not change the set of γ-consistent elements of X_F. Note that the condition $\gamma(g_1) = \gamma(g_2)$ can be effectively tested since by [12, Theorem 9.3.28] the word problem on group H is decidable.

We can now assume that all patterns in F are γ-consistent. For $p \in A^D$ in F we can effectively construct the pattern $p' \in A^{\gamma(D)}$ where $p'(\gamma(g)) = p(g)$ for all $g \in D$. Let F' be the set of finite patterns p' obtained in this manner from $p \in F$.

It suffices to prove that $X_F \subseteq A^G$ contains an γ-consistent configuration if and only if $X_{F'} \subseteq A^H$ is non-empty:

- If $c \in X_F$ is γ-consistent then consider the configuration $c' \in A^H$ where for all $g \in G$ we have $c'(\gamma(g)) = c(g)$. Note how the γ-consistency of c means that c' is well-defined, and how the surjectivity of γ means that c' is unique. If c' contains a pattern $p' \in F'$ in position $\gamma(g)$ then c contains the corresponding pattern $p \in F$ in position $g \in G$. This is not possible so that $c' \in X_{F'}$.
- Conversely, if $c' \in X_{F'}$ exists then the configuration $c = c' \circ \gamma$ is γ-consistent and in X_F. □

From Lemma 16 and the inseparability result of Lemma 15 we obtain the following

Proposition 17. *Let $m, n \geq 1$. Let H be a finitely generated group such that there is a group homomorphism $\gamma : BS(m,n) \longrightarrow H$ and that (α, β) factors through γ in the sense that*

$$\gamma(g_1) = \gamma(g_2) \implies \begin{cases} \alpha(g_1) & = & \alpha(g_2), \\ \beta(g_1) & = & \beta(g_2). \end{cases}$$

Then the domino problem is undecidable on H.

Proof. By the inseparability result of Lemma 15 it is undecidable whether a given SFT on $BS(m,n)$ contains an γ-consistent configuration. Indeed, the empty SFT does not contain an γ-consistent configuration but any SFT that contains a (α, β)-consistent configuration also contains an γ-consistent one. The result thus follows from Lemma 16. □

As a first application of Proposition 17 we take $\gamma = \Phi$. By Remark 2, the function (α, β) factors through Φ in the case $|m| \neq |n|$.

Corollary 18. *Let $m, n \geq 1$ and $m \neq n$. The domino problem is undecidable on the matrix group $\langle \Phi(a), \Phi(t) \rangle$ generated by the 2×2 matrices $\Phi(a)$ and $\Phi(t)$ in (1).*

This is interesting because Φ erases the grid structure so that $\mathbb{Z}^2$ does not embed in $\langle \Phi(a), \Phi(t) \rangle$ even when $m, n > 1$. In other words, our proof of Theorem 12 does not take advantage of the subgroup $\mathbb{Z}^2$ inside $BS(m,n)$.

As a second corollary we take care of the case of negative parameters n.

Corollary 19. *The domino problem is undecidable on $B(m,n)$ for all $m, n \neq 0$.*

Proof. Let $m, n \geq 1$ and consider $H = BS(m, -n) = \langle a', t' \mid (a')^m t' = t'(a')^{-n} \rangle$. (Theorem 12 covers all other cases because $BS(m,n)$ and $BS(-m,-n)$ are isomorphic.) Define $G = BS(m^2, n^2) = \langle a, t \mid a^{m^2} t = t a^{n^2} \rangle$. We have $m^2, n^2 \geq 1$. Mapping $\gamma(a) = a'$ and $\gamma(t) = (t')^2$ defines a group homomorphism $\gamma : G \longrightarrow H$ because $\gamma(a^{m^2} t) = (a')^{m^2} (t')^2 = (t')^2 (a')^{n^2} = \gamma(t a^{n^2})$.

Let Φ, α and β be the functions of Subsection 2.3 for G, and let Φ', α' and β' be the analogous functions for H. From the definition of homomorphism γ we get $\beta'(\gamma(g)) = 2\beta(g)$, so that β factors through γ.

We also have $\Phi(a) = \Phi'(a')$, and $\Phi(t) = \Phi'(t')^2$ because $\Phi(t) = diag(m^2/n^2, 1)$ and $\Phi'(t') = diag(m/(-n), 1)$. This means that Φ and $\Phi' \circ \gamma$ are identical on a and t so that $\Phi = \Phi' \circ \gamma$. Thus Φ, and therefore also α, factors through γ.

We have proved that

$$\gamma(g_1) = \gamma(g_2) \implies \begin{cases} \alpha(g_1) & = & \alpha(g_2), \\ \beta(g_1) & = & \beta(g_2). \end{cases}$$

It now follows from Proposition 17 that the domino problem is undecidable on $H = BS(m, -n)$. □

5. A tile set that only admits non-recursive tilings of $BS(m,n)$

Let $\mathbb{G} = BS(m,n)$ for any fixed $m, n \geq 1$. Let us call a configuration $x \in A^{\mathbb{G}}$ *recursive* if there is an algorithm that returns x_g when given as input a representation of $g \in \mathbb{G}$ as a word w over the generating set $\{a, t, a^{-1}, t^{-1}\}$. The following theorem was stated at the end of [1] without a detailed proof.

Theorem 20. *There exists a tile set on $\mathbb{G}$ that admits a tiling but it admits no recursive tiling of G.*

The result was previously known in the case $m = n = 1$ (that is, for $\mathbb{G} = \mathbb{Z}^2$) [18]. In this section we prove it for arbitrary $m, n \geq 1$.

Our starting point is an analogous result for Turing machines, and its conversion to piecewise affine maps. Recall that a Turing machine uses two finite sets: a tape alphabet A and a state set Q. A tape configuration $c \in A^{\mathbb{Z}}$ assigns a tape symbol to each cell of a two-way infinite tape. An instantaneous description (ID) of the Turing machine is a triple $(c, q, k) \in A^{\mathbb{Z}} \times Q \times \mathbb{Z}$ consisting of a tape configuration c, current state $q \in Q$ of the machine and its position $k \in \mathbb{Z}$ on the tape. The Turing machine is a partial transformation $g : A^{\mathbb{Z}} \times Q \times \mathbb{Z} \longrightarrow A^{\mathbb{Z}} \times Q \times \mathbb{Z}$ of the ID, determined by a local transition rule of the machine. If $g(c, q, k)$ is undefined then (c, q, k) is called a *halting* ID. We call ID $d = (c, q, k)$ *immortal* if g can be iterated starting with d without ever reaching a halting ID. The *orbit* of an immortal ID d is the sequence of ID's $d_i = (c_i, q_i, k_i)$ for $i \in \mathbb{N}$ where $(c_0, q_0, k_0) = d$ and $(c_{i+1}, q_{i+1}, k_{i+1}) = g(c_i, q_i, k_i)$ for all $i \in \mathbb{N}$, and its *trace* is the sequence $(q_i, c_i(k_i))_{i\in\mathbb{N}} \in (Q \times A)^{\mathbb{N}}$ of consecutive pairs of the states/tape symbols scanned by the machine. We say that the trace is recursive if there is an algorithm that outputs $(q_i, c_i(k_i))$ for any given $i \in \mathbb{N}$. Note that a Turing machine has traces if and only if it has immortal IDs.

Theorem 21 *([19]). There exists a Turing machine that has traces but none of the traces is recursive.*

Recall from Section 3.3 the concept of families $\{(U_i, f_i) \mid i \in I\}$ of affine maps and the function $f : U \longrightarrow \mathbb{R}^2$ they define where $U = \cup_{i\in I} U_i$ and $f(\vec{x}) = f_i(\vec{x})$ for $\vec{x} \in U_i$. To each immortal point $\vec{x} \in U$ we associate the infinite sequence $\sigma \in I^{\mathbb{N}}$ of indices such that $f^i(\vec{x}) \in U_{\sigma(i)}$ for all $i \in \mathbb{N}$. We call σ a *trace*. Note that there are traces if and only if there are immortal points.

In [15, Theorem 3.1] a detailed conversion is given of any Turing machine g into a family $\{(U_i, f_i) \mid i \in I\}$ of affine maps. The conversion is such that g has immortal IDs if and only if the family $\{(U_i, f_i) \mid i \in I\}$ has immortal points. In fact it is easy to see that the conversion preserves recursiveness of traces: from a recursive trace of the family $\{(U_i, f_i) \mid i \in I\}$ one can also read a recursive trace of the Turing machine g. Thus executing the conversion on the machine of Theorem 21 provides the following corollary.

Corollary 22. *There exists a finite family $\{(U_i, f_i) \mid i \in I\}$ of rational affine maps that has traces but none of the traces is recursive.*

Now we can proceed with the proof of Theorem 20.

Proof of Theorem 20. Consider the tile set T on $BS(m, n)$ constructed in Section 3.3 for the family $\{(U_i, f_i) \mid i \in I\}$ of Corollary 22. Since there is a trace there is an immortal point and thus by Lemma 9 the tile set T admits a tiling of $BS(m, n)$.

Let us then show that the tile set T does not admit a recursive tiling. Suppose the contrary: there is a recursive configuration c that only contains tiles of T. Then there is an algorithm that returns for any given $i \in \mathbb{N}$ the color of the edge $t^i \to t^{i+1}$ in the configuration c. Let $\sigma(i) \in I$ be the additional label (iii) of the edge $t^i \to t^{i+1}$ that identifies the index in I such that the tile in position $t^i \in \mathbb{G}$ belongs to the tile set $T_{\sigma(i)}$. The sequence σ is recursive. But σ is a trace of $\{(U_i, f_i) \mid i \in I\}$, which contradicts the fact that there are no recursive traces. Indeed, the proof of Lemma 11 shows that a vector $\vec{x}$ coded by the top colors on the horizontal line of tiles through $t^0 \in \mathbb{G}$ is immortal and, moreover, its trace is the sequence σ. □

6. Conclusions and perspectives

Baumslag-Solitar groups are important groups with a marked role in combinatorial and geometric group theory. They are canonical examples of HHN-extensions, a fundamental construction in combinatorial group theory that, for instance, play a key role in Higman embedding theorem (every finitely generated recursively presented group can be embedded as a subgroup of some finitely presented group) and also appear in Stalling's theorem about ends of groups. A natural extension to our results would be to investigate more complicated HNN-extensions, to see how far our proof method can be adapted. There are also other interesting finitely generated groups to consider. For example, it is not known whether the domino problem is decidable on the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$.

The classical domino problem plays a role in proofs of undecidability for various properties of endomorphisms of full shifts on the group $\mathbb{Z}^2$, that is, two-dimensional cellular automata. Some of these proofs readily extend to the Baumslag-Solitar groups. However, the techniques in [20] to show that both injectivity and surjectivity of endomorphisms are undecidable properties on $\mathbb{Z}^2$ do not immediately seem to work for full shifts on all Baumslag-Solitar groups. As we have pointed out, if $|m| > 1$ and $|n| > 1$ the group $BS(m, n)$ has $\mathbb{Z}^2$ as a subgroup (see [13] or Remark 1), and in these cases any endomorphism f of $A^{\mathbb{Z}^2}$ can be effectively converted into an endomorphism of the full shift $A^{\mathbb{G}}$ over the group $\mathbb{G} = BS(m, n)$ that executes f on the cosets of $\mathbb{Z}^2$. As injectivity and surjectivity properties of f are preserved by this conversion, we obtain the desired undecidability of injectivity and surjectivity in these cases. However, the basic cases $m = 1, n > 1$ are not covered by this reasoning.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This work was partially supported by the ANR project CoCoGro (ANR-16-CE40-0005).

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