Optimal identifying codes in cycles and paths

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Abstract

The concept of identifying codes in a graph was introduced by Karpovsky, Chakrabarty and Levitin in 1998. These codes have been studied in several types of graphs such as hypercubes, trees, the square grid, the triangular grid, cycles and paths. In this paper, we determine the optimal cardinalities of identifying codes in cycles and paths in the remaining open cases.

Running title: Identification in cycles and paths

1 Introduction

Let $G = (V, E)$ be a simple connected and undirected finite graph with V as the set of vertices and E as the set of edges. Let u and v be vertices in V. If u and v are adjacent to each other, then the edge between u and v is denoted by uv. The *distance* $d(u, v)$ denotes the number of edges in any shortest path between u and v. We say that u *r-covers* v if the distance $d(u, v)$ is at most r. The *ball of radius r centered at u* is defined as

$$B_r(u) = \{x \in V \mid d(u, x) \leq r\}.$$

A nonempty subset of V is called a *code*, and its elements are called *codewords*. Let $C \subseteq V$ be a code and u be a vertex in V. An *I-set* of the vertex u with respect to the code C is defined as

$$I_r(C; u) = I_r(u) = B_r(u) \cap C.$$

We say that a code $T \subseteq V$ is a *transversal* of G if for each edge $e = uv \in E$ the vertex u or the vertex v belongs to T. A transversal is also sometimes called a *vertex cover* [10, p. 102] or an *edge-covering set* [11] of G.

*Research supported by the Academy of Finland under grant 210280.

Definition 1.1. Let r be a positive integer. A code $C \subseteq V$ is said to be r-*identifying* in G if for all $u, v \in V$ the set $I_r(C;u)$ is nonempty and

$$I_r(C;u) \neq I_r(C;v).$$

Let X and Y be subsets of V. The *symmetric difference* of X and Y is $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$. We say that the vertices u and v are r-*separated* by a code $C \subseteq V$ if the symmetric difference $I_r(C;u) \triangle I_r(C;v)$ is nonempty. The definition of r-identifying codes can now be reformulated as follows: $C \subseteq V$ is an r-identifying code in G if and only if for all $u, v \in V$ the vertex u is r-covered by a codeword of C and

$$I_r(C;u) \triangle I_r(C;v) \neq \emptyset.$$

The smallest cardinality of an r-identifying code in G is denoted by $M_r(G)$. An r-identifying code attaining the smallest cardinality is called *optimal*.

Codes which identify vertices in a graph were introduced by Karpovsky, Chakrabarty and Levitin in [5] for fault diagnosis in multiprocessor systems. For an application to sensor networks see [7]. Identifying codes in many different kinds of underlying graphs have been examined (see [6]). Among them are cycles and paths [1, 3, 8, 11]; see also [2, 4, 9].

Let n be an integer such that $n \geq 3$. A cycle $\mathcal{C}_n = (V_n, E_n)$ is a graph such that the set of vertices $V_n = \{v_i \mid i \in \mathbb{Z}_n\}$ and the set of edges

$$E_n = \{v_i v_{i+1} \mid i = 0, 1, \ldots, n-2\} \cup \{v_{n-1} v_0\}.$$

The exact values of $M_1(\mathcal{C}_n)$ and $M_2(\mathcal{C}_n)$ have been presented in [3] and [8], respectively. For general r, the following results are known:

- If n is even and $n \geq 2r+4$, then $M_r(\mathcal{C}_n) = n/2$. Moreover, we have $M_r(\mathcal{C}_{2r+2}) = 2r+1$. [1]
- If $n = 2r+3$, then $M_r(\mathcal{C}_n) = \lfloor 2n/3 \rfloor$ [3].
- If n is odd, $3r+2 \leq n \leq 8r+1$, $n \neq 4r+3$ and $\gcd(2r+1, n) = 1$, then $M_r(\mathcal{C}_n) = (n+1)/2$. Moreover, we have $M_r(\mathcal{C}_{4r+3}) = 2r+3$. [3]
- If n is odd, $n \geq 3r+2$ and $\gcd(2r+1, n) > 1$, then by [3]

$$M_r(\mathcal{C}_n) = \gcd(2r+1, n) \left\lceil \frac{n}{2\gcd(2r+1, n)} \right\rceil.$$

- Assume that n is odd, $n \geq 3r+2$ and $\gcd(2r+1, n) = 1$. If $n = 2m(2r+1) + 1$ or $n = (2m+1)(2r+1) + 2r$ for $m \geq 1$, then $M_r(\mathcal{C}_n) = (n+1)/2 + 1$, else $M_r(\mathcal{C}_n) = (n+1)/2$ [11].

In conclusion, what remains to be shown is the exact values of $M_r(\mathcal{C}_n)$ when n is odd and $2r+5 \leq n \leq 3r+1$. (Notice that there are no r-identifying codes in $\mathcal{C}_n$ when $n \leq 2r+1$.) These remaining cases are solved in Section 2.

Let n be a positive integer. For $n \geq 3$, a path $\mathcal{P}_n = (V_n, E'_n)$ is a graph such that the set of vertices V_n is the same as with the cycles and the set of edges $E'_n = E_n \setminus \{v_{n-1}v_0\}$. Furthermore, we define the path $\mathcal{P}_1 = (V_1, E'_1)$, where $E'_1 = \emptyset$, and the path $\mathcal{P}_2 = (V_2, E'_2)$, where $E'_2 = \{v_1 v_2\}$. The exact values of $M_1(\mathcal{P}_n)$ and $M_2(\mathcal{P}_n)$ have been presented, respectively, in [1] and [8]. An infinite family of optimal r-identifying codes have been introduced in [1, Theorem 5] giving the following values: $M_r(\mathcal{P}_{2k(2r+1)+1}) = k(2r+1)+1$, where k is a non-negative integer and $r \geq 2$. In Section 3, we solve the exact values of $M_r(\mathcal{P}_n)$ for general r.

2 Identifying codes in cycles

Let r be a positive integer. In this section, we first study r-identifying codes in cycles $\mathcal{C}_n = (V_n, E_n)$, where n is an odd integer and $2r+5 \leq n \leq 3r+1$. We end the section by a short discussion of identification, where modified balls are used instead of (the usual balls) $B_r(x)$ ($x \in V_n$). Throughout the section, the indices of the vertices v_i of $\mathcal{C}_n$ are calculated modulo n. Let t be a positive integer. For the following considerations, we define a graph $\mathcal{C}'_{(n,t)} = (V_n, F_n)$, where $F_n = \{v_i v_{i+t} \mid i \in \mathbb{Z}_n\}$. Notice that if C is an r-identifying code of $\mathcal{C}_n$, then C is also a transversal of $\mathcal{C}'_{(n,2r+1)}$ since the adjacent vertices v_i and v_{i+1} ($i \in \mathbb{Z}_n$) have to be r-separated by C. We also define $Q_t(i) = \{v_i, v_{i+1}, \ldots, v_{i+t-1}\}$.

The following lower bound on identifying codes in cycles $\mathcal{C}_n$ have been presented in [3, Theorem 1].

Theorem 2.1. *Let r be a positive integer and $n \geq 2r+2$. Then*

$$M_r(\mathcal{C}_n) \geq \gcd(2r+1, n) \left\lceil \frac{n}{2\gcd(2r+1, n)} \right\rceil.$$

Let n be an odd integer such that $2r+5 \leq n \leq 3r+1$. Then n can be written as follows: $n = 2r+1+p$, where p is an even integer such that $4 \leq p \leq r$. The following lemma provides a new way to characterize r-identifying codes in cycles with small order compared to r. Notice that in the following lemma for all $i \in \mathbb{Z}_n$ we have $V_n \setminus B_r(v_i) = Q_p(i+r+1)$, i.e. that the set $Q_p(i+r+1)$ denotes the complement of the ball $B_r(v_i)$.

Lemma 2.2. *Let r be a positive integer and $n = 2r+1+p$, where p is an even integer such that $4 \leq p \leq r$. Let T be a transversal of $\mathcal{C}'_{(n,2r+1)}$. If u and v are vertices of $\mathcal{C}_n$ such that $d(u,v) \leq p$, then u and v are r-separated by T. Moreover, the transversal T is an r-identifying code in $\mathcal{C}_n$ if and only if there does not exist $i, j \in \mathbb{Z}_n$ such that*

$$Q_p(i) \cap Q_p(j) = \emptyset \text{ and } T \cap (Q_p(i) \cup Q_p(j)) = \emptyset. \tag{1}$$

Proof. Let u and v be vertices of $\mathcal{C}_n$ such that $d(u,v) = d \leq p$. Without loss of generality, we may assume that $u = v_k$ and $v = v_{k+d}$ for some $k \in \mathbb{Z}_n$. Clearly, $v_{k-r} \in B_r(u) \setminus B_r(v)$ and $v_{k+r+1} \in B_r(v) \setminus B_r(u)$. Since T is a transversal of $\mathcal{C}'_{(n,2r+1)}$, then $v_{k-r} \in T$ or $v_{k+r+1} \in T$. Hence, the vertices u and v are r-separated by T.

Assume first that the transversal T is an r-identifying code in $\mathcal{C}_n$. Assume to the contrary that there exist $i, j \in \mathbb{Z}_n$ such that $Q_p(i) \cap Q_p(j) = \emptyset$ and $T \cap (Q_p(i) \cup Q_p(j)) = \emptyset$. Since $B_r(v_{i-r-1}) \triangle B_r(v_{j-r-1}) = Q_p(i) \cup Q_p(j)$, then $I_r(T; v_{i-r-1}) \triangle I_r(T; v_{j-r-1}) = \emptyset$ (a contradiction). Recall from the previous that $Q_p(i)$ and $Q_p(j)$ denote the complement of the balls $B_r(v_{i-r-1})$ and $B_r(v_{j-r-1})$, respectively. Therefore, the condition (1) holds.

Assume then that the condition (1) holds. Let $u = v_i$ ($i \in \mathbb{Z}_n$). Let us then show that v_i is r-covered by a vertex of T. Assume to the contrary that $I_r(T; v_i) = \emptyset$. Now $T \cap (Q_p(i-p) \cup Q_p(i)) \subseteq I_r(T; v_i)$ and $Q_p(i-p) \cap Q_p(i) = \emptyset$ (a contradiction). Hence, we have $I_r(T; u) \neq \emptyset$. In addition, the first part of the proof shows that vertices $u, v \in V_n$ such that $d(u, v) \leq p$ are r-separated by T. Let then $u \in V_n$ and $v \in V_n$ be vertices such that $d(u, v) > p$. Now we have $B_r(u) \triangle B_r(v) = Q_p(i) \cup Q_p(j)$ for some $i, j \in \mathbb{Z}_n$. Since $Q_p(i) \cap Q_p(j) = \emptyset$, we obtain by the condition (1) that $I_r(T; u) \triangle I_r(T; v) \neq \emptyset$. Thus, T is an r-identifying code in $\mathcal{C}_n$. $\square$

The following theorem provides exact values for $M_r(\mathcal{C}_n)$ when $2r + 5 \leq n \leq 3r + 1$ and $\gcd(2r + 1, n) = 1$.

Theorem 2.3. *Let r be a positive integer and $n = 2r+1+p$, where p is an even integer such that $4 \leq p \leq r$. Assume that $\gcd(2r + 1, n) = 1$. If $n = 2mp + 1$ or $n = (2m + 1)p + p - 1$ with $m \geq 2$, then $M_r(\mathcal{C}_n) = (n + 1)/2 + 1$, else $M_r(\mathcal{C}_n) = (n + 1)/2$.*

Proof. Recall first that $M_r(\mathcal{C}_n) \geq (n + 1)/2$, by Theorem 2.1. Since $\gcd(2r + 1, n) = 1$, the graph $\mathcal{C}'_{(n,2r+1)}$ is actually a cycle with n vertices. It is also easy to see that $\mathcal{C}'_{(n,2r+1)} = \mathcal{C}'_{(n,p)}$. As is stated earlier, in order for a code to be r-identifying in $\mathcal{C}_n$, it has to contain a transversal of $\mathcal{C}'_{(n,p)}$. Clearly, a code

$$T = \{v_{ip} \mid 0 \leq i \leq n - 1, i \text{ is even}\}$$

is a transversal of $\mathcal{C}'_{(n,p)}$. Furthermore, T is up to rotations the only transversal of $\mathcal{C}'_{(n,p)}$ with size $(n + 1)/2$.

Assume first that $n \leq 4p - 1$. Let us then show that there does not exist $i, j \in \mathbb{Z}_n$ such that $Q_p(i) \cap Q_p(j) = \emptyset$ and $T \cap (Q_p(i) \cup Q_p(j)) = \emptyset$. Assume to the contrary that such i and j exist. Since $T \cap Q_p(i) = \emptyset$ and T is a transversal of $\mathcal{C}'_{(n,p)}$, the sets $Q_p(i - p) \subseteq T$ and $Q_p(i + p) \subseteq T$. The fact that $n \geq |Q_p(i-p) \cup Q_p(i) \cup Q_p(i+p) \cup Q_p(j)| = 4p$ implies a contradiction. Therefore, by Lemma 2.2, T is an r-identifying code in $\mathcal{C}_n$. Hence, $M_r(\mathcal{C}_n) = (n + 1)/2$ when $n \leq 4p-1$. Notice that the cases $n = 2mp$ or $n = (2m+1)p$ are impossible since n is odd. Now the rest of the proof divides into the following four cases.

1) Assume then that $n = 2mp + x$ with $m \geq 2$ and $2 \leq x \leq p - 1$. Let us then show that $T \cap Q_p(i) \neq \emptyset$ for any $i \in \mathbb{Z}_n$. Assume to the contrary that $k \in \mathbb{Z}_n$ is such that $T \cap Q_p(k) = \emptyset$. Since $v_k \notin T$ and $v_{k+1} \notin T$, then $v_{k+p} \in T$ and $v_{k+p+1} \in T$. If the vertex v_{k+p} is such that $v_{k+p+ip} \neq v_0$ for any $i = 0, 1, \ldots, 2m$, then $v_{k+p+2mp} = v_{k+p-x} \in T$ (a contradiction). Otherwise, the vertex v_{k+p+1} is such that $v_{k+p+1+ip} \neq v_0$ for any $i = 0, 1, \ldots, 2m$. Then $v_{k+p+1+2mp} = v_{k+p+1-x} \in T$ (a contradiction). Thus, there does not exist $k \in \mathbb{Z}_n$ such that $T \cap Q_p(k) = \emptyset$. Hence, by Lemma 2.2, T is an r-identifying code in $\mathcal{C}_n$ and $M_r(\mathcal{C}_n) = (n + 1)/2$.

2) Assume now that $n = (2m + 1)p + x$, where $m \geq 2$ and $1 \leq x \leq p - 2$. Since $n = (2m + 2)p - (p - x)$, we can write $n = (2m + 2)p - x'$, where $2 \leq x' \leq p - 1$. In what follows, we show that $T \cap Q_p(i) \neq \emptyset$ for any $i \in \mathbb{Z}_n$. Assume to the contrary that $k \in \mathbb{Z}_n$ is such that $T \cap Q_p(k) = \emptyset$. Then, clearly, $v_{k-1} \in T$ and $v_{k-2} \in T$. If the vertex v_{k-1} is such that $v_{k-1+ip} \neq v_0$ for any $i = 0, 1, \ldots, 2m + 2$, then $v_{k-1+(2m+2)p} = v_{k-1+x'} \in T$ (a contradiction). Otherwise, the vertex v_{k-2} is such that $v_{k-2+ip} \neq v_0$ for any $i = 0, 1, \ldots, 2m+2$.

Then $v_{k-2+(2m+2)p} = v_{k-2+x'} \in T$ (a contradiction). Hence, by Lemma 2.2, T is an r-identifying code in $\mathcal{C}_n$ and $M_r(\mathcal{C}_n) = (n+1)/2$.

3) Consider then the case $n = 2mp+1$ with $m \geq 2$. It is easy to conclude that

$$T = \{v_0\} \cup \bigcup_{i=1}^{m} Q_p((2i-1)p+1).$$

Therefore, $V_n \setminus T = \bigcup_{i=0}^{m-1} Q_p(2ip+1)$. Thus, by Lemma 2.2, the transversal T is not an r-identifying code in $\mathcal{C}_n$. Since T is the unique transversal of $\mathcal{C}'_{(n,p)}$ with size $(n+1)/2$ and every r-identifying code of $\mathcal{C}_n$ is also a transversal of $\mathcal{C}'_{(n,p)}$, we have $M_r(\mathcal{C}_n) \geq (n+1)/2+1$.

Define first sets $A_k = \{v_{k+1}, v_{k+2}, \ldots, v_{k+p-2}, v_{k+p}, v_{k+2p-1}\}$, where k is an integer such that $0 \leq k \leq 2(m-1)p$. Define then a code

$$C_1 = \{v_0, v_{2mp}\} \cup \bigcup_{i=0}^{m-1} A_{2ip}.$$

It is straightforward to verify that C_1 is a transversal of $\mathcal{C}'_{(n,p)}$ and that $T \cap Q_p(i) \neq \emptyset$ for any $i \in \mathbb{Z}_n$. Hence, C_1 is an r-identifying code in $\mathcal{C}_n$. Since $|C_1| = (n+1)/2+1$, we have $M_r(\mathcal{C}_n) = (n+1)/2+1$.

4) Finally, assume that $n = (2m+1)p+p-1$ with $m \geq 2$. Now $T = \bigcup_{i=0}^{m} Q_p(2ip)$ and $V_n \setminus T = \bigcup_{i=0}^{m-1} Q_p((2i+1)p) \cup Q_{p-1}((2m+1)p)$. Then, using similar arguments as in the previous case, we have $M_r(\mathcal{C}_n) \geq (n+1)/2+1$. Define first sets $B_k = \{v_{k+p-3}, v_{k+p}, v_{k+p+1}, \ldots, v_{k+2p-4}, v_{k+2p-2}, v_{k+2p-1}\}$, where k is an integer such that $0 \leq k \leq 2(m-1)p$. Define also a set $B' = \{v_{(2m+1)p-3}, v_{(2m+1)p}, v_{(2m+1)p+1}, \ldots, v_{(2m+1)p+p-2}\}$. Define then a code

$$C_2 = \{v_0\} \cup B' \cup \bigcup_{i=0}^{m-1} B_{2ip}.$$

It is straightforward to verify that C_2 is a transversal of $\mathcal{C}'_{(n,p)}$ and that the set $C_2 \cap Q_p(i)$ is nonempty for any $i \in \mathbb{Z}_n$. Hence, C_2 is an r-identifying code in $\mathcal{C}_n$. Since $|C_2| = (n+1)/2+1$, we have $M_r(\mathcal{C}_n) = (n+1)/2+1$. $\square$

The following theorem provides exact values for $M_r(\mathcal{C}_n)$ when $2r+5 \leq n \leq 3r+1$ and $\gcd(2r+1,n) > 1$. The proof of the theorem is similar to the one of in [3, Theorem 9].

Theorem 2.4. *Let r be a positive integer and $n = 2r+1+p$, where p is an even integer such that $4 \leq p \leq r$. If $\gcd(2r+1,n) > 1$, then*

$$M_r(\mathcal{C}_n) = \gcd(2r+1,n) \left\lceil \frac{n}{2\gcd(2r+1,n)} \right\rceil.$$

Proof. Let $d = \gcd(2r+1,n) = \gcd(p,n)$ and $n' = n/d$. Notice that n' is odd and $d \geq 3$ since $2 \nmid n$. Recall that $\mathcal{C}'_{(n,2r+1)} = \mathcal{C}'_{(n,p)}$. The graph $\mathcal{C}'_{(n,p)}$ consists of the disjoint union of d cycles on n' vertices. For all $j \in \mathbb{Z}_d$ define the sets

$$T_j = \{v_{j+kp} \mid 0 \leq k \leq n'-1, k \text{ is even}\}$$

and

$$T'_j = \{v_j\} \cup \{v_{j+kp} \mid 0 \leq k \leq n'-1, k \text{ is odd}\}.$$

Since n' is odd, we have $|T_j| = |T'_j| = \lceil n'/2 \rceil$. Now define

$$T = T_0 \cup T'_1 \cup \bigcup_{j=2}^{d-1} T_j.$$

Since each T_j and T'_j is a transversal of one of the disjoint subcycles of $\mathcal{C}'_{(n,p)}$, which together form the whole $\mathcal{C}'_{(n,p)}$, the set T is a transversal of $\mathcal{C}'_{(n,p)}$. Furthermore, the number of vertices in T is equal to $\gcd(2r+1, n) \lceil n/(2\gcd(2r+1, n)) \rceil$.

Let us then show that there does not exist $i \in \mathbb{Z}_n$ such that $T \cap Q_p(i) = \emptyset$. Notice that $d \leq p$. Hence, there exists $k \in \mathbb{Z}_{n'}$ such that $\{v_{kp}, v_{kp+1}\} \subseteq Q_p(i)$ or $\{v_{kp+1}, v_{kp+2}\} \subseteq Q_p(i)$. Thus, by the construction of T, we have $T \cap Q_p(i) \neq \emptyset$ for any $i \in \mathbb{Z}_n$. Therefore, by Lemma 2.2, T is an r-identifying code in $\mathcal{C}_n$. Thus, the claim follows. $\square$

One fact that follows from the result of Theorem 2.1 is that an r-identifying code in a cycle $\mathcal{C}_n$ always has the size at least $n/2$ no matter what n or r are. A natural question is whether there exist identifying codes with less than $n/2$ codewords, when we change the ball slightly; we retain the assumptions that the new ball $B'_s(x)$ has the same size (equal to s) and shape for all $x \in \mathbb{Z}_n$.

It is essential to notice that now we have to be more careful with the definition of an I-set; namely, $I_s(x) = \{c \in C \mid x \in B'_s(c)\}$. Indeed, the fact that $x \in B'_s(c)$ if and only if $c \in B'_s(x)$ need not to be true anymore.

Since the size of the new ball is s, the bound [5, Theorem 2] implies that any identifying code has then at least $2n/(s+1)$ codewords in a cycle $\mathcal{C}_n$. Can this be reached for some $s > 1$ and n with $2n/(s+1) < n/2$? The answer is positive as pointed out in the next example.

It is easy to see that then the parameters are related as $s^2 + s \leq 2n$ (to attain the bound [5, Theorem 2] there must be $|C|$ vertices covered by a single codeword and all the others must be covered by exactly two codewords of C), and to obtain the best ratio of $|C|/n$, we could try $s^2 + s = 2n$.

Example 2.5. Let us consider the cycle $\mathcal{C}_{15}$. Indeed, choose $B'_5(x) = \{x-2, x-1, x, x+1, x+5\}$ for any $x \in \mathbb{Z}_{15}$. Notice that this new ball is very similar to the usual ball $B_2(x)$, only one element is different. By selecting $C = \{2, 5, 8, 11, 14\}$ it is easy to verify that this code is identifying with respect to the new ball. It clearly attains the bound [5, Theorem 2] and has the ratio $|C|/n = 1/3$ which is strictly less than $1/2$ (the best ratio that can be obtained by usual r-identifying codes).

Another example can be found in the cycle $\mathcal{C}_{28}$. Let $B'_7(x) = \{x, x+2, x+7, x+14, x+15, x+17, x+21\}$ for every $x \in \mathbb{Z}_{28}$. Now the code $C = \{0, 10, 11, 16, 19, 22, 27\}$ forms an identifying code attaining the bound [5, Theorem 2] and gives the ratio $|C|/n = 1/4 < 1/2$.

3 Identifying codes in paths

In this section, we study r-identifying codes in paths $\mathcal{P}_n = (V_n, E'_n)$. For the following considerations, we define a graph $\mathcal{P}'_{(n,t)} = (V_n, F'_n)$, where t is a positive integer and $F'_n = \{v_i v_{i+t} \mid 0 \leq i \leq n-t-1\}$. Define also sets $A_1(n) = \{v_{r+1}, v_{r+2}, \ldots, v_{2r}\}$ and $A_2(n) = \{v_{n-2r-1}, v_{n-2r}, \ldots, v_{n-r-2}\}$.

The following lemma characterizes identifying codes in paths.

Lemma 3.1. *Let r be a positive integer and $n \geq 2r+1$. A code $C \subseteq V_n$ is r-identifying in $\mathcal{P}_n$ if and only if the following conditions hold:*

(i) All vertices of V_n are r-covered by a codeword of C.

(ii) The code C is a transversal of $\mathcal{P}'_{(n,2r+1)}$.

(iii) The sets $A_1(n)$ and $A_2(n)$ are subsets of C.

Proof. Assume first that C is an r-identifying code in $\mathcal{P}_n$. By the definition, each vertex of V_n is r-covered by a codeword of C. For $i = r, r+1, \ldots, n-r-2$, the vertices $v_i \in V_n$ and $v_{i+1} \in V_n$ are r-separated by C. Therefore, C is a transversal of $\mathcal{P}'_{(n,2r+1)}$. For $i = 0, 1, \ldots, r-1$, we have $B_r(v_i) \triangle B_r(v_{i+1}) = \{v_{i+r+1}\}$. Hence, $A_1(n)$ is a subset of C. It can be proved similarly that $A_2(n) \subseteq C$.

Assume then that C is a code satisfying the conditions (i), (ii) and (iii). Let u and v be vertices of V_n. In order to prove that C is an r-identifying code in $\mathcal{P}_n$, it is enough to show that the vertices u and v are r-separated by C. Without loss of generality, we may assume that $B_r(u) \cap B_r(v)$ is nonempty and that $u = v_i$ and $v = v_j$ with $i < j$. If $0 \leq i \leq r-1$, then the codeword v_{i+r+1} belongs to $B_r(v_i) \triangle B_r(v_j)$. If $n-r \leq j \leq n-1$, then the codeword v_{j-r-1} belongs to $B_r(v_i) \triangle B_r(v_j)$. Therefore, we may assume that $r \leq i < j \leq n-r-1$. Now the vertices v_{i-r} and v_{i+r+1} belong to $B_r(v_i) \triangle B_r(v_j)$. Since C is a transversal of $\mathcal{P}'_{(n,2r+1)}$, then $v_{i-r} \in C$ or $v_{i+r+1} \in C$. Thus, u and v are r-separated by C. □

For any path $\mathcal{P}_n = (V_n, E'_n)$, define the following subsets of V_n:

$$K_1(\mathcal{P}_n) = \{v_i \mid 0 \leq i \leq n-1, i \text{ is even}\}$$

and

$$K_2(\mathcal{P}_n) = \{v_i \mid 0 \leq i \leq n-1, i \text{ is odd}\}.$$

The following lemma provides a lower bound on the size of a transversal of $\mathcal{P}_n$. The lemma is easy to prove.

Lemma 3.2. *Let n be a positive integer. If T is a transversal of $\mathcal{P}_n$, then*

$$|T| \geq \left\lfloor \frac{n}{2} \right\rfloor.$$

Moreover, if n is odd, then the unique transversal of $\mathcal{P}_n$ attaining the lower bound is $K_2(\mathcal{P}_n)$.

The following theorem provides exact values for $M_r(\mathcal{P}_n)$ when $n \geq 4r+3$.

Theorem 3.3. *Let r be a positive integer and $n = q(2r+1)+p$, where $q \geq 2$ and $1 \leq p \leq 2r+1$. Then we have the following results:*

(i) Assume that q is even. If $1 \leq p \leq r+1$, then $M_r(\mathcal{P}_n) = q(2r+1)/2 + p$, else $M_r(\mathcal{P}_n) = q(2r+1)/2 + p - 1$.

(ii) *Assume that q is odd. If $1 \leq p \leq 2r$, then $M_r(\mathcal{P}_n) = (q+1)(2r+1)/2$, else $M_r(\mathcal{P}_n) = (q+1)(2r+1)/2 + 1$.*

Proof. Let C be an r-identifying code in $\mathcal{P}_n$. For a lower bound on $|C|$, we first consider more closely the graph $\mathcal{P}'_{(n,2r+1)}$. Rename the vertices of V_n as follows:

$$w_k^{(j)} = v_{j+k(2r+1)},$$

where j and k are non-negative integers such that $0 \leq j \leq 2r$ and $0 \leq j + k(2r+1) \leq n-1$. For $j = 0, 1, \ldots, p-1$, define

$$W_j(n) = \{w_k^{(j)} \mid 0 \leq k \leq q\} \setminus (A_1(n) \cup A_2(n))$$

and, for $j = p, p+1, \ldots, 2r$, define

$$W_j(n) = \{w_k^{(j)} \mid 0 \leq k \leq q-1\} \setminus (A_1(n) \cup A_2(n)).$$

Let j be an integer such that $0 \leq j \leq 2r$. Define then a graph $\mathcal{S}_j(n) = (W_j(n), H_j(n))$, where the set of edges

$$H_j(n) = \{uv \in F'_n \mid u \in W_j(n), v \in W_j(n)\}.$$

In other words, $\mathcal{S}_j(n)$ is an induced subgraph of $\mathcal{P}'_{(n,2r+1)}$ determined by the vertex set $W_j(n)$. Since only the first or the last vertex of $\{w_k^{(j)} \mid 0 \leq k \leq q\}$ or $\{w_k^{(j)} \mid 0 \leq k \leq q-1\}$ can belong to $A_1(n) \cup A_2(n)$, the induced subgraph $\mathcal{S}_j(n)$ is actually a path.

By Lemma 3.1, the r-identifying code C is a transversal of $\mathcal{P}'_{(n,2r+1)}$. Therefore, $C \cap W_j(n)$ is a transversal of $\mathcal{S}_j(n)$. Since $\mathcal{S}_j(n)$ is a path, we have that $|C \cap W_j(n)| \geq \lfloor |W_j(n)|/2 \rfloor$ by Lemma 3.2. Since the pairwise intersections of the vertex sets $W_j(n)$ are empty, we have

$$|C| \geq |A_1(n)| + |A_2(n)| + \sum_{i=0}^{2r} \left\lfloor \frac{|W_i(n)|}{2} \right\rfloor = 2r + \sum_{i=0}^{2r} \left\lfloor \frac{|W_i(n)|}{2} \right\rfloor. \tag{2}$$

Thus, in order to provide a lower bound on $M_r(\mathcal{P}_n)$, we need to calculate the number of vertices in the sets $W_j(n)$.

Let $n = q(2r+1) + p$, where $q \geq 2$ and $1 \leq p \leq 2r+1$. Now we have the following two cases to consider.

1) Assume first that $1 \leq p \leq r+1$. By straightforward calculations, we now have the following results:

(a) For $i = 0, \ldots, p-1$, we have $W_i(n) = \{w_0^{(i)}, \ldots, w_q^{(i)}\}$ and $|W_i(n)| = q+1$.

(b) For $i = p, \ldots, r$, we have $W_i(n) = \{w_0^{(i)}, \ldots, w_{q-2}^{(i)}\}$ and $|W_i(n)| = q-1$.

(c) For $i = r+1, \ldots, p+r-1$, we have $W_i(n) = \{w_1^{(i)}, \ldots, w_{q-2}^{(i)}\}$ and $|W_i(n)| = q-2$.

(d) For $i = p+r, \ldots, 2r$, we have $W_i(n) = \{w_1^{(i)}, \ldots, w_{q-1}^{(i)}\}$ and $|W_i(n)| = q-1$.

PSfrag replacements

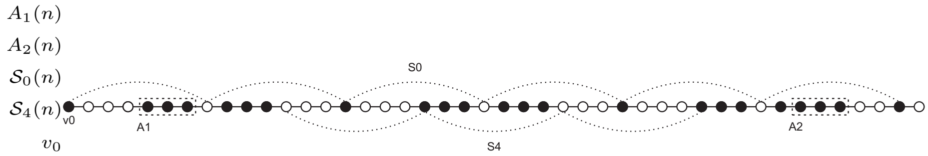

Figure 1: The code D_1 illustrated when $r = 3$, $q = 6$, $p = 2$ and $n = 44$. The black dots represent the codewords of D_1.

Notice that the cases (b) and (d) are empty when $p = r + 1$ and the case (c) is empty when $p = 1$. These facts do not affect the calculations of the equation (2). Notice also that Lemma 3.2 still applies when q is equal to 2 or 3, even though the lengths of the paths $\mathcal{S}_j(n)$ are equal to 0 or 1.

Assume then that q is even. By the equation (2) and the previous calculations, we have $|C| \geq q(2r+1)/2+p-1$. Assume that C attains this lower bound. Then the sets $C \cap W_i(n)$ are uniquely determined in the cases (a), (b) and (d), by Lemma 3.2. Therefore, since $W_i(n) \cap B_r(v_0) = \emptyset$ in the case (c), the vertex $v_0 \in V_n$ cannot be r-covered by a codeword of C. Hence, $|C| \geq q(2r+1)/2 + p$.

Let us then construct an r-identifying code in $\mathcal{P}_n$ attaining the lower bound. Define

$$D_1 = A_1(n) \cup A_2(n) \cup K_1(\mathcal{S}_0(n)) \cup \bigcup_{i=1}^{2r} K_2(\mathcal{S}_i(n)).$$

The code D_1 is illustrated in Figure 1 when $n = 44$ and $r = 3$. Clearly, the code D_1 satisfies the conditions (ii) and (iii) of Lemma 3.1. Therefore, it is enough to show that each vertex of V_n is r-covered by a codeword of D_1. By the definitions of $K_1(\mathcal{S}_0(n))$, $K_2(\mathcal{S}_{r+1}(n))$ and $K_2(\mathcal{S}_1(n))$, we know that $k(4r+2) \in D_1$, $k(4r+2)+r+1 \in D_1$ and $k(4r+2)+2r+2 \in D_1$, respectively, when k is an integer such that $1 \leq k \leq q/2-1$. Thus, each vertex $v_i \in V_n$ with $3r+2 \leq i \leq (q-2)(2r+1)+3r+2$ is r-covered by a codeword. Since $A_1(n)$ and $A_2(n)$ are subsets of D_1, we also obtain that $v_i \in V_n$ is r-covered by a codeword when $0 \leq i \leq 3r$ or $n-3r-1 \leq i \leq n-1$, respectively. Hence, we have shown that all the vertices of V_n except v_{3r+1} are r-covered by a codeword of D_1. Thus, since v_{3r+1} is r-covered by $v_{2r+2} \in K_2(\mathcal{S}_1(n)) \subseteq D_1$, the condition (i) of Lemma 3.1 is satisfied. Hence, D_1 is an r-identifying code in $\mathcal{P}_n$. Moreover, D_1 attains the lower bound. Hence, we have $M_r(\mathcal{P}_n) = q(2r+1)/2 + p$.

Assume now that q is odd. By the equation (2) and the previous results, we have $|C| \geq (q+1)(2r+1)/2$. The code D_1 again satisfies the conditions (ii) and (iii) of Lemma 3.1. By considering the set of codewords $K_1(\mathcal{S}_0(n))$, $K_2(\mathcal{S}_1(n))$ and $K_2(\mathcal{S}_{r+1}(n))$ as in the previous case, it can be shown that each vertex of V_n is r-covered by a codeword of D_1. Thus, D_1 is an r-identifying code in $\mathcal{P}_n$ and it attains the obtained lower bound. Hence, we have $M_r(\mathcal{P}_n) = (q+1)(2r+1)/2$.

2) Assume then that $r+2 \leq p \leq 2r+1$. By straightforward calculations, we have the following results:

(a) For $i = 0, \ldots, p-r-2$, we have $W_i(n) = \{w_0^{(i)}, \ldots, w_{q-1}^{(i)}\}$ and $|W_i(n)| = q$.

(b) For $i = p-r-1, \ldots, r$, we have $W_i(n) = \{w_0^{(i)}, \ldots, w_q^{(i)}\}$ and $|W_i(n)| = q+1$.

(c) For $i = r+1, \ldots, p-1$, we have $W_i(n) = \{w_1^{(i)}, \ldots, w_q^{(i)}\}$ and $|W_i(n)| = q$.

(d) For $i = p, \ldots, 2r$, we have $W_i(n) = \{w_1^{(i)}, \ldots, w_{q-2}^{(i)}\}$ and $|W_i(n)| = q-2$.

The fact that the case (d) is empty when $p = 2r+1$ does not affect the calculation of the equation (2).

Assume first that q is even. By the equation (2) and the previous results, we have $|C| \geq q(2r+1)/2 + p - 1$. Define

$$D_2 = A_1(n) \cup A_2(n) \cup \bigcup_{i=0}^{p-r-2} K_1(\mathcal{S}_i(n)) \cup \bigcup_{i=p-r-1}^{p-1} K_2(\mathcal{S}_i(n)) \cup \bigcup_{i=p}^{2r} K_1(\mathcal{S}_i(n)).$$

Clearly, the condition (ii) and (iii) of Lemma 3.1 are satisfied by D_2. Since $K_1(\mathcal{S}_0(n))$, $K_2(\mathcal{S}_{p-r-1}(n))$ and $K_2(\mathcal{S}_{r+1}(n))$ are subsets of D_2, it can be shown using similar arguments as before that $I_r(D_1; v_i) \neq \emptyset$ for each $i = 0, 1, \ldots, n-1$. Thus, D_2 is an r-identifying code and it attains the obtained lower bound. Hence, we have $M_r(\mathcal{P}_n) = q(2r+1)/2 + p - 1$.

Assume then that q is odd. Now we have $|C| \geq (q+1)(2r+1)/2$. Further, assume that $r + 2 \leq p \leq 2r$. Define

$$D_3 = A_1(n) \cup A_2(n) \cup \bigcup_{i=0}^{p-r-2} K_2(\mathcal{S}_i(n)) \cup K_1(\mathcal{S}_{p-r-1}(n)) \cup \bigcup_{i=p-r}^{2r} K_2(\mathcal{S}_i(n)).$$

Clearly, the conditions (ii) and (iii) of Lemma 3.1 are satisfied by D_3. Since $K_2(\mathcal{S}_0(n))$, $K_1(\mathcal{S}_{p-r-1}(n))$ and $K_2(\mathcal{S}_{p-r}(n))$ are subsets of D_3, it can be shown that $I_r(D_1; v_i) \neq \emptyset$ for each $i = 0, 1, \ldots, n-1$. Thus, D_3 is an r-identifying code attaining the lower bound. Therefore, we have $M_r(\mathcal{P}_n) = (q+1)(2r+1)/2$.

Finally, let q be odd and $p = 2r + 1$. Assume that the r-identifying code C attains the previously obtained lower bound, i.e. $|C| = (q+1)(2r+1)/2$. Then the sets $C \cap W_i(n)$ are uniquely determined in the cases (a) and (c), by Lemma 3.2. Since $p = 2r + 1$, the only graph contained in the case (b) is $\mathcal{S}_r(n)$ and the case (d) is empty. Hence, the only case that may contribute a codeword of C to the balls $B_r(v_0)$ and $B_r(v_{n-1})$ is the case (b). Since C attains the lower bound, we have $|C \cap W_r(n)| = |W_r(n)|/2$. Therefore, at least one of the sets $I_r(C; v_0)$ or $I_r(C; v_{n-1})$ is empty. Thus, $|C| \geq (q+1)(2r+1)/2 + 1$. Define then

$$D_4 = A_1(n) \cup A_2(n) \cup K_1(\mathcal{S}_0(n)) \cup \bigcup_{i=1}^{2r} K_2(\mathcal{S}_i(n)).$$

Clearly, the conditions (ii) and (iii) of Lemma 3.1 are satisfied by D_4. Since $K_1(\mathcal{S}_0(n))$, $K_2(\mathcal{S}_1(n))$ and $K_2(\mathcal{S}_{r+1}(n))$ are subsets of D_4, it can be shown that $I_r(D_1; v_i) \neq \emptyset$ for each $i = 0, 1, \ldots, n-1$. Thus, D_4 is an r-identifying code in $\mathcal{P}_n$ and it attains the obtained lower bound. Hence, we have $M_r(\mathcal{P}_n) = (q+1)(2r+1)/2 + 1$. □

Consider the r-identifying codes in $\mathcal{P}_n$ with $n \leq 4r+2$. Trivially, $M_r(\mathcal{P}_1) = 1$ for any positive integer r. If $2 \leq n \leq 2r$, then there are no r-identifying codes in $\mathcal{P}_n$. The following theorem provides exact values for $M_r(\mathcal{P}_n)$ when $2r + 1 \leq n \leq 4r + 2$.

Theorem 3.4. *Let r be a positive integer. Then we have $M_r(\mathcal{P}_{2r+1}) = 2r$ and $M_r(\mathcal{P}_{4r+2}) = 2r + 2$. If $2r + 2 \leq n \leq 4r + 1$, then $M_r(\mathcal{P}_n) = 2r + 1$.*

Proof. Let C be an r-identifying code in $\mathcal{P}_n$. Assume first that $n = 2r + 1$. By Lemma 3.1, we have $A_1(n) \cup A_2(n) \subseteq C$. Since $A_1(n) \cup A_2(n) = V_n \setminus \{v_r\}$, then $|C| \geq 2r$. Furthermore, it is easy to conclude that the set $A_1(n) \cup A_2(n)$ is actually an r-identifying code in $\mathcal{P}_n$. Therefore, we have $M_r(\mathcal{P}_{2r+1}) = 2r$.

Let then $n = 2r + 1 + p$, where $1 \leq p \leq r$. Now we have

$$A_1(n) \cup A_2(n) = \{v_p, v_{p+1}, \dots, v_{2r}\}.$$

Hence, $|A_1(n) \cup A_2(n)| = 2r - p + 1$. The set of edges of $\mathcal{P}'_{(n,2r+1)}$ is equal to $F'_n = \{v_0v_{2r+1}, v_1v_{2r+2}, \dots, v_{p-1}v_{2r+p}\}$. Therefore, by Lemmas 3.1 and 3.2, we have

$$|C| \geq |A_1(n) \cup A_2(n)| + |F'_n| = 2r + 1.$$

By Lemma 3.1, the code $A_1(n) \cup A_2(n) \cup \{v_0, v_1, \dots, v_{p-1}\}$ is r-identifying in $\mathcal{P}_n$ attaining the lower bound. Thus, we have $M_r(\mathcal{P}_n) = 2r + 1$.

Let now $n = 3r + 1 + p$, where $1 \leq p \leq r$. We have

$$A_1(n) \cup A_2(n) = \{v_{r+1}, v_{r+2}, \dots, v_{2r+p-1}\}.$$

Therefore, $|A_1(n) \cup A_2(n)| = r + p - 1$. For $i = p - 1, p, \dots, r$, we know that the edges v_iv_{i+2r+1} are such that $v_i \notin A_1(n)$ and $v_{i+2r+1} \notin A_2(n)$. Hence, by similar arguments as before, we have $|C| \geq |A_1(n) \cup A_2(n)| + (r - p + 2) = 2r + 1$. By Lemma 3.1, the set $A_1(n) \cup A_2(n) \cup \{v_p, v_{p+1}, \dots, v_r, v_{2r+p}\}$ is an r-identifying code in $\mathcal{P}_n$ attaining the obtained lower bound. Thus, we have $M_r(\mathcal{P}_n) = 2r+1$.

Finally, assume that $n = 4r + 2$. We have

$$A_1(n) \cup A_2(n) = \{v_{r+1}, v_{r+2}, \dots, v_{3r}\}.$$

Notice that the sets $B_r(v_0) \cap (A_1(n) \cup A_2(n))$ and $B_r(v_{4r+1}) \cap (A_1(n) \cup A_2(n))$ are empty. Hence, we have $|C| \geq |A_1(n) \cup A_2(n)| + 2 = 2r + 2$. On the other hand, the set $\{v_r, v_{3r+1}\} \cup A_1(n) \cup A_2(n)$ is an r-identifying code in $\mathcal{P}_n$ attaining the lower bound. Thus, we have $M_r(\mathcal{P}_{4r+2}) = 2r + 2$. □

It is obvious that a cycle $\mathcal{C}_n$ and a path $\mathcal{P}_n$ are closely related to each other. Indeed, the path $\mathcal{P}_n$ only misses the edge $v_{n-1}v_0$. Therefore, a natural question arising is whether there is a link between an r-identifying code in $\mathcal{C}_n$ and $\mathcal{P}_n$. The following theorem concentrates on this question.

Theorem 3.5. *Let $n \geq 4r + 2$. We have $M_r(\mathcal{P}_n) \geq M_r(\mathcal{C}_n) - 1$.*

Proof. Let C be an r-identifying code in a path $\mathcal{P}_n$ of the optimal size $M_r(\mathcal{P}_n)$. Join the ends v_0 and v_{n-1} of the path with an edge forming a cycle $\mathcal{C}_n$. Now consider the code C in the cycle; we obtain $I_r(x) \neq I_r(y)$ for any $x \neq y$ except $x = v_0$ and $y = v_{n-1}$. Indeed, any two vertices $x, y \in \{v_r, \dots, v_{n-r-1}\}$ have distinct I-sets; their balls are not affected by the new edge. Any vertex $x = v_i \in \{v_0, \dots, v_{r-1}\}$ is also distinguished from any $y = v_j$ as long as $i < j$ and $j \neq n - 1$ since $I_r(y)$ contains a codeword not belonging to $I_r(x)$. Therefore, by symmetry, $I_r(x) = I_r(y)$ implies that $x = v_0$ and $y = v_{n-1}$. These can be distinguished by adding (if necessary) one more codeword to v_r or v_{n-1-r} giving an r-identifying code of size at most $M_r(\mathcal{P}_n) + 1$ in a cycle. □

The bound of the previous theorem can be met (infinitely many times) with equality when n is odd and $\gcd(2r + 1, n) = 2r + 1$. However, we usually have $M_r(\mathcal{P}_n) > M_r(\mathcal{C}_n) - 1$.

Acknowledgements

We would like to thank the anonymous referee for valuable comments and suggestions.

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