Spontaneous focusing on quantitative relations as a predictor of rational number and algebra knowledge

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Abstract

Spontaneous Focusing On quantitative Relations (SFOR) has been found to predict the development of rational number conceptual knowledge in primary students. Additionally, rational number knowledge has been shown to be related to later algebra knowledge. However, it is not yet clear: (a) the relative consistency of SFOR across multiple measurement points, (b) how SFOR tendency and rational number knowledge are inter-related across multiple time points, and (c) if SFOR tendency also predicts algebra knowledge. A sample of 140 third to fifth graders were followed over a four-year period and completed measures of SFOR tendency, rational number conceptual knowledge, and algebra knowledge. Results revealed that the SFOR was relatively consistent over a one-year period, suggesting that SFOR is not entirely context-dependent, but a more generalizable tendency. SFOR tendency was in a reciprocal relation with rational number conceptual knowledge, each being uniquely predictive of the other over a four-year period. Finally, SFOR tendency predicted algebra knowledge three-years later, even after taking into account non-verbal intelligence and rational number knowledge. The results of the present study provide further evidence that individual differences in SFOR tendency may have an important role in the development of mathematical knowledge, including rational numbers and algebra.

Keywords: Spontaneous focusing on quantitative relations; rational numbers; algebra; middle school; fractions

Spontaneous focusing on quantitative relations in late primary school predicts rational number and algebra knowledge four years later

The world is full of opportunities for mathematical interpretations. However, there are often competing features which can be focused on in these situations and there can be a lack of clear signals as to when mathematical aspects are relevant. Instead, it is often dependent on the individual to recognize the potential mathematics that surrounds them in their everyday life (Gunderson & Levine, 2011; Reyna & Brainerd, 2007). The lack of signals of the mathematical aspects of everyday life stands in contrast to how mathematics are encountered in most formal educational settings. Recent evidence has revealed that there are substantial inter-individual differences in the tendency to notice the mathematics of situations that are not explicitly mathematical (e.g. Hannula & Lehtinen, 2005). These differences have been found to have a long-lasting relation with formal mathematical development and may have an influence on students' success with some of the most challenging components of mathematics learning in basic education (Hannula-Sormunen, Lehtinen, & Räsänen, 2015), including rational numbers (McMullen, Hannula-Sormunen, Laakkonen, & Lehtinen, 2016; Van Hoof et al., 2016).

There has been an increasing amount of study on inter-individual differences of two types of such tendencies – namely, Spontaneous Focusing On Numerosity (SFON) and Spontaneous Focusing On quantitative Relations (SFOR). SFON refers to a separate attentional process, whereby children spontaneously (i.e., self-initiated, not prompted by others) focus their attention on the exact number of a set of items or incidents and make use of this exact numerosity information in their action (Hannula-Sormunen, 2015; Hannula & Lehtinen, 2005; Hannula, Lepola, & Lehtinen, 2010). SFON tendency indicates the amount of self-initiated practice with exact enumeration a child engages in within her or his everyday surroundings (Hannula, Mattinen, & Lehtinen, 2005; Lehtinen, Hannula-Sormunen, McMullen, & Gruber, 2017; Mattinen, 2006). SFOR tendency is the spontaneous, (i.e. unguided) focusing of attention on quantitative relations and the use of these relations in non-explicitly mathematical situations. SFOR tendency involves focusing on the quantifiable relation(s) between of two or more (sub-)sets of items or quantities, rather than only focusing on the numerosity of separate items or quantifiable amounts (McMullen et al., 2016). In comparison, SFON refers to focusing on an exact number of objects (or incidences), such as *there are six apples in that bowl*. Thus far, SFOR has referred to focusing on the multiplicative relations between two or more quantities or sets of objects, such as *there are twice as many red apples as green apples in the bowl*.

Existing research has suggested that SFOR tendency may play an important role in the development of rational number knowledge (McMullen et al., 2016; McMullen, Hannula-Sormunen, & Lehtinen, 2014; Van Hoof et al., 2016). Rational number knowledge has been found to be linked to later success with algebra in secondary school (Siegler et al., 2012), and the relational features of rational numbers have been suggested to support algebra development (DeWolf, Bassok, & Holyoak, 2015). Given the interweaving relations between algebra, rational numbers, and SFOR tendency, and their common foundation in quantitative relational reasoning, in the present study we utilize a longitudinal path model to examine (a) the relative consistency of SFOR across multiple measurement points, (b) SFOR tendency's iterative relation with rational number knowledge, and (c) how SFOR tendency and rational number knowledge predict algebra knowledge three years later.

Relational thinking: The development of algebraic reasoning

Recent evidence provides a potential link between rational number knowledge and algebra knowledge (Booth, Newton, & Twiss-Garrity, 2014; DeWolf et al., 2015; Hurst & Cordes, 2017; Siegler et al., 2012), with success on rational numbers predictive of later success with algebra (Siegler et al., 2012). Within this discussion, the importance of relational reasoning in both rational numbers and algebraic concepts is highlighted (DeWolf et al., 2015), suggesting that this link is not entirely dependent on general achievement within mathematics.

Recent evidence has aimed to look for deeper links between rational number knowledge and algebra knowledge. Booth and colleagues (2014) investigated the link between students' numerical magnitude knowledge and their algebraic knowledge development. They found that it was fraction magnitude knowledge and not whole number knowledge that predicted success with learning about equation problem solving. However, this relation is not consistent across specific aspects of algebra knowledge and it is not clear why such inconsistencies exist. Furthermore, when looking at the role of different types of magnitude estimation, whole number knowledge is not collapsed across different number line endpoints (0-1,000,000 and 0-62,571), potentially diminishing the relation between whole number competences and algebra knowledge. More recently, Hurst and Cordes (2017) have found that rational number arithmetic knowledge may mediate the relation between rational number magnitude knowledge and algebra ability.

DeWolf and colleagues (2015) found that knowledge of the relational nature of fractions is a strong predictor of later rational number knowledge. As well, decimal magnitude understanding was also predictive of later algebra knowledge. One possible explanation provided by the authors is that these two variables separately measure the acts of (a) determining the magnitude of a fraction (using an analogous $\frac{a}{b} = a \div b$) and (b) representing it as a magnitude on the number line. These are described as two core features of reasoning about rational numbers. When included in the same regression, fraction number line estimation did not explain any unique variance over these two measures. The interesting possibility is that decimal magnitude estimation is a better indicator of magnitude mapping than fraction estimation (which confounds the two processes). However, even according to the authors, it still is not entirely clear just why these two components were better predictors of later algebra than fraction number line estimation alone.

In all, it is apparent that some form of rational number knowledge is crucial for coming to learn about algebra. Since quantitative relational reasoning may provide a key link between the two topics, it is possible that there are other potential mediators of the development of rational number and algebra knowledge. Those students with a higher SFOR tendency are more likely to recognize and use quantitative relations in various non-explicitly mathematical tasks (Lehtinen et al., 2017). This extra practice with mathematical relations may not only support rational number development (McMullen et al., 2016), but may also provide a benefit to students when learning about algebra.

SFOR Tendency

Previous research has shown that SFOR tendency is a distinct aspect of students' mathematical behavior, which is not entirely explained by other cognitive skills or knowledge (McMullen et al., 2014; 2016). Thus, it can be said that there are substantial individual differences in the degree to which students pay attention and use multiplicative relations in situations that are not explicitly mathematical. These individual differences are captured by measures of SFOR tendency. In multiple studies, it has been found that a substantial portion of variance in SFOR tendency is independent from the ability to recognize and describe quantitative relations on tasks in which students have been guided to pay attention the same quantitative relations as were in the SFOR tasks (McMullen et al., 2014, 2016).

Key to the validation of SFOR tendency is the distinction between SFOR tendency and other cognitive and non-cognitive constructs in research on cognitive and mathematical development. It is important to note that when referring to *spontaneous* in this context, we do not argue for the spontaneous development of the tendency, but rather refer to the spontaneous (i.e. unguided) recognition and use of quantitative relations in a mathematically unspecified situation. Thus, like SFON tendency (e.g. Hannula & Lehtinen, 2005; Hannula et al., 2005), it is expected that SFOR tendency can be influenced through direct intervention or increases in related mathematical knowledge. Many general cognitive concepts proposed in the developmental psychology and cognitive traditions are relevant in defining the nature of SFOR tendency. For example, Piaget used the term functional assimilation to refer children’s tendency to exercise existing schemes (Piaget, 1952), so that these existing schemes act as a lens to carve up aspects of particular contexts. Alexander (2017) notes the possibility that relational components of thought seem to be relevant at both an unconscious and deliberate levels of attention. Pascual-Leone and Johnson (2011) have described spontaneous attention which is effortless attention driven by the salience of novel perceptual experiences, which can be automatized habits that effortlessly call the direction of attention and are intrinsically self-motivating. On the other hand, there is rich research tradition showing the role of interest in triggering attention (Renninger & Hidi, 2016), suggesting that these processes may be more goal-directed than perceptually directed.

In fact, previous research has suggested that SFOR instances may be more effortful than the recognition of unique numerosities, potentially involving multiple goal-directed steps (McMullen, Hannula-Sormunen, & Lehtinen, 2011). Thus, an instance of providing a SFOR response on a task seems to take place across a series of intermediate quantitative recognition and reasoning stages forming a multistep process, unlike a SFON response which is more perceptual-conceptual recognition process primarily happening during the initial encoding of a situation (Hannula et al., 2009). This seems reasonable considering that a SFOR response will require attending to multiple quantities, identifying that there is a relation between these quantities, and determining what this relation is. Such an effortful process is not expected to be automatized, nor, given previous evidence of the primacy of number even when proportional relations are relevant (e.g. Jeong, Levine, Huttenlocher, 2007), is SFOR tendency aligned exclusively with intuitive reasoning, as described in dual-process theories (see a recent issue on inhibition in mathematics development, Van Dooren & Inglis, 2015). In fact, while in some cases, a SFOR response is in conflict with natural number features (e.g. matching amounts of bread, not the number of different sized pieces, McMullen et al., 2014) or other relational features (e.g. Degrande, Verschaffel, & Van Dooren, 2017), other SFOR tasks make it possible to use both exact number and relational aspects of a situation in the same response (McMullen et al., 2016; Van Hoof et al., 2016).

Previous research has indicated that SFOR tendency has been related to general mathematical achievement, arithmetic skills, non-verbal intelligence, and rational number knowledge (McMullen et al., 2014, 2016; Van Hoof et al., 2016). However, these other cognitive factors do not entirely explain individual differences in SFOR tendency. Yet, they do suggest that more success with formal mathematical skills may support a higher SFOR tendency. This is line with research which showed that SFON tendency was in a reciprocal relation with enumeration skills (Hannula & Lehtinen, 2005). Not only did SFON at the age of four years predict enumeration skills at the age of five years, but enumeration at the age of five subsequently predicted SFON tendency at the age of six. Until now, there has not been an investigation into whether SFOR tendency and rational number knowledge are also in a reciprocal relationship.

Alongside the *inter*-individual differences found in SFOR tendency, there is the expectation that their *intra*-individual differences in SFOR are not entirely explained by task context or formal mathematical development. SFOR tendency is expected to be a somewhat stable person-centered tendency. Thus, while relative to the affordances of different situations (Gibson, 1969) and affected by the development of formal mathematical knowledge and skills (Hannula & Lehtinen, 2005), a student's SFOR tendency is expected to still be relatively consistent across different measures in comparison to their peers. In this way, we expect that students' SFOR may vary across tasks, but the relative level of SFOR tendency should remain stable (i.e. there should be rank-order stability) within a sample. The generalization of behavior beyond specific task contexts allows for the distinction of SFOR as a legitimate tendency, which can then be extrapolated to more situations, potentially including everyday situations. For both SFON and SFOR, there is evidence of consistency across multiple task contexts, indicating that indeed such tendencies exist (e.g. Hannula & Lehtinen, 2005; McMullen et al., 2016). However, there is no evidence of the long-term consistency of SFOR measures, as has been found with SFON tendency (Hannula & Lehtinen, 2005).

That said, it is not expected that SFOR tendency would be a fixed trait that is impervious to instruction. Rather, in the current educational context, it is possible that the influencing factors on SFOR tendency reside both within and outside of the classroom. As most mathematical instruction in primary and secondary schools tends to constrain mathematical content to the formal mathematics within the designated times for math, many of the influences on SFOR tendency may come from socialization in family and peer-group contexts. While these influences are expected to be mostly stable across a one-year period, at least during late primary school, this does not preclude the possibility to enhance SFOR tendency through explicit instructional activities. Indeed preliminary evidence has suggested that it is possible to do so (McMullen et al., submitted).

SFOR tendency and mathematical development

SFOR tendency has been found to predict the development of rational number knowledge (McMullen et al., 2014, 2016; Van Hoof et al., 2016). Students, and even well-educated adults, face a great deal of difficultly in overcoming a natural number bias when learning about fractions and decimals (Merenluoto & Lehtinen, 2004; Ni & Zhou, 2005; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Vamvakoussi, Christou, Mertens, & Van Dooren, 2011; Van Hoof, Janssen, Verschaffel, & Van Dooren, 2014). Late primary school students with a higher SFOR tendency improved more with their reasoning about rational numbers, becoming more effective in overcoming their natural number bias regarding the size, density, and operations of rational numbers

(McMullen et al., 2016; Van Hoof et al., 2016). There is also limited evidence to suggest that early SFOR tendency at Grade 1 predicts fraction knowledge at Grade 4, when fractions are first introduced (McMullen et al., 2014).

Richland and colleagues (2017) suggest that explicit cues may be necessary for supporting relational reasoning in all students when teaching mathematical topics that are negatively affected by students‘ misconceptions. They argue that previous research has shown that only high-performers benefit from relational comparisons without explicit cues, while when misconceptions are explicitly highlighted all students benefit. As well, learners own noticing, or not, of relevant features to complex mathematical situations may explain some individual differences in performance on such tasks (Lobato, Rhodehamel, & Hohensee, 2012) These results suggest that individual differences in noticing of key relational features of complex situations, whether explicitly mathematical or not, may have an impact on conceptual development, especially with processes of conceptual change needed in coming to understand rational numbers (e.g. Stafylidou & Vosniadou, 2004) and algebra (e.g. Booth et al., 2015).

The proposed relation between SFOR and mathematical development is thought to be produced by increased self-initiated practice with multiplicative relations (cf. deliberate practice: Ericsson, 2006). The notion being that those with higher SFOR tendencies recognize opportunities to use their newly acquired mathematical skills or knowledge more often than their peers in their day-to-day activities outside of the math classroom, much like budding experts have been found to do (Ericsson & Lehmann, 1996). They then practice these newly acquired skills or knowledge more frequently and in a more advantageous way (by recognizing a more mathematically appropriate aspect of a situation), leading to better outcomes in formal mathematical attainment. Just as SFON tendency has been found to not only contribute to enumeration skills, but also support the transition to arithmetic skills (Hannula et al., 2010), we aim in the present study to examine how SFOR tendency may also contribute to the development of algebra knowledge, which has been shown to be related to prior rational number knowledge (Siegler et al., 2012).

The Present Study

Our aims in the present study are to further fill in some of the above described gaps in the existing literature on SFOR tendency and the development of mathematical knowledge. First, the consistency of the SFOR measures included in the present study will be examined. As well, the reciprocal nature of the relation between SFOR tendency and rational number knowledge will be examined. Finally, the relation between SFOR tendency and later rational number and algebra knowledge will be examined. To this end the following research questions will be addressed by testing a hypothesized path model (Figure 1) :

What is the relative consistency SFOR across multiple time points? SFOR tendency has been previously found to be consistent across multiple task contexts (McMullen et al., 2014, 2016). However, there is no existing evidence of its long-term consistency across time and task context, which would indicate that it is a more general tendency not entirely dependent on task context or short-term effects. There is evidence of the relative consistency of SFON measures across a two-year period (Hannula & Lehtinen, 2005), suggesting that there may be something similar for SFOR tendency. Thus, we expect that SFOR will be relatively consistent across measurement time points (Hypothesis 1a). Relatedly it is expected that, even after taking into account rational number knowledge (Path 1b) and non-verbal intelligence (Path 1c), SFOR tendency at the first time point will predict SFOR tendency one year later (Figure 1, Hypothesis 1b; Path 1a),.

What is the developmental relation between SFOR tendency and rational number knowledge? Previously, SFON tendency and early numeracy were found to be in a reciprocal relation during development; SFON tendency predicted later numeracy, which in turn predicted later SFON tendency (Hannula & Lehtinen, 2005). SFOR tendency has been found to predict the development of rational number knowledge over a year-and-a-half period (McMullen et al., 2016). It is expected that, even after taking into account prior rational number knowledge (Paths 2a–2c) and non-verbal intelligence (Paths 2d–2e), SFOR tendency and rational number knowledge are expected to have a reciprocal relation (Hypothesis 2; Paths 2f–2i).

Does SFOR tendency predict algebra knowledge? Given previous evidence of its relation with mathematical development, it is relevant to further consider the role SFOR tendency plays in mathematical development. Relational reasoning and rational number knowledge have been found to be important for the development of algebra knowledge (e.g. DeWolf et al., 2015). It is possible then that SFOR tendency may also be related to later algebra knowledge. Thus, we expect to find that SFOR tendency is a predictor of algebra knowledge in lower secondary school (3a), even after taking into account prior rational number knowledge and non-verbal intelligence (Hypothesis 3; 3b-3c).

Methods

Participants

Students from two schools in Finland (N = 140, 74 female) took part in the study, which was carried out over a four year period. At the first time point, participants were in either Third (n = 49), Fourth (n = 45), or Fifth (n = 46) grade. The sample was representative of the urban Finnish population, including students from diverse ethnic backgrounds (roughly 84% of participants had a Finnish background and 16% non-Finnish background, primarily from the Middle East, North Africa, Russia, and Southeast Asia). All participants had parental permission to participate in the study and ethical board approval was granted for this study. Participants were taken from a larger sample of students who participated in a longitudinal study covering the first three time points. Those included in the present study did not differ from the sample as a whole in knowledge of rational numbers at any of the first three time points, nor on SFOR tendency, guided focusing on quantitative relations, or non-verbal intelligence, all Fs < 3.79, ps > .05.

Procedure

All tests were completed in the students' normal classrooms using paper-and-pencil instruments, which were read aloud and projected on a screen at the front of the classroom. At the first time point, prior to their rational number instruction for the year (February 2012), students completed measures of rational number conceptual knowledge, SFOR tendency, and guided focusing on multiplicative relations. At the second time point three months later (May 2012), after their rational number instruction was done for the year, students completed measures of rational number knowledge and non-verbal intelligence. The next year, at the third time point, prior to their rational number instruction (January 2013), students completed measures of their SFOR tendency and rational number knowledge. Three years later, at the final time point (May 2016), students completed measures of their rational number and algebra knowledge. SFOR tendency tests were given prior to any mathematical tasks, and no mention of the mathematical nature of the tasks was made prior to completion of these tasks. At the third time point (January 2013), a new tester who the students could not associate with the previous year's tests administered the SFOR task.

Measures

SFOR tasks Participants completed three measures of SFOR tendency across the first and third time points. SFOR tasks were picture description tasks (McMullen et al., 2016; Van Hoof et al., 2016), in which the students were asked to describe either how objects had changed during a transformation (Teleportation task), what was on a plate of food (Plate task), or how purchased items were different from what was intended to be bought (Shopping task). Two other SFOR tasks were discarded because of an extremely low number of SFOR responses on these tasks. In all SFOR tasks, there is no mentioning of mathematical or quantitative aspects, neither prior to nor during testing (see Hannula-Sormunen, 2015 for more methodological considerations). SFOR tasks are always the first activities the students' complete and they are not presented to students during their normal mathematics class. Pilot testing of all SFOR measures was made prior to large-scale testing, which indicated that the measures were able to capture students' spontaneous activities with multiple interpretations being possible for describing the situations presented in the tasks. In this way, any time a student responds by using or describing the multiplicative relations within the situation, he/she must have payed attention to the relations spontaneously, without explicit guidance. Since the recognition and use of the multiplicative relations happened without any guidance, it is considered a SFOR response (McMullen et al., 2016).

The Teleportation and Plate tasks were presented to participants at the first time point. The teleportation task involved three sets of objects which were teleported to distant space colonies. During teleportation the objects changed color (e.g. shades of red to blue), quality (elongated or different food type), and number of items (e.g. multiplied by three). The participants were first asked to "describe in as many ways as possible" how the objects had changed, and on a subsequent trial were asked to draw what they expected to arrive based on what happened the time before (the initial number of items was adjusted from the previous trial). The plate task asked participant to describe a meal in "as many ways as possible"; plates were filled with food that was arranged in a proportional manner (e.g. half of the area of the plate consisted of salad). Altogether, there were four teleportation and four plate task items, for a total of six writing items and two drawing items. As the reliability across all eight items for these two tasks was good (α = .81), a sum score of all items was used in further analysis.

The Shopping task was presented to participants at the third time point. As can be seen in Figure 2, the shopping task took a similar form as the Teleportation task, as both tasks elicit a comparison of three sets of discrete objects with systemic differences, including a multiplicative relation, and systematic form and color changes. In this case, a woman was planning a trip to the store and had a set of items she wished to purchase. However, she ended up buying a different set of items, which had uniformly changed in item type (e.g. balls instead of books), color (shades of blue to shade of red), and number (divided by 3). Once again, participants were asked to first, "describe in as many ways as possible" how the items were different from what she planned to purchase, and then draw what they expected her to purchase from a similar set of items based on what happened the time before. There were four Shopping items, two writing items and two drawing items. For the Shopping tasks, the four items showed good reliability (α = .78), and were therefore used in further analysis as a sum score

Participants' responses to the SFOR items were scored for each item independent of the other items, based on the precision with which they described the quantitative relations present in the task (e.g. the multiplicative relation between sets of items, the fractional relation between food item and whole plate). Two points were given for a description that included exact proportional or multiplicative relations (e.g. "Half of the plate is salad", "She bought three times less than she planned to") or drawing the correct number of items based on the multiplicative relation (e.g. three times less items). One point was given for non-exact, but mathematically specific, relational descriptions (“They multiplied”), describing an incorrect relation, drawing two of three items correctly, or drawing based on a multiplicative, but incorrect, relation (e.g. all items divided by 2, not three). All other responses, including non-specific quantitative relations (e.g. “more”, “less”) were given zero points.

Rational number conceptual knowledge At all time points, students completed measures of their knowledge of rational numbers concepts (Martinie, 2007; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2010), which had been either explicitly or implicitly covered in their previous instruction. At the first three time points, items examined students’ conceptual knowledge of fraction and decimal size and the density of the set of rational numbers. Items included multiple choice and short-answer questions in which participants were asked (1) to name which fraction or decimal was bigger (e.g. Circle the larger number: 0.36 or 0.5), (2) order fractions and decimals by size (“Put the numbers in order from smallest to largest”: 6/8; 2/2; 1/3) or (3) provide a response on the density of the set of fractions and decimals (e.g. Are there any fractions between 3/5 and 4/5? If so, how many?). Participants were given 1 point for a correct answer on the comparison and ordering items. Density items were scored from 0-2, with 2 points given for responses that included the concept of an infinitely dense number line and 1 point given for responses that note a large number of numbers existing between the two given numbers. The maximum score for the whole test was 28 points.

Additional items of students fraction magnitude estimation were included at the second time point. Students were asked to place a fraction at the correct location on a number line that had endpoints of 0 and 1 (items: ¼ and 3/5) or 0 and 5 (items: 2 ½ and 6/5). Since this was a specific test of their concept of rational number size, students were scored as being correct if they were within ±10% of the correct location (Rittle-Johnson, Siegler, & Alibali, 2001).

At the last time point (May 2016), the rational number test examined students’ conceptual knowledge of the size and density of rational numbers, along with their conceptual knowledge of rational number representations. Items were again multiple choice and short answer in which participants were asked to (1) order fractions and/or decimals by size (e.g. “Put numbers in order from smallest to largest: 0.5; $\frac{1}{4}$; $\frac{5}{100}$; 0.356.”), (2) describe the density of the set of rational numbers with both open ended and multiple-choice questions (response options taken from Vamvakoussi & Vosniadou, 2010), (3) explain if and how whole numbers, decimals, and fractions were related (e.g. “Can all numbers be written as a fraction? Why or why not?”; “Are there any other ways to exactly write 0.3? If so, give some examples.”). There were 22 fraction and/or decimal comparisons that were incongruent with natural number features embedded in the ordering items, which were scored as correct or incorrect. The four fraction density items and four decimal density items were scored as 0 points for stating/choosing that there are no numbers between the two fractions or decimals, 1 point for stating/choosing there were a limited number of numbers, 2 points for stating/choosing that there were an infinite number of fractions or decimals, and 3 points for stating/choosing that there were an infinite number of numbers including fractions and decimals. The five representation items were scored as 0 points for stating that there was no relation between whole numbers, fractions, and decimals, 1 point for a limited understanding of the relation between whole numbers, fractions, and decimals (e.g. all numbers can be written as fractions), and 2 points for mathematically correct understanding of the relation between whole numbers, fractions, and decimals (e.g. not all numbers can be written as fractions).

Algebra knowledge At the last time point students completed a measure of their procedural and conceptual knowledge of algebra. Items were multiple choice and chosen from a previously established instrument covering these two types of knowledge (Star et al., 2015). This instrument was chosen as it is relatively curriculum neutral, covering a wide range of topics in algebra. As well, the test did not include any items with fractions or decimals in the given problems. There were altogether six items covering procedural knowledge with algebra (e.g. solving one- or two-step equations) and seven items covering conceptual knowledge of algebra (e.g. determining the slope of a linear equation). Each item was scored as correct (1 point) or incorrect (0 points).

Guided focusing on quantitative relations In order to examine students’ ability to reason about the mathematical aspects imbedded in the SFOR tasks at the first time point (for more information on the use of guided measures in spontaneous focusing tendencies in general, see Hannula & Lehtinen, 2005), after completing the Teleportation and Plate SFOR tasks, students were asked to repeat one item from each task with new instructions to specifically describe the multiplicative relations within the image (for more information on these guided measures with SFOR tendency specifically, see McMullen et al., 2016). Students were given one point for correctly describing the multiplicative relation (e.g. there were twice as many items as before, half the plate is salad, etc.) for each item.

Non-verbal intelligence. At the second time point, Raven’s Colored Progressive Matrices (Raven, 1976) were used as a measure of non-verbal intelligence. This measure was particular useful due to its similarity in form to the SFOR teleportation and shopping tasks (both involving spatial patterns), thus adding a more stringent measure of confirmation that SFOR tendency (and its relation to mathematical development) is not entirely dependent on non-verbal intelligence. In total, 20 items were used: two items from Set B (B1 and B2), the whole of Set C (C1 – C12), and six items from Set D (D1 – D6). Items were scored as 1 point for a correct answer or 0 points for incorrect answers, with the maximum score of 20.

Statistical Analysis

First, average measure intra-class correlation (ICC) was used in order to examine the rank-order stability of SFOR across the multiple time points (Berchtold, 2016); ICC values can be interpreted similarly than Cronbach’s alpha, with values closer to one representing better consistency across items. In order to investigate the relations between algebra knowledge, rational number knowledge, and SFOR tendency a path model was estimated using Mplus 7.2 (Muthén & Muthén, 1998 - 2016). Path modeling provides a useful tool to examine the inter-relations among variables across multiple time points (Cirino, Tolar, Fuchs, & Huston-Warren, 2016), and is particularly useful for our purposes of examining cross-lag relations between SFOR tendency and rational number knowledge. Maximum likelihood robust estimation, which is a full information approach, was employed to account for the missing-at-random data (of the 140 students included in the model, there were 19 students missing at time 1, 21 students missing at Time 2, and 22 students missing at Time 3, due to being absent on the day of testing or having moved to another school). In order to account for the repeated measures used in the model (Freedman, 1987), the error structure of the model was defined by constraining the error terms for rational number knowledge in May 2012 and January 2013 to be equal (Cole & Maxwell, 2003). Chi-square test and fit indices were used to determine the goodness of fit for the model. The comparative fit index (CFI) and Tucker-Lewis Index (TLI) threshold was set at > .95 and the standardized root mean square residual (SRMR) at < .08 (Byrne, 2012; Hu & Bentler, 1999).

Results

In order to confirm that performance on the SFOR tasks was not entirely dependent on the skill needed to solve the tasks using quantitative relations (McMullen et al., 2014), performance on the SFOR and guided Teleportation and Plate tasks is examined. Results indicate that only 7.4% of the sample were unable to use quantitative relations on either the spontaneous or guided versions of the tasks. This proportion decreases to only 5% of the sample when including the Shopping task from January 2013. Thus, it can be said that the SFOR tasks fell within the competence range of the participants in terms of the ability to recognize and describe the multiplicative relations in the task, when guided to do so. This suggests that variation in SFOR responses are not entirely dependent on the requisite skills needed to solve the task and rather are reflections of students tendency to spontaneously (i.e. without guidance) recognize and use multiplicative relations in situations that are not-explicitly mathematical and can therefore be said to measure SFOR tendency (Hannula-Sorumnen, 2015; McMullen et al., 2016).

In order to examine the consistency of SFOR across the one-year period, the reliability of all 12 SFOR items from the Teleportation, Plate, and Shopping task was calculated. The resulting ICC was .87; such a value indicates that, across the one-year period and three tasks, these SFOR measures were relatively consistent (Nunnally, 1978), confirming Hypothesis 1a.

Table 1 provides descriptive statistics for all variables. Overall, the multivariate measures that will be used in subsequent analysis appear to be fairly reliable as sum variables, with Cronbach’s alphas above .70. As well, all variables show a normal distribution, based on Skewness and Kurtosis values, allowing for the use of standard parametric analyses and conforming to the assumptions of path modelling (Freedman, 1987).

Table 1
Descriptive statistics for each variable across the whole sample.

	Mean	SD	Skewness	Kurtosis	Range	Reliability (Cronbach's α)
February 2012						
SFOR – Teleportation and Plate	4.03	4.04	1.05	.42	0-16	.81
Rational number test	7.89	6.00	.57	-.74	0-23	.91
May 2012						
Rational number test	10.66	7.08	.51	-.60	1-32	.90
Non-verbal intelligence	14.36	3.50	-.88	.47	4-20	.78
January 2013						
Rational number test	10.17	6.65	.32	-.79	0-28	.92
SFOR – Shopping	1.64	2.28	1.48	1.22	0-8	.78
May 2016						
Rational number test	24.56	11.05	.63	-.44	6-52	.89
Algebra knowledge	5.05	3.00	.64	-.18	0-13	.70

Table 2 describes the relations between all indicator variables. In general, there were moderate to strong relations between all variables. The patterns of relations between the variables suggest that there will be significant longitudinal connections across the different measures. In order to test this more explicitly, a longitudinal path model was run. Statistical results indicate an acceptable fit of the hypothesized model $\chi^2(10) = 17.21$, $p = .07$; CFI = .99; TLI = .97; SRMR = .02. The resulting significant coefficients can be found in Figure 3.

Table 2
Parametric correlations between all variables (n = 140)

	1	2	3	4	5	6	7
January 2012							
1. SFOR – Teleportation and Plate	-						
2. Rational number test	.55	-					
May 2012							
3. Rational number test	.58	.88	-				
4. Non-verbal intelligence	.49	.42	.41	-			
January 2013							
5. Rational number test	.58	.84	.88	.52	-		
6. SFOR – Shopping	.66	.52	.61	.41	.57	-	
May 2016							
7. Rational number test	.57	.60	.64	.42	.70	.49	-
8. Algebra knowledge	.40	.54	.56	.40	.60	.49	.59

Note: All values significant at $p < .001$

As evidence by the estimated model, all or parts of each of the three hypotheses were confirmed. First, SFOR tendency at the first time point tendency was found to predict SFOR tendency one year later (Path 1a: $\beta = .43$, $p < .001$), even after taking into account non-verbal intelligence and rational number knowledge (Paths 1b and 1c). In general, about half of the variance of SFOR tendency in January 2013 was explained by these factors ($R^2 = .48$).

Second, SFOR tendency and rational number knowledge were found to be a reciprocally related across the four time points, with SFOR at the first time point predicting rational number knowledge at the second time point (Path 2g: $\beta = .14$, $p < .01$), which predicted SFOR a year later (Path 2h: $\beta = .31$, $p < .001$), which predicted rational number knowledge three years later (Path 2i: $\beta = .16$, $p < .05$). These relations hold even after controlling for prior knowledge of rational numbers, which was found to strongly predict later rational number knowledge at each time point (Paths 2a-2c: βs > .57, ps < .001), and non-verbal intelligence, which only was a significant predictor of rational number knowledge between May 2012 and January 2013 (Path 2d: $\beta = .15$, $p < .01$), and did not predict rational number knowledge four years later in May 2016 (Path 2e: $p > .05$). Overall, a large amount of variance in rational number knowledge was explained in the model, especially in May 2012 and January 2013, with over three-quarters of the variance in students' performance on these test explained (mostly by prior knowledge). Given the three year gap, a substantial amount of variance in performance on the rational number test in May 2016 was still explained by prior knowledge and SFOR tendency ($R^2 = .49$).

Finally, SFOR tendency was found to predict algebra knowledge three years later (Path 3a: $\beta = .24$, $p < .01$), even after taking into account prior rational number knowledge (Path 3b: $\beta = .44$, $p < .001$) and non-verbal intelligence (Path 3c: $p > .05$). Roughly two-fifths of variance in algebra knowledge ($R^2 = .39$) could be explained by prior SFOR tendency and rational number knowledge. Again, however, the large gap in time, and also the distance between topics, may account for the decrease in explained variance in comparison with the other outcome measures included in the model.

Discussion

The present study provides new evidence that SFOR tendency is an indicator of future mathematical performance in a number of topics within mathematical development. First, the present study indicates that SFOR was relatively consistent over a one-year period, suggesting that the measures are indicative of a more general tendency that is not entirely explained by task-specific behavior (Hypothesis 1a). As well, even after taking into account rational number knowledge and non-verbal intelligence, SFOR tendency predicted later SFOR tendency (Hypothesis 1b). Furthermore, like in previous students (McMullen et al., 2016; Van Hoof et al., 2016), SFOR tendency was found to be related to later rational number knowledge, even after taking into account prior knowledge and other factors. The present study extends previous evidence that SFOR tendency predicts later rational number knowledge in showing that SFOR tendency and rational number knowledge were related in a reciprocal manner (Paths 2f-2i), with each predicting later levels of the other (Hypothesis 2). Furthermore, the present study indicates that this was the case throughout late primary school and even into lower secondary school over a 4.5 year period. The present study is also the first to suggest that SFOR tendency may also be a predictor of algebra knowledge over this same time period (Hypothesis 3).

Our expectation was that SFOR tendency would be more of a consistent personal tendency, rather than being entirely dependent on specific tasks or context (cf. Geary, 2015). In previous studies, SFOR measures were relatively consistent across multiple task-contexts (McMullen et al., 2013; 2014; 2016). The present study provides the first evidence of the longitudinal consistency of SFOR measures, across both verbal and non-verbal task contexts and time. These results suggest that like SFON tendency (e.g. Hannula & Lehtinen, 2005), SFOR tendency is not entirely explained by task context nor the development of formal mathematical knowledge and skills. In addition, non-verbal intelligence did not predict later SFOR tendency after taking into account prior SFOR tendency and rational number knowledge. While the present study does contribute to our understanding of SFOR as a more person-centered tendency, further investigation is needed in order to better understand the developmental predictors of SFOR tendency itself. Future studies should include more possible explanations for differences in SFOR tendency, along with more distinct task contexts across time. In particular, it would be important to examine how spontaneous mathematical focusing tendencies (e.g. SFON, SFOR with additive relations) are related to each other both concurrently and across time.

The hypothesized mechanism by which SFOR tendency and rational number knowledge interact with each other is through students' own mathematical activities both in and out of school contexts. Despite early skills with proportional reasoning (Boyer, Levine, & Huttenlocher, 2008; Spinillo & Bryant, 1999), students face serious difficulties with coming to understand many concepts of rational numbers (Booth & Newton, 2012; Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2010). Those students with a higher SFOR tendency may gain more experience with mathematical relations that exist and are meaningful features of their environments, as they may be more likely to spontaneously notice the mathematical features that exists around them. A higher SFOR tendency is therefore hypothesized to account for an increase in students' self-initiated practice with these mathematical relations. This increase in the quantity and quality of practice with mathematical relations then would lead to more improvement in understanding rational numbers.

The present study indicates that the relation between SFOR tendency and relational reasoning in mathematics development is not necessarily unidirectional. The noticing of opportunities to practice newly acquired knowledge mirrors activities of future experts (Ericsson & Lehmann, 1996). Thus, as new skills with rational numbers and relational reasoning increase, students are able to see more opportunities to reason using quantitative relations (Lindahl & Samuelsson, 2002), as indexed by an increase in SFOR tendency. This virtuous cycle may explain why SFOR tendency, and SFON tendency elsewhere (e.g. Hannula-Sormunen et al., 2015), are regularly found to be strong predictors of mathematical skills and knowledge many years later. While the present study provides even stronger evidence for a relation between rational number knowledge and SFOR tendency, a randomized-control experiment aimed at enhancing SFOR tendency is necessary to determine if there is truly a causal relation between SFOR tendency and rational number knowledge.

With rational numbers, the expectation is that a higher SFOR tendency helps students overcome their everyday intuitive reasoning about number when faced with rational numbers that are in conflict with their natural number features (e.g. Van Hoof et al., 2014). Faced with more opportunities to experience the limitations of natural numbers (e.g. sharing 3 cookies among 2 friends), those students with a higher SFOR tendency may be better aware of these limitations when learning about fractions and decimals in the classroom. Likewise, those students who have a better understanding of the nature of rational numbers may also be better prepared to recognize opportunities to use them outside of formal mathematical situations (Lobato et al., 2012; Ohlsson & Lehtinen, 1997) . The multiplicative relational nature of many everyday situations may also then provide opportunities to uncover and reason with the specific mathematical relations between quantities that underlie algebraic thinking, whether it in the form of for example, “more A, more B” or “more A, less B”. Thus, the role of SFOR tendency in mathematical development is theoretically longer lasting than an initial advantage. Instead, SFOR tendency and rational number knowledge seem to form an iterative loop, each amplifying the other, thereby creating an advantage for those with a high SFOR tendency over time also in related areas such as algebra. Previous research suggests that relational features of both rational numbers and algebra may explain the link between the two in terms of students’ success over time (DeWolf et al., 2015; Siegler et al., 2012). Such a long-lasting impact of SFOR tendency, with the cross-pollination between SFOR and formal mathematical development, resembles previous evidence of the interaction between SFON tendency and numerical skills in early childhood (Hannula & Lehtinen, 2005).

Limitations and Future Directions

The main concern with the present study is that despite it providing a clearer picture of the relation between SFOR and mathematical development, this relation remains correlational and a causal relation cannot be determined. In particular, a measure of general mathematical achievement has been found to explain some shared variance between SFOR tendency and rational number knowledge (Van Hoof et al., 2016), although this did not entirely account for the relation between SFOR and rational number knowledge. Any future correlational studies should take into account other measures that might influence SFOR tendency, such as mathematical motivation and overall mathematical achievement.

The present study cannot provide clear prescription for changes to classroom activities. However, it can help researchers and educators understand more clearly exactly what the most successful students are doing that may provide them an advantage over their peers in learning some of the most difficult mathematical topics in compulsory education. The link between success in learning rational numbers and success in learning algebra (e.g. Siegler et al., 2012) is further confirmed in the present study, suggesting that it might be necessary to make sure students have a solid foundation in rational numbers when learning algebra. The shared context of quantitative relations between rational numbers, SFOR tendency, and algebra also suggests that making students more aware of the relational aspects of their everyday life may support learning both of these topics in the classroom. At the very least, more emphasis on the relational nature of rational numbers may be useful in helping students come to understand the relational nature of algebra as well.

In the end, the present study strengthens the evidence for the long-term relevance of SFOR tendency in mathematical development. Importantly, it further builds the case that the relational features of our everyday world hold potential for mathematical discovery and development. Educators and researchers interested in looking for potential solutions to the widening achievement gap in mathematics education could benefit from examining how students' own self-initiated activities influence their mathematical attainment (Lehtinen & Hannula, 2006; Lobato et al., 2012). While further evidence is needed to better understand the relevance of SFOR for mathematics educators and researchers, the present study suggests that students' own tendencies to pay attention and use the mathematics outside of formal mathematical situations are worth consideration.

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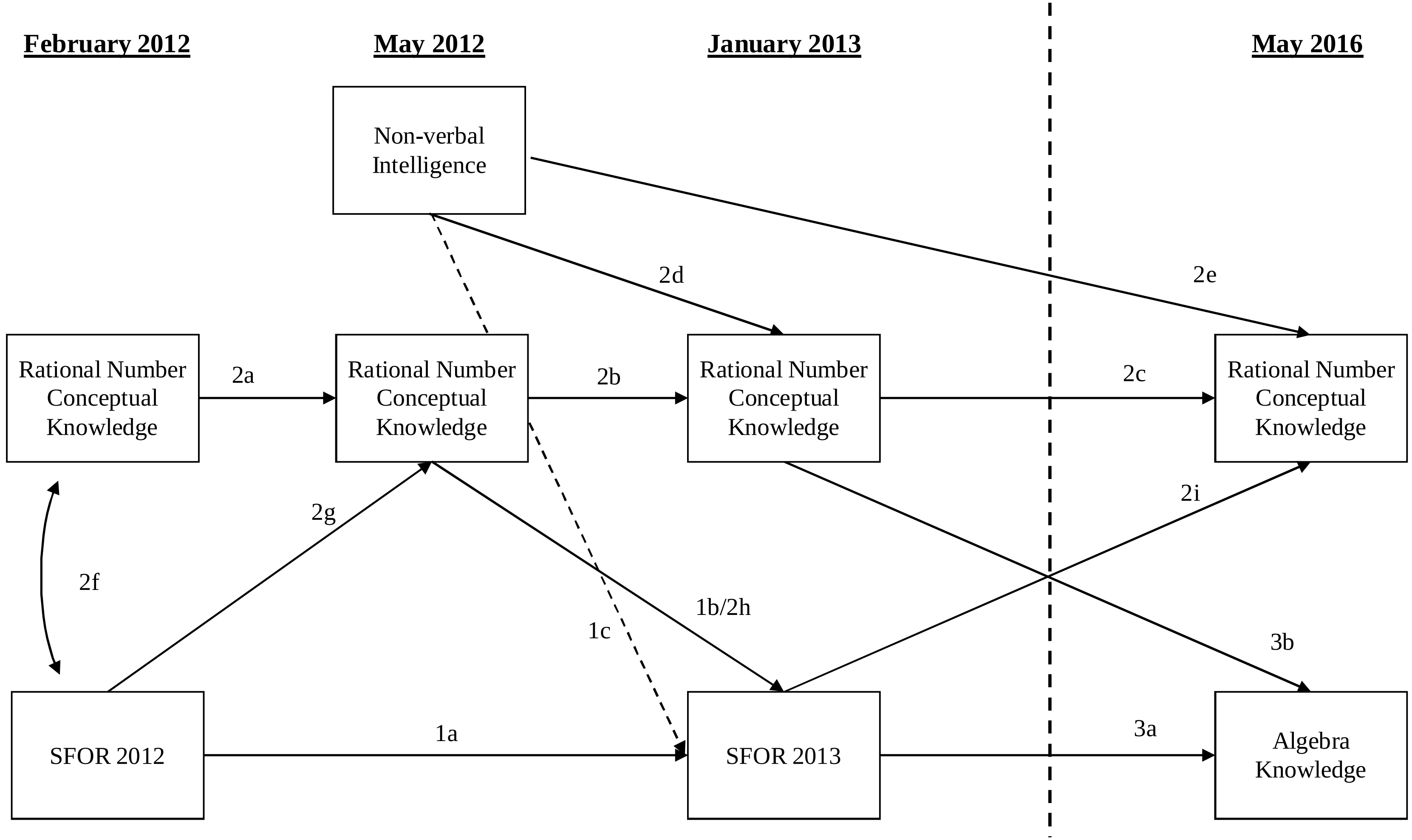

Figure 1 Hypothesized model. Solid lines represent expected relations, dotted lines represent non-relations (that are tested in the model)

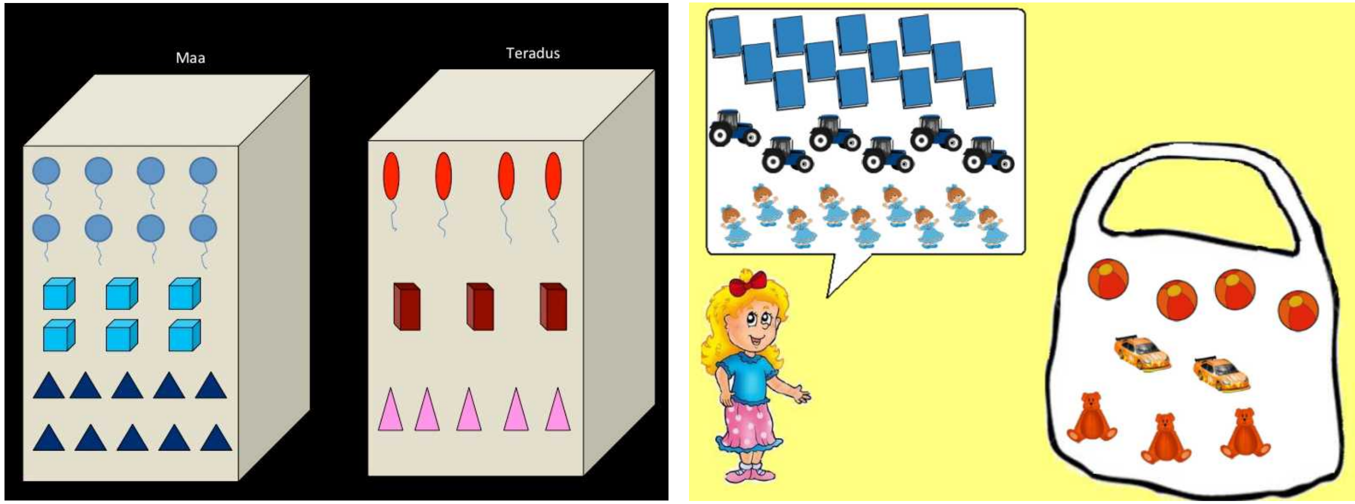

Figure 2 Examples from Teleportation (Left) and Shopping (Right) SFOR Tasks, from the first and third time points respectively. In both instances students were asked to “describe in as many ways as possible” how the items “had changed” (Teleportation) or “were different” (Shopping).

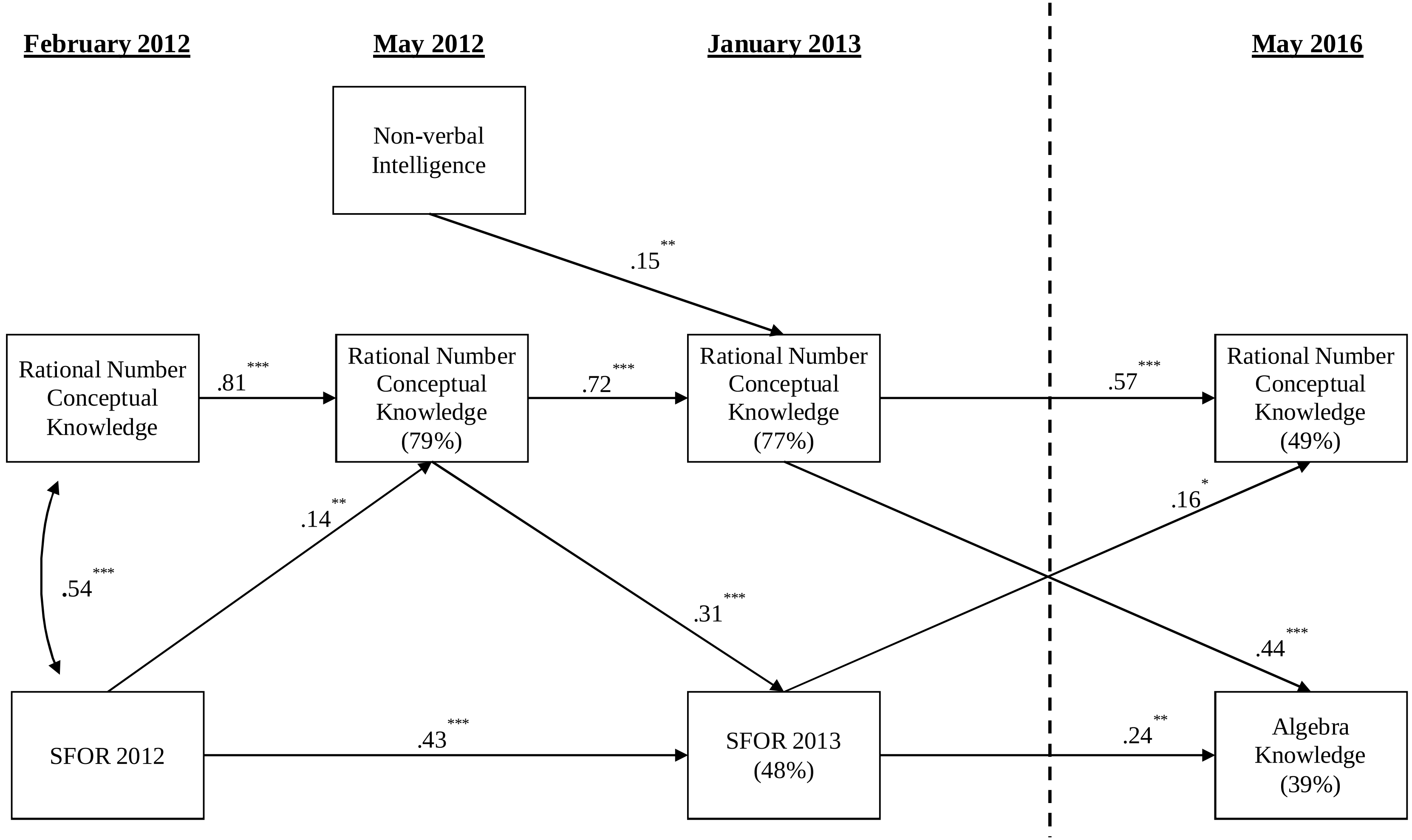

Figure 3 Correlations and regression coefficients of the significant relations in the resulting model of the development of relational reasoning from SFOR through rational number knowledge to algebra knowledge. Values in parentheses are percentage of variance explained in outcome measures (R^2)