Adaptive number knowledge and its relation to arithmetic and pre-algebra knowledge

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Abstract

Traditionally measured skills with arithmetic are not related to later algebra success at levels that would be expected given the close conceptual relation between arithmetic and algebra. However, adaptivity with arithmetic may be one aspect of arithmetic competences that can account for additional variation in algebra attainment. With this in mind, the present study aims to present evidence for the existence and relevance of a newly acknowledged component of adaptivity with arithmetic, namely, adaptive number knowledge. In particular, we aim to examine whether there are substantial individual differences in adaptive number knowledge and to what extent these differences are related to arithmetic and pre-algebra skills and knowledge. Adaptive number knowledge is defined as the well-connected knowledge of numerical characteristics and relations. A large sample of 1065 Finnish late primary school students completed measures of adaptive number knowledge, arithmetic conceptual knowledge, and arithmetic fluency. Three months later they completed a measure of pre-algebra skills. Substantial individual differences in adaptive number knowledge were identified using latent profile analysis. The identified profiles were related to concurrent arithmetic skills and knowledge. As well, adaptive number knowledge was found to predict later pre-algebra skills, even after taking into account arithmetic conceptual knowledge and arithmetic fluency. These results suggest that adaptive number knowledge is a relevant component of mathematical development, and may help account for disparities in algebra development.

Keywords: adaptive number knowledge; algebra; pre-algebra; mathematical development; latent profile analysis

1. Introduction

In the transition from arithmetic to algebra, it is yet unclear exactly what types of basic skills are needed for later success. Traditional skills of calculation – the rapid algorithmic solving of typical problems – have been deemphasized in many circles of mathematics education. Indeed, it is not clear that basic whole-number arithmetic calculation skills (e.g. 5 + 12 = ?) have a strong impact on later success with algebra when compared with other aspects of mathematical development such as rational number knowledge (Siegler et al., 2012), despite their logical connection.

Instead, many researchers of children's mathematical development have turned their concern to an alternate form of mathematical knowledge and skills. This new form sits in contrast to the static and calcified knowledge, with little transferability to new situations, referred to as *routine expertise*. Instead, a more malleable and interconnected set of knowledge and skills that is easily applied to new situations, referred to as *adaptive expertise*, is desired (Baroody, 2003; Hatano & Oura, 2003).

Within the frames of whole-number arithmetic, a core feature of adaptive expertise is adaptivity with arithmetic problem solving strategies. Adaptivity (which this study will focus on) refers to choosing and using the arithmetic problem solving strategy that is the most situationally-appropriate strategy for that person who is solving that particular problem (Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009). Adaptivity with arithmetic has been linked with later success with mathematics, including algebra (Kieran, 1992). In particular, one cornerstone of adaptivity has been well-researched over the past twenty years, namely, the flexible switching between multiple strategies (e.g. Lemaire & Siegler, 1995; Torbeyns, Ghesquière, & Verschaffel, 2009). However, given that flexibility in terms of strategy choice may only make up a portion of what entails full adaptivity (Threlfall, 2009; Verschaffel et al., 2009), it is just as important to look more generally at what makes adaptivity with whole number arithmetic possible. This more general view of adaptivity with whole number arithmetic may also include the well-connected knowledge of numerical characteristics and relations, or adaptive number knowledge (McMullen et al., 2016). Thus, the present study aims to examine the nature of individual differences in adaptive number knowledge in late primary school students and how adaptive number knowledge is related to arithmetic and pre-algebra knowledge and skills.

1.1. Adaptive number knowledge

If adaptivity with arithmetic requires the choosing of the most appropriate strategy at the time, for that person, one must be able to flexibly switch between numerous problem solving strategies (Siegler & Lemaire, 1997). However, one must also be able to recognize the relevant numerical characteristics and relations within the problem that would suggest the most appropriate strategy. The well-connected network of numerical characteristics and relations that defines adaptive number knowledge is required for this recognition to happen (Threlfall, 2009). Previously, procedural flexibility has been suggested to be a necessary, but not sufficient, component of adaptivity (Threlfall, 2009; Verschaffel et al., 2009). Recently, McMullen and colleagues (2016) have argued that the advanced representation of numerical relations, which constitutes adaptive number knowledge, is a key requirement for adaptivity with arithmetic problem solving in varying situations.

The existence of a set of solution strategies to be drawn from when confronted by an arithmetic problem has been called into question (Threlfall, 2002, 2009). As an alternative, it is suggested that when students are faced with an arithmetic problem they formulate a solution strategy in-situ. Research in various domains of mathematical development suggest that there are large individual differences in the types of knowledge that students rely on, with some more reliant on procedural knowledge and others relying more on conceptual knowledge (Bempeni & Vamvakoussi, 2015; Hallett, Nunes, & Bryant, 2010). Within the realm of adaptivity with arithmetic, it is also possible that students rely on these different meta-strategies in different ways, with some students more reliant on choosing from a set of existing strategies, while others are actively using idiosyncratic strategies developed during the problem solving process.

With the in-situ creation of strategies, it is clear that one must first recognize the different relations and characteristics of the numbers that exist in the problem in order to determine the most effective solution strategy. Expert mathematicians were found to rely on "nice" numbers that had specific features and relations (such as the proximity of 59 to a multiple of 20) instead of standard algorithms when mentally solving arithmetic problems (Dowker, 1992). Likewise, even if these solution strategies come from an existing repertoire, as suggested by some research (Hickendorff, van Putten, Verhelst, & Heiser, 2010; Torbeyns, de Smedt, Ghesquière, & Verschaffel, 2009), the adaptive choice of the most appropriate strategy still requires recognizing the characteristics of the numbers present (e.g. numbers close each other across decades, like in 41 - 39). For example, there are larger individual differences in flexible strategy use with adaptive composition (e.g. $4 + 7 + 6 = 10 + 7 = 17$) than other strategies (Canobi, Reeve, & Pattison, 2003). One reason for this may be the need to focus on the numerical relations in order to recognize the opportunity to use this strategy. In general, the lack of connection between students' knowledge of potential solution strategies and their actual use (Blöte, Klein, & Beishuizen, 2000), especially when not explicitly guided to do so (Gaschler, Vaterrodt, Frensch, Eichler, & Haider, 2013), suggests that it is not sufficient to know about a strategy. Instead, students must be able to independently recognize when a particular strategy is appropriate, in relation to their own skills with that strategy, the socio-cultural context in which the problem exists, and the numerical features of that particular problem (Verschaffel et al., 2009).

1.2. Arithmetic knowledge and skills

There are two components of adaptive number knowledge that have been roughly defined previously, 'numerical knowledge and skills' and 'arithmetic calculation knowledge and skills' (McMullen et al., 2016). These closely connected features of mathematics learning inform the extent and interconnectedness of students' knowledge of numerical characteristics and relations, which describes their adaptive number knowledge.

The numerical knowledge and skills that impact a students' adaptive number knowledge include knowledge of the natural number system, including its base-ten structure. This includes the ability to recognize important and useful arithmetical relations between numbers and determine the key characteristics of numbers that would be useful for problem solving (Geary, Hoard, Byrd-Craven, & DeSoto, 2004). For example, these relations and characteristics include the amount of factors a number has (e.g. 24 has more factors than 25), its proximity to other useful or "nice" numbers (e.g. 123 is close to 11^2), and estimates of multiples of the number. In particular, precision in magnitude representation and estimation is expected to play a key role in adaptive number knowledge (Gallistel & Gelman, 1992; Siegler & Lortie-Forgues, 2014). In general, such skills are closely related to advanced number sense, as defined in much work on the later development of mathematical competences (Mou et al., 2016). Adaptive number knowledge expands this notion by integrating numerical skills and knowledge with calculation skills and knowledge.

The relation between adaptive number knowledge and students' arithmetic calculation knowledge and skills has been tentatively established in a previous study (McMullen et al., 2016). In particular, ninth graders' arithmetic fluency and conceptual knowledge have been found to be related to their adaptive number knowledge. Arithmetic fluency reflects the ability to rapidly work with the four arithmetic operations in a conventional form, and can be seen as a procedural fluency that mainly requires recall or algorithmic solution strategies (Rittle-Johnson, Siegler, & Alibali, 2001; Schneider, Rittle-Johnson, & Star, 2011). This ability to rapidly and accurately complete one-step arithmetic problems informs adaptive number knowledge by allowing for the quick assessment of potential arithmetic relations between numbers, for example in recognizing useful multiples (e.g. 12 and 3 are related through 3 x 4 = 12). Arithmetic conceptual knowledge – including knowledge of the order of operations, the commutativity principle, and the associativity principle – has also been shown to be related to adaptive number knowledge (McMullen et al., 2016). In order to have strong adaptive number knowledge, students must know the allowances and constraints of how numbers and operations can be used in arithmetic. This is particularly true with more complex arithmetic relations, such as using both additive and multiplicative operations to relate numbers (e.g. 30 • 2 – 1 = 59).

1.3. Adaptive Number Knowledge and Pre-algebra Skills

Previous research suggests that there is a relation between students' adaptive number knowledge and their arithmetic fluency and knowledge (McMullen et al., 2016). Similarly, procedural flexibility with algebra has been found to be related to both procedural fluency and conceptual knowledge (Schneider et al., 2011). It has been suggested that procedural flexibility may allow for more successful transfer to novel situations (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Star & Rittle-Johnson, 2008). This relation hints at the possibility that adaptive number knowledge may be related to pre-algebra skills (cf. Kieran, 1992). While the term pre-algebra has no widely agreed definition, in this article we operationalize it as including tasks where students have to deal with equivalence and solve for unknown numbers (Herscovics & Linchevski, 1994: McNeil & Alibali, 2005). Having well developed adaptive number knowledge suggests being able to manipulate arithmetic relations in complex ways, often using inverse and multi-step relations. These skills are similar to what is need to be able to solve for unknowns in basic algebraic sentences. Taking the above example of the relation between 12 and 3, having well developed adaptive number knowledge would suggest being able to see numbers as a system of relations and recognize not only what 12/3 is, but also what times 3 equals 12, and 12 divided by what equals 3 (and knowledge how these three relations are related to each other).

It has been argued that the transition from arithmetic thinking to algebraic thinking requires similar shifts in thought processes than other major transitions in mathematics (Carraher, Schliemann, & Brizuela, 2001). This cognitive gap may cause particular difficulties with understanding algebra, as thinking algebraically requires considering relations instead of quantities (Nunes, Bryant, & Watson, 2006). However, students have been shown to be successful with solving missing-value problems before formal algebra instruction, when the missing values are not represented as an "unknown" (Herscovics & Linchevski, 1994). Thus, others have argued that algebra learning is not constrained by readiness defined developmentally, but by a lack of preparation for reasoning algebraically (Carraher, Schliemann, Brizuela, & Earnest, 2006). The problem is that traditional arithmetic instruction is focused too narrowly on computation that limits the understanding of relations between numbers and operations that are necessary for algebraic reasoning (Schliemann, Carraher, & Brizuela, 2006).

It is precisely these numerical and operational relations that are expected to unify adaptive number knowledge and algebraic reasoning. Anghileri and colleagues (2002) found that when students were given the opportunities to explore division calculation strategies on their own they were able to show signs of algebraic thinking. Likewise, the use of more mathematically advanced mental calculation strategies was linked with better skills with algebraic symbols (Britt & Irwin, 2008). These results suggest that that the role of relational thinking in algebra, as opposed to a calculational or operational approach (McNeil, Rittle-Johnson, Hattikudur, & Petersen, 2010; Nunes et al., 2006), is strongly related to the role of numerical characteristics and relations in our view of adaptive number knowledge. Thus, it is expected that adaptive number knowledge will be related to pre-algebra skills, especially in solving missing-value problems, which require a more advanced understanding of arithmetic operations and having a relational view of equations (e.g. McNeil et al., 2010).

1.4. The Present Study

There has been a long tradition of examining procedural flexibility in the frames of adaptivity with arithmetic problem solving (Heinze, Star, & Verschaffel, 2009; Hickendorff et al., 2010; Siegler & Lemaire, 1997; Torbeyns, Ghesquière, et al., 2009). While these studies have been successful in outlining the calculation procedure components of adaptivity (conceptualized as flexibility), there seems to be a need to also examine the nature of the numerical knowledge that is required for adaptivity with arithmetic. While there is some evidence for individual differences in the spontaneous use of more adaptive problem solving strategies (Gaschler et al., 2013; Haider, Eichler, Hansen, Vaterrodt, & Frensch, 2014), these studies rely on relatively broad measures, basing their conclusions of strategy use on overall differences in the speed of task completion over many items. As well, the relations between components of early adaptivity with arithmetic, such as procedural flexibility or adaptive number knowledge, and algebra skills have not been systematically examined.

Previous research suggests that individual differences in adaptive number knowledge can be captured with a task in which participants create arithmetic sentences using a set of given numbers to equal a target number (Brezovszky et al., 2015; McMullen et al., 2016). These individual differences were found among primary school, lower secondary school, and university students (McMullen et al., 2016). However, these studies had relatively small samples, which did not allow for broad conclusions to be drawn nor robust statistical techniques to be used. The present study aims to further this examination, by looking at individual differences in adaptive number knowledge in a large sample with which Latent Profile Analysis can be used to examine the nature of these individual differences. Thus, the first question the present study aims to answer is: what are the qualitative and/or quantitative individual differences in late primary school students' adaptive number knowledge?

The results of previous research suggest that procedural flexibility in mathematics is related to both procedural skills and conceptual knowledge (Schneider et al., 2011). As well, there is preliminary evidence that adaptive number knowledge is related to both knowledge of arithmetic concepts and arithmetic fluency (McMullen et al., 2016), though this evidence is based on a small sample of ninth grade students. Based on this previous evidence it would be expected that adaptive number knowledge is a relevant component of arithmetic knowledge and skills. Therefore the second question posed by the present study is: How are profiles of adaptive number knowledge related to concurrent arithmetic knowledge and skills in a large sample of primary school students?

Finally, there is little evidence about how features of adaptivity are related to algebra skills and knowledge. Flexibility with arithmetic procedures has previously been argued to be related to algebra skills (Kieran, 1992), but to the best of our knowledge, no such studies exist that explicitly test this link. The nature of adaptive expertise with arithmetic has been argued to be related to more advance mathematics, especially in transferring knowledge to new contexts (Baroody, 2003; Verschaffel et al., 2009). The well-connected numerical characteristics and relations needed for strong adaptive number knowledge may also support pre-algebra reasoning (Nunes et al., 2006). Thus, the final aim of the present study is to examine how adaptive number knowledge is related to pre-algebra skills by asking: Are profiles of adaptive number knowledge related to pre-algebra skills, even after taking into account arithmetic conceptual knowledge and arithmetic fluency?

2. Methods

2.1. Participants

Participants were 1065 fourth to sixth grade primary school students (498 female) from small to medium urban, suburban, and rural areas in the south of Finland. Participants were recruited to participate in a large-scale experiment investigating the effects of an educational game on enhancing adaptive number knowledge. There were 123 fourth graders (M_{age}=10.18, SD=0.42), 549 fifth graders (M_{age}=11.14, SD=0.38) and 393 sixth graders (M_{age}=12.20, SD=0.45), from a total of 59 classrooms. All teachers from these classes were volunteers. Participation was voluntary; informed consents from parents and assent of the students were obtained before data gathering. Ethical guidelines of the university were followed.

2.2. Procedures

Data from the present study was collected first during the pre-testing before any intervention was carried out, at the beginning of the spring term 2014. As well, at the final time point, three and a half months later at the end of the spring term 2014, when all students completed a parallel post-test set of measures (see Sections 2.3 and 2.4 for more details on similarities between measures). At both time points, participants completed paper-and-pencil tests in their regular classrooms, supervised by trained testers who made use of a timed presentation to ensure uniformity in all testing procedures.

A two-stage staggered experimental design was used in the overall study, in which participants in the initial control condition completed the intervention after the students in the experimental group. Participants from both conditions are included in the present study, as they all participated in the intervention activities at some point in between the pre- and post-tests. During the intervention, students' normal mathematics instruction was enriched with a digital mathematics involving the navigation of a boat through a virtual hundreds-square using arithmetic operations game (for a full description of the game, see Lehtinen et al., 2015). Average playing time was 5.7 hours (SD = 3.3 hours) during the three month in between the pre- and post-tests.[1]

2.3. Measures at Pre-test

2.3.1. Adaptive Number Knowledge

Adaptive number knowledge was measured using the Arithmetic Production Task; extensive details and analysis of this task can be found elsewhere (McMullen et al., 2016). Participants are asked to form as many valid arithmetic sentences as they can, using a set of five given numbers, which equal a target number. Participants are told they can use the given numbers and the four arithmetic operations in any combination and as many times as they want. After an example item, there were four test items (Table 1). Participants were given 90 seconds to complete each item. Reliability for the total number of correct answers across the four items was acceptable (Cronbach's alpha = .70).

Two types of items were used on the task (see McMullen et al., 2016 for a detailed explanation of item types). First, *dense* items included a numbers where there were a large amount of arithmetical relations which can easily be identified between the given and target numbers. Second, *sparse* items have a relatively small number of obvious relations between the given and target numbers.

Table 1

Items of the Arithmetic Production Task during Pre- and Post-test.

Item	Given	Target	Type
1	2, 4, 8, 12, 32	= 16	Dense
2	1, 2, 3, 5, 30	= 59	Sparse
3	2, 4, 6, 16, 24	= 12	Dense
4	3, 5, 30, 120, 180	= 12	Sparse

[1] Overall time playing the game was not related to any variables used in the present study: *p*s > .44.

For the purposes of examining potential qualitative differences in students adaptive number knowledge participants' responses were coded as either *Simple* or *Complex*. Simple answers were either solely additive, containing only addition or subtraction, or solely multiplicative, containing only multiplication or division. Complex answers contained both additive and multiplicative operations (e.g. both addition and multiplication, $2{\cdot}4+8=16$). Sum scores for Simple and Complex responses were calculated for the dense items and sparse items separately.

2.3.2. Arithmetic Fluency

Arithmetic fluency was measured using the Woodcock-Johnson mathematics fluency sub-test (Woodcock, McGrew, & Mather, 2001). Participants are asked to complete as many single-step basic arithmetic problems out of a two-page set of 160 items as they can in three minutes. Reliability values of this instrument in this age range have previously been reported as above .90 (Schrank, McGrew, & Woodcock, 2001). Test-retest reliability over a five-week period in the present sample shows similar reliability (Pearson's correlation = .89).

2.3.3. Arithmetic Conceptual Knowledge

Participants' arithmetic conceptual knowledge was examined with a collection of multiple-choice items examining their knowledge of arithmetic concepts (based on corresponding algebra items from Schneider et al., 2011) and missing-value equation solving. The six arithmetic concept items covered concepts such as commutativity, associativity, and inverse/complementary operations. The six missing-value equation problems involved choosing the correct missing number from a two-step equation (e.g. 6 x __ = 2 x 15; Options; 5, 24, 6, 4). Due to the novel nature of these items, they were considered to measure students' arithmetic conceptual knowledge (Rittle-Johnson et al., 2001). Therefore at Time 1, these items were combined with the Knowledge of arithmetic concept items to create a general measure of students' initial arithmetic conceptual knowledge. At Time 1, reliability was acceptable across all 12 of these items (Cronbach's alpha = .73).

2.4. Measures at Post-Test

2.4.1. Pre-Algebra skills

Participants completed open-ended versions of similar missing value equivalence problems as were presented on the pre-test (e.g. 18 ÷ __ = 54 – 48). There were eight items in total on this measure, which had high reliability (Cronbach's alpha = .82).

2.5. Analysis

Latent Profile Analysis (LPA) was run using Mplus version 7.0 (Muthén & Muthén, 1998-2012). LPA was chosen because it can be seen as more useful than other cluster techniques, as it does not require variables to meet many of the traditional modeling assumptions, such as distributional normality (Magidson & Vermunt, 2003). The estimation method was maximum likelihood with robust standard errors (MLR); a full information approach that is able to handle missing-at-random data. The LPA analyses were carried out as mixture models, in which 1000 and 100 random start values were used in the first and second steps of model estimation, respectively, to ensure the validity of the solution (Geiser, 2013). Model fits were evaluated through a combination of a) statistical indicators (Nylund, Asparouhov, & Muthén, 2007) and b) substantive theory, in order to determine the most suitable number of latent classes for the best fitting model. Low values for AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion) indicate a better fit, relative to nested models. Entropy values that approach 1 signify more certainty in the classification, with a rough cut off of .80 (Collins & Lanza, 2010). Finally, a significant result on the BLRT (Parametric Bootstrapped Likelihood Ratio Test) supports the *k*-class solution in comparison with the *k–1*-class solution. Analysis of variance tests were run in SPSS version 21.

3. Results

Variance Component Analyses was conducted in order to explore whether there were substantial classroom level differences in the measures of adatptive number knowledge, arithmetic fluency, conceptual knowledge, and pre-algebra knowledge. In all cases Intraclass Correlation Coefficients (ICC) were below the 0.25 threshold indicating that multi-level analysis is not needed (Bowen & Guo, 2011; Kreft, 1996). The ICC for total correct number of responses on the Arithmetic Production Task was .10, total correct arithmetic fluency responses was .13, for total correct on the conceptual knowledge at the pre-test measure the ICC was .14, and post-test pre-algebra knowledge scores’ ICC was .14.

Table 2
Descriptive statistics for whole sample and by grade level for all measures.

Grade Level	**Arithmetic Production Task Responses**				**Arithmetic Fluency**	**Conceptual Knowledge**	**Pre-Algebra (Post-test)**
	Dense Simple	Dense Complex	Sparse Simple	Sparse Complex			
All	8.06	.71	.79	1.43	70.53	8.41	5.72
	(3.14)	(1.06)[a]	(1.04)	(1.29)	(17.40)	(2.62)	(2.28)
4	6.77	.46	.62	1.17	65.34	7.69	4.81
	(2.70)	(.74)	(.80)	(1.11)	(17.82)	(2.81)	(2.59)
5	8.22	.79	.80	1.45	70.84	8.54	5.49
	(3.13)	(1.13)	(1.07)	(1.30)	(17.35)	(2.63)	(2.27)
6	8.24	.68	.82	1.50	71.83	8.44	6.31
	(3.20)	(1.05)[a]	(1.08)	(1.32)	(17.09)	(2.52)	(2.03)

[a]Skewness>2 and Kurtosis>7

Table 2 presents the means and standard deviations for adaptive number knowledge, arithmetic fluency, and arithmetic conceptual knowledge for the whole sample and by grade-level. Overall, there remains variability in adaptive number knowledge even within grade level. One-way ANOVAs indicated grade-level differences in all variables except simple responses on sparse items: Fs $(2, 1062) > 3$, ps $\leq .05$. Further analyses are now conducted to examine possible qualitative differences in adaptive number knowledge.

3.1. Profiles of Adaptive Number Knowledge

Based on recommendations (Collins & Lanza, 2010; Nylund et al., 2007), the LPA of adaptive number knowledge indicated that the five-class model was statistically appropriate while also having an acceptable entropy value (see Table 3 for a full account of statistical indicators). First, the five-class model was considered more advantageous because the six-class model had two extremely small classes (<5% of sample), which is not recommended (Nylund et al., 2007). Thus, the value afforded by the additional class in the six-class model was considered inconsequential. As well, the additional class added in the five class model when compared to the four-class model had a strong theoretical justification, in that it represented those students who had an extremely high number of all types of correct responses (i.e. the “High” group detailed in the next paragraph).

Table 3
Statistical indicators for the 2 to 8 class LPA models.

Number of classes	Log-likelihood	AIC	BIC	Entropy	VLMR p-value	Average latent class posterior probabilities
2	-5627	11296	11400	.89	.20	.97/.87
3	-5496	11048	11188	.75	<.0001	.90/.85/.93
4	-5442	10953	11127	.75	.64	.88/.82/.93/.83
5	-5311	10706	10916	.81	.04	.88/.82/.88/.96/.98
6	-5266	10630	10873	.81	.02	.87/.90/.81/.89/.96/.97
7	-5312	10737	11016	.77	.32	.84/.92/.76/.78/.85/.92/.97
8	-5227	10580	10893	.80	.28	.90/.84/.94/.76/.76/.88/.96/.96

Table 4 outlines the mean adaptive number knowledge responses for the five classes. The “Basic” group (n=535) had below average values for all response types. The “Simple” group (n=54) had relatively high simple responses on both the dense and sparse items, but few complex responses. The “Complex” group (n=158) had above average complex responses on both dense and sparse items, but few simple responses. The “Strategic” group (n=293) had above average simple responses on dense items and above average complex responses on the sparse items. The “High” group (n=24) had above average responses for all response types.

Table 4
Mean values for the five adaptive number knowledge profiles across response type.

	Percent of sample	Dense Simple	Dense Complex	Sparse Simple	Sparse Complex
Basic (n=535)	50.2%	6.41	.22	.60	.87
Simple (n=54)	5.1%	10.14	.49	3.63	.71
Complex (n=158)	14.9%	9.00	2.27	.76	2.29
Strategic (n=293)	27.5%	10.32	.49	.65	2.08
High (n=24)	2.3%	9.48	4.89	1.40	2.96

Figure 1 details the mean values for correct responses and complex responses for each item for the five classes. The mean values for correct responses between the Simple, Complex, and Strategic groups are relatively similar. However, despite having similar numbers of complex solutions on the sparse items there is a wide disparity between these three groups with the use of complex solutions on the dense items (on which more simple solutions were easily identifiable).

-Insert Figure 1 about here-

A chi-squared test indicated a positive association between profile membership and grade-level, $\chi^2(8)=58.89$, $p<.001$. Standardized adjusted residuals above an absolute value of 2 reveal the profiles which significantly differ from what would be expected. There was a clear relation between adaptive number knowledge profile membership and grade level. However, there remained substantial within grade level differences in profile membership. There were more 4th and 5th grade students (adjusted residual = 5.2 and 2.6) and less 6th grade students (adjusted residual = -6.1) in the Basic group than expected by chance. As well, there were less 4th and 5th grade students (adjusted residual = -3.1 and -3,5) and more 6th grade students (adjusted residual = 5.7) in the Strategic group than would be expected. Finally, there were less 4th grade students (adjusted residual = -2.2) in the Complex group than would be expected. There were no substantial differences in the grade-level makeup of the Simple and High groups, mostly likely due to the small sizes of these profiles.

3.2. Adaptive number knowledge and arithmetic skills and knowledge

In order to explore the relation between adaptive number knowledge and arithmetic skills and knowledge two ANCOVAs were run to examine group level differences among the five adaptive number knowledge profiles in arithmetic fluency and arithmetic conceptual knowledge, while taking into account grade-level differences. Overall, profile membership and grade-level accounted for 40% of the variance of arithmetic fluency and 25% of conceptual knowledge. Adaptive number knowledge profile membership was related to both arithmetic fluency: $F(4, 1051)=98.72$, $p<.001$, $\eta_p^2=.27$; and arithmetic conceptual knowledge: $F(4, 1051)=44.26$, $p<.001$, $\eta_p^2=.14$. Grade level showed weaker relations with arithmetic fluency: $F(2, 1051)=8.54$, $p<.001$, $\eta_p^2=.02$; and arithmetical conceptual knowledge: $F(2, 1051)=4.89$, $p<.01$, $\eta_p^2=.01$. There was no interaction effect between profile membership and grade-level.

-Insert Figure 2 about here-

Planned post-hoc analyses revealed substantial differences between the different profiles. Figure 2 presents the mean values for arithmetic fluency and conceptual knowledge for the different adaptive number knowledge profiles. As shown in Figure 3, there were difference in arithmetic fluency scores between all profiles, and differences in arithmetic conceptual knowledge between most profiles.

-Insert Figure 3 about here-

3.3. Adaptive number knowledge and pre-algebra skills

In order to examine how adaptive number knowledge was related to pre-algebra skills an ANCOVA was run to determine group level differences among the five adaptive number knowledge profiles in pre-algebra skills at the post-test, while controlling for grade level, pre-test arithmetic fluency and conceptual knowledge. Overall, profile membership, grade level, and pre-test scores accounted for 44% of the variance in post-test pre-algebra skills. Even after taking into account the strong effect of pre-test conceptual knowledge on post-test pre-algebra (η_p^2=.20, p<.001), the moderate effect of pre-test arithmetic fluency (η_p^2=.03, p<.001), and the small effect of grade level (η_p^2=.005, p=.03), adaptive number knowledge profile membership still explained a moderate portion of the variance in post-test pre-algebra, $F(4, 970)=9.14$, $p<.001$, $\eta_p^2=.04$. As can be seen in Figure 4, there were substantial differences between the profiles. In general, those students in the Basic and Simple groups had lower performance on the post-test, while the students in the Complex group were most successful with pre-algebra skills on the post-test.

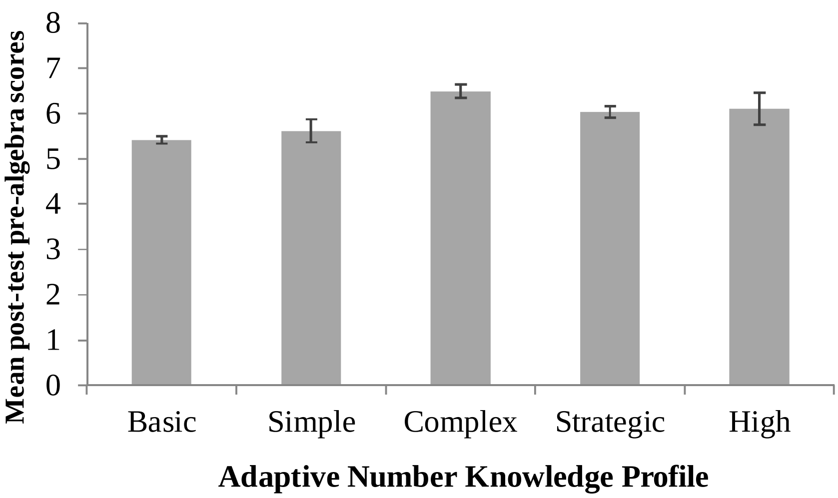

Figure 4 Estimated marginal means for post-test pre-algebra skills by adaptive number knowledge profile, controlling for pre-test conceptual knowledge and arithmetic fluency. Error bars = ±2 S.E, range determined by ±2 standard deviations from sample mean.

4. Discussion

The main aims of the present study were to examine (a) the qualitative and/or quantitative nature of individual differences in students' knowledge of numerical characteristics and relations, (b) how these individual differences are related to arithmetic fluency and arithmetic conceptual knowledge, and (c) how these differences were related to pre-algebra skills. The results of the present study provide the first substantial evidence, in a large sample, which shows that there are indeed such individual differences in students' knowledge of numerical characteristics and relations. Thus, it is possible to distinguish, within students' existing mathematical competences, a relevant component of arithmetic problem solving – referred to as adaptive number knowledge (McMullen et al., 2016). Adaptive number knowledge is shown in the present study to be related to other arithmetic knowledge and skills in a large sample of primary school students. Finally, adaptive number knowledge is uniquely related to pre-algebra skills, even after taking into account prior conceptual knowledge and arithmetic fluency.

4.1. Adaptive number knowledge

The approach of the present study was to determine the type of patterns students display when completing the Arithmetic Production Task (Brezovszky et al., 2015; McMullen et al., 2016). Using LPA modeling, it was apparent that there are substantial individual differences in students' responses on this task, not only in terms of the number of solutions, but also the types of solutions the students produced. In particular, it was apparent that some students were more adaptive in their solution types, choosing solution strategies depending on the type of problems that were presented (Strategic, High), while others were more fixed in the type of solutions they designed no matter the problem type (Complex, Simple). These qualitative distinctions do not appear to be entirely explained by grade level, nor by other arithmetic skills, suggesting that the different approaches to the task may be due to differences in other factors, particularly adaptive number knowledge.

The results of the LPA suggest that there are qualitative differences in students' adaptive number knowledge that cannot entirely be taken into account through simple sum scores. For example, the Simple group actually had on average more correct responses in total on the Arithmetic Production Task than every other group but the High group. However, the almost non-existent use of arithmetically complex solutions (i.e. both additive and multiplicative operations in one solution) in the Simple group potentially indicates a lower level of adaptive number knowledge. Those students in other groups (i.e. Strategic, Complex, High) were able to more consistently apply more mathematically complex calculations to reach the target number, suggesting more well-developed knowledge of numerical characteristics and relations. These differences were most apparent on sparse items, in which students in the more advanced groups were able to successfully find solutions that did not require repeated addition and instead utilized the complex numerical relations between the numbers (e.g. $2 \times 30 \approx 59$). It should be noted, however, that the Simple group was relatively small, and may simply reflect a particular strategy for solving these task, which could be adjusted with different instructions, and not a particular level of adaptive number knowledge. Thus, it may be sufficient to use sum scores in order to access general adaptive number knowledge in studies in which the particular focus is not on examining the nature of adaptive number knowledge itself. However, the relevance of uncovering such qualitative differences in adaptive number knowledge is also supported by the obvious hierarchy of the five profiles in regards to other arithmetic skills and knowledge.

4.2. Relation between adaptive number knowledge and arithmetic skills and knowledge

Prior evidence provided tentative support that individual differences in adaptive number knowledge were related to other arithmetic skills and knowledge (McMullen et al., 2016), however this finding was in a small sample of ninth grade students, and did not necessary suggest that adaptive number knowledge would be relevant during the formative years of arithmetic development in late primary school. The present study confirms these results among primary school students, and reveals that adaptive number knowledge is a relevant component of arithmetic knowledge in primary school.

There were clear distinctions between all groups in terms of arithmetic fluency; overall procedural fluency with basic arithmetic increases across the groups in the order of Basic, Simple, Complex, Strategic, and High. While it is no surprise that the ability to fluently calculate simple arithmetic problems is related to a time-sensitive task such as this one, this is particularly informative when examining differences between the three middle groups of Simple, Complex, and Strategic. While, in total, these groups had similar number of correct answers on the Arithmetic Production Task, their differences in arithmetic fluency shows an interesting relation with the particular nature of their solutions. Those students who were included in the Simple group rely almost exclusively on simple, single operation solutions and also had the lowest arithmetic fluency besides the Basic group. As well, despite both groups being fairly successful with using mathematically complex solutions with both additive and multiplicative operations, those students in the Strategic group outperformed those in the Complex group on the measure of arithmetic fluency. The only major difference between the Complex and Strategic groups was that the Complex group used complex solutions on the dense items, while the Strategic group mostly relied on simple solutions on the dense items (both used complex solutions on the sparse items, and had a similar number of correct answers for both item types). This suggests that having the ability to quickly and accurately calculate simple solutions allows for adapting one's strategy for producing solutions based on the problem type – using more simple solutions on dense items, when they are readily accessible, and more complex solutions on sparse items, when they are procedurally advantageous.

While there were distinct differences between all groups with arithmetic fluency, there were fewer differences among the more advanced groups for conceptual knowledge. These results suggest that conceptual knowledge of arithmetic can be seen as necessary, but not sufficient, for advanced adaptive number knowledge. In particular, knowing the rules of how to arithmetically combine numbers using multiple operations to form other numbers requires a basic understanding of some arithmetic principles, such as commutativity. However, once these rules are understood other factors play a larger role, for example arithmetic fluency, and possibly numerical knowledge and skills. Schneider and colleagues (2011) found that procedural and conceptual knowledge predicted algebra flexibility in similar ways. A full battery of tests covering all features of arithmetic adaptivity, including numerical knowledge and advanced number sense (e.g. Geary et al., 2009) and flexibility measures (e.g. Hickendorff et al., 2010; Torbeyns, Smedt, Ghesquière, & Verschaffel, 2009) would be important for examining the inter-relations among this diverse set of skills and knowledge.

4.3. Adaptive number knowledge and pre-algebra skills

The present study provides the first evidence that adaptive number knowledge is related to pre-algebra skills, even after taking into account arithmetic conceptual knowledge, arithmetic fluency, and grade level. These results are the first to provide evidence of the role adaptive number knowledge plays in the larger development of adaptive expertise with arithmetic, and mathematics in general. Importantly, these results align with theoretical accounts of mathematical development that have been proposed previously, but have had little empirical support until now (e.g. Carpenter et al., 1998; Kieran, 1992).

Previously, evidence has found that many aspects of arithmetic knowledge, including skills with addition, subtraction, and multiplication are not strong predictors of algebra knowledge (Siegler et al., 2012). As has been proposed previously, core features of adaptive expertise, such as flexible problem solving, are expected to have a larger impact on learning since they allow for more well developed conceptual knowledge and seem to support the transfer of knowledge to new contexts (Baroody, 2003; Star & Rittle-Johnson, 2008). The findings of the present study are in line with this conjecture, as those students with more well developed adaptive number knowledge were more successful in solving complex multiple-operation missing-value problems than their peers. One possibility is that the relational thinking required for the missing value problems (e.g. Nunes et al., 2006) aligns closely with the knowledge of arithmetic relations needed on the Arithmetic Production Task. Being able to fluidly align the numerical relations and characteristics for producing arithmetic sentences on this task required similar kind of knowledge of arithmetic relations as is needed to solve missing-value problems. An alternative explanation is that those students who have a stronger set of arithmetic skills and a wider range of arithmetic knowledge also can adapt these advantages to complete these missing-value problems. However, given that the relation between adaptive number knowledge and the development of pre-algebra skills held even after controlling for arithmetic conceptual knowledge and arithmetic fluency, it seems to be the case that this relation is fairly unique.

4.4. Limitations

One of the major limitations of the present study is that since the two measurement points were interceded by an intervention involving students playing an education game aimed at enhancing adaptive number knowledge the identified relations may be potentially overstated. However, while the game was meant to enhance adaptive number knowledge, the Arithmetic Production Task was quite dissimilar to the mathematical activities covered in the game, thus only distant transfer effects were expected. The measure of pre-algebra knowledge used on the post-test was even further removed from the activities completed by the students in the game. Thus, along with the fact that gameplay should equal effect on all participants, we do not expect this limitation to have a substantial impact on the results of this study.

There appears to be a fairly large imbalance in the groups identified in the LPA model, with half of the participants being included in the Basic group. However, it is not entirely surprising that most of the participants were only able to provide a small number of relatively simple solutions given that the adaptive number knowledge measure was a fairly atypical task in comparison to most mathematics classroom activities. While a scalar measure of adaptive number knowledge may be acceptable in other circumstances, using an LPA model can allow for a more in-depth understanding of the nature of individual differences, by more accurately describing heterogeneity in individuals' response patterns (e.g. Collins & Lanza, 2010), as are found in the Arithmetic Production Task in the present study.

Finally, the present study provides evidence that adaptive number knowledge is related to other arithmetic skills, and even pre-algebra knowledge. However, it is possible that these correlations may be explained by confounding factors. While classroom level differences were not found, nor did students receive explicit instruction in such solving missing-value problems prior to testing, it is possible that students' general cognitive ability may be a confounding variable that explains the relation between adaptive number knowledge and other arithmetic and pre-algebra skills and knowledge. However, if this were the case we would expect to see even larger differences between grade levels on these measures, yet grade-level explained only a minor amount of variance in post-test pre-algebra scores. As well, adaptive number knowledge was found to explain a moderate level of variance in pre-algebra knowledge even after taking into account prior knowledge of arithmetic fluency, grade level, and even arithmetic concepts – which were partially measured using very similar tasks as the post-test pre-algebra measure. This suggests that any confounding by other more general cognitive measure may be minimal. However, future studies of the nature of adaptive number knowledge should confirm this by taking into account the potential for domain-general and domain-specific cognitive measures to have a potential mediating effect on the relation between adaptive number knowledge and arithmetic and pre-algebra knowledge.

4.5. Educational implications

Adaptivity with arithmetic has been connected with the movement in mathematics education towards focusing on adaptive expertise, instead of routine expertise (Baroody, 2003; Hatano & Oura, 2003; Verschaffel et al., 2009). Within this framework, the emphasis is placed on focusing instruction and learning in arithmetic away from calcified knowledge and skills that have limited scope beyond a small set of known problems, onto knowledge and skills that can be readily applied to novel tasks in flexible ways. The present study provides one task which may prove useful for assessing this type of knowledge and skills with arithmetic problem solving. By placing the task of arithmetic sentence creation on the student, the Arithmetic Production Task is able to assess their ability to adaptively apply their knowledge of numerical characteristics and relations in a new but relevant task. The numerical features of many mathematical situations, both in later mathematics classes and real life, are manifold and require a deep understanding of their potential connections (Lobato, Rhodehamel, & Hohensee, 2012; Reyna & Brainerd, 2007). It is possible that those students with stronger adaptive number knowledge may be more able to navigate these situations in recognizing the most relevant numerical features, both within explicitly mathematical contexts and outside of them (e.g. Hannula-Sormunen, 2015; McMullen, Hannula-Sormunen, Laakkonen, & Lehtinen, 2015).

While more evidence is needed, the results of this study and previous studies suggest that adaptive number knowledge could be considered for inclusion in the instructional content of arithmetic teaching. Recent studies also suggest that adaptive number knowledge can be enhanced with game-based learning (Brezovszky et al., 2015). The use of a more open learning environment to allow for students to explore the numerical connections between numbers seems to be a promising mode for the development of adaptive number knowledge. Integrating such an approach into traditional arithmetic instruction may help lessen the gaps between different students' adaptive number knowledge found in the present study.

4.6. Conclusions

Solution strategies in the context of arithmetic flexibility and adaptivity may be regularly re-born within the problem solving process (Threlfall, 2002, 2009), or they may be selected from a set of known strategy options (Siegler & Lemaire, 1997; Verschaffel et al., 2009). Whatever the case, it is now clear that some students are much more capable than others of creating arithmetic sentences from a given set of numbers that equal a target number. Not only can they do so with different frequency of responses, but also with different levels of mathematical complexity. This approach of examining students' own creative abilities aims to explore the issue of adaptivity with arithmetic problem solving from a different direction than research on flexibility and supposes that in order to be truly adaptive with arithmetic one must be able to approach arithmetic problems with a well-connected network of numerical characteristics and the arithmetic relations between these numbers (Lehtinen et al., 2015; McMullen et al., 2016). Whether creating strategies in-situ or choosing the most appropriate for the task at hand, it is now apparent that adaptive number knowledge may be an important aspect of mathematical development.

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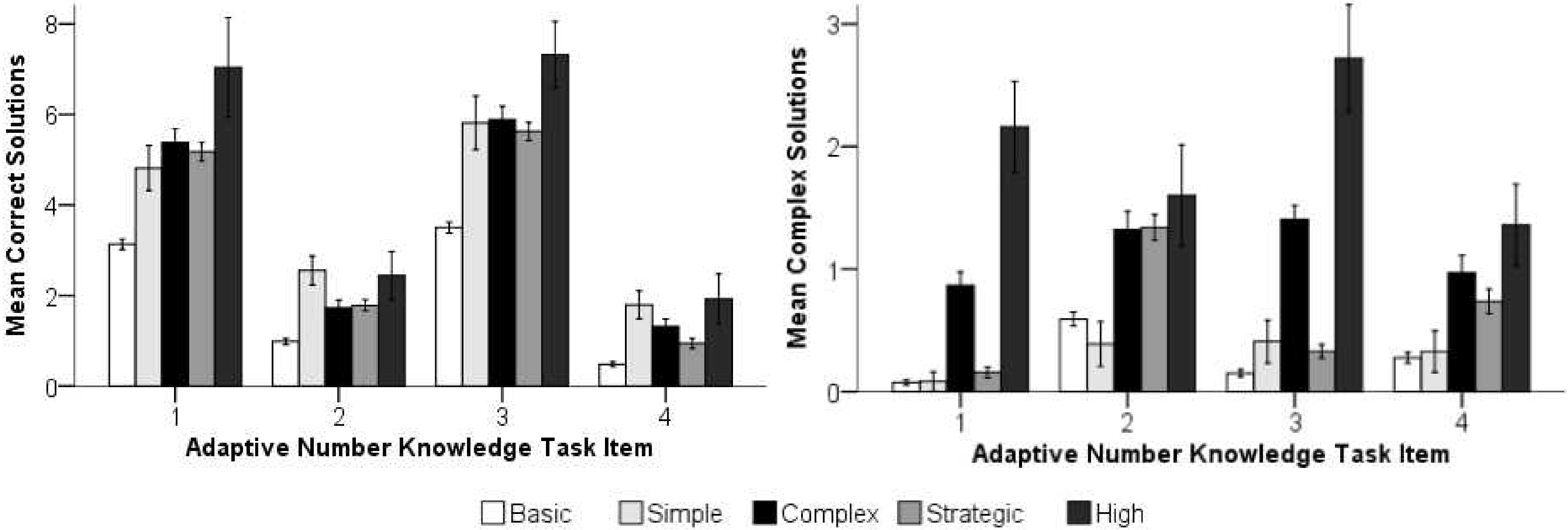

Figure 1 Mean correct and complex solutions for each item by adaptive number knowledge profile. Items 1 and 3 are dense items, items 2 and 4 are sparse items. Error bars represent ±2 standard error.

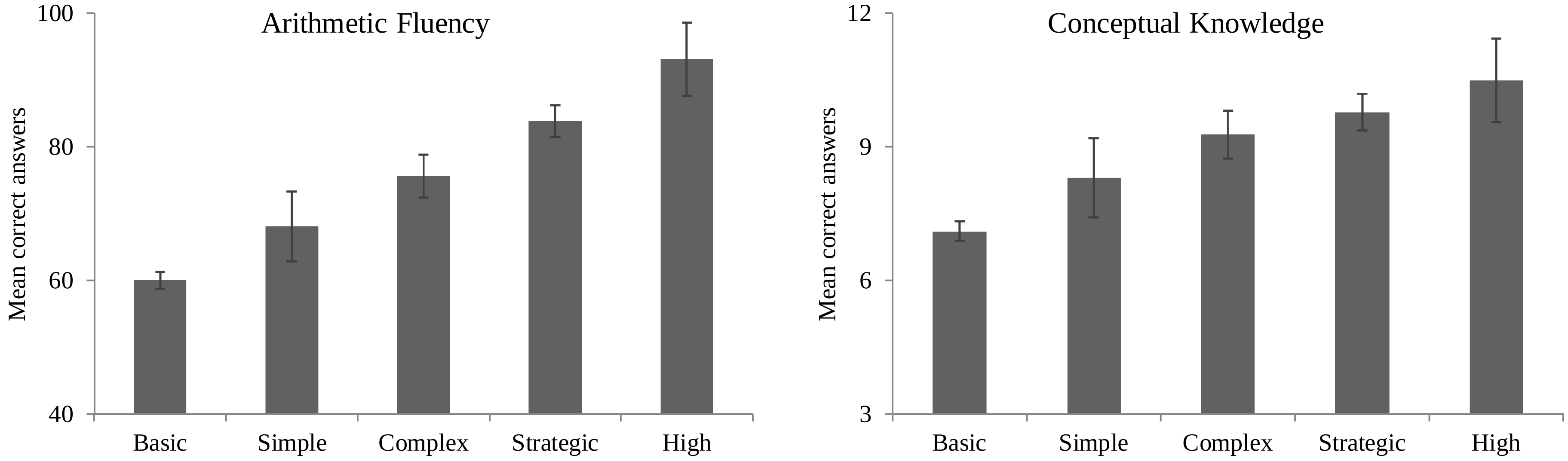

Figure 2 Mean correct answers for arithmetic fluency and arithmetic conceptual knowledge by adaptive number knowledge profile. Error bars represent ±2 standard error, range determined by ±2 standard deviation from sample mean.

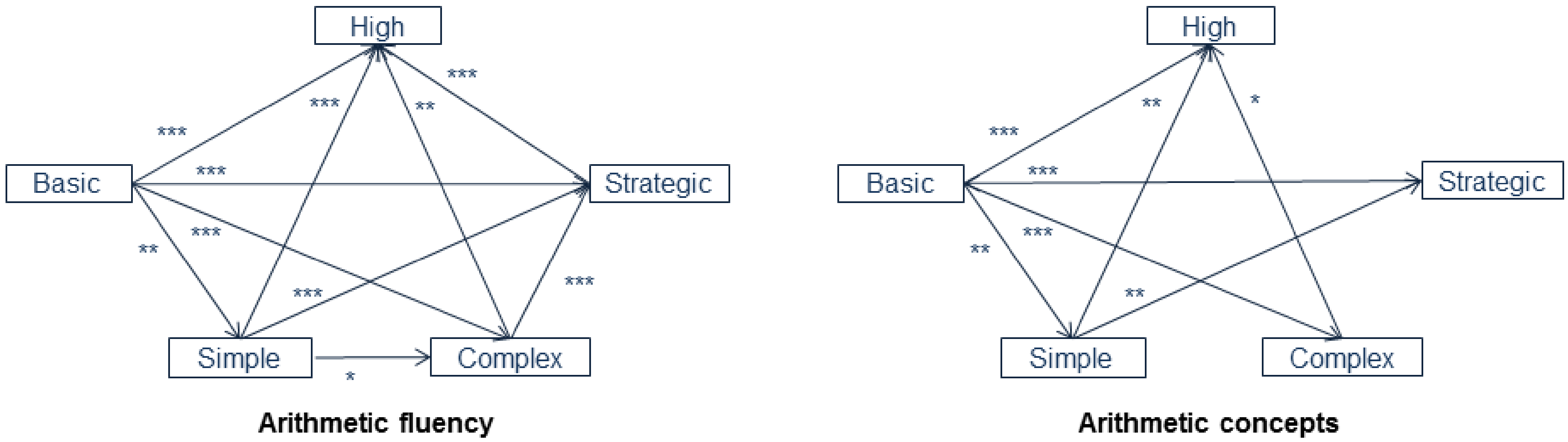

Figure 3 Post-hoc comparisons between adaptive number knowledge profiles. Arrows indicate direction of significant differences. *p<.05; **p<.01; ***p<.001