# The Solid-Metric Dimension

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**Abstract**

Resolving sets are designed to locate an object in a network by measuring the distances to the object. However, if there are more than one object present in the network, this can lead to wrong conclusions. To overcome this problem, we introduce the concept of solid-resolving sets. In this paper, we study the structure and constructions of solid-resolving sets. In particular, we classify the forced vertices with respect to a solid-resolving set. We also give bounds on the solid-metric dimension utilizing concepts like the Dilworth number, the boundary of a graph, and locating-dominating sets. It is also shown that deciding whether there exists a solid-resolving set with a certain number of elements is an NP-complete problem.

**Keywords:** resolving set, metric dimension, solid-metric dimension, detection of several objects.

## 1 Introduction

The graphs considered in this paper are simple, undirected, and finite. Let $G$ be such a graph. We denote the vertex set of $G$ by $V(G)$ and the edge set by $E(G)$ (or simply $V$ and $E$ if the graph in question is clear from the context). The *distance* $d_G(v,u)$ between two vertices $v$ and $u$ is the length of any shortest path between them. The *diameter* of the graph $G$ is denoted by $\operatorname{diam}(G)$ and defined as $\operatorname{diam}(G) = \max_{v,u\in V}\{d_G(v,u)\}$. If $d_G(v,u) = 1$, we say that $v$ and $u$ are *adjacent*, and denote $v \sim u$. The *open neighbourhood* of a vertex $v$ is defined as $N_G(v) = \{u \in V(G) \mid v \sim u\}$, and the *closed neighbourhood* as $N_G[v] = N_G(v) \cup \{v\}$. The *degree* $\deg_G(v)$ of a vertex $v$ is the cardinality of the open neighbourhood $N_G(v)$. We omit the subscripts if the graph in question is clear from the context. A *clique* is a subset of vertices whose elements are all (pairwise) adjacent. The *path* and the *complete graph* with $n$ vertices are denoted by $P_n$ and $K_n$, respectively.

The *Cartesian product* of $G$ and $H$ is denoted by $G\square H$ and defined as follows: the set of vertices is $V(G\square H) = V(G) \times V(H) = \{au \mid a \in V(G),\ u \in V(H)\}$ and there is an edge between $x = x_1x_2 \in V(G\square H)$ and $y = y_1y_2 \in V(G\square H)$ if either 1) $x_1 = y_1$ and $x_2 \sim y_2$, or 2) $x_1 \sim y_1$ and $x_2 = y_2$. Notice that $d_{G\square H}(au, bv) = d_G(a,b) + d_H(u,v)$.

Consider a graph $G$ with the vertex set $V$. Let $R = \{r_1,\dots,r_k\} \subseteq V$. The *distance array* of a vertex $v$ with respect to $R$ is defined as $\mathcal{D}_R(v) = (d(r_1,v),\dots,d(r_k,v))$. If each vertex has a unique distance array, then the set $R$ is called a *resolving set* of $G$. Resolving sets were introduced independently by Slater [19] and Harary and Melter [10] in the 1970's, and since then many new variations have been presented. Resolving sets have applications, for example, in robot navigation [14], chemistry [3], and network discovery and verification [1].

Let $X$ be a nonempty set of vertices with unknown elements. If $|X| = 1$, we can determine the elements of $X$ with a resolving set. What happens when $X$ has two or more elements? First of all, we define the distance (see [15, 9]) from a vertex $v$ to the set $X$ as $d(v,X) = \min_{x\in X}\{d(v,x)\}$. The distance array of the set $X$ with respect to $R$ is $\mathcal{D}_R(X) = (d(r_1,X),\dots,d(r_k,X))$. Furthermore, we denote $\mathcal{D}_R(\{v_1,\dots,v_n\}) = \mathcal{D}_R(v_1,\dots,v_n)$. Let us consider the square grid $P_9\square P_7$ in Figure 1 with the resolving set $R = \{r_1,r_2\}$ (see [14]). For $X = \{x,y\}$ we have $\mathcal{D}_R(X) = (7,5)$. However, for the vertex $u$ we also have $\mathcal{D}_R(u) = (7,5)$. Now the use of a resolving set leads us to thinking

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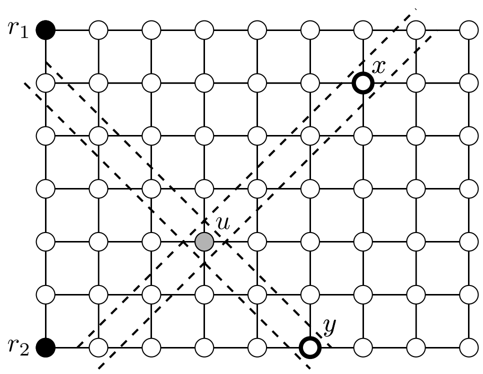



Figure 1: The square grid $P_9 \square P_7$. The vertices that are at the same distance as $x$ from $r_1$ or $y$ from $r_2$ are outlined with the dashed lines.

that $X = \{u\}$, which is false. For these situations, we define a new class of resolving sets that can differentiate one vertex from a set of vertices with two or more elements.

**Definition 1.1.** Let $G$ be a graph with the vertex set $V$. A set $S \subseteq V$ is a *solid-resolving set* of $G$ if for all vertices $x \in V$ and nonempty subsets $Y \subseteq V$,

$$\mathcal{D}_S(x) = \mathcal{D}_S(Y)$$

implies that $Y = \{x\}$. The minimum cardinality of a solid-resolving set is called the *solid-metric dimension* of $G$, and it is denoted by $\beta^s(G)$.

It is clear that $V(G)$ is a solid-resolving set of $G$. Indeed, to determine the elements of $X$ we only need to check which elements of $\mathcal{D}_V(X)$ are zeros. Thus, every graph has a solid-resolving set, and we can focus on finding the solid-metric dimension of the graph. It is easy to see from the previous definition that a solid-resolving set is always a usual resolving set. Thus, a solid-resolving set can determine one vertex accurately. If the distance array given by the solid-resolving set does not fit any one vertex, then we know that there are at least two objects in the graph. Similar questions have been studied for identifying codes [12] and locating-dominating sets [13].

We will find an easy characterisation for solid-resolving sets in Section 2. Moreover, we discuss the vertices which are forced (if any) to be in a solid-resolving set. In Section 3, we consider the relations between solid-resolving sets and other concepts such as locating-dominating sets, the boundary of a graph, and other classes of resolving sets. We also consider the solid-metric dimensions of trees and cycles. In Section 4, we study the solid-metric dimension of the $n$-dimensional hypercube. We establish a connection of solid-resolving sets to the vicinal preorder and the Dilworth number in Section 5. In Section 6, we prove bounds for the solid-metric dimensions of Cartesian and strong product of graphs. Then, in Section 7, we give upper bounds for the maximum degree and clique number of the graph when the solid-metric dimension is given. Finally, in Section 8, we prove that deciding whether there exists a solid-resolving set of given size is an NP-complete problem.

## 2 Basics

Before we begin to consider the solid-metric dimension or the connection to other concepts, we make some observations on the structure of solid-resolving sets. In the following remark, we first discuss the case where the considered graph is disconnected.

**Remark 2.1.** Let $G$ be a disconnected graph with $k$ connected components $G_1, \ldots, G_k$. Let $S$ be a solid-resolving set of $G$, and denote $S_i = S \cap V(G_i)$ for all $i \in \{1, \ldots, k\}$. The set $S_i$ must be nonempty for all $i$. Otherwise there is a vertex $u \in S_i$ such that $d(s, u) = \infty$ for all $s \in S$ and we have $\mathcal{D}_S(x) = \mathcal{D}_S(x, u)$ for all $x \in V \setminus \{u\}$. Furthermore, each $S_i$ must be a solid-resolving set of $G_i$.

By the previous remark, it suffices from here on to consider nontrivial connected graphs, that is, connected graphs with at least two vertices. The next result gives us a useful characterisation for solid-resolving sets.

**Theorem 2.2.** *Let $G$ be a nontrivial connected graph with the vertex set $V$. A set $S \subseteq V$ is a solid-resolving set of $G$ if and only if for all distinct $x, y \in V$ there is an element $s \in S$ such that*

$$d(s, x) < d(s, y). \tag{1}$$

*Proof.* Assume that $S \subseteq V$ satisfies (1). Let $x \in V$ and let $Y$ be a nonempty subset of $V$. Assume that $Y \neq \{x\}$, i.e., there exists an element $y \in Y$ such that $y \neq x$. Since $S$ satisfies (1), there is an element $s \in S$ such that $d(s, y) < d(s, x)$. Now $d(s, Y) < d(s, x)$, and clearly we have $\mathcal{D}_S(x) \neq \mathcal{D}_S(Y)$. Thus, the set $S$ is a solid-resolving set of $G$.

Let $S$ be a solid-resolving set. Assume to the contrary that (1) does not hold. Then there are two distinct vertices $x, y \in V$ such that $d(s, x) \leq d(s, y)$ for all $s \in S$. Let $Y = \{x, y\}$. Now $d(s, x) = d(s, Y)$ for all $s \in S$, and therefore $\mathcal{D}_S(x) = \mathcal{D}_S(Y)$, a contradiction. □

Consider again the example in Figure 1. Clearly, we have $d(r_2, u) < d(r_2, x)$. However, since $d(r_1, u) = d(r_1, x)$, there does not exist any $r_i \in R$ such that $d(r_i, x) < d(r_i, u)$. Now (1) is not satisfied and the set $R$ is not a solid-resolving set of $P_9 \square P_7$ according to Theorem 2.2.

The following result is an immediate corollary of the previous characterisation.

**Corollary 2.3.** *Let $G$ be a nontrivial connected graph. Then $\beta^s(G) \geq 2$.*

In light of Theorem 2.2, it is easy to see that a superset of a solid-resolving set is also a solid-resolving set. Indeed, if we add elements to a solid-resolving set, the set still satisfies (1). The structure of a graph may demand that a specific vertex must be included in any solid-resolving set. Next we will define these vertices and study their properties.

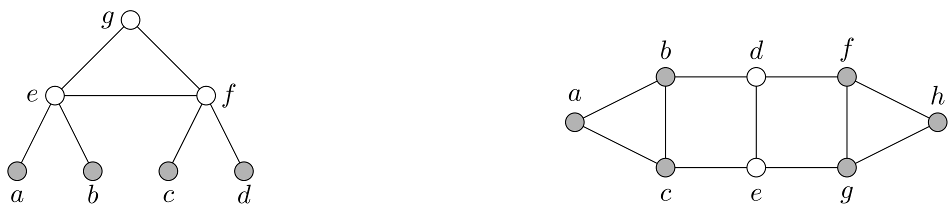
**Definition 2.4.** A vertex $u \in V$ is called a *forced vertex* of a solid-resolving set of $G$ if it must be included in any solid-resolving set of $G$.

**Theorem 2.5.** *A vertex $u \in V$ is a forced vertex of a solid-resolving set $G$ if and only if there exists a vertex $v \in V \setminus \{u\}$ such that $N(u) \subseteq N[v]$.*

*Proof.* Consider distinct vertices $u, v \in V$ such that $N(u) \subseteq N[v]$. We will show that $S = V \setminus \{u\}$ is not a solid-resolving set. Assume to the contrary that $S$ is a solid-resolving set. According to Theorem 2.2, there exists an element $s \in S$ such that $d(s, u) < d(s, v)$. Since $s \neq u$, a shortest path from $s$ to $u$ must go through an element of $N(u)$, say $x$. Now we have $d(s, u) = d(s, x) + 1$. Since $x \in N[v]$, we have $d(s, v) \leq d(s, x) + 1 = d(s, u)$ (a contradiction). Thus, the set $S$ and all its subsets are not solid-resolving sets according to Theorem 2.2.

Assume then that $u$ is a vertex such that $N(u) \not\subseteq N[v]$ for all $v \in V \setminus \{u\}$. We will show that $S = V \setminus \{u\}$ is a solid-resolving set by utilizing Theorem 2.2. Clearly, for any distinct $v_1, v_2 \in S$ we have $d(v_1, v_1) < d(v_1, v_2)$ and $d(v_2, v_2) < d(v_2, v_1)$. We need only to show that (1) holds for $u$ and any $x \in S$. It is clear that $d(x, x) < d(x, u)$. Furthermore, for each $x \in S$ there exists $y \in N(u)$ such that $y \notin N[x]$. Now $d(y, u) = 1$ and $d(y, x) \geq 2$, and thus $d(y, u) < d(y, x)$. Now $S$ is a solid-resolving set according to Theorem 2.2. □

Distinct vertices $u$ and $v$ are called *true twins* if $N[u] = N[v]$, and *false twins* if $N(u) = N(v)$. According to Theorem 2.5 both true and false twins are forced vertices of a solid-resolving set. However, as we have seen in the previous theorem, there also exist other kinds of forced vertices.




(a) The graph $G_1$. The gray vertices are the elements of the set $K$.

(b) The graph $G_2$. The gray vertices are the boundary vertices of $G_2$.

Figure 2: Examples for the $k$-metric dimension and the boundary of a graph.

# 3 Related concepts

In this section, we explore the connections between solid-resolving sets and some other types of resolving sets. We also consider other similar concepts, namely, the boundary of a graph and locating-dominating sets.

A vertex set $R \subseteq V$ is a *doubly resolving set* of $G$ if for any distinct $v, u \in V$ we have $d(v,r) - d(u,r) \neq d(v,s) - d(u,s)$ for some $r, s \in R$. Doubly resolving sets were first introduced and studied in [2]. According to Theorem 2.2, a solid-resolving set is also a doubly resolving set. However, a doubly resolving set is not necessarily a solid-resolving set. Indeed, consider the graph $G_1$ in Figure 2(a). It is easy to see that the set $K = \{a, b, c, d\}$ is a doubly resolving set of $G$. These vertices are also forced vertices of a solid-resolving set. However, the vertex $g$ is also a forced vertex, and thus the set $K$ cannot be a solid-resolving set.

The set $R \subseteq V$ is a *$k$-resolving set* of $G$ if for any distinct $v, u \in V$ we have $d(r,v) \neq d(r,u)$ for at least $k$ distinct $r \in R$ (see [7, 6]). Again, by Theorem 2.2, it is clear that a solid resolving set is a 2-resolving set. However, when we consider the graph $G_1$, we notice that the set $K$ is a 2-resolving set, but not a solid-resolving set. In general, if we have two vertices $x$ and $y$ with distance arrays, say, $\mathcal{D}_R(x) = (2,2,2)$ and $\mathcal{D}_R(y) = (3,3,3)$, these distance arrays are acceptable for a 3-resolving set. However, they do not satisfy (1), since for all $r \in R$ we have $d(r,y) > d(r,x)$. Thus, these cannot be distance arrays given by a solid-resolving set. Furthermore, there are graphs for which there does not exist any $k$-resolving set when $k \geq 3$ (see [7, 6]). On the other hand, each graph $G$ has at least one solid-resolving set, namely $V(G)$.

The set $R \subseteq V$ is an *$\{\ell\}$-resolving set* of $G$, where $1 \leq \ell \leq |V|$, if for any nonempty subsets $X, Y \subseteq V$ such that $|X| \leq \ell$ and $|Y| \leq \ell$ we have $\mathcal{D}_R(X) = \mathcal{D}_R(Y)$ if and only if $X = Y$. The smallest possible cardinality of an $\{\ell\}$-resolving set is called the *$\{\ell\}$-metric dimension* of $G$, and it is denoted by $\beta_\ell(G)$. Notice that if $\ell = 1$, then the definition of a $\{1\}$-resolving set is equivalent to the definition of a resolving set. The $\{\ell\}$-metric dimension has been studied in [15, 9]. It is clear by definition that a solid-resolving set is not an $\{\ell\}$-resolving set when $\ell \geq 2$. Indeed, although a solid-resolving set can distinguish a vertex from a larger vertex set, it cannot necessarily differentiate vertex sets with two elements from each other. However, an $\{\ell\}$-resolving set with $\ell \geq 2$ always satisfies (1). Otherwise, there exist two vertices $x$ and $y$ such that the sets $\{y\}$ and $\{x, y\}$ have the same distance array. Therefore, an $\{\ell\}$-resolving set is a solid-resolving set when $\ell \geq 2$. Consequently, $\beta^s(G) \leq \beta_\ell(G)$ for all $\ell \geq 2$.

In the following theorem, we show that the metric dimension of a graph is always strictly smaller than the solid-metric dimension.

**Theorem 3.1.** *Let $S$ be a solid-resolving set of $G$. The set $S \setminus \{s\}$ is a resolving set of $G$ for any element $s \in S$. Thus, $\beta_1(G) \leq \beta^s(G) - 1$.*

*Proof.* According to Theorem 2.2, for each pair of distinct vertices $u, v \in V$ there are two elements $s, t \in S$ such that $d(s,u) < d(s,v)$ and $d(t,u) > d(t,v)$. Since $s \neq t$, we can remove either one of them, and still be able to distinguish between $u$ and $v$ as well as between any two distinct vertices. ☐

## 3.1 The boundary of a graph

**Definition 3.2.** The *boundary* of a connected graph $G$ is the set

$$\partial(G) = \{v \in V \mid \exists\, u \in V,\ \forall\, w \in N(v) : d(u,w) \leq d(u,v)\}.$$

The elements of the boundary are called *boundary vertices.*

Boundary vertices were first introduced in [4]. The vertex $v$ is called an *extreme vertex* if the induced subgraph $G[N(v)]$ is a clique. Extreme vertices are not only boundary vertices but also forced vertices of a solid-resolving set since for any $u \in N(v)$ we have $N(v) \subseteq N[u]$.

In the following theorem, we consider connections of a boundary and a solid-resolving set of a graph.

**Theorem 3.3.** *Let $G$ be a finite nontrivial connected graph.*

*(i) The boundary of $G$ is a solid-resolving set of $G$.*

*(ii) If $v$ is a forced vertex of a solid-resolving set, then it is a boundary vertex.*

*(iii) There are graphs which have solid-resolving sets of minimum cardinality that are not subsets of the boundary.*

*Proof.* (i) We will show that $\partial(G)$ satisfies (1). Consider distinct vertices $v, u \in V$. If $v \in \partial(G)$, then we have $d(v,v) < d(v,u)$. Assume that $v \notin \partial(G)$. Then there is a vertex $w \in N(v)$ such that $d(w,u) > d(v,u)$. Now $d(w,u) = d(w,v) + d(v,u)$ and $d(w,v) < d(w,u)$. If $w \in \partial(G)$, then we are done. However, if $w \notin \partial(G)$, then we repeat the same procedure for $w$. That is, since $w \notin \partial(G)$, there is a vertex $w' \in N(w)$ such that $d(u,w') > d(u,w)$. The vertex $w'$ cannot be $v$ or one of its neighbours, since otherwise $d(u,w') \leq d(u,w)$. Thus, we have $d(u,w') = d(u,w) + d(w,w') = d(u,v) + d(v,w')$ and $d(w',v) < d(w',u)$. Again, if $w' \in \partial(G)$, then we are done. Otherwise, we continue in this fashion, and since $G$ is finite, we will eventually find a vertex $x \in \partial(G)$ such that $d(x,v) < d(x,u)$ since the boundary of the graph is always nonempty. Indeed, if the vertices $a, b \in V$ are such that $d(a,b) = \operatorname{diam}(G)$, then $a, b \in \partial(G)$ and $|\partial(G)| \geq 2$.

(ii) Since $v$ is a forced vertex, there exists a vertex $u \in V \setminus \{v\}$ such that $N(v) \subseteq N[u]$. Now $d(u,w) \leq 1$ for all $w \in N(v)$. However, $d(u,v) \geq 1$ and thus $d(u,w) \leq d(u,v)$ for all $w \in N(v)$ and $v$ is a boundary vertex.

(iii) Consider the graph $G_2$ in Figure 2(b). Clearly, $\partial(G_2) = \{a,b,c,f,g,h\}$. First we will show that $\beta^s(G_2) \geq 4$. The vertices $a$ and $h$ are forced vertices of a solid-resolving set, however, it is easy to see that the set $\{a,h\}$ is not a solid-resolving set. The set $A = \{a,b,h\}$ is not a solid-resolving set, since $d(x,c) \geq d(x,b)$ for all $x \in A$. Similarly, the set $B = \{a,d,h\}$ is not a solid-resolving set, since $d(x,e) \geq d(x,d)$ for all $x \in B$. All other vertex sets with $a$, $h$, and a third element are isomorphic to either $A$ or $B$. Thus, we have $\beta^s(G_2) \geq 4$. It is easy to verify that the set $S = \{a,d,e,h\}$ is a solid-resolving set of $G_2$. However, the set $S_2$ is not a subset of $\partial(G_2)$. Thus, a solid-resolving set (of minimum cardinality) is not necessarily a subset of the boundary. □

**Remark 3.4.** According to Theorem 3.3, we now have $\beta^s(G) \leq |\partial(G)|$. It was shown in [11] that the boundary of a graph is a resolving set, and consequently $\beta_1(G) \leq |\partial(G)|$. However, we now improve this to $\beta_1(G) \leq |\partial(G)| - 1$ according to Theorem 3.1. This bound is attained for paths and complete graphs; indeed, we have $\beta_1(P_n) = 1 = |\partial(P_n)| - 1$ and $\beta_1(K_n) = n - 1 = |\partial(K_n)| - 1$.

We denote by $G - v$ the graph we obtain by removing from $G$ the vertex $v$ and all edges adjacent to it. A vertex $v$ of a connected graph $G$ is called a *cut-vertex* if the graph $G - v$ is disconnected.

**Theorem 3.5.** *Let $G$ be a nontrivial connected graph with a cut-vertex $v$. If $S \subseteq V$ is a solid-resolving set of $G$ such that $v \in S$, then $S \setminus \{v\}$ is also a solid-resolving set of $G$. Specifically, a solid-resolving set of minimum cardinality does not contain any cut-vertices.*

*Proof.* We will show that we can satisfy (1) with $S \setminus \{v\}$. The subgraph $G - v$ has at least two connected components. Let $G_1$ and $G_2$ be any two such components. It is easy to see (as in Remark 2.1) that if $V(G_i) \cap S = \emptyset$, then (1) does not hold for $S$. Thus, we have $V(G_1) \cap S \neq \emptyset$ and $V(G_2) \cap S \neq \emptyset$. To show that $S \setminus \{v\}$ satisfies (1) we first need to compare two vertices in the same connected component as well as two vertices in different connected components of $G - v$. Let $s_1 \in V(G_1) \cap S$ and $s_2 \in V(G_2) \cap S$. In what follows, we consider pairs of vertices for which $v$ satisfies (1) for $S$. If $u_1, u_2 \in V(G_1)$ are such that $d(u_1, v) < d(u_2, v)$, then

$$d(u_1, s_2) = d(u_1, v) + d(v, s_2) < d(u_2, v) + d(v, s_2) = d(u_2, s_2).$$

If for some $u \in V(G_1)$ and $w \in V(G_2)$ we have $d(u, v) < d(w, v)$, then

$$d(u, s_1) \leq d(u, v) + d(v, s_1) < d(w, v) + d(v, s_1) = d(w, s_1).$$

Now, we only need to check that (1) is satisfied for $v$. If $u \in V(G_1)$, then we have $d(s_2, v) < d(s_2, u)$ (and similarly for other connected components). Thus, $S \setminus \{v\}$ is a solid-resolving set of $G$. □

Next we will demonstrate how the previous results can be used to determine the solid-metric dimension of trees.

**Theorem 3.6.** *Let $T$ be a tree with leaves $L$, i.e., $L = \{v \in V \mid \deg(v) = 1\}$. The set $L$ is the unique minimal solid-resolving set of $T$. Furthermore, if $G$ is a nontrivial connected graph, then $\beta^s(G) = 2$ if and only if $G = P_n$.*

*Proof.* According to Theorem 2.5, a leaf is a forced vertex. If a vertex of $T$ is not a leaf, then it is a cut-vertex, and thus is not needed in any solid-resolving set according to Theorem 3.5. Therefore, a set $S \subseteq V$ is a solid-resolving set of $T$ if and only if $L \subseteq S$ and the first claim follows.

Consider then a nontrivial connected graph $G$. Clearly, if $G$ is a path, then we have $\beta^s(G) = 2$. Conversely, if $\beta^s(G) = 2$, then according to Theorem 3.1 we have $\beta_1(G) = 1$. A well-known result states that $\beta_1(G) = 1$ if and only if $G = P_n$, see [14, 3]. Therefore, the graph $G$ must be a path. □

It is clear that for a cycle $C_n$ we have $\partial(C_n) = V$. However, as we will see next, the solid-metric dimension of a cycle is very small. Thus, the difference between $\beta^s(G)$ and $|\partial(G)|$ can be very large for some graphs.

**Theorem 3.7.** *Let $C_n$ be a cycle of $n \geq 3$ vertices. Then*

$$\beta^s(C_n) = \begin{cases} 4, & \textit{when } n = 4, \\ 3, & \textit{otherwise.} \end{cases}$$

*Proof.* In [14], it was shown that $\beta_1(C_n) = 2$ for all $n \geq 3$. Thus, according to Theorem 3.1, we have $\beta^s(C_n) \geq 3$ for all $n \geq 3$. This is all we need for the case $n = 3$. Consider then the cycle of four vertices. Let $x, y \in V$ such that $d(x, y) = 2$. Since $N(x) \subseteq N[y]$, $x$ is a forced vertex according to Theorem 2.5. Thus, we have $\beta^s(C_4) = 4$.

To complete our proof, we need to show that $\beta^s(C_n) \leq 3$ when $n \geq 5$. We will prove this by finding a solid-resolving set with three elements. Define $S = \{s_1, s_2, s_3\}$ in such a way that its elements are spread evenly along the cycle. In other words, if we denote $d(s_1, s_2) = d$, then without loss of generality we may assume that $d \leq d(s_2, s_3) \leq d(s_1, s_3) \leq d + 1$.

We will show that $S$ is a solid-resolving set by considering pairs of vertices. Consider distinct vertices $x, y \in V$. The proof now divides into the following cases depending on whether $x$ and $y$ belong to $S$ or not.

- If $x, y \in S$, then the vertices themselves satisfy (1).
- If $x \in S$ and $y \notin S$, then clearly $d(x, x) < d(x, y)$. Denote $S = \{s_i, s_j, x\}$. If $d(s_i, y) \geq d(s_i, x)$ and $d(s_j, y) \geq d(s_j, x)$, then $y = x$ (since $n \geq 5$). Thus, $d(s_i, y) < d(s_i, x)$ or $d(s_j, y) < d(s_j, x)$.

- If $x, y \notin S$ and they both are along the same shortest path from $s$ to $t$ for some $s, t \in S$, then we have $d(s, x) < d(s, y)$ and $d(t, y) < d(t, x)$ (or $d(t, x) < d(t, y)$ and $d(s, y) < d(s, x)$). Assume that $x, y \notin S$ are such that they are not along the same shortest path from one element of $S$ to another. Denote $S = \{s_i, s_j, s_k\}$, and let $x$ be along the path from $s_i$ to $s_j$ and $y$ along the path from $s_j$ to $s_k$. If $d(s_i, x) \geq d(s_i, y)$, then $d(s_i, s_k) \leq d(s_i, s_j) - 2$, which is a contradiction. Now we have $d(s_i, x) < d(s_i, y)$, and similarly $d(s_k, y) < d(s_k, x)$.

Thus, $S$ is a solid-resolving set of $C_n$ according to Theorem 2.2. □

## 3.2 Locating-dominating sets

Locating-dominating sets were first introduced by Slater in [20, 21, 17]. For more literature on location-domination and related subjects we refer to [16]. Let $C$ be a subset of $V$ and denote $I(C; u) = N[u] \cap C$, where $u \in V$.

**Definition 3.8.** A set $C \subseteq V$ is *locating-dominating* in $G$ if for all distinct $u, v \in V \setminus C$ we have $I(C; u) \neq \emptyset$ and $I(C; u) \neq I(C; v)$. The cardinality of an optimal locating-dominating set, i.e., a locating-dominating set with the minimum size, is denoted by $\gamma^{LD}(G)$.

The following definition of self-locating-dominating sets is due to [13].

**Definition 3.9.** A set $C \subseteq V$ is *self-locating-dominating* in $G$ if for all $u \in V \setminus C$ we have $I(C; u) \neq \emptyset$ and

$$\bigcap_{c \in I(C;u)} N[c] = \{u\}.$$

The minimum cardinality of a self-locating-dominating set of $G$ is denoted by $\gamma^{SLD}(G)$.

**Theorem 3.10.** *Let $G$ be a nontrivial connected graph and $C$ a subset of $V$.*

*(i) If $C$ is locating-dominating, then it is a resolving set of $G$.*

*(ii) If $C$ is self-locating-dominating, then it is a solid-resolving set of $G$.*

*(iii) If $\operatorname{diam}(G) = 2$ and $S \subseteq V$ is a solid-resolving set of $G$, then $S$ is also a self-locating-dominating set in $G$. Consequently, $\beta^s(G) = \gamma^{SLD}(G)$ if $\operatorname{diam}(G) = 2$.*

*Proof.* (i) Let $C$ be a locating-dominating set in $G$. Then for all distinct $u, v \in V \setminus S$ we have $I(C; u) \neq I(C; v)$. Thus, there exists $c \in C$ such that $d(c, u) \neq d(c, v)$. Furthermore, if $u \in C$, then clearly $d(u, u) \neq d(u, v)$ for all $v \in V \setminus \{u\}$. Thus, $C$ is a resolving set.

(ii) Let $C$ be a self-locating-dominating set in $G$. Consider a vertex $u \in V \setminus C$. If for some $v \in V$ we have $d(c, u) \geq d(c, v)$ for all $c \in I(C; u)$, then $d(c, v) \leq 1$ for all $c \in I(C; u)$. Consequently, we have

$$v \in \bigcap_{c \in I(C;u)} N[c].$$

Since $C$ is self-locating-dominating, this implies that $v = u$. Thus, if $v \neq u$, then for some $c \in I(C; u)$ we have $d(c, u) < d(c, v)$ and the condition (1) holds. Furthermore, if $u \in C$, then (1) is immediately satisfied. Now, the set $C$ is a solid-resolving set of $G$ according to Theorem 2.2.

(iii) Let $G$ be a graph with $\operatorname{diam}(G) = 2$ and $S$ be a solid-resolving set of $G$. Consider distinct vertices $u \in V \setminus S$ and $v \in V$. Since $S$ is a solid-resolving set, there exists a vertex $s \in S$ such that $d(s, u) < d(s, v)$. Since $u \notin S$ and $\operatorname{diam}(G) = 2$, we have $d(s, u) = 1$ and $d(s, v) = 2$. Now $u \in N[s]$ and $v \notin N[s]$. Therefore, we have

$$\bigcap_{s \in I(S;u)} N[s] = \{u\},$$

and $S$ is self-locating-dominating in $G$. □

By the previous theorem, for the graphs $G$ with $\mathrm{diam}(G) = 2$, we know that $S$ is a solid-resolving set of $G$ if and only if $S$ is self-locating-dominating in $G$. The analogous result for locating-dominating and resolving sets also almost holds. However, if $S$ is a resolving set of $G$, then there might exist one vertex $v$ such that $\mathcal{D}_S(v) = (2, \ldots, 2)$, i.e., $I(S; v) = \emptyset$. In this case, $S$ is not a locating-dominating set, but $S \cup \{v\}$ is. Thus, together with the previous theorem, we have $\gamma^{LD}(G) - 1 \leq \beta_1(G) \leq \gamma^{LD}(G)$, when $\mathrm{diam}(G) = 2$. For the rook's graph $K_m \Box K_n$ we have $\mathrm{diam}(K_m \Box K_n) = 2$, and these bounds are achieved; for exact values of $\beta_1(K_m \Box K_n)$ and $\gamma^{LD}(K_m \Box K_n)$ see [2] and [13], respectively. According to Theorem 3.10, we have $\beta^s(K_m \Box K_n) = \gamma^{SLD}(K_m \Box K_n)$. In [13], the authors determined $\gamma^{SLD}(K_m \Box K_n)$, and the next result is immediate.

**Corollary 3.11.** *For the rook's graph $K_m \Box K_n$ we have*

$$\beta^s(K_m \Box K_n) = \begin{cases} n, & n \geq 2m, \text{ or } m = 1, \\ 2m, & 2m > n > m \geq 2, \\ 2m - 1, & n = m > 2, \\ 4, & n = m = 2. \end{cases}$$

## 4 The solid-metric dimension of the hypercube

Let $\mathbb{F}_2^n = \{0, 1\}^n$. We define the *$n$-dimensional hypercube* $Q_n$ as a graph with $V(Q_n) = \mathbb{F}_2^n$, and two vertices are adjacent if and only if they differ in exactly one coordinate place. The *weight* of $v \in V(Q_n)$ is defined as the number of 1's in $v$ and it is denoted by $w(v)$.

**Theorem 4.1.** *For the hypercube $Q_n$, with $n \geq 3$, we have*

*(i)* $\beta^s(Q_n) \leq n + 1$,

*(ii)* $\beta^s(Q_n) \leq 2 \cdot \beta_1(Q_n)$,

*(iii)* $2 \leq \lim_{n \to \infty} \dfrac{\beta^s(Q_n) \log_2 n}{n} \leq 4$.

*Proof.* (i) Let $S = \{v \in V \mid w(v) = 1 \text{ or } w(v) = n\}$. We will show that (1) holds for $S$. Consider distinct $u, v \in V$. Without loss of generality, we can assume that $w(u) \leq w(v)$.

If $w(v) > w(u)$, then $d(s, v) < d(s, u)$ for $s \in S$ such that $w(s) = n$. If $w(u) \geq 1$, then there clearly exists a vertex $s \in S$ such that $w(s) = 1$ and $d(s, u) < d(s, v)$ ($s$ has a common 1 with $u$). Assume that $w(u) = 0$. If $1 \leq w(v) \leq 2$, then there exists a vertex $s \in S$ such that $w(s) = 1$ and $d(s, v) \geq 2$ (since $n \geq 3$). Now we have $d(s, u) < d(s, v)$. If $w(v) \geq 3$, then $d(s, u) < d(s, v)$ for all $s \in S$ such that $w(s) = 1$.

If $0 < w(u) = w(v) < n$, then for some integer $i$ the $i$th symbol of $u$ is 1 and the $i$th symbol of $v$ is 0. Similarly, for some integer $j$ the $j$th symbol of $u$ is 0 and the $j$th symbol of $v$ is 1. Let $s, t \in S$ such that $w(s) = w(t) = 1$ and the $i$th symbol of $s$ is 1 and the $j$th symbol of $t$ is 1. Now $d(s, u) < d(s, v)$ and $d(t, v) < d(t, u)$. Thus, $S$ is a solid-resolving set of $Q_n$ according to Theorem 2.2, and $\beta^s(Q_n) \leq |S| = n + 1$.

(ii) Let $R$ be a resolving set of $Q_n$. Let $R'$ consist of the unique vertices $r' \in V$ such that $d(r', r) = \mathrm{diam}(Q_n) = n$ for some $r \in R$. We will show that the set $S = R \cup R'$ is a solid-resolving set of $Q_n$.

Consider distinct $u, v \in V$. Since $R$ is a resolving set, we have $d(r, u) \neq d(r, v)$ for some $r \in R$. Without loss of generality we can assume that $d(r, u) < d(r, v)$. Let $s \in R'$ be such that $d(s, r) = n$. Now

$$d(s, u) = n - d(r, u) > n - d(r, v) = d(s, v).$$

Thus, the set $S$ satisfies (1) and is a solid-resolving set of $Q_n$. Now we have $\beta^s(Q_n) \leq |S| \leq 2 \cdot \beta_1(Q_n)$.

(iii) In [18], it was shown (by an existence proof, not by a construction) that

$$\lim_{n\to\infty} \frac{\beta_1(Q_n)\cdot \log_2 n}{n} = 2.$$

Now the claim follows immediately from (ii). $\square$

# 5 Bounds from the vicinal preorder

A preorder is a binary relation, which is reflexive and transitive. In the set of vertices of a graph, the relation

$$x \preccurlyeq y \quad \text{if and only if} \quad N(x) \subseteq N[y]$$

gives a preorder. This is called the *vicinal preorder*, see [8]. A subset $B \subseteq V$ is an *antichain* if neither $x \preccurlyeq y$ nor $y \preccurlyeq x$ for any distinct $x, y \in B$. The maximal cardinality of an antichain in $V$ with respect to the vicinal preorder is the *Dilworth number* $\nabla(G)$ of a graph, see [5]. We denote $x \prec y$ if $x \preccurlyeq y$ and not $y \preccurlyeq x$. A vertex $x \in V$ is *maximal*, if there does not exist an element $y \in V \setminus \{x\}$ such that $x \prec y$. A finite graph has at least one maximal vertex.

Next we will give bounds on the solid-metric dimension utilizing the vicinal preorder.

**Theorem 5.1.** *Let $G$ be a nontrivial connected graph with $n$ vertices. We have*

*(i) $n - \nabla(G) \leq \beta^s(G)$,*

*(ii) $\beta^s(G) = n$ if and only if every maximal vertex has a true or false twin.*

*Proof.* (i) Assume that $S \subseteq V$ is a solid-resolving set of cardinality $\beta^s(G)$. Denote $B = V \setminus S$. We claim that the set $B$ is an antichain with respect to the vicinal preorder. Let $x, y \in B$ and $x \neq y$. Assume to the contrary that $x \preccurlyeq y$. Consequently, $N(x) \subseteq N[y]$ and $x$ is a forced vertex according to Theorem 2.5. This contradicts the fact that $S$ is a solid-resolving set. Hence, $B$ is an antichain. Utilizing the Dilworth number, this yields that

$$n - \beta^s(G) = |V \setminus S| \leq \nabla(G).$$

(ii) Recall that in the proof of Theorem 2.5 we saw that if a vertex $v$ is not a forced vertex, then $V \setminus \{v\}$ is a solid-resolving set. Thus, if $\beta^s(G) = n$, then every vertex is a forced vertex of a solid-resolving set. Due to Theorem 2.5, this implies that for any $x \in V$ there exists $y \in V \setminus \{x\}$ such that $N(x) \subseteq N[y]$, i.e., $x \preccurlyeq y$. This is true also for a maximal vertex $x$ and, thus, $x$ has a true or false twin $y$. Assume next that each maximal vertex has a true or false twin. Let $x \in V$. If $x$ is not maximal, then there exists a vertex $y$ such that $x \preccurlyeq y$. Due to the assumption, this holds also when $x$ is maximal. Consequently, $N(x) \subseteq N[y]$ and $x$ is a forced vertex. Therefore, $\beta^s(G) = n$. $\square$

The bounds in the previous theorem can be attained as will be seen next. A *threshold graph* is a graph which is constructed by starting with a single vertex and using repeatedly one of the following two operations: either 1) add an isolated vertex to the graph, or 2) add a vertex which is adjacent to all of the previous vertices, i.e., a dominating vertex. Denote by $X_n$ a threshold graph and by $V_n = \{v_1, v_2, \dots, v_n\}$ its set of vertices where the vertices have been added in the given order. Since the threshold graph with *at most one edge* has clearly $\beta^s(X_n) = n$ (recall Remark 2.1), we assume that there is at least two edges in $X_n$.

**Corollary 5.2.** *Let $X_n$ be a (possibly disconnected) threshold graph with at least two edges and denote by $k'$ the maximum $k \in \{1, \dots, n\}$ such that $v_k$ was added as a dominating vertex. If $v_{k'-1}$ was not added as a dominating vertex, we have $\beta^s(X_n) = n - 1$, and otherwise, $\beta^s(X_n) = n$. In particular, $\beta^s(K_n) = n$.*

*Proof.* It is well known that the Dilworth number of a threshold graph satisfies $\nabla(X_n) = 1$. Hence, Theorem 5.1(i) states that $\beta^s(X_n) \geq n-1$. Next we show that $\beta^s(X_n) = n-1$ if $v_{k'-1}$ was not added as a dominating vertex. To that end, we verify that $S = V \setminus \{v_{k'}\}$ is a solid-resolving set of $X_n$. Since $v_{k'-1}$ was not added as a dominating vertex, the vertex $v_{k'}$ is a cut-vertex. Now the set $S$ is a solid-resolving set of $X_n$ according to Theorem 3.5, and thus $\beta^s(X_n) = n-1$. Assume then that $v_{k'-1}$ was added as a dominating vertex. Now $v_{k'}$ and $v_{k'-1}$ (and all dominating vertices added directly before these) are the only maximal vertices and also true twins. By Theorem 5.1(ii), we have $\beta^s(X_n) = n$. □

# 6 The solid-metric dimension of graph products

We consider two widely studied graph products; the Cartesian product and the strong product. We give upper and lower bounds for the solid-metric dimensions and characterise the forced vertices in both products. If the subscript is omitted (e.g. $N[ab]$), we mean the product graph.

## 6.1 The Cartesian product

We consider first the forced vertices of the Cartesian product.

**Theorem 6.1.** *Let $G$ and $H$ be nontrivial connected graphs. A vertex $v = v_1v_2 \in V(G\square H)$ is a forced vertex of a solid-resolving set of $G\square H$ if and only if $\deg_G(v_1) = 1$ and $\deg_H(v_2) = 1$.*

*Proof.* Assume first that $\deg_G(v_1) = 1$ and $\deg_H(v_2) = 1$. Let $N_G(v_1) = \{u_1\}$ and $N_H(v_2) = \{u_2\}$. Now $N(v_1v_2) = \{v_1u_2, u_1v_2\} \subseteq N[u_1u_2]$ and $v$ is a forced vertex.

Assume then that $\deg_G(v_1) \geq 2$ (the case where $\deg_H(v_2) \geq 2$ goes similarly). Now $N_G(v_1)$ contains at least two distinct vertices $a_1$ and $a_2$. Let $b$ be a vertex in $N_H(v_2)$ (indeed, such a vertex exists as $H$ is a nontrivial connected graph). Then $\{v_1b, a_1v_2, a_2v_2\} \subseteq N(v)$. Assume that there exists a vertex $u = u_1u_2 \in V(G\square H)$ such that $u \neq v$ and $N(v) \subseteq N[u]$.

- If $u_1 = v_1$, then $a_1v_2 \in N[u]$ implies that $u_2 = v_2$, and thus $u = v$.
- If $u_2 = v_2$, then $v_1b \in N[u]$ implies that $u_1 = v_1$, and thus $u = v$.
- If $u_1 \neq v_1$ and $u_2 \neq v_2$, then $a_1v_2 \in N[u]$ and $a_2v_2 \in N[u]$ together imply that $u_1 = a_1 = a_2$.

Thus, $v$ is not a forced vertex. □

In the next theorem, we give bounds for the solid-metric dimension of a Cartesian product of graphs. The *projection* of a set $U \subseteq V(G\square H)$ onto $G$ consists of vertices $u \in V(G)$ such that $uv \in U$ for some $v \in V(H)$.

**Theorem 6.2.** *Let $G$ and $H$ be nontrivial connected graphs.*

*(i) Let $S$ be a solid-resolving set of $G\square H$. Then the projection of $S$ onto $G$ is a solid-resolving set of $G$. Similarly, the projection of $S$ onto $H$ is a solid-resolving set of $H$.*

*(ii) Let $T$ and $U$ be solid-resolving sets of $G$ and $H$, respectively. Then the set $S = T \times U$ is a solid-resolving set of $G\square H$.*

*(iii) We have $\max\{\beta^s(G), \beta^s(H)\} \leq \beta^s(G\square H) \leq \beta^s(G) \cdot \beta^s(H)$.*

*Proof.* (i) We will prove the claim for the projection onto $G$, the latter part of the claim follows from the isomorphicity of $G\square H$ and $H\square G$. Consider distinct vertices $av, bv \in V(G\square H)$. Since $S$ satisfies (1), there exists a vertex $st \in S$ such that $d(st, av) < d(st, bv) \Leftrightarrow d_G(s,a) + d_H(t,v) < d_G(s,b) + d_H(t,v) \Leftrightarrow d_G(s,a) < d_G(s,b)$. Thus, the projection of $S$ onto $G$ is a solid-resolving set of $G$.

(ii) Consider vertices $a, b \in V(G)$ and $v, w \in V(H)$ such that $a \neq b$ and $v \neq w$. Since $T$ is a solid-resolving set of $G$, there exists a vertex $t \in T$ such that $d_G(a,t) < d_G(b,t)$. Similarly, since

$U$ is a solid-resolving set of $H$, there exists a vertex $u \in U$ such that $d_H(v,u) < d_H(w,u)$. Since $d(av,tu) = d_G(a,t) + d_H(v,u)$, we have

$$\begin{aligned} d(av,tu) &< d_G(b,t) + d_H(v,u) = d(bv,tu), \\ d(av,tu) &< d_G(a,t) + d_H(w,u) = d(aw,tu), \\ d(av,tu) &< d_G(b,t) + d_H(w,u) = d(bw,tu). \end{aligned}$$

This shows that $S$ is a solid-resolving set of $G \square H$ according to Theorem 2.2.

(iii) The bounds follow immediately from (i) and (ii). □

In Section 4, we saw that $\beta^s(Q_3) \leq 4$ (see Theorem 4.1). Since we can define the 3-dimensional hypercube as the Cartesian product $Q_3 = C_4 \square P_2$, we have $\max\{\beta^s(C_4), \beta^s(P_2)\} \leq \beta^s(Q_3)$. Now, $\beta^s(Q_3) = 4$, according to Theorem 3.7, and the lower bound in Theorem 6.2 (iii) is obtained. The upper bound can be attained with trees as explained in the following corollary.

**Corollary 6.3.** *Let $T_1$ and $T_2$ be trees. Then $\beta^s(T_1 \square T_2) = \beta^s(T_1) \cdot \beta^s(T_2)$. In particular, $\beta^s(P_m \square P_n) = 4$.*

*Proof.* By the proof of Theorem 3.6, we immediately know that the leaves of the trees are the forced vertices of $T_1$ and $T_2$. Moreover, the leaves of a tree always form a solid-resolving set. Therefore, by Theorem 6.1, there are $\beta^s(T_1) \cdot \beta^s(T_2)$ forced vertices in $T_1 \square T_2$. Thus, the claim follows due to Theorem 6.2. □

## 6.2 The strong product

The *strong product* of $G$ and $H$ is denoted by $G \boxtimes H$. The set of vertices is $V(G \boxtimes H) = V(G) \times V(H) = \{au \mid a \in V(G),\ u \in V(H)\}$. There is an edge between $x = x_1x_2 \in V(G \boxtimes H)$ and $y = y_1y_2 \in V(G \boxtimes H)$ if we have either 1) $x_1 = y_1$ and $x_2 \sim y_2$, or 2) $x_1 \sim y_1$ and $x_2 = y_2$, or 3) $x_1 \sim y_1$ and $x_2 \sim y_2$. Notice that $d(au,bv) = \max\{d_G(a,b), d_H(u,v)\}$. It is clear by the definition that $N_{G \boxtimes H}[av] = N_G[a] \times N_H[v]$.

**Theorem 6.4.** *Let $G$ and $H$ be nontrivial connected graphs.*

*(i) Let $S$ be a solid-resolving set of $G \boxtimes H$. Then the projection of $S$ onto $G$ is a solid-resolving set of $G$. Similarly, the projection of $S$ onto $H$ is a solid-resolving set of $H$.*

*(ii) Let $T$ and $U$ be solid-resolving sets of $G$ and $H$, respectively. Then the set $S = \{tu \in V(G \boxtimes H) \mid t \in T \text{ or } u \in U\}$ is a solid-resolving set of $G \boxtimes H$.*

*(iii) Let $\Lambda(G,H) = \beta^s(G) \cdot |V(H)| + \beta^s(H) \cdot |V(G)| - \beta^s(G) \cdot \beta^s(H)$. Then we have*

$$\max\{\beta^s(G), \beta^s(H)\} \leq \beta^s(G \boxtimes H) \leq \Lambda(G,H).$$

*Proof.* (i) It is enough to consider the projection onto $G$. Consider distinct vertices $av, bv \in V(G \boxtimes H)$. Since (1) holds for $S$, there exists a vertex $st \in S$ such that

$$d(st,av) < d(st,bv) \Leftrightarrow \max\{d_G(s,a), d_H(t,v)\} < \max\{d_G(s,b), d_H(t,v)\}.$$

Now if $d(st,av) = d_G(s,a)$, then $d_H(t,v) \leq d_G(s,a) < d(st,bv) = d_G(s,b)$. Similarly, if $d(st,av) = d_H(t,v)$, then $d_G(s,a) \leq d_H(t,v) < d(st,bv) = d_G(s,b)$. Thus, we have $d_G(s,a) < d_G(s,b)$, and the projection of $S$ onto $G$ is a solid-resolving set of $G$.

(ii) Consider vertices $a,b \in V(G)$ and $v,w \in V(H)$ such that $a \neq b$ and $v \neq w$. Since $T$ is a solid-resolving set of $G$, there exists a vertex $t \in T$ such that $d_G(a,t) < d_G(b,t)$. Similarly, since $U$ is a solid-resolving set of $H$, there exists a vertex $u \in U$ such that $d_H(v,u) < d_H(w,u)$. Now we have

$$\begin{aligned} d(av,tv) &= d_G(a,t) < d_G(b,t) = d(bv,tv), \\ d(av,au) &= d_H(v,u) < d_H(w,u) = d(aw,au), \\ d(av,tu) &= \max\{d_G(a,t), d_H(v,u)\} < \max\{d_G(b,t), d_H(w,u)\} = d(bw,tu). \end{aligned}$$

Since $tv, au, tu \in S$, the set $S$ is a solid-resolving set of $G \boxtimes H$ according to Theorem 2.2.

(iii) The lower bound follows immediately from (i). If we choose $T$ and $U$ in (ii) to be minimal solid-resolving sets, then we have $|S| = \Lambda(G, H)$, and the upper bound follows. □

To attain the upper bound we consider the forced vertices of $G \boxtimes H$. We will show that if $G$ or $H$ has a forced vertex of a specific type, then every instance of that vertex produces a forced vertex into the product graph. Notice the difference between the next theorem and Theorem 2.5.

**Theorem 6.5.** *Let $G$ and $H$ be nontrivial connected graphs. A vertex $av \in V(G \boxtimes H)$ is a forced vertex of a solid-resolving set of $G \boxtimes H$ if and only if $N_G[a] \subseteq N_G[b]$ for some $b \in V(G) \setminus \{a\}$ or $N_H[v] \subseteq N_H[u]$ for some $u \in V(H) \setminus \{v\}$.*

*Proof.* Let $a, b \in V(G)$ such that $a \neq b$ and $N_G[a] \subseteq N_G[b]$. We have

$$N[av] = N_G[a] \times N_H[v] \subseteq N_G[b] \times N_H[v] = N[bv].$$

Thus, $av$ is a forced vertex for any $v \in V(H)$. Similarly, if $N_H[v] \subseteq N_H[u]$ for some distinct $v, u \in V(H)$, then $av$ is a forced vertex for any $a \in V(G)$.

Consider then vertices $a \in V(G)$ and $v \in V(H)$ such that $N_G[a] \nsubseteq N_G[b]$ for all $b \in V(G) \setminus \{a\}$ and $N_H[v] \nsubseteq N_H[u]$ for all $u \in V(H) \setminus \{v\}$. Assume that the vertex $av$ is a forced vertex. Then, by Theorem 2.5, $N(av) \subseteq N[st]$ for some vertex $st \in V(G \boxtimes H)$ such that $s \neq a$ or $t \neq v$. If $s \neq a$, then there is a vertex $a' \in N_G[a]$ such that $a' \notin N_G[s]$. Let $v' \in N_H(v)$. Now $a'v' \in N(av)$ and $a'v' \notin N_G[s] \times N_H[t] = N[st]$. Thus, $N(av) \nsubseteq N[st]$. Similarly, if $t \neq v$, then $N(av) \nsubseteq N[st]$. Therefore, $av$ is not a forced vertex. □

Now we can reach the upper bound of Theorem 6.4(iii) as follows. If $G$ and $H$ are graphs with minimum solid-resolving sets consisting of the special types of forced vertices described in the previous theorem, then we have $\beta^s(G \boxtimes H) = \Lambda(G, H)$. For example, if $T_1$ and $T_2$ are trees, then by (the proof of) Theorem 3.6 the trees $T_1$ and $T_2$ meet the previous requirements and we have $\beta^s(T_1 \boxtimes T_2) = \Lambda(T_1, T_2)$.

# 7 Maximum degree and clique number

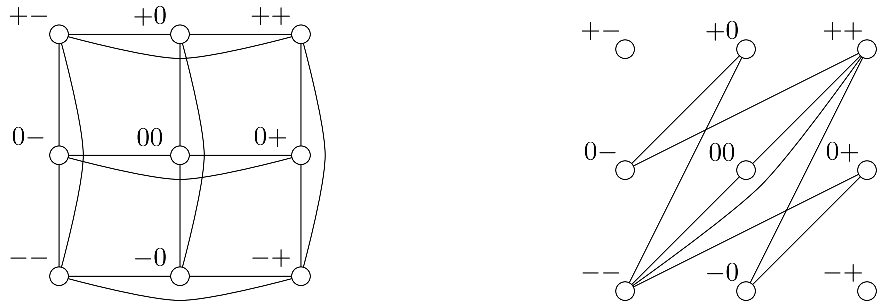
We denote by $\Delta(G)$ the *maximum degree* of the graph $G$, that is, $\Delta(G) = \max_{v \in V}\{\deg(v)\}$. The *clique number* $\omega(G)$ of $G$ is the cardinality of the largest clique in $G$. A vertex set is *independent* if no two elements are adjacent. An independent set is *maximal* if it does not have a proper independent superset. The *independence number* $\alpha(G)$ of $G$ is the cardinality of any largest possible maximal independent set of $G$. A *maximum independent set* is an independent set of cardinality $\alpha(G)$.

Our aim is to determine how large $\Delta(G)$ and $\omega(G)$ can be when we know $\beta^s(G)$. To that end, we fix a vertex $u$ and compare the distance arrays of $u$ and its neighbours. Assume that $\beta^s(G) = k$ and let $S = \{s_1, \ldots, s_k\}$ be a solid-resolving set of $G$. We denote $D_u(v) = \mathcal{D}_S(v) - \mathcal{D}_S(u)$ and $D_u(X) = \{D_u(x) \mid x \in X\}$. If $v \in N(u)$ and $d(s_i, v) = d(s_i, u)$ for some $i \in \{1, \ldots, k\}$, then the $i$th element of $D_u(v)$ is 0. Otherwise, either $d(s_i, v) = d(s_i, u) + 1$ or $d(s_i, v) = d(s_i, u) - 1$, and the $i$th element of $D_u(v)$ is 1 or $-1$, respectively. When we use the notation $\{-1, 0, 1\} = \{-, 0, +\}$, we have $D_u(N[u]) \subseteq \{-, 0, +\}^k$.

We define the relation $\lhd$ on $\{-, 0, +\}^k$ as follows:

- when $a, b \in \{-, 0, +\}$, we have $a \lhd b$ if $a = b$ or $(a, b) \in \{(-, 0), (0, +), (-, +)\}$,
- when $a, b \in \{-, 0, +\}^k$, where $k \geq 2$, $a = a_1 \ldots a_k$, and $b = b_1 \ldots b_k$, we have $a \lhd b$ if $a_i \lhd b_i$ for all $i = 1, \ldots, k$.

This relation is a coordinatewise partial order on $\{-, 0, +\}^k$. Consider distinct $x, y \in N[u]$. If $D_u(x) \lhd D_u(y)$, then $d(s, x) \leq d(s, y)$ for all $s \in S$, and (1) is not satisfied. Thus, the set $D_u(N[u])$ must be an antichain with respect to this relation. Since $D_u(u) = 0 \ldots 0 \in D_u(N[u])$,




(a) $K_3^2$ (b) $\mathcal{N}_2 - E(K_3^2)$

Figure 3: The graph $\mathcal{N}_2$.

the maximum cardinality of an antichain that contains $0 \dots 0$ gives us an upper bound for $|N[u]|$ and consequently for $\Delta(G)$.

We define a graph $\mathcal{N}_k$ whose vertices correspond to the elements of $\{-, 0, +\}^k$, and distinct vertices $x$ and $y$ are adjacent if and only if $x \lhd y$ or $y \lhd x$. Now, finding the largest antichain with $0 \dots 0$ is the same as finding the maximum independent set containing $0 \dots 0$.

The exact structure of $\mathcal{N}_k$ is rather complicated. Therefore, we focus on its substructures that are easier to handle. We denote by $G^n$ the Cartesian product of $n$ copies of $G$.

**Lemma 7.1.** *The graph $K_3^k$ is (isomorphic to) a spanning subgraph of $\mathcal{N}_k$.*

*Proof.* Let $V(K_3) = \{-, 0, +\}$. Obviously, we have $K_3 = \mathcal{N}_1$. By the definition of the Cartesian product, we have $V(K_3^k) = \{-, 0, +\}^k = V(\mathcal{N}_k)$, and distinct vertices $x = x_1 \dots x_k$ and $y = y_1 \dots y_k$ are adjacent in $K_3^k$ if and only if $x_i \sim y_i$ for exactly one $i \in \{1, \dots, k\}$ and $x_j = y_j$ for all $j \neq i$. Consequently, either $x \lhd y$ or $y \lhd x$. Thus, $K_3^k$ is a spanning subgraph of $\mathcal{N}_k$. □

Notice that if $H$ is a spanning subgraph of $G$, then $\alpha(G) \leq \alpha(H)$. It is easy to see that $\mathcal{N}_k$ consists of three copies of $\mathcal{N}_{k-1}$ (by considering the last coordinate). Furthermore, $\mathcal{N}_2$ and $\mathcal{N}_3$ are induced subgraphs of any $\mathcal{N}_k$, where $k \geq 3$.

**Theorem 7.2.** *If $\beta^s(G) = 2$, then $\Delta(G) \leq 2$. If $\beta^s(G) = k \geq 3$, then $\Delta(G) \leq 7 \cdot 3^{k-3} - 1$.*

*Proof.* Consider first the case where $\beta^s(G) = 2$. We will show that $\alpha(\mathcal{N}_2) = 3$ by using Lemma 7.1. Consider the subgraph $K_3^2$, see Figure 3(a). Each row or column of vertices can contain at most one element of an independent set. Thus, we have $\alpha(\mathcal{N}_2) \leq 3$. The set $N = \{+-, 00, -+\}$ is an independent set of $K_3^2$. From Figure 3(b), we can see that $N$ is also an independent set of $\mathcal{N}_2 - E(K_3^2)$. Thus, the set $N$ is an independent set of $\mathcal{N}_2$ containing 00, and we have $\alpha(\mathcal{N}_2) = 3$. Now, we have $|D_u(N[u])| \leq |N|$ for every $u \in V(G)$, and thus $\Delta(G) \leq 2$.

Assume then that $\beta^s(G) = k \geq 3$. The vertices of $\mathcal{N}_k$ can be partitioned into $3^{k-3}$ sets, each of which induces a copy of $\mathcal{N}_3$. Each copy of $\mathcal{N}_3$ can be partitioned further into three vertex sets, each of which induces a copy of $\mathcal{N}_2$. From Figure 3, it is easy to see that the set $N$ is the unique maximum independent set of $\mathcal{N}_2$. If in one copy of $\mathcal{N}_3$ there are two copies of $\mathcal{N}_2$ that have the same "pattern" $N$, the set is not independent. Thus, we can choose at most seven vertices from one copy of $\mathcal{N}_3$, and we have $\Delta(G) \leq 7 \cdot 3^{k-3} - 1$. □

The upper bound in Theorem 7.2 is attained for the graph in Figure 4. If the induced subgraph $G[N[u]]$ is a clique, then each coordinate of the elements of $D_u(N[u])$ can contain either 0 and + or 0 and −. Indeed, if $u$ has neighbours $v$ and $w$ such that the $i$th symbol of $D_u(v)$ is + and the $i$th symbol of $D_u(w)$ is −, then we must have $v \nsim w$. If we permute the coordinates of the elements of $D_u(N[u])$, the partial ordering $\lhd$ is preserved and so is the structure of $\mathcal{N}_k$. Thus, we can assume that $D_u(N[u])$ is a subset of $\{+, 0\}^p \times \{-, 0\}^{k-p}$ for some $0 \leq p \leq k$. Let $\mathcal{M}_{k,p}$ be the induced subgraph of $\mathcal{N}_k$ with the vertex set $V(\mathcal{M}_{k,p}) = \{+, 0\}^p \times \{-, 0\}^{k-p}$ where $0 \leq p \leq k$. If $D_u(N[u])$

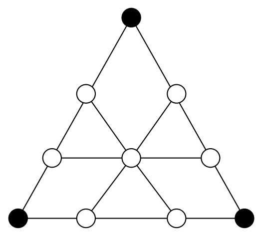


Figure 4: A graph for which the upper bounds in Theorems 7.2 and 7.4 are sharp. The black vertices form a minimum solid-resolving set of the graph.

is not independent in $\mathcal{M}_{k,p}$, then (1) does not hold. Now, finding the maximum independent set containing $0\dots0$ in $\mathcal{M}_{k,p}$ gives us an upper bound for $\omega(G)$.

**Lemma 7.3.** *The graph $K_2^k$ is (isomorphic to) a spanning subgraph of $\mathcal{M}_{k,p}$.*

*Proof.* Denote by $K^+$ the complete graph with the vertex set $\{0,+\}$ and by $K^-$ the complete graph with the vertex set $\{0,-\}$. Consider the Cartesian product

$$G = \underbrace{K^+\square\dots\square K^+}_{p \text{ times}}\square\underbrace{K^-\square\dots\square K^-}_{k-p \text{ times}},$$

where the parameters $p$ and $k$ are the same as for the graph $\mathcal{M}_{k,p}$. Now, we have $V(G) = V(\mathcal{M}_{k,p})$ and $G$ is isomorphic to $K_2^k$. Consider distinct vertices $x = x_1\dots x_k$ and $y = y_1\dots y_k$ of $G$. By the definition, $x$ and $y$ are adjacent if and only if $x_i \sim y_i$ for exactly one $i \in \{1,\dots,k\}$ and $x_j = y_j$ for all $j \neq i$. Now either $x \lhd y$ or $y \lhd x$ and $G$ is a spanning subgraph of $\mathcal{M}_{k,p}$. □

**Theorem 7.4.** *If $\beta^s(G) = 2$, then $\omega(G) \leq 2$. If $\beta^s(G) = k \geq 3$, then $\omega(G) \leq 3\cdot 2^{k-3}$.*

*Proof.* Denote $\beta^s(G) = k$ and $z = 0\dots0 \in V(\mathcal{M}_{k,p})$. If $p = 0$ or $p = k$, then we have $z \sim v$ for all $v \in V(\mathcal{M}_{k,p})$, $v \neq z$. Thus, we have $1 \leq p \leq k-1$.

Assume first that $k = 2$. Then we have $p = 1$ and $V(\mathcal{M}_{2,1}) = \{00, +0, 0-, +-\}$. Clearly, the maximum independent set containing 00 is $\{00, +-\}$, and thus we have $\omega(G) \leq 2$.

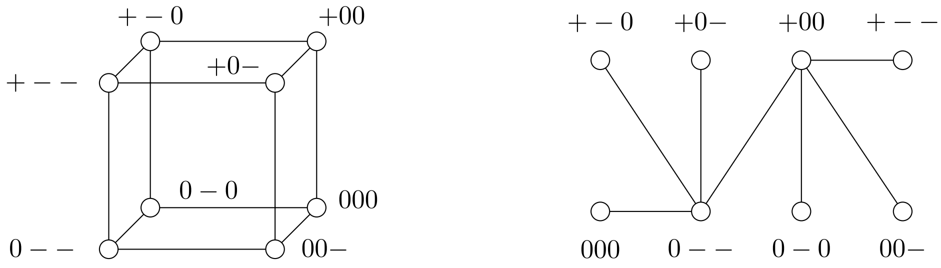
Assume then that $k \geq 3$ and $1 \leq p \leq k-2$. Now we can permute the labels of the vertices of $\mathcal{M}_{k,p}$ such that the first coordinate contains only 0 or $+$ and the next two 0 or $-$. Utilizing Lemma 7.3, we can partition the vertices of $\mathcal{M}_{k,p}$ into $2^{k-3}$ sets, each of which induces a copy of $\mathcal{M}_{3,1}$ (see Figure 5(a)). It is clear that for $K_2^3$ we have $\alpha(K_2^3) = 4$, and all the maximum independent sets are $\{+-0, +0-, 0--, 000\}$ and $\{+--, +00, 0-0, 00-\}$. However, neither of these is an independent set of $\mathcal{M}_{3,1}$. Indeed, from Figures 5(a) and 5(b) we notice that the vertices $0--$ and $+00$ are adjacent to all other vertices of $\mathcal{M}_{3,1}$. Thus, the maximal independent sets of $\mathcal{M}_{3,1}$ containing either of these are $\{0--\}$ and $\{+00\}$. Therefore, we have $\alpha(\mathcal{M}_{3,1}) \leq 3$. Now, each copy of $\mathcal{M}_{3,1}$ can contain at most three elements of the maximum independent set of $\mathcal{M}_{k,p}$, and we have $\omega(G) \leq 3\cdot 2^{k-3}$.

If $p = k-1$, then we can switch all pluses into minuses and vice versa, and consider the same case as above. □

Again, the upper bound in Theorem 7.4 can be attained for the graph in Figure 4. Our next result shows how we can construct graphs with very large cliques.

**Theorem 7.5.** *There exists a graph $G$ such that $\beta^s(G) = k$ and*

$$\omega(G) = \binom{k}{\lfloor k/2\rfloor}.$$




(a) $K_2^3$ (b) $\mathcal{M}_{3,1} - E(K_2^3)$

Figure 5: The graph $\mathcal{M}_{3,1}$.

*Proof.* Let $p = \lfloor k/2 \rfloor$. We construct a set $X \subseteq \{0,+\}^p \times \{0,-\}^{k-p}$ by combining a prefix from $\{0,+\}^p$ with $i$ pluses with a suffix from $\{0,-\}^{k-p}$ with $i$ minuses. We have

$$|X| = \sum_{i=0}^{p} \binom{p}{i}\binom{k-p}{i} = \binom{p+k-p}{p} = \binom{k}{\lfloor k/2 \rfloor}.$$

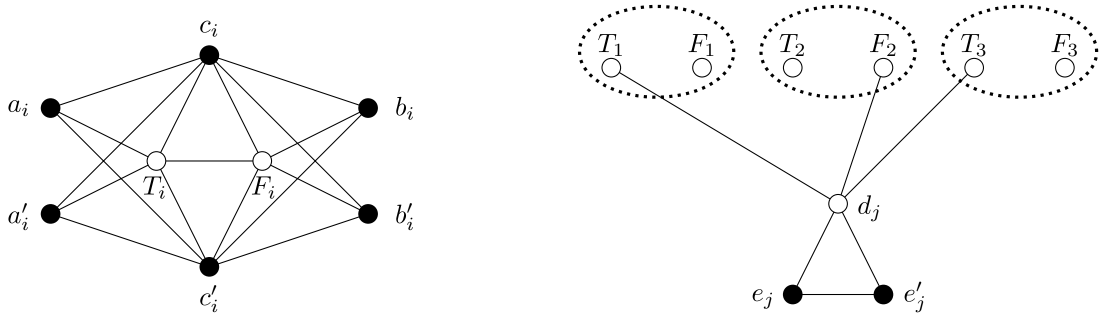
We will show that the set $X$ is (i) an antichain with respect to the partial order $\lhd$ and (ii) realisable as a graph.

(i) Assume to the contrary that for some distinct $x, y \in X$ we have $x \lhd y$. Denote by $x_+$ and $y_+$ the prefixes and by $x_-$ and $y_-$ the suffixes of $x$ and $y$. Since $x \lhd y$, we have $x_+ \lhd y_+$ and $x_- \lhd y_-$. Thus, if $y_+$ has $i$ pluses and $y_-$ has $i$ minuses, then $x_+$ has at most $i$ pluses and $x_-$ has at least $i$ minuses. Since $x_+$ has the same amount of pluses as $x_-$ has minuses (by the definition of $X$), $x_+$ has exactly $i$ pluses and $x_-$ has exactly $i$ minuses. Now $x_+ \lhd y_+$ and $x_- \lhd y_-$ if and only if $x_+ = y_+$ and $x_- = y_-$ (a contradiction).

(ii) We construct the graph $G$ as follows:

- $V(G) = X \cup S$, where $S = \{s_1, \ldots, s_k\}$,
- for all distinct $x, y \in X$ we have $x \sim y$,
- for all distinct $s_i, s_j \in S$ we have $s_i \nsim s_j$,
- when $i \leq p$, $x \in X$, and $s_i \in S$, we have $x \sim s_i$ if and only if the $i$th coordinate of $x$ is 0,
- when $i > p$, $x \in X$, and $s_i \in S$, we have $x \sim s_i$ if and only if the $i$th coordinate of $x$ is $-$.

The induced subgraph $G[X]$ is a clique. Since $N(s_i) \subseteq N[x]$ for all $s_i \in S$ and $x \in X$, all elements of $S$ are forced vertices (of a solid-resolving set) according to Theorem 2.5. Thus, we have $\beta^s(G) \geq k$. To conclude our proof, we will show that $S$ is a solid-resolving set of $G$ by Theorem 2.2. For all distinct $x, y \in X$ the condition (1) holds, since otherwise we have $x \lhd y$ or $y \lhd x$, which contradicts (i). We then compare an element of $X$ and an element of $S$. If $k = 2$ or $k = 3$, each element of $X$ is adjacent to exactly one element of $S$ and vice versa. Thus, $d(s, x) \leq 2$ for all $s \in S$ and $x \in X$, and $d(s_i, s_j) = 3$ for all distinct $s_i, s_j \in S$. Therefore, $d(s_i, x) < d(s_i, s_j)$ for all distinct $s_i, s_j \in S$ and $x \in X$. If $k \geq 4$, each element of $X$ is adjacent to at least two elements of $S$. Indeed, the prefixes and suffixes are now long enough that either the prefix has at least two zeros, the suffix has at least two minuses, or the prefix has exactly one zero and the suffix has exactly one minus. Now, if $x \in X$ and $s, t \in S$ such that $s, t \in N(x)$, then we have $d(s, x) < d(s, t)$ and $d(t, x) < d(t, s)$. Thus, the set $S$ satisfies (1) and is a solid-resolving set of $G$. □




(a) Variable gadget of $x_i$



(b) Clause gadget of $\mathcal{C}_j = x_1 \vee \overline{x}_2 \vee x_3$

Figure 6: The gadgets of the graph $G'$ illustrated.

# 8 Algorithmic complexity

In this section, we consider the algorithmic complexity of determining the solid-metric dimension of a graph. As we have seen earlier in the paper, this problem is algorithmically easy, i.e., there exists a polynomial (or even linear) time algorithm for solving the problem, in various graph families such as paths, cycles, trees, rook's graphs, complete graphs and threshold graphs. However, in what follows, we show that in general graphs the problem is NP-complete as is the case of determining the (regular) metric dimension of a graph (see [14]). More precisely, we prove that deciding whether the solid-metric dimension of a graph is at most $k \in \mathbb{N}$ is NP-complete by showing that a polynomial time algorithm solving the decision problem would also provide an efficient solution to the 3-satisfiability (3-SAT) problem, which is well known to be NP-complete.

**Theorem 8.1.** *If $G$ is a graph and $k$ is an integer, then deciding whether the solid-metric dimension $\beta^s(G) \leq k$ is an NP-complete problem.*

*Proof.* It is immediate by Theorem 2.2 that the problem of deciding whether $\beta^s(G) \leq k$ belongs to NP. In what follows, we prove that the problem is also NP-complete by a polynomial time reduction of the 3-satisfiability (3-SAT) problem to the decision problem regarding the solid-metric dimension of a graph.

For the 3-SAT problem, denote the set of variables by $X = \{x_1, \ldots, x_n\}$ and the set of literals by $U = \{x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n\}$, where $\overline{x}_i$ denotes the negation of the variable $x_i$. Let $F$ be an instance of the 3-SAT problem; more precisely, let $F$ be a formula $F = \mathcal{C}_1 \wedge \ldots \wedge \mathcal{C}_m$, where each clause $\mathcal{C}_j$ contains exactly three literals, i.e., each clause is of the form $\mathcal{C}_j = u_{j,1} \vee u_{j,2} \vee u_{j,3}$ ($u_{j,1}, u_{j,2}, u_{j,3} \in U$). Based on the given formula $F$, we form a graph $G' = (V', E')$ as follows:

- For each variable $x_i \in X$, we construct a variable gadget of $x_i$ with vertices $a_i$, $a'_i$, $b_i$, $b'_i$, $c_i$, $c'_i$, $T_i$ and $F_i$ and edges as described in Figure 6(a).
- For each clause $\mathcal{C}_j = u_{j,1} \vee u_{j,2} \vee u_{j,3}$, we construct a clause gadget $\mathcal{C}_j$ with vertices $d_j$, $e_j$ and $e'_j$ and edges $d_je_j$, $d_je'_j$ and $e_je'_j$. Moreover, if $u_{j,1} = x_i$, then $d_j$ is adjacent to $T_i$, else $u_{j,1} = \overline{x}_i$ and $d_j$ is adjacent to $F_i$. Analogous edges are also added with respect to the other literals $u_{j,2}$ and $u_{j,3}$ (the clause gadget is illustrated in Figure 6(b)).
- Finally, we add a *universal vertex* $u$ which is adjacent to the vertices $a_i$, $a'_i$, $b_i$, $b'_i$, $c_i$ and $c'_i$ for every $i$ in the variable gadgets and to the vertices $d_j$, $e_j$ and $e'_j$ for every $j$ in the clause gadgets.

It is immediate that the graph $G'$ can be constructed in polynomial time.

Let $S$ be a solid-resolving set of the graph $G'$. Let us first present some useful observations concerning $S$:

- The vertices $a_i$ and $a'_i$ are forced vertices of $S$ (by Theorem 2.5) since $N(a_i) = \{u, c_i, c'_i, T_i\} = N(a'_i)$. Analogously, $b_i$ and $b'_i$ also belong to $S$. Similarly, the vertices $c_i$ and $c'_i$ as well as $e_j$ and $e'_j$ are forced since $N(c_i) = \{u, a_i, a'_i, b_i, b'_i, T_i, F_i\} = N(c'_i)$ and $N[e_j] = \{u, d_j, e_j, e'_j\} = N[e'_j]$, respectively.
- At least one of the vertices $T_i$ and $F_i$ in the variable gadget of $x_i$ belongs to $S$ since otherwise for any vertex $s \in S$ the distance $d(s, u)$ is less than or equal to the distance between $s$ and $T_i$ or $F_i$. (However, notice that $T_i$ or $F_i$ are not forced vertices in the sense of Definition 2.4 although at least one of them belongs to $S$.)

By the previous observations, we immediately obtain that any solid-resolving set contains at least $7n + 2m$ vertices. In what follows, we show that the solid-metric dimension of $G'$ is $7n + 2m$ if and only if the formula $F$ is satisfiable.

Let us first show that if the formula $F$ is satisfiable, then $\beta^s(G') = 7n + 2m$. Let $A$ be a satisfiable truth assignment of $F$. Construct then a set $C$ as follows: $C$ consists of all the forced vertices and if the assignment of $x_i$ is *true* in $A$, then $T_i$ belongs to $C$, and otherwise $F_i$ is in $C$. By the previous observations, we immediately notice that $C$ contains exactly $7n + 2m$ vertices. Notice that the only vertices that do not belong to $C$ are $u$, $d_j$, and either $T_i$ or $F_i$ depending on the assignment of $x_i$ in $A$. In what follows, we show that for any distinct $x, y \in V$ there exists a vertex $c \in C$ such that $d(c, x) < d(c, y)$:

- Suppose first that either $x = T_i$ or $x = F_i$. Without loss of generality, we may assume that $x = T_i$. Recall that now $N(T_i) \cap C = \{u, a_i, a'_i, c_i, c'_i, F_i\}$. Hence, it is straightforward to verify that there exists no vertex $v \in V$ such that $N(T_i) \cap C \subseteq N[v]$. Therefore, there always exists a vertex $c \in C$ such that $1 = d(c, x) < d(c, y)$.
- Suppose then that $x = d_j$. Recall that at least one of the three vertices adjacent to $d_j$ in the variable gadgets belongs to $C$; denote that vertex by $w$. Hence, we have $\{w, e_j, e'_j\} \subseteq N(d_j) \cap C$. It is straightforward to verify that there exists no vertex $v \in V$ such that $N(d_j) \cap C \subseteq N[v]$. Therefore, there always exists a vertex $c \in C$ such that $1 = d(c, x) < d(c, y)$.
- Finally, suppose that $x = u$. Recall that by the construction of $G'$ the universal vertex is adjacent to the vertices $a_i$, $a'_i$, $b_i$, $b'_i$, $c_i$ and $c'_i$ of $C$ in the variable gadgets and to $e_j \in C$ and $e'_j \in C$ in the clause gadgets. Hence, analogously to the previous cases, there always exists a vertex $c \in C$ such that $1 = d(c, x) < d(c, y)$.

Thus, by Theorem 2.2, $C$ is a solid-resolving set of $G'$ with $7n + 2m$ vertices, and we have $\beta^s(G') = 7n + 2m$.

Let us then show that if the solid-metric dimension of $G'$ is $7n + 2m$, then the formula $F$ is satisfiable. Let $C$ be a solid-resolving set in $G'$ with $7n + 2m$ vertices. Due to the previous observations, we know that for each $i$ exactly one of the vertices $T_i$ and $F_i$ belongs to $C$. Form then a truth assignment $A$ of $F$ as follows: if $T_i \in C$, then set the variable $x_i$ to be *true*, else set $x_i = \textit{false}$. In what follows, we show that the truth assignment $A$ satisfies the formula $F$. Suppose to the contrary that a clause $\mathcal{C}_j$ is not satisfied by $A$. This implies that the vertex $d_j$ in the clause gadget is not adjacent to any vertices of $C$ in the variable gadgets. Hence, it is straightforward to verify that $d(c, u) \leq d(c, d_j)$ for any $c \in C$. However, this contradicts with the characterisation of Theorem 2.2. Thus, the truth assignment $A$ satisfies the formula $F$.

In conclusion, we have shown that the solid-metric dimension of $G'$ is $7n + 2m$ if and only if the formula $F$ is satisfiable. Thus, as the graph $G'$ can be constructed in polynomial time and the 3-SAT problem is NP-complete, the problem of deciding whether $\beta^s(G) \leq k$ is also NP-complete. □

## 9 Conclusions

In this paper, we introduced a new class of resolving sets called solid-resolving sets. We gave a very useful characterisation for solid-resolving sets and showed that (depending on the graph) there are vertices that must be included in any solid-resolving set. We considered the connection between solid-resolving sets and other resolving sets, the boundary of a graph and locating-dominating sets. We proved bounds on the solid-metric dimension of a graph utilizing the Dilworth number, maximum degree and clique number. Finally, we showed that deciding whether the solid-metric dimension of a graph is at most a given number is an NP-complete problem.

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