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Locating-dominating sets: From graphs to oriented graphs [☆]

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ABSTRACT

A locating-dominating set of an undirected graph is a subset of vertices S such that S is dominating and for every $u, v \notin S$, the neighbourhood of u and v on S are distinct (i.e. $N(u) \cap S \neq N(v) \cap S$). Locating-dominating sets have received a considerable attention in the last decades. In this paper, we consider the oriented version of the problem. A locating-dominating set in an oriented graph is a set S such that for each $w \in V \setminus S$, $N^-(w) \cap S \neq \emptyset$ and for each pair of distinct vertices $u, v \in V \setminus S$, $N^-(u) \cap S \neq N^-(v) \cap S$. We consider the following two parameters. Given an undirected graph G, we look for $\overrightarrow{\gamma}_{LD}(G)$ ($\overrightarrow{\Gamma}_{LD}(G)$) which is the size of the smallest (largest) optimal locating-dominating set over all orientations of G. In particular, if D is an orientation of G, then $\overrightarrow{\gamma}_{LD}(G) \leq \gamma_{LD}(D) \leq \overrightarrow{\Gamma}_{LD}(G)$ where $\gamma_{LD}(D)$ is the minimum size of a locating-dominating set of D.
For the best orientation, we prove that, for every twin-free graph G on n vertices, $\overrightarrow{\gamma}_{LD}(G) \leq n/2$ which proves a "directed version" of a widely studied conjecture on the location-domination number. As a side result we obtain a new improved upper bound for the location-domination number in undirected trees. Moreover, we give some bounds for $\overrightarrow{\gamma}_{LD}(G)$ on many graph classes and drastically improve the value $n/2$ for (almost) d-regular graphs by showing that $\overrightarrow{\gamma}_{LD}(G) \in O(\log d/d \cdot n)$ using a probabilistic argument.
While $\overrightarrow{\gamma}_{LD}(G) \leq \gamma_{LD}(G)$ holds for every graph G, we give some graph classes such as outerplanar graphs for which $\overrightarrow{\Gamma}_{LD}(G) \geq \gamma_{LD}(G)$ and some for which $\overrightarrow{\Gamma}_{LD}(G) \leq \gamma_{LD}(G)$ such as complete graphs. We also give general bounds for $\overrightarrow{\Gamma}_{LD}(G)$ such as $\overrightarrow{\Gamma}_{LD}(G) \geq \alpha(G)$. Finally, we show that for many graph classes $\overrightarrow{\Gamma}_{LD}(G)$ is polynomial on n but we leave open the question whether there exist graphs with $\overrightarrow{\Gamma}_{LD}(G) \in O(\log n)$.

1. Introduction

A *dominating set* of an undirected graph G is a subset S of its vertices such that each vertex of G not in S has a neighbour in S. The *domination number* of G, denoted by $\gamma(G)$ is the size of a smallest dominating set of G. Domination theory is one of the main topics of graph theory, see for example the two reference books [21,22]. Among the variations

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of domination, the *location-domination*, introduced by Slater [29], has been extensively studied. A *locating-dominating set* of an undirected graph G is a dominating set S such that all vertices not in S have pairwise distinct neighbourhoods in S. The *location-domination number* of G, denoted by $\gamma_{LD}(G)$, is the size of a smallest locating-dominating set of G. Since $V(G)$ is always a locating-dominating set, $\gamma_{LD}(G)$ is well-defined. Structural and algorithmic properties of locating-dominating sets have been widely studied (see e.g. [26] for an online bibliography). Location-domination in directed graphs was briefly mentioned in several articles (see e.g. [7,28]) and further studied in [13]. A *locating-dominating set* of a directed graph D is a subset S of its vertices such that two vertices not in S have distinct and non-empty *in-neighbourhoods* in S. The *directed location-domination number* of D, denoted by $\gamma_{LD}(D)$, is the size of a smallest locating-dominating set of D.

Two oriented graphs with the same underlying graph can have a very different behaviour towards locating-dominating sets. Let us illustrate it on *tournaments* that are oriented complete graphs. Transitive tournaments (i.e. acyclic tournaments) have directed location-domination number $\lceil n/2 \rceil$ whereas one can construct locating-dominating sets of size $\lceil \log n \rceil$ for a well-chosen orientation of K_n [28]. Following the idea of Caro and Henning for domination [6] and the work started by Skaggs [28], we study in this paper the best and worst orientations of a graph for locating-dominating sets. Orientation of graph G is considered to be *best* (resp. *worst*) if it minimizes (resp. maximizes) the location-domination number over all the orientations of G. A similar line of work has been recently initiated for the related concepts of identifying codes [9] and metric dimension [2].

The two parameters that are considered in this paper are the following. The *lower directed location-domination number* of an undirected graph G, denoted by $\overrightarrow{\gamma}_{LD}(G)$, is the minimum directed location-domination number over all the orientations of G. The *upper directed location-domination number* of an undirected graph G, denoted by $\overrightarrow{\Gamma}_{LD}(G)$, is the maximum directed location-domination number over all the orientations of G.

1.1. Outline of the paper

Basic definitions, some background and first results are given in Section 2. Section 3 is dedicated to the study of the best orientations whereas Section 4 focuses on the worst orientations.

Main results on best orientations We first give basic results on $\overrightarrow{\gamma}_{LD}(G)$ and relations with classical parameters of graphs. Skaggs [28] proved in 2007 that for any graph G, $\overrightarrow{\gamma}_{LD}(G) \leq \gamma_{LD}(G)$. We refine this inequality by proving that, in graphs without cycles of size 4 (as a subgraph), $\overrightarrow{\gamma}_{LD}(G)$ and $\gamma_{LD}(G)$ coincide. As a consequence, computing $\overrightarrow{\gamma}_{LD}(G)$ is NP-complete.

Two vertices are *twins* if they have the same open or closed neighbourhood. Twins play an important role in locating-dominating sets since any locating-dominating set must contain at least one vertex of each pair of twins. As a consequence, if G is a star on n vertices, then $\overrightarrow{\gamma}_{LD}(G) = n - 1$. In Section 3.3, we prove that this function can be drastically improved when the graph G is twin-free, which is one of the main contributions of our paper.

Theorem 1. *Let G be a twin-free graph of order n with no isolated vertices. Then, $\overrightarrow{\gamma}_{LD}(G) \leq n/2$.*

The fact that any twin-free graph of order n satisfies $\gamma_{LD}(G) \leq n/2$ is a notorious conjecture, left open in [12,16] for instance.

Conjecture 2 *([16]). If G is a twin-free graph of order n, then $\gamma_{LD}(G) \leq n/2$.*

The proof of Theorem 1 holds in two steps. First, we show in Section 3.2 that $\overrightarrow{\gamma}_{LD}(G)$ is the smallest undirected location-domination number among all the (connected) spanning subgraphs of G. Then, we prove in Section 3.3 that there exists a spanning subgraph for which the condition is satisfied. In particular, our result implies a weakening of Conjecture 2 since we prove that any twin-free connected graph G on n vertices admits a spanning subgraph H with $\gamma_{LD}(H) \leq n/2$. As a side result we obtain a new improved upper bound for the location-domination number in trees. We also give a characterization for trees attaining this new upper bound.

We then focus on (almost) regular graphs in Section 3.4 and prove, using a probabilistic argument, that there exists a constant c_d such that, if G is d-regular,

$$\overrightarrow{\gamma}_{LD}(G) \leq c_d \cdot \frac{\log d}{d} \cdot |V(G)|.$$

We continue this subsection by giving some bounds using independence and matching numbers.

Main results on worst orientations In Section 4.1, we give some examples and relate $\overrightarrow{\Gamma}_{LD}(G)$ with some classical graph parameters. In particular, we prove that $\overrightarrow{\Gamma}_{LD}(G) \geq \gamma_{LD}(G)$ if G does not have any cycle of length 4 as a (not necessarily induced) subgraph. Moreover we prove that if G is a C_4-free bipartite graph (which in particular, contains the class of trees), then $\overrightarrow{\Gamma}_{LD}(G) = \alpha(G)$ where $\alpha(G)$ is the independence number of G.

In Section 4.2, we prove that $\overrightarrow{\Gamma}_{LD}(G) \geq \gamma_{LD}(G)$ is satisfied for other graph classes such as bipartite graphs, cubic graphs, and outerplanar graphs. Somehow surprisingly at first glance, $\overrightarrow{\Gamma}_{LD}(G) \geq \gamma_{LD}(G)$ is not always true. In [13], Foucaud et al. have shown that for a complete graph K_n on n vertices we have $\overrightarrow{\Gamma}_{LD}(K_n) = \lceil n/2 \rceil$ but $\gamma_{LD}(K_n) = n-1$. We prove that the existence of twins is not the reason why this inequality fails since we exhibit a family of twin-free graphs for which the ratio $\overrightarrow{\Gamma}_{LD}(G)/\gamma_{LD}(G)$ tends to 1/2. We did not succeed to bound this ratio by a constant. However, we prove that $\overrightarrow{\Gamma}_{LD}(G) \geq \gamma_{LD}(G)/\lceil \log_2(\Delta(G))+1 \rceil$. We leave the existence of a constant bounding $\overrightarrow{\Gamma}_{LD}(G)/\gamma_{LD}(G)$ as an open problem.

Finally, in Section 4.3, we provide some lower bounds on $\overrightarrow{\Gamma}_{LD}(G)$ using the number of vertices. For numerous classes of graphs, we actually have $\overrightarrow{\Gamma}_{LD}(G) \geq c_1 \cdot n^{c_2}$ where c_1 and c_2 are constant. This is true for perfect graphs (with $c_2 = 1/2$), C_3-free graphs, claw-free graphs and actually for any χ-bounded class of graphs with a polynomial χ-bounding function. However, we leave as an open problem the existence of a graph G on n vertices such that $\overrightarrow{\Gamma}_{LD}(G)$ is logarithmic on n.

Note that we did not find the complexity of computing $\overrightarrow{\Gamma}_{LD}(G)$. In particular, it is not clear that this problem belongs to NP.

2. Preliminaries

2.1. Notations

We give in this subsection the main definitions and notations we are using along the paper. The reader may refer to some classical graph theory books like [4] for missing definitions.

Let $G = (V, E)$ be an undirected and simple graph. We usually denote by n the number of vertices of G. We denote by $N_G(u)$ (or $N(u)$ when G is clear from context) the *open neighbourhood* of a vertex u, that is the set of neighbours of u. And we denote by $N_G[u]$ (abbreviated into $N[u]$) the *closed neighbourhood* of u that is $N(u) \cup \{u\}$. Two vertices u and v are *twins* if $N(u) = N(v)$ or $N[u] = N[v]$. The *degree* of a vertex u, denoted by $d(u)$, is the size of $N(u)$. The minimum and maximum degree of G are respectively denoted by $\delta(G)$ and $\Delta(G)$. A *leaf* is a vertex of degree 1.

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H is *induced* if for any pair of vertices of H, (x, y) is an edge of H if and only if it is an edge of G. A graph G is *H-free* if it does not contain H has an induced subgraph. We say that a graph G is *without H* if G does not contain H as a subgraph (not necessarily induced). A subgraph H is a *spanning* subgraph if $V(H) = V(G)$.

The complete graph on n vertices is denoted by K_n. The complete bipartite graph with size n and m is denoted by $K_{n,m}$. A *star* is a graph isomorphic to $K_{1,m}$. The star with three leaves, $K_{1,3}$, is also called a *claw*. The cycle on n vertices is denoted by C_n whereas the path on n vertices is denoted by P_n. The *girth* of a graph G is the length of a shortest cycle in G. If G does not contain any cycle we say that G has infinite girth. A set S of vertices is *independent* if they are pairwise non-adjacent. A set S is an *edge cover* if every edge has at least one endpoint in S. A set of edges M is a *matching* if no two edges in M share an endpoint. In a graph G, we denote the cardinalities of maximum independent sets and matchings by $\alpha(G)$ and $\alpha'(G)$, respectively. Moreover, the cardinality of a minimum edge cover is denoted by $\beta(G)$. The clique number of a graph G, denoted by $\omega(G)$, is the maximal order of a complete subgraph of G.

Let S be a subset of V. Set S is a *dominating set* of G if any vertex of G is either in S or adjacent to a vertex of S. The minimum size of a dominating set is denoted by $\gamma(G)$. We denote by $I_G(S; u)$ ($I(u)$ for short) the set $N_G(u) \cap S$ that is the neighbours in S of a vertex u. Note that S is a *locating-dominating set* if for each vertex $u \in V(G) \setminus S$, $I(u)$ is non-empty (since S is a dominating set) and for each pair of distinct vertices $u, v \in V \setminus S$, we have $I(u) \neq I(v)$. We say that a vertex $s \in S$ *separates* u and v if s is in exactly one of sets $I(u)$ and $I(v)$. Note that any locating-dominating set must intersect any pair of twins. The minimum size of a locating-dominating set of G is denoted by $\gamma_{LD}(G)$.

These notions are similarly defined for directed graphs. In this paper, we mostly consider directed graphs derived from orienting an undirected graph. A *directed graph* (also called *digraph*), is a pair $D = (V, E)$, where V is a set whose elements are called vertices, and A is a set of ordered pairs of vertices, called *arcs*. Let $G = (V, E)$ be a simple undirected graph. An *orientation* of G is a directed graph (*oriented graph*) D on V where every edge uv of G is either oriented from u to v (resulting to the arc (u, v) in D) or from v to u (resulting to the arc (v, u)). In particular, all the directed graphs considered are oriented and simple: if (u, v) is an arc then (v, u) is not. The undirected graph G is called the *underlying* graph of D. Unless otherwise stated, "graph" means "undirected graph". A *tournament* is an orientation of a complete graph. The open out-neighbourhood and in-neighbourhood of a vertex u of D are denoted by $N_D^+(u)$ and $N_D^-(u)$ whereas the closed out- and in-neighbourhood are denoted by $N_D^+[u]$ and $N_D^-[u]$. The maximum out- and in-degree are denoted by $\Delta^+(G)$ and $\Delta^-(G)$. A *source* is a vertex with no in-neighbours. Locating dominating sets are defined similarly as in the undirected case by considering the in-neighbourhoods. We denote by $I_D(S; u)$ (or $I(u)$ for short) the set $N_D^-(u) \cap S$, that is, the in-neighbours of u that are in a set S of vertices. The set S is a locating-dominating set of D if all the sets $I_D(S; u)$ are non-empty and distinct for $u \notin S$. The minimum size of a locating-dominating set of D, called the minimum directed location-domination number, is denoted by $\gamma_{LD}(D)$.

We finally recall the two main parameters that we are considering along this paper. The *lower directed location-domination number* of an undirected graph G, denoted by $\overrightarrow{\gamma}_{LD}(G)$, is the minimum directed location-domination number over all the orientations of G. Formally, we have

$$\overrightarrow{\gamma}_{LD}(G) = \min\{\gamma_{LD}(D) \mid D \text{ is an orientation of } G\}.$$

The *upper directed location-domination number* of an undirected graph G, denoted by $\overrightarrow{\Gamma}_{LD}(G)$, is the maximum directed location-domination number over all the orientations of G. Formally, we have

$$\overrightarrow{\Gamma}_{LD}(G) = \max\{\gamma_{LD}(D) \mid D \text{ is an orientation of } G\}.$$

2.2. Preliminary results and examples

Let D be a digraph and u be a non-source vertex of D. Then, $V(D) \setminus \{u\}$ is a locating-dominating set of D. In particular, for any directed graph containing at least one edge, $\Gamma_d(D) \leq n-1$. In [13], the authors characterized those digraphs reaching this extremal value. This characterization is useful for studying the extremal values of $\overrightarrow{\gamma}_{LD}(G)$ and $\overrightarrow{\Gamma}_{LD}(G)$. A *directed star* is a (non-necessarily simple) directed graph such that the underlying graph is a star. A *bi-directed clique* is a directed graph that contains all the possible arcs between two vertices.

Theorem 3 *([13], Theorem 6). Let D be a connected (non necessarily simple) digraph of order $n \geq 2$. Then, $\gamma_{LD}(D) = n-1$ if and only if at least one of the following conditions holds:*

1. *$n = 3$;*
2. *D is a directed star;*
3. *$V(D)$ can be partitioned into three (possibly empty) sets S_1, C and S_2, where S_1 and S_2 are independent sets, C is a bi-directed clique, and the remaining arcs in D are all the possible arcs from S_1 to $C \cup S_2$ and those from C to S_2.*

In particular, any orientation of a star has location-domination number $n-1$.

Corollary 4. *Let G be a star on n vertices. Then, $\overrightarrow{\gamma}_{LD}(G) = \overrightarrow{\Gamma}_{LD}(G) = n-1$.*

In [13], the authors also proved a tight upper bound for tournament:

Theorem 5 *([13]). Let D be a tournament on n vertices. Then, $\gamma_{LD}(D) \leq \lceil n/2 \rceil$. Moreover, $\gamma_{LD}(D) = \lceil n/2 \rceil$ if D is transitive.*

As a consequence, the upper directed location-domination number of complete graphs is known:

Corollary 6. *Let $n \geq 2$ be an integer. Then, $\overrightarrow{\Gamma}_{LD}(K_n) = \lceil n/2 \rceil$.*

Concerning the best orientation of a complete graph, Skaggs proved in his thesis [28] that one can obtain the best possible number for $\overrightarrow{\gamma}_{LD}(G)$. For the sake of completeness, we add a short proof of this result.

Theorem 7 *([28], Proposition 5.4). Let $n \geq 2$ be an integer. Let k be the smallest integer such that $n \leq k + 2^k - 1$. Then, $\overrightarrow{\gamma}_{LD}(K_n) = k$.*

Proof. Let S be a set of k vertices of K_n. Then, consider an injective map f from the other vertices of K_n (there are at most $2^k - 1$ of them) to the non-empty subsets of S. Let $u \notin S$ and $v \in S$. Orient edge uv from v to u if $v \in f(u)$ and from u to v otherwise. Orient all the other edges in any direction. Then, S is a locating-dominating set for this orientation of K_n. □

3. Best orientation

In this section we focus on the best orientation. We first give basic results and links with classical parameters. Then, we give another definition of $\overrightarrow{\gamma}_{LD}(G)$ using spanning subgraphs and use this definition to show that $\overrightarrow{\gamma}_{LD}(G) \leq n/2$ if G is twin-free. We finally improve this last result in the case of almost regular graphs.

3.1. Basics

Theorem 8. *Let G be a graph of order n. Then,*

1. *[28, Proposition 5.3]* $\overrightarrow{\gamma}_{LD}(G) \le \gamma_{LD}(G)$.
2. $\overrightarrow{\gamma}_{LD}(G) \le n - \alpha'(G)$.

Proof. Claim (1) is proved in [28], for completeness, we include a short proof here. Consider a graph G and a locating dominating set S of size $\gamma_{LD}(G)$ of G. Then, orient all the edges uv between S and $V \setminus S$ from S to $V \setminus S$ and all the other edges in any way. Then, S is a locating-dominating set for this orientation.

Let us next prove (2). Let G be a graph on n vertices and let M be a maximum matching of G. Let V_M be a subset of vertices containing exactly one vertex from each edge of M and C_M be the set of vertices which are not endpoints of edges in M. Let $C = V_M \cup C_M$. Note that $|C| = n - \alpha'(G)$. Choose any orientation D' of G where the edges in M have their tails in C and all the other edges between $V \setminus C$ and C are oriented from $V \setminus C$ to C. Now, C is a locating-dominating set in D' since all the vertices of $V \setminus C$ have exactly one in-neighbour in V_M and all of them are pairwise distinct. □

We show that these bounds are tight in Corollary 9 and Theorem 10. Using Theorem 3, we next provide a characterization of graphs reaching the extremal value $\overrightarrow{\gamma}_{LD}(G) = n - 1$.

Corollary 9. *For any connected graph G of order $n \ge 2$, $\overrightarrow{\gamma}_{LD}(G) = n - 1$ if and only if either $n = 3$ or G is a star.*

Proof. Let G be a graph of order $n \ge 2$ with $\overrightarrow{\gamma}_{LD}(G) = n - 1$. If either $n = 3$ or G is a star, then, $\overrightarrow{\gamma}_{LD}(G) = n - 1$ by Corollary 4.

Otherwise, let D be an orientation of G. Since $\Gamma_d(D) \le n - 1$ we must actually have $\Gamma_d(D) = n - 1$. Since G is not at star, then D must have the structure of the third condition of Theorem 3.

Thus, $V(G)$ can be partitioned to sets S_1, C and S_2 satisfying the third condition of Theorem 3. Since C is a bi-directed clique in Theorem 3, we have $|C| \le 1$ because D is an oriented graph. Assume first that $|C| = 1$. If S_1 or S_2 are empty, then G is a star. If both of them are not empty, then G contains a triangle and there is an orientation D' of G with an oriented cycle. Then, by Theorem 3, $\Gamma_d(D') \le n - 2$, a contradiction.

If $C = \emptyset$, then G is a star if either $|S_i| = 1$ for $i \in \{1, 2\}$ and disconnected if either is an emptyset. But if $|S_i| \ge 2$, then again G contains a cycle and an orientation with an oriented cycle which is against the conditions of Theorem 3. Hence, the claim follows. □

Theorem 8 ensures that, for every graph G, $\overrightarrow{\gamma}_{LD}(G) \le \gamma_{LD}(G)$. Let us prove that if G is without C_4 as a (not necessarily induced) subgraph, then it is actually an equality.

Theorem 10. *Let G be a graph without C_4 as a subgraph. Then,*

$$\overrightarrow{\gamma}_{LD}(G) = \gamma_{LD}(G).$$

Proof. To prove this equality, let us show that, any locating-dominating set S of an orientation D of a graph G is also a locating-dominating set for G. Let D be an arbitrary orientation of G and S be a locating-dominating set of D. First note that S is indeed a dominating set of G. Thus, if S is not locating-dominating in G, then there exist $u, v \notin S$ such that $I_G(u) = I_G(v)$. Moreover, we have $|I_G(u)| = |I_G(v)| \ge 2$ since $|I_G(u)| \ge |I_D(u)|$ and $|I_G(v)| \ge |I_D(v)|$. Thus, if $|I_G(u)| = 1$, then $|I_G(v)| = |I_D(v)| = |I_D(u)| = 1$ and hence, $I_D(u) = I_G(u) = I_G(v) = I_D(v)$, a contradiction. Let $\{c_1, c_2\} \subseteq I_G(u)$. But then u, c_1, v and c_2 induce a cycle on four vertices, a contradiction. □

In particular, Theorem 10 means that $\overrightarrow{\gamma}_{LD}(T) = \gamma_{LD}(T)$ for any tree T. Let us complete this warm-up part by proving that finding the value of $\overrightarrow{\gamma}_{LD}(G)$ is NP-hard.

Locating-Dominating-Set
Instance: A graph G, an integer k.
Question: Is it true that $\gamma_{LD}(G) \le k$?

Lower-Directed-LD-Number
Instance: A graph G, an integer k.
Question: Is it true that $\overrightarrow{\gamma}_{LD}(G) \le k$?

Theorem 11. Locating-Dominating-Set *and* Lower-Directed-LD-Number *are NP-complete for planar graphs of maximum degree 5 without C_4 as a subgraph.*

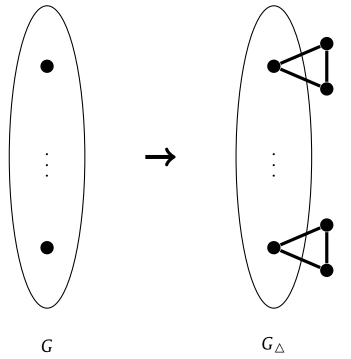

Fig. 1. Reduction from Dominating-Set to Locating-Dominating-Set.

Proof. Both problems are in NP. For Lower-Directed-LD-Number, a polynomial certificate for $\overrightarrow{\gamma}_{LD}(G) \leq k$ is an orientation D of G and a locating-dominating set of D of size at most k.

By Theorem 10, both values are equal in the class of graphs without C_4. Thus, we just prove the result for Locating-Dominating-Set. We reduce it from Dominating-Set.

Dominating-Set
Instance: A graph G, an integer k.
Question: Is it true that $\gamma(G) \leq k$?

We use the reduction of Gravier *et al.* [18, Figure 7]. Consider an instance (G, k) of Dominating-Set. Let $G_{\triangle}$ be the graph obtained by adding to each vertex of the graph a pendant triangle (see Fig. 1). Then it is proved in [18] that G has a dominating set of size k if and only if $G_{\triangle}$ has a locating-dominating set of size $k+n$ (where n is the number of vertices of G). Indeed, each triangle must contain at least one of the new vertices in a locating-dominating set and if there is exactly one vertex in a triangle, the vertex of the original graph must be dominated in the original graph.

Dominating-Set has been proved to be NP-complete even for planar graphs of maximum degree 3 and girth at least 5 [33]. If G is planar of maximum degree 3 and girth at least 5, then $G_{\triangle}$ is planar, of maximum degree 5, and does not contain C_4 as a subgraph. This implies our result. $\square$

3.2. Relation to spanning subgraphs

In this section, we prove a simple but important lemma that links $\overrightarrow{\gamma}_{LD}(G)$ with optimal locating-dominating sets of spanning subgraphs of G. This result is used to prove several important results all along the section, but we illustrate its interest by first giving several simple lower bounds on $\overrightarrow{\gamma}_{LD}(G)$.

Lemma 12. *Let G be an undirected graph. Then,*

$$\overrightarrow{\gamma}_{LD}(G) = \min\{\gamma_{LD}(H) \mid H \text{ is a spanning subgraph of } G\}.$$

Proof. Let us show first that $\overrightarrow{\gamma}_{LD}(G) \leq \gamma_{LD}(H)$ holds for each spanning subgraph H of G. Let S be a locating-dominating set of a spanning subgraph H of G. We next construct an orientation D of G such that an edge e between S and $V \setminus S$ is oriented away from the vertex in S if $e \in E(H)$ and if $e \notin E(H)$, then we orient edge e towards the vertex in S. Other edges can be oriented in any way. Observe that we have $I_D(S; w) = I_H(S; w)$ for each vertex $w \notin S$ and hence, S is locating-dominating in D. Thus, $\overrightarrow{\gamma}_{LD}(G) \leq \min\{\gamma_{LD}(H) \mid H$ is a spanning subgraph of $G\}$.

Let us then show that for any orientation D' of G, there exists a spanning subgraph H' of G such that $\gamma_{LD}(H') \leq \gamma_{LD}(D')$. Let S be a locating-dominating set in D'. Let us construct a spanning subgraph H' from the graph G by having $V(H') = V(G)$ and $e = uv \in E(H')$ if and only if either $u \in S$ and the edge is oriented away from u in D' or $v \in S$ and the edge is oriented away from v in D'. Observe that now $I_{D'}(S; w) = I_{H'}(S; w)$ for each vertex $w \notin S$ and hence, S is locating-dominating in H'. Thus, $\min\{\gamma_{LD}(H') \mid H'$ is a spanning subgraph of $G\} \leq \overrightarrow{\gamma}_{LD}(G)$ and the claim follows. $\square$

In the following theorem, we apply the previous lemma on classes of graphs which are closed under (spanning) subgraphs. In particular, general lower bounds for undirected location-domination numbers in such classes also hold when we orient graphs.

Lemma 13. *Let $\mathcal{G}$ be a class of graphs closed under subgraphs. If there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for each graph $G \in \mathcal{G}$ with n vertices we have $\gamma_{LD}(G) \geq f(n)$, then*

$$\overrightarrow{\gamma}_{LD}(G) \geq f(n).$$

Proof. Assume by contradiction that $\overrightarrow{\gamma}_{LD}(G) < f(n)$ for some $G \in \mathcal{G}$. By Lemma 12, there exists a spanning subgraph H such that $\gamma_{LD}(H) = \overrightarrow{\gamma}_{LD}(G)$. So $H \in \mathcal{G}$ and $\gamma_{LD}(H) < f(n)$, a contradiction. □

As proven in [27], planar graphs satisfy $\gamma_{LD}(G) \geq \frac{n+10}{7}$ and outerplanar graphs satisfy $\gamma_{LD}(G) \geq \frac{2n+3}{7}$. Since planar and outerplanar graphs are closed under subgraphs, the following is a consequence of Lemma 13.

Corollary 14. *Let G be a planar graph on n vertices. Then,*

$$\overrightarrow{\gamma}_{LD}(G) \geq \frac{n+10}{7}.$$

Let G' be an outerplanar graph on n vertices. Then,

$$\overrightarrow{\gamma}_{LD}(G') \geq \frac{2n+3}{7}.$$

Lemma 15. *Let G be a graph of order n. Then,*

$$\overrightarrow{\gamma}_{LD}(G) \geq \frac{2n}{\Delta(G)+3}.$$

Proof. Let G be a graph of order n. In [31, Theorem 2] Slater has given general lower bound $\gamma_{LD}(G) \geq 2n/(d+3)$ for a locating-dominating set in a d-regular graph G on n vertices. Moreover, it is easy to generalize the proof for non-regular graphs, giving $\gamma_{LD}(G) \geq 2n/(\Delta(G)+3)$. For completeness, we include the proof here. Let G be a graph on n vertices with a locating-dominating set S. We give one *share unit* for each vertex. Next, we shift $1/|I(v)|$ share from each vertex $v \in V(G) \setminus S$ to every vertex in $I(v)$. After this shift, total share over all vertices remains as n. Let s denote the largest share in any vertex $u \in S$. Notice that $s|S| \geq n$ and hence, $|S| \geq n/s$. Moreover, we have that $s \leq 2 + (\Delta(G)-1)/2$. Indeed, vertex u has share of 1 at the beginning. After which, we shift at most $1 + (\Delta(G)-1)/2$ share to u since there is at most one adjacent vertex v with $|I(v)| = 1$. Thus, $|S| \geq n/s \geq 2n/(\Delta(G)+3)$.

Moreover, we also have $\gamma_{LD}(H) \geq 2n/(\Delta(H)+3) \geq 2n/(\Delta(G)+3)$ for each spanning subgraph H of G since $\Delta(H) \leq \Delta(G)$. Thus, the claim follows from Lemma 13 with graph class $\mathcal{G}_G = \{H \mid H \text{ is a subgraph of } G\}$. □

3.3. *Conjecture 2 holds for graph orientations*

The main goal of this section is to prove Theorem 1 we restate here:

Theorem 1. *Let G be a twin-free graph of order n with no isolated vertices. Then, $\overrightarrow{\gamma}_{LD}(G) \leq n/2$.*

We first need some auxiliary definitions and results.

Let $G = (V, E)$ be an undirected graph of order $n \geq 3$. A vertex adjacent to a leaf is called a *support vertex* and a non-leaf, non-support vertex u which has only support vertices as neighbours is called a *support link*. The number of support vertices, leaves and support links in G are denoted by respectively $s(G)$, $\ell(G)$ and $sl(G)$. Moreover, let us denote by $L(G)$, $S(G)$ and $SL(G)$ the sets of leaves, support vertices and support links, respectively, in G. By convention, for the path P_2 we assume that one of its two vertices is a support vertex and the other is a leaf.

We first introduce a useful lemma. The result has been previously discussed in [3] and Claim 2 has been proven in [3, Lemma 2.1].

Lemma 16. *Let T be a tree, $s \in S(T)$ with k leaves $v_1, \ldots, v_k$ attached to s. Then:*

1. *Every locating-dominating set C in T contains at least k vertices in $\{s, v_1, \ldots, v_k\}$.*
2. *There exists a minimum locating-dominating set C in T which contains all the vertices $s \in S(T)$ and for each $s \in S(T)$ there is exactly one leaf attached to s which is not in C.*

Proof. Let C be a locating-dominating set. Let $s \in S(T)$. If $s \notin C$, then all the leaves attached to s are in C. Otherwise, C is not dominating. Let v be a leaf attached to s. We claim that $C' = \{s\} \cup C \setminus \{v\}$ is a locating-dominating set. Indeed, we have

$I(C'; v) = \{s\}$ and if $I(C'; u) = \{s\}$ for any $v \neq u \in V \setminus C'$, then $I(C; u) = \emptyset$. Thus, C' is a locating-dominating set. So, for every C, there exists a locating-dominating set of the same size containing s. We assume that $S(T) \subseteq C$ holds in the rest of the proof.

Assume by contradiction that $|\{s, v_1, \dots, v_k\} \cap C| \leq k-1$. Since $s \in C$, there are $v_i, v_j \notin C$ with $i \neq j$. But then $I(v_i) = I(v_j) = \{s\}$, a contradiction. So the first point holds.

Assume next that $N(s) \cap L(T) \subseteq C$. Let $v \in N(s) \cap L(T)$. Since C has minimum size, there exists a vertex $u \notin L(T) \cup C$ such that $I(u) = \{s\}$ (otherwise v can be safely removed from C contradicting the minimality of C). However, if we now consider the set $C' = \{u\} \cup C \setminus \{v\}$, then we notice immediately that C' is locating-dominating and the claim follows. □

Locating-dominating sets in trees have been widely studied. Blidia et al. proved in [3] that

$$\gamma_{LD}(T) \leq \frac{n + \ell(T) - s(T)}{2}. \tag{1}$$

Let us prove a slight improvement of this result that is needed in the proof of the main result of this section. As this is the best known upper bound for locating-dominating sets in trees, we have included a complete characterization of trees attaining it in Theorem 18.

Theorem 17. *Let T be a tree of order $n \geq 2$. Then,*

$$\gamma_{LD}(T) \leq \frac{n + \ell(T) - s(T) - sl(T)}{2}.$$

Proof. Let T be a tree and let $F = T - SL(T)$. The set F induces a forest without isolated vertices. Moreover $S(T) = S(F)$ and $L(T) = L(F)$ (by choosing the right vertex in L and S if the component is a P_2). Let C be an optimal locating-dominating set in F such that $S(F) \subseteq C$. Observe that now C is also a locating-dominating set in T. Indeed, if $u \in SL(T)$, then $I(u) \subseteq S(T)$ and $|I(u)| \geq 2$. Moreover, if $I(v) = I(u)$, then we have a cycle. Finally, if $u, v \notin SL(T)$, then $I_T(u) = I_T(v)$ implies that $I_F(u) = I_F(v)$. Thus, $\gamma_{LD}(T) \leq |C| = \gamma_{LD}(F) \leq \frac{n - sl(T) + \ell(T) - s(T)}{2}$. The last inequality is due to bound (1). □

As a slight side-step from proving Theorem 1, we first give a characterization for trees reaching the upper bound of Theorem 17. For this, we need some definitions. Let $\mathcal{T}$ be a family of trees such that $T \in \mathcal{T}$ if and only if $\gamma_{LD}(T) = \frac{n + \ell(T) - s(T)}{2}$ where $n = |V(T)|$ or if $T = P_2$. This family has been characterized in [3]. We say that trees $T_1, T_2, \dots, T_k$, where $k \geq 2$ are *support linked* into tree T and we note $T = \mathcal{SL}(T_1, T_2, \dots, T_k)$ if there are vertices $v_i \in S(T_i)$ and $w \notin \bigcup_{i=1}^k V(T_i)$ such that $V(T) = \bigcup_{i=1}^k V(T_i) \cup \{w\}$ and $E(T) = \bigcup_{i=1}^k E(T_i) \cup \{v_i w \mid 1 \leq i \leq k\}$. Let us denote by $\mathcal{T}_{SL}$ the closure of $\mathcal{T}$ under $\mathcal{SL}$.

Theorem 18. *Let T be a tree. We have $\gamma_{LD}(T) = \frac{n + \ell(T) - s(T) - sl(T)}{2}$ if and only if $T \in \mathcal{T}_{SL}$.*

Proof. Notice first that if $sl(T) = 0$, then $\gamma_{LD}(T) = \frac{n + \ell(T) - s(T) - sl(T)}{2}$ if and only if $T \in \mathcal{T} \subseteq \mathcal{T}_{SL}$.

Let us assume first that there exists $T \in \mathcal{T}_{SL}$ such that $\gamma_{LD}(T) < \frac{n + \ell(T) - s(T) - sl(T)}{2}$. Let T be a tree satisfying these properties with the least number of vertices. By the previous remark, we can assume that $sl(T) > 0$ and thus, that T can be written as $T = \mathcal{SL}(T_1, \dots, T_k)$, where $k \geq 2$, with $T_i \in \mathcal{T}_{SL}$ for both i. Let w be the vertex in $V(T) \setminus \bigcup_{i=1}^k V(T_i)$ and let $v_i \in N(w) \cap V(T_i)$ for each $i \in \{1, \dots, k\}$. Notice that for any i, we have $v_i \in S(T)$ and $v_i \in S(T_i)$. Furthermore, by the minimality of T, $\gamma_{LD}(T_i) = \frac{|V(T_i)| + \ell(T_i) - s(T_i) - sl(T_i)}{2}$ for each $i \in \{1, \dots, k\}$. Moreover, let C, be a locating-dominating set of minimum size in T and C_i be a locating-dominating set of minimum size in T_i. Notice that

$$\sum_{i=1}^{k} \frac{|V(T_i)| + \ell(T_i) - s(T_i) - sl(T_i)}{2} = \frac{|V(T)| + \ell(T) - s(T) - sl(T)}{2}.$$

Indeed, we have $|V(T)| = 1 + \sum_{i=1}^k |V(T_i)|$, $\ell(T) = \sum_{i=1}^k \ell(T_i)$, $s(T) = \sum_{i=1}^k s(T_i)$ and $sl(T) = 1 + \sum_{i=1}^k sl(T_i)$. By Lemma 16, we may assume that $S(T) \subseteq C$ and $S(T_i) \subseteq C_i$ for each $i \in \{1, \dots, k\}$.

Since $|C| < \frac{n + \ell(T) - s(T) - sl(T)}{2}$, we have $|C \cap V(T_i)| < \frac{|V(T_i)| + \ell(T_i) - s(T_i) - sl(T_i)}{2}$ for some $i \in \{1, \dots, k\}$. Since $v_i \in C \cap V(T_i)$ and since C is a locating-dominating set in T, $C \cap V(T_i)$ is a locating-dominating set in T_i, a contradiction. Thus, any tree in $\mathcal{T}_{SL}$ satisfies the claim.

Let us then show that no tree outside of $\mathcal{T}_{SL}$ can satisfy the claim. Let us consider a tree T of minimum size satisfying $\gamma_{LD}(T) = \frac{n + \ell(T) - s(T) - sl(T)}{2}$ and $T \notin \mathcal{T}_{SL}$. Observe that $sl(T) > 0$, otherwise we would have $T \in \mathcal{T} \subseteq \mathcal{T}_{SL}$. Thus, we may assume that $T = \mathcal{SL}(T_1, \dots, T_k)$, where $k \geq 2$, for some trees T_i, where $T_1 \notin \mathcal{T}_{SL}$ and $w \in V(T) \setminus \bigcup_{i=1}^k V(T_i)$. Let C be a minimum size locating-dominating set in T such that $S(T) \subseteq C$ (we may assume this by Lemma 16). Since $S(T) \subseteq C$ and since C is of minimum size, we have $w \notin C$. Since $|V(T)| = 1 + \sum_{i=1}^k |V(T_i)|$, $\ell(T) = \sum_{i=1}^k \ell(T_i)$, $s(T) = \sum_{i=1}^k s(T_i)$ and $sl(T) =$

$1+\sum_{i=1}^{k} sl(T_i)$ and $\sum_{i=1}^{k} \frac{|V(T_i)|+\ell(T_i)-s(T_i)-sl(T_i)}{2} = \frac{|V(T)|+\ell(T)-s(T)-sl(T)}{2}$, we have $|C \cap V(T_1)| = \frac{n+\ell(T_1)-s(T_1)-sl(T_1)}{2}$. Indeed, since $|C \cap V(T_i)| \leq \frac{n+\ell(T_i)-s(T_i)-sl(T_i)}{2}$, we would otherwise have $|C| < \frac{|V(T)|+\ell(T)-s(T)-sl(T)}{2}$. However, this is a contradiction on the minimality of T. Thus, $T \in \mathcal{T}_{SL}$. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let T be a spanning tree of G such that $\ell(T)-s(T)$ is minimal among all the spanning trees of G. If T has $\ell(T)=s(T)$, then we are done by Lemma 12 and Lemma 17.

First, we claim that any leaf of T adjacent in T to a support vertex s such that $|N(s) \cap L(T)| \geq 2$, is adjacent, in G, only to vertices which are support vertices in T. Observe that if u and v are two leaves of T adjacent to the same support vertex s, then either u or v has another neighbour in G since G is twin-free. Moreover, if $s' \in N_G(u)$, then s' is a support vertex in T. Indeed, if s' is a leaf in T, then the spanning tree $T' = T - us + us'$ satisfies $\ell(T') - s(T') < \ell(T) - s(T)$, a contradiction with the minimality of T. Moreover, if s' is a non-leaf, non-support vertex, then we have $s(T') = s(T)+1$ and $\ell(T') = \ell(T)$, a contradiction.

We next construct an auxiliary graph G' as follows. First we add to the tree T every edge $e = uv \in E(G)$ such that $u \in L(T)$, $v \in S(T)$ and there is a support vertex $s \in S(T)$ in $N_T(u)$ such that $|N_T(s) \cap L(T)| \geq 2$. Then, we delete some of the newly added extra edges so that there is exactly one leaf adjacent to every vertex in $S(T)$. The resulting graph is denoted by G'. Observe that, because G is twin-free, none of the vertices in $L(T)$ are pairwise twins in G'.

Let C' be an optimal locating-dominating set in T such that every support vertex is included in it and for each $s \in S(T)$ there exists a leaf $u \in N(s) \cap L(T)$ such that $u \notin C'$. By Lemma 16 such a set exists. Let us now denote $C'' = C' \setminus L(T)$. Now, Lemma 17 and Lemma 16 together imply that $|C''| \leq n/2$. Indeed,

$$|C''| = |C'| - (\ell(T) - s(T)) \leq \frac{n - \ell(T) - sl(T) + s(T)}{2}.$$

Finally, we create the locating-dominating set C by adding to set C'' all vertices in $SL(T)$ that have a twin in G'. Let us denote their set by W. Observe that if $v \in SL(T)$ has a twin u in G', then v and u belong to a cycle in G'. Moreover, since $N_T(v) \subseteq S(T)$, we have $u \in L(T)$. Furthermore, vertices u and v may only have one twin in G' and for each $s \in S(T) \cap N(u)$ we have exactly one adjacent leaf in G' (which is not u). Thus, $\ell(T) \geq s(T) + |W|$. Hence, $|C| = |C''| + |W| \leq \frac{n-|W|-sl(T)}{2} + |W| \leq \frac{n}{2}$.

Next, we show that C is a locating-dominating set in G'. First of all, because none of the vertices in $L(T)$ are pairwise twins in G' and because $S(T) \subseteq C$, all the vertices in $L(T)$ are dominated and pairwise separated by C. Moreover, because we removed only leaves from C', which is a locating-dominating set in T, and because each support vertex is in C, all the non-leaf vertices are dominated and pairwise separated. Finally, there is the case with $I_{G'}(C;u) = I_{G'}(C;v)$ where $u \in L(T)$ and $v \in V(T) \setminus (L(T) \cup S(T) \cup SL(T) \cup C)$. We have $|I(v)| \geq 2$, otherwise we would have $I_T(C';v) = I_T(C';u')$ for some leaf $u' \notin C'$. Moreover, since $I(u) \subseteq S(T)$, we also have $I(v) \subseteq S(T)$. Let us denote $I(u) = \{u_1, \dots, u_t\}$, $t \geq 2$, and assume without loss of generality that $uu_1 \in E(T)$. Observe that because $v \notin SL(T) \cup S(T) \cup L(T)$, there exists $w \in N_T(v) \setminus N_{G'}(u)$ and w is not a leaf in T. Let us next consider the tree $T'' = T - u_1 v + uu_2$. We notice that no new leaves are created since $\{w, u_2\} \subseteq N_{T''}(v)$ and u_1 has at least three neighbours in T, namely v, u and at least one other leaf. Moreover, the number of support vertices does not decrease. Indeed, $u_2 \in S(T'')$ and $u_1 \in S(T'')$. Finally, $u \in L(T)$ but $u \notin L(T'')$. Thus, we have $\ell(T'') - s(T'') < \ell(T) - s(T)$, a contradiction and hence, C is a locating-dominating set in G', a spanning subgraph of G and the claim follows by Lemma 12. □

The bound $n/2$ is asymptotically tight even for graphs with large minimum degree as we can see in the next subsection (see Lemma 23). However, it can be improved in many cases, even without the twin-freeness assumption. Let us provide two simple classes for which we can improve it.

Remark 19. Let G be a graph on n vertices with a twin-free spanning subgraph G' with no isolated vertices. Then, $\overrightarrow{\gamma}_{LD}(G') \leq n/2$ by Theorem 1 and by Lemma 12, we have $\gamma_{LD}(G) \leq \overrightarrow{\gamma}_{LD}(G')$. Hence, the existence of a twin-free spanning subgraph G' is enough for Theorem 1 to hold.

Lemma 20. *Let G be a graph on n vertices with a Hamiltonian path. Then,*

$$\overrightarrow{\gamma}_{LD}(G) \leq \left\lceil \frac{2n}{5} \right\rceil.$$

Proof. The Hamiltonian path is a spanning subgraph. Since $\gamma_{LD}(P_n) = \left\lceil \frac{2n}{5} \right\rceil$ as proven in [30], Lemma 12 ensures that $\overrightarrow{\gamma}_{LD}(G) \leq \left\lceil \frac{2n}{5} \right\rceil$. □

We say that a graph G has a $P_{\geq t}$*-factor* (or t*-path factor*) if it has a spanning subgraph containing only paths of length at least t as its components.

Lemma 21. *Let G be a claw-free graph with minimum degree $\delta \geq 5t+3$, where t is a positive integer, on n vertices. Then,*

$$\overrightarrow{\gamma}_{LD}(G) \leq \frac{2t+4}{5t+8}n.$$

Proof. Let G be a claw-free graph with minimum degree $\delta \geq 5t+3$, where t is a positive integer, on n vertices. Ando et al. proved in [1] that every claw-free graph with minimum degree δ has a $P_{\geq \delta+1}$-factor. Let $P_1, \dots, P_q$ be the paths in the $P_{\geq \delta+1}$-factorization where $m_i = |P_i| \geq \delta + 1$. As proven in [30], each of these paths has a locating-dominating set of size exactly $\lceil 2m_i/5 \rceil$. Hence, by Lemma 12, we have $\overrightarrow{\gamma}_{LD}(G) \leq \sum_{i=1}^{q} \lceil 2m_i/5 \rceil = \sum_{i=1}^{q} (\lceil 2m_i/5 \rceil - 2m_i/5) + \sum_{i=1}^{q} 2m_i/5$.

Observe that we have $\lceil 2m_i/5 \rceil - 2m_i/5 \leq 4/5$ and this value is attained whenever $m_i = 3 \mod 5$. It is easy to check that the sum is upper bounded by the case where each $m_i = 5(t+1)+3$ because each $m_i \geq 5t+4 \geq 9$ and the larger each m_i is the smaller q is. Hence, we have $\sum_{i=1}^{q} (\lceil 2m_i/5 \rceil - 2m_i/5) + \sum_{i=1}^{q} 2m_i/5 \leq n/(5(t+1)+3) \cdot 4/5 + 2n/5 = n(2t+4)/(5t+8)$. □

3.4. (Almost) regular graphs

The goal of this section is to prove that the $n/2$ bound can be drastically improved when the graph is (almost) regular. The proof is based on a probabilistic argument. Namely we prove that, if we select a random subset of vertices of the graph, then we can find an orientation where it is "almost" a locating-dominating set. That is, with positive probability, we can obtain a locating-dominating set from a random set by simply adding a small well-chosen subset of vertices to this random set.

A graph G is d-regular if all the vertices of G have degree exactly d. A class of graphs $\mathcal{G}$ is *k-almost regular* if for every graph $G \in \mathcal{G}$, we have $\Delta(G) \leq \delta(G)^k$.

Theorem 22. *Let $\mathcal{G}$ be a class of k-almost regular graphs. Then, there exists a constant $c_{\mathcal{G},k}$ such that, for every $G \in \mathcal{G}$,*

$$\overrightarrow{\gamma}_{LD}(G) \leq c_{\mathcal{G},k} \cdot \frac{\log \delta}{\delta} \cdot n.$$

Before proving Theorem 22, let us make a couple of remarks. First notice that the bound is tight up to a constant multiplicative factor since, by Theorem 7, $\Theta(\log n)$ vertices are needed for cliques.

Another hypothesis of Theorem 22 asserts that there is a polynomial gap between the minimum and maximum degree. One can wonder if a similar result holds if we only have some assumptions on the minimum degree of the graph. We can prove that it is not true:

Lemma 23. *Let $d, n_0 \in \mathbb{N}$, and $\epsilon > 0$ be a real. Then, there exists a twin-free graph G of minimum degree at least d, order $n \geq n_0$ such that*

$$\overrightarrow{\gamma}_{LD}(G) \geq (\frac{1}{2} - \epsilon)n$$

Proof. Let p and q be two integers with $p \geq q \geq 4$. We define the graph $G_{p,q}$ of order $n = 4p+q$ as a disjoint union of p paths on four vertices complete to a set $\{v_1, v_2, ..., v_q\}$ of size q such that the subgraph induced by $\{v_1, v_2, ..., v_q\}$ is a cycle. An example is given by Fig. 2. As $p \geq q$ the minimal degree is $\delta(G) = q+1$ and one can check $G_{p,q}$ is twin-free.

Let us prove that $\overrightarrow{\gamma}_{LD}(G_{p,q}) \geq 2p - 2^{4q+2}$ which is enough to obtain the lemma since then, $\overrightarrow{\gamma}_{LD}(G_{p,q})/n$ will tend to $\frac{1}{2}$, when $p \to \infty$.

Let D be an orientation of $G_{p,q}$ and let S be an optimal locating-dominating set of D. Let $G_1 = G_{p,q}[p_1, p_2, p_3, p_4]$, $G_2 = G_{p,q}[q_1, q_2, q_3, q_4]$ and $G_3 = G_{p,q}[r_1, r_2, r_3, r_4]$ be three P_4 of $G_{p,q}$ which belongs to the disjoint union of P_4's. If, for every $1 \leq i \leq 4$ and every $1 \leq j \leq q$, the edges $p_i v_j$, $q_i v_j$ and $r_i v_j$ have the same orientation in D, then $|S \cap V(G_1)| \geq 2$ or $|S \cap V(G_2)| \geq 2$ or $|S \cap V(G_3)| \geq 2$. Indeed, if there is at most one vertex of S in each subgraph, then in each subgraph G_i one extremity have no neighbour in $G_i \cap S$. Hence we can assume this is the case for p_1 and q_1. Then, p_1 and q_1 have the same neighbourhood in S, a contradiction.

There are 2^{q^4} orientations of edges between a set of four vertices and a set of q vertices so at least $p - 2 \times 2^{q^4} = p - 2^{q^4+1}$ paths of the disjoint union contain at least two elements of S. So $\overrightarrow{\gamma}_{LD}(G_{p,q}) \geq 2p - 2^{4q+2}$. □

The rest of this section is devoted to prove Theorem 22. Let G be a graph in $\mathcal{G}$. We can assume that G has minimum degree at least e^2. (For graphs of degree less than e^2, the conclusion indeed follows since we can modify the constant to guarantee that $c_{\mathcal{G},k} \cdot \frac{\log \delta}{\delta} \cdot n$ is at least n). The proof is based on a probabilistic argument. We will select a subset of vertices

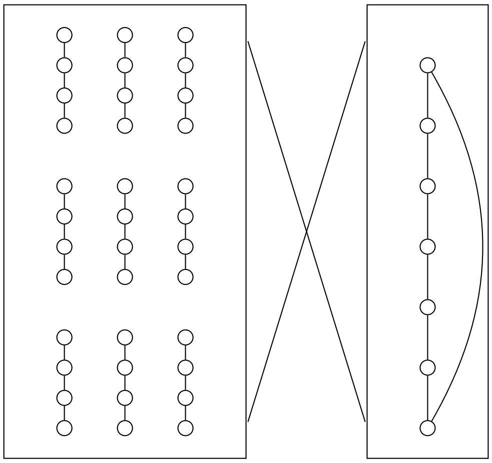

Fig. 2. Example of $G_{9,7}$ of Lemma 23.

at random and prove that, by only modifying it slightly (with high probability), we can construct an orientation of G such that this set is a locating-dominating set.

Let us first recall the Chernoff inequality.

Lemma 24. *[Chernoff] Let $X = \sum_{i=1}^{n} X_i$ where $X_i = 1$ with probability p and 0 otherwise and where all the X_i are independent. Let $\mu = \mathbb{E}(X)$ and $r > 0$. We have*

$$\mathbb{P}(X \leq (1-r)\mu) \leq e^{-\mu \cdot r^2/2}.$$

Also recall the Markov's inequality: If X is a random variable taking non-negative values and $a > 0$, then:

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.$$

In order to prove Theorem 22, we also need the following general lemma:

Lemma 25. *Let G be a graph and X be a subset of vertices such that every vertex v not in X is adjacent to at least $\log \Delta + 1$ vertices of X. Then, there exists an orientation D of G where X is a locating-dominating set.*

Proof. Let $V' = \{v_1, \ldots, v_t\}$ be an arbitrary ordering of $V \setminus X$. Let us prove that we can associate to each vertex v_i of V' a non-empty subset S_i of $X \cap N(v_i)$ such that, for every $i \neq j$, $S_i \neq S_j$.

Let us prove that such a collection of sets S_i can be found greedily. Since v_1 is adjacent to at least $\log \Delta + 1 \geq 1$ vertex of X, we can indeed find such a set for v_1. Assume that we have already selected $S_1, \ldots, S_r$. Let us prove that we can select a set for v_{r+1}. Let $Y_{r+1} = N(v_{r+1}) \cap X$ and $u \in Y_{r+1}$. The number of subsets of Y_{r+1} containing u is $2^{|Y_{r+1}|-1} \geq 2^{\log \Delta + 1} \geq \Delta$. So at least one of them has not been selected since a subset S_j can contain u only if $v_j u$ is an edge. We arbitrarily select a subset of Y_{r+1} containing u that is distinct from $S_1, \ldots, S_r$, which completes the first part of the proof.

Next, for every $x \in X$ in $N(v_i)$, we orient the edges from v_i to x_j if $x \notin S_i$ and orient from x to v_i if $x \in S_i$. One can easily check that X is a locating-dominating set of this orientation of the graph. □

We now have all the ingredients to prove Theorem 22.

Proof of Theorem 22. Let us first start with the following claim:

Claim 26. *Let $c \geq 2$ be constant. For every graph G of minimum degree δ, there exists a subset X of $25c \cdot (\log \delta)/\delta \cdot n$ vertices[2] of G such that all the vertices of $V \setminus X$ have at least $c \log \delta$ neighbours in X.*

Proof. Start with a set X which is empty and add each vertex in X with probability $6c \cdot (\log \delta)/\delta$. So $\mathbb{E}(|X|) = 6c \cdot \frac{\log \delta}{\delta} \cdot n$. Moreover $\mathbb{P}(|X| \geq 24c \cdot \frac{\log \delta}{\delta} \cdot n) \leq \frac{1}{4}$ by Markov's inequality.

[2] All the logarithms of the paper have to be understood base 2.

Let u be a vertex of G. Since $N(u)$ has size at least δ, $\mathbb{E}(|X \cap N(u)|) \geq 6c \cdot \log \delta$. Thus, Lemma 24 ensures that

$$\mathbb{P}(|X \cap N(u)| \leq c \cdot \log \delta) = \mathbb{P}(|X \cap N(u)| \leq (1 - 5/6)6c \cdot \log \delta) \leq e^{-6c \cdot \log \delta \cdot (5/6)^2/2} \leq \frac{1}{\delta^3}$$

as long as $c \geq 2$.

Let us next enrich X with all the vertices u such that $|X \cap N(u)|$ is less than $c \log \delta$. By union bound, the average number of vertices that are added in X is at most n/δ^3. Moreover, using again Markov's inequality, we know that, with probability at least 1/2, the number of vertices that are added in X is at most $2 \cdot n/\delta^3 \leq c \cdot \frac{\log \delta}{\delta} \cdot n$.

So, with probability at least 1/4, the size of X is at most $24c \cdot \frac{\log \delta}{\delta} \cdot n$ before X is enriched and we add at most $c \cdot \frac{\log \delta}{\delta} \cdot n$ vertices in X during the second phase. So there exists a set X of size at most $25c \log \delta/\delta \cdot n$ such that all the vertices are either in X or have at least $c \log \delta$ neighbours in X. □

Let $c = 2k$. By Claim 26, G admits a subset of vertices X of size $50k \cdot \log \delta/\delta \cdot n$ such that every vertex v is either in X or has at least $2k \cdot \log \delta$ neighbours in X. We claim that we can orient the edges between X and $V \setminus X$ to guarantee that all the vertices of $V \setminus X$ have a different neighbourhood in X. It follows from Lemma 25 and the fact that $\log \Delta + 1 \leq 2k \cdot \log \delta$ since $\log \Delta \leq k \log \delta$. □

Let us complete the results of this section with additional results on regular graphs or based on Lemma 25.

A set S is *k-dominating* in G if we have for each $v \in V \setminus S$ that $|N(v) \cap S| \geq k$. Let us denote with $\gamma_k(G)$ the cardinality of a minimum k-dominating set of G. The following lemma is a simple consequence of Lemma 25.

Lemma 27. *Let G be a graph with maximum degree Δ. If $k \geq \log \Delta + 1$, then*

$$\overrightarrow{\gamma}_{LD}(G) \leq \gamma_k(G).$$

By [10, Corollary 14], $\gamma_k(G) \leq n - \alpha(G)$ while $k \leq \delta$. Then, the inequality is an immediate consequence of Lemma 27.

Corollary 28. *Let G be a graph with maximum degree Δ and minimum degree $\delta \geq \log \Delta + 1$. Then,*

$$\overrightarrow{\gamma}_{LD}(G) \leq n - \alpha(G).$$

A similar result holds for locating-dominating sets, when G is twin-free [16, Corollary 4.5].

Let M be a matching in a graph G. We say that a vertex $u \in V(G)$ is *M-unmatched* if u is not an endpoint of any edge in M.

Theorem 29. *Let G be a d-regular graph with $d \geq 3$. Then,*

$$\overrightarrow{\gamma}_{LD}(G) \leq \alpha'(G).$$

Proof. Let G be a d-regular graph and M be a maximum matching in G. Moreover, let us construct the set D_M by choosing for each edge $uv \in M$ the vertex u to D_M if only u has an adjacent M-unmatched vertex. If neither u or v, or both u and v have an adjacent (common) M-unmatched vertex, then we arbitrarily add one of them to D_M. In the latter case, the M-unmatched vertex is common to u and v by the maximality of M.

Observe that D_M is a dominating set in G and each M-unmatched vertex is 2-dominated by D_M. First of all, each M-matched vertex is dominated by another M-matched vertex. Secondly, no two M-unmatched vertices can be adjacent because M is a maximum matching. Moreover, since $d \geq 3$, each M-unmatched vertex is adjacent to the endpoints of at least two different edges in M. Now, due to the structure of D_M, each M-unmatched vertex is at least 2-dominated.

Let us next construct graph G' by removing each edge $e \in E(G) \setminus M$ with both endpoints in M-matched vertices. Now, $|I_{G'}(D_M; u)| = 1$ for each M-matched vertex in $V(G)$ and $|I_{G'}(D_M; v)| \geq 2$ for each M-unmatched vertex v. Thus, M-matched vertices have unique I-sets in G'.

Let w and w' be two M-unmatched vertices with identical I-sets. If $2 \leq |I(D_M; w)| = |I(D_M; w')| \leq d - 1$, then w is adjacent to vertices u and v with $uv \in M$ and, say, $u \in D_M$ and $v \notin D_M$. Moreover, we also have $u \in N(w')$. But now we could have chosen uw' and vw in our matching M which is a contradiction to the maximality of M.

Let us then assume that $|I(D_M; w)| = |I(D_M; w')| = d$ and $I(D_M; w) = I(D_M; w') = \{u_1, \dots u_d\}$. Thus, w and w' are twins. Let us then count the maximum number, N, of M-unmatched vertices which are adjacent to at least two of vertices in $I(D_M; w)$. Each vertex in $I(D_M; w)$ is adjacent in G to at least one M-matched vertex, u and v. Hence, there might be at most $d - 3$ other adjacent M-unmatched vertices. Hence, we have $N \leq d(d - 3)/2 + 2$. Furthermore, there are exactly $2^d - d - 1$ subsets of $I(D_M; w)$ of cardinality at least two. Since $d \geq 3$, we have $2^d - d - 1 > d(d - 3)/2 + 2$. Thus, we may go through each M-unmatched vertex one by one and if an M-unmatched vertex w has an I-set identical to some other (M-unmatched) vertex, then there exists a set of adjacent edges which can be removed so that w has a unique I-set afterwards. Therefore, we may construct a spanning subgraph G'' with the property $\gamma_{LD}(G'') \le \alpha'(G)$. Hence, the claim follows by Lemma 12. □

4. Worst orientation

We next focus on the worst possible orientation. We again start with basic results. Then, we study the lower bound $\overrightarrow{\Gamma}_{LD}(G) \ge \gamma_{LD}(G)/2$ that we prove to be true for several classes of graphs and let it open in general. Finally, we consider lower bounds using the number of vertices.

4.1. Basic results

Let us start by first showing some lower bounds that are used all along the section. The *maximum average degree* of a graph G, denoted by $mad(G)$ is the maximum quantity $\frac{2|E(H)|}{|V(H)|}$ over all the subgraphs H of G.

Lemma 30. *Let G be a graph of order n. Then,*

1. $\overrightarrow{\Gamma}_{LD}(G) \ge \alpha(G)$,
2. $\overrightarrow{\Gamma}_{LD}(G) \ge \lceil \omega(G)/2 \rceil$,
3. $\overrightarrow{\Gamma}_{LD}(G) \ge 2n/\lceil mad(G)/2+3 \rceil$.

Proof. Let G be a graph on n vertices. Point 1 has already been noticed for the worst orientation for dominating sets (see [6]) and thus, is still true for locating-dominating sets. We repeat here the argument. Take an independent set X of size $\alpha(G)$ and orient all the edges with an endpoint in X from X to $V \setminus X$. Then, all the vertices in X are sources and thus, must be in any locating dominating set.

Let us next prove the second point. Let K_m be a clique of G. Consider an orientation D such that each edge is oriented away from K_m and the edges inside K_m are oriented in a transitive way. In a locating-dominating set S of D, no vertices outside K_m can be in the in-neighbourhoods of the vertices of K_m. Thus, S must induce a locating-dominating set in K_m. Since K_m is oriented in a transitive way, by [13], we necessarily have at least $\lceil m/2 \rceil$ vertices in $V(K_m) \cap S$ and so in S.

Let us prove the last point. To do so, let us show that $\gamma_{LD}(D) \ge 2n/(\Delta^+(D)+3)$ for any orientation D of G. Let C be a locating-dominating set of D. For each vertex $c \in C$, let $s(c) = \sum_{v \in N^+[c]} 1/|N^-[v]|$. Since C is dominating in D, we have $\sum_{c \in C} s(c) = n$. Moreover, for any $c \in C$, at most two vertices in $N^+[c]$ have only c in their I-sets (at most one vertex outside c and maybe c). Thus, $s(c) \le 2 + (\Delta^+(D)-1)/2$. Now,

$$n = \sum_{c \in C} s(c) \le |C| \frac{3+\Delta^+(D)}{2}.$$

Hence, $|C| \ge 2n/(3+\Delta^+(D))$. So Point 3 follows since each graph has an orientation D' such that $\Delta^+(D') \le \lceil \mathrm{mad}(G)/2 \rceil$ by [20]. □

Observe that all the bounds are tight. Indeed, we see in Corollary 35 that for some bipartite graphs $\overrightarrow{\Gamma}_{LD}(G) = \alpha(G)$. Moreover, for a complete graph K_n, we have $\overrightarrow{\Gamma}_{LD}(K_n) = \lceil n/2 \rceil$, by Corollary 6. Finally, we will see (Corollary 36) that for a cycle on n vertices we have $\overrightarrow{\Gamma}_{LD}(C_n) = \lceil n/2 \rceil$.

We now present three general upper bounds for $\overrightarrow{\Gamma}_{LD}(G)$. We denote by $ad(G)$ the average degree of G and by $\alpha_2(G)$ the maximum size of an independent set at 2-distance, that is a set of vertices such that any two vertices of the set are at distance greater than 2.

Lemma 31. *Let G be a graph of order n. Then,*

1. $\overrightarrow{\Gamma}_{LD}(G) \le n - \alpha_2(G)$;
2. $\overrightarrow{\Gamma}_{LD}(G) \le n - \left\lfloor \frac{\omega(G)}{2} \right\rfloor$;
3. $\overrightarrow{\Gamma}_{LD}(G) \le n - \left\lfloor \frac{n}{2n-2ad(G)} \right\rfloor$.

Proof. Let G be a graph on n vertices and D be an orientation of G such that $\gamma_{LD}(D) = \overrightarrow{\Gamma}_{LD}(G)$. Let S be a maximum independent set at 2-distance in G. Observe that, for any two distinct vertices $u, v \in S$, we have $N[u] \cap N[v] = \emptyset$. Let us construct set S' by adding, for each vertex $u \in S$, either u to S' if u has no out-neighbours in D or an out-neighbour of u if u has one in D. Now, one can easily check that $C = V \setminus S'$ is a locating-dominating set of G of size $n - \alpha_2(G)$.

Let us next prove 2. Let K be a maximal clique in G and let C_K be an optimal locating-dominating set in $D[K]$. Now, $C = C_K \cup (V(G) \setminus K)$ is a locating-dominating set of D. Furthermore, $|C_K| \leq \lceil \omega(G)/2 \rceil$ by Theorem 5 and hence, the claim follows.

Let us finally prove the third bound. We have

$$\overrightarrow{\Gamma}_{LD}(G) \leq n - \lfloor \omega(G)/2 \rfloor = n - \lfloor \alpha(\overline{G})/2 \rfloor \leq n - \lfloor n/(2ad(\overline{G}) + 2) \rfloor = n - \lfloor n/(2n - 2ad(G)) \rfloor.$$

Here the second inequality is due to Caro-Wei lower bound for independence number [5,32] and the last equality is due to equality $ad(G) + ad(\overline{G}) = n - 1$. □

All these bounds are tight: the first bound is tight for stars and the two others for complete graphs.

We still have $\overrightarrow{\Gamma}_{LD}(G) \leq n - 1$ as soon as G has at least one edge. As in the case of $\overrightarrow{\gamma}_{LD}(G)$, we can characterize the set of graphs reaching $\overrightarrow{\Gamma}_{LD}(G) = n - 1$ using Theorem 3.

Lemma 32. *For a connected graph G, $\overrightarrow{\Gamma}_{LD}(G) = n - 1$ if and only if at least one of the following conditions holds:*

1. *$n = 3$;*
2. *G is a star;*
3. *G consists of a complete bipartite graph and possibly a single universal vertex.*

Proof. By Theorem 3, we have $\overrightarrow{\Gamma}_{LD}(G) = n - 1$ if $n = 3$ or G is a star. Moreover, since we consider oriented graphs, the third condition of Theorem 3 implies that C must be of size one. Thus, the claim follows. □

Cycles on four vertices have a special role for best orientations. It is also the case for worst orientations, as illustrated by the following results.

Lemma 33. *Let G be a graph without C_4 as a subgraph. Then, $\overrightarrow{\Gamma}_{LD}(G) \leq \overrightarrow{\Gamma}_{LD}(G - e)$ for any edge $e \in E(G)$.*

Proof. Let G be a graph without C_4 and with at least one edge. Let D be an orientation such that $\gamma_{LD}(D) = \overrightarrow{\Gamma}_{LD}(G)$. By contradiction, assume that $\overrightarrow{\Gamma}_{LD}(G - e) < \overrightarrow{\Gamma}_{LD}(G)$. Then, we have $\gamma_{LD}(D - e) < \gamma_{LD}(D)$.

Let S be an optimal locating-dominating set in $D - e$. Since $\gamma_{LD}(D - e) < \gamma_{LD}(D)$, S cannot be a locating-dominating set in D. Because S is dominating in $D - e$, S is also dominating in D. Thus, there are vertices $u, v \in V(G)$ such that $I_D(v) = I_D(u)$. Moreover, we have $|I_D(v)| = |I_D(u)| \geq 2$. Let $\{c_1, c_2\} \subseteq I_D(v)$. But now we have a cycle on four vertices u, c_1, v and c_2. □

Note that Lemma 33 does not hold for C_4. We have $\overrightarrow{\Gamma}_{LD}(C_4) = 3$ and $\overrightarrow{\Gamma}_{LD}(P_4) = 2$. The bounds in the following lemma are tight for example for stars.

Lemma 34. *Let G be a graph without C_4 as a subgraph. Then,*

$$\gamma_{LD}(G) \leq \overrightarrow{\Gamma}_{LD}(G) \leq n - \alpha'(G).$$

Proof. Let $G = (V, E)$ be a graph without C_4 as a subgraph. The lower bound follows from Theorem 10. Let us prove the upper bound. Let M be a maximum matching in G and G' be a graph we get from G by removing each edge not belonging to M. By Lemma 33, we have $\overrightarrow{\Gamma}_{LD}(G) \leq \overrightarrow{\Gamma}_{LD}(G')$. Moreover, the graph G' consists of isolated vertices and components isomorphic to P_2. Thus, a set S consisting of isolated vertices and a single vertex for each P_2-component is locating-dominating in G' and $\overrightarrow{\Gamma}_{LD}(G') = \gamma_{LD}(G') = |S| \leq n - \alpha'(G)$. □

Together with some classical results of König and Gallai, Lemma 34 permits to determine the exact value of $\overrightarrow{\Gamma}_{LD}(G)$ for bipartite graphs without C_4 (which in particular include all trees).

Corollary 35. *Let G be a bipartite graph without C_4 as a subgraph. Then, $\overrightarrow{\Gamma}_{LD}(G) = \alpha(G)$.*

Proof. Let G be a bipartite graph without C_4. We have $\overrightarrow{\Gamma}_{LD}(G) \geq \alpha(G)$ by Lemma 30. By [25], we have $\alpha'(G) = \beta(G)$ since G is bipartite. Moreover, by [15], we have $\alpha(G) + \beta(G) = n$. Hence, $\alpha(G) = n - \alpha'(G)$. Now, we have, by Lemma 34, $\overrightarrow{\Gamma}_{LD}(G) \leq n - \alpha'(G) = \alpha(G)$. Thus, $\alpha(G) \leq \overrightarrow{\Gamma}_{LD}(G) \leq \alpha(G)$. □

Corollary 36. *Let C_n be a cycle on n vertices. Let $n = 3$ or $n \geq 5$. Then,*

$$\overrightarrow{\Gamma}_{LD}(C_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Proof. By Lemma 33, we have $\overrightarrow{\Gamma}_{LD}(C_n) \leq \overrightarrow{\Gamma}_{LD}(P_n)$ by Corollary 35 applied to P_n (where $\alpha(P_n) = \lceil \frac{n}{2} \rceil$). Moreover, if we take a cyclic orientation of C_n, the set of vertices with an odd index number forms an optimal locating-dominating set. □

Observe that, for the path on n vertices, P_n, we have $\gamma_{LD}(P_n) = \lceil 2n/5 \rceil$ for paths [30] while we have $\overrightarrow{\Gamma}_{LD}(P_n) = \alpha(P_n) = \lceil n/2 \rceil$. As we mentioned above, there exist graphs without C_4 with $\overrightarrow{\Gamma}_{LD}(G) > \gamma_{LD}(G)$. However, we are not aware of any graph G without C_4 which does not attain the upper bound of Lemma 34.

Open problem 37. Does there exist a graph G without C_4 as a subgraph with $\overrightarrow{\Gamma}_{LD}(G) < n - \alpha'(G)$?

4.2. Lower bound with $\gamma_{LD}(G)$

In Section 4.1, we have seen that $\overrightarrow{\Gamma}_{LD}(G) \geq \gamma_{LD}(G)$ if G is without C_4 subgraphs. One can easily remark that this equality does not hold in general. For example, for complete graphs we have $\gamma_{LD}(K_n) = n - 1$ and $\overrightarrow{\Gamma}_{LD}(K_n) = \lceil n/2 \rceil$ by Corollary 6. However the clique example is somehow unsatisfactory since all the vertices are twins. One can wonder if we can also provide an example of twin-free graphs where $\overrightarrow{\Gamma}_{LD}(G) < \gamma_{LD}(G)$. We will prove (Theorem 43) that there are graphs for which $\overrightarrow{\Gamma}_{LD}(G)/\gamma_{LD}(G)$ is arbitrarily close to $1/2$. Moreover, we strengthen the result that $\overrightarrow{\Gamma}_{LD}(G) \geq \gamma_{LD}(G)$ on graphs without C_4 to a wider class of graphs (Lemma 41).

Despite our efforts, we were not able to find graphs for which $\overrightarrow{\Gamma}_{LD}(G) < \gamma_{LD}(G)/2$. We leave as an open problem the following question:

Open problem 38. Is it true that for every graph G, $\overrightarrow{\Gamma}_{LD}(G) \geq \gamma_{LD}(G)/2$?

We were actually not able to prove the existence of any constant c such that, for any graph G, $\overrightarrow{\Gamma}_{LD}(G) \geq c \cdot \gamma_{LD}(G)$. However, in the following theorem we present a bound with $\Delta(G)$.

Theorem 39. *Let G be a graph. Then,*

$$\overrightarrow{\Gamma}_{LD}(G) \geq \frac{\gamma_{LD}(G)}{\lceil \log_2 \Delta(G) \rceil + 1}.$$

Proof. Let D be an orientation of G such that $\overrightarrow{\Gamma}_{LD}(G) = \gamma_{LD}(D)$. Moreover, let S be an optimal locating-dominating set in D. Observe, that for each subset I of S, the set

$$S^I = \{v \in V(G) \setminus S \mid I_G(S; v) = I\}$$

contains at most $\Delta(G)$ vertices: $|S^I| \leq \Delta(G)$. Let us next construct a new orientation D_1 by first taking for each set S^I, $\lfloor |S^I|/2 \rfloor$ disjoint vertex pairs within the set S^I, that is, as many disjoint vertex pairs as possible. Then, we number each vertex of $V(G)$ as u_i, $1 \leq i \leq |V(G)|$ so that each pair has consecutive numbers. Finally we orient each edge from u_i to u_j where $i < j$.

Let S_1 be an optimal locating-dominating set for orientation D_1. Notice that $|S_1| \leq \overrightarrow{\Gamma}_{LD}(G)$. Moreover, $S_1' = S \cup S_1$ is a locating-dominating set in D and D_1. Furthermore, S_1 separates each paired pair of vertices in D_1. Thus, if for a pair u_i, u_{i+1}, vertex x separates u_i and u_{i+1} in D_1, then either $x = u_{i+1}$ or it separates also u_i and u_{i+1} in G. Moreover, for each $I' \subseteq S_1'$ such that $I = I' \cap S$, we have that $S_1'^{I'} \subseteq S^I$. Since S_1 separates the pairs in G, we have that $|S_1'^{I'}| \leq \lfloor |S^I|/2 \rfloor \leq \lfloor \Delta(G)/2 \rfloor$.

If we now iterate this process $\lceil \log_2(\Delta(G)) \rceil$ times, each time creating a new orientation with a new numbering and a new optimal locating-dominating set for the orientation, then we finally get set $S_t' = S \cup \bigcup_{i=1}^{t} S_i$, where $t = \lceil \log_2(\Delta(G)) \rceil$, with $|S_t'| \leq \lceil \log_2(\Delta(G)) + 1 \rceil \overrightarrow{\Gamma}_{LD}(G)$. Moreover, because we (almost) halve the number of vertices with the same I-set in

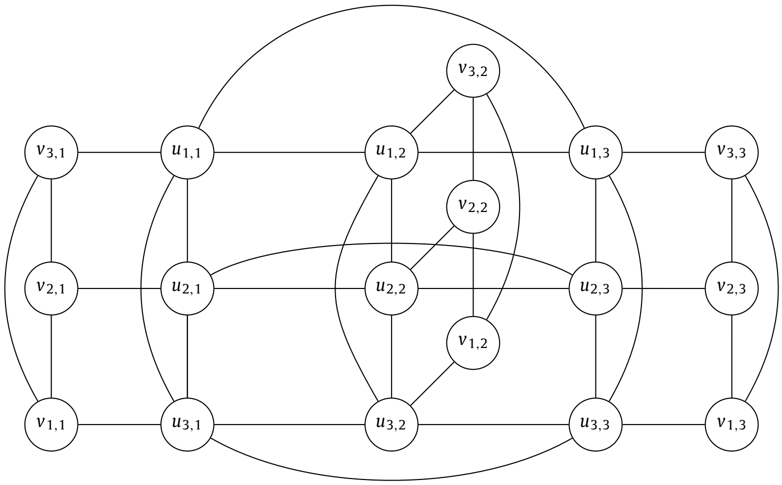

Fig. 3. Example of graph G of Theorem 43 with $t = k = 3$.

G each time, no vertices in $V \setminus S'_t$ share the same I-set with the set S'_t in G. Thus, S'_t is locating-dominating in G and $\gamma_{LD}(G) \le |S'_t| \le \lceil \log_2(\Delta(G)) + 1 \rceil \overrightarrow{\Gamma}_{LD}(G)$. □

4.2.1. Graphs for which $\overrightarrow{\Gamma}_{LD}(G) \ge \gamma_{LD}(G)$

Lemma 40. *Let G be a graph and D be an orientation of G such that no C_4 in G contains a directed path of length 4 in D. Then, any locating-dominating set of D is a locating-dominating set of G. In particular, $\overrightarrow{\Gamma}_{LD}(G) \ge \gamma_{LD}(G)$.*

Proof. Let G be a graph and D be an orientation of G such that no C_4 in G contains a directed path of length 4 in D. Let S be locating-dominating in D. Let us assume that S is not locating-dominating in G. Set S is clearly dominating in G. Let $v, u \in V \setminus S$ be vertices with $I_G(v) = I_G(u)$. Since $I_D(v) \ne I_D(u)$, we have $|I_G(v)| \ge 2$. Let us assume that $c_1 \in I_D(v) \setminus I_D(u)$ and $c_2 \in I_D(u)$. But now we have a directed path c_2uc_1v, a contradiction. □

Let M and R be subgraphs of G. An (M, R)-*WORM colouring* [17] of graph G, is a colouring of the vertices of G where no subgraph of G isomorphic to M is monochromatic and no subgraph of G isomorphic to R is heterochromatic (i.e. has all its vertices of different colours). The following lemma gives us a tool for applying Lemma 40.

Lemma 41. *If G admits a (K_2, C_4)-WORM colouring, then $\overrightarrow{\Gamma}_{LD}(G) \ge \gamma_{LD}(G)$.*

Proof. Let G be a graph which admits a (K_2, C_4)-WORM colouring c using colours $\{1, \dots, k\}$. Let D be the orientation such that we have an edge from u to v if $c(u) < c(v)$. Since c is a (K_2, C_4)-WORM colouring, it defines an orientation for any edge and $\overrightarrow{\Gamma}_{LD}(G) \ge \gamma_{LD}(D)$. Hence, it is enough to show that $\gamma_{LD}(D) \ge \gamma_{LD}(G)$.

We claim that no C_4 in D contains a directed path of length 4. Indeed, if there is a directed path $u_1u_2u_3u_4$, then $c(u_1) < c(u_2) < c(u_3) < c(u_4)$ and if this path is contained in a C_4, then this C_4 is heterochromatic, a contradiction. Then, the claim follows from Lemma 40. □

Observe that any proper colouring with at most three colours is also a (K_2, C_4)-WORM colouring. Hence, we get the following corollary (where $\chi(G)$ denotes the chromatic number of G).

Corollary 42. *Let G be a graph with $\chi(G) \le 3$. Then, $\overrightarrow{\Gamma}_{LD}(G) \ge \gamma_{LD}(G)$.*

4.2.2. Worst examples

The following theorem ensures that there exist examples of twin-free graphs where we almost reach the ratio $\frac{1}{2}$ for $\overrightarrow{\Gamma}_{LD}(G)/\gamma_{LD}(G)$.

Theorem 43. *There exists an infinite family of twin-free graphs G such that*

$$\frac{\overrightarrow{\Gamma}_{LD}(G)}{\gamma_{LD}(G)} \overset{n\to\infty}{\longrightarrow} \frac{1}{2}.$$

Proof. Let $k, t \geq 2$ be integers and let $H_{k,t}$ be the graph with vertex set

$$V(H_{k,t}) = \{v_{i,j}, u_{i,j} \mid 1 \leq i \leq k, 1 \leq j \leq t\}$$

and edge set

$$E(H_{k,t}) = \{u_{i,j}u_{i',j} \mid i \neq i'\} \cup \{u_{i,j}u_{i,j'} \mid j \neq j'\} \cup \{v_{i,j}v_{i',j} \mid i \neq i'\} \cup \{v_{i,j}u_{i,j}\}$$

where we have $1 \leq i \leq k, 1 \leq j \leq t$ for each i and j. We illustrate graph $H_{3,3}$ in Fig. 3.

In other words, the set of vertices $\{v_{i,j} \mid 1 \leq i \leq k\}$ induces a clique V_t^j for every j. Similarly, the set of vertices $\{u_{i,j} \mid 1 \leq i \leq k\}$ induces a clique U_t^j for every j and the set of vertices $\{u_{i,j} \mid 1 \leq j \leq t\}$ induces a clique U_k^i for each i. In fact, the set of vertices $u_{i,j}$, for $1 \leq i \leq k, 1 \leq j \leq t$, forms the Cartesian product $K_t \square K_k$. Observe that $H_{t,k}$ is twin-free since each vertex $u_{i,j}$ has a unique neighbour $v_{i,j}$ and vice versa.

Let C be a locating-dominating set of $H_{t,k}$. If we have $\{v_{i,j}, u_{i,j}, v_{i',j}, u_{i',j}\} \cap C = \emptyset$. Then, $I(v_{i,j}) = I(v_{i',j})$ and hence, we have a contradiction. Thus, $\gamma_{LD}(H_{t,k}) \geq (k-1)t$. On the other hand, the set $\{v_{i,j} \mid 1 \leq i \leq k, 1 \leq j \leq t\}$ forms a locating-dominating set and hence,

$$kt \geq \gamma_{LD}(H_{t,k}) \geq (k-1)t.$$

Let us then consider the oriented locating-dominating sets. Let D be an orientation of $H_{t,k}$ with $\overrightarrow{\Gamma}_{LD}(G) = \gamma_{LD}(D)$.

Let S_j be a 2-dominating set in the tournament U_t^j for each j (i.e. each vertex outside S_j is dominated twice). Let S'_i be a dominating set in the tournament $U_k^i \setminus \bigcup_{j=1}^t S_j$ for each i in the orientation D. Observe that $|S_j| \leq 2\log(t+1)$ and $|S'_i| \leq \log(k+1)$ for each i and j by [11]. Moreover, let C_j be an optimal locating-dominating set in the tournament V_t^j. We have $|C_j| \leq t/2$.

Consider next the set $C = \bigcup_{j=1}^t C_j \bigcup_{i=1}^k S'_i \bigcup_{j=1}^t S_j$. We have

$$|C| \leq kt/2 + 2k\log(t+1) + t\log(k+1).$$

Observe that for each i and j, vertex $u_{i,j}$ is now 3-dominated by $\bigcup_{a=1}^k S'_a \cup \bigcup_{b=1}^t S_b$ and $I_D(u_{i,j}) \neq I_D(u_{i',j'})$ where $(i,j) \neq (i',j')$. Moreover, each vertex $v_{i,j}$ is located by the set C_j. Thus, C is a locating-dominating set of D and

$$\overrightarrow{\Gamma}_{LD}(H_{k,t}) = \gamma_{LD}(D) \leq kt/2 + 2k\log(t+1) + t\log(k+1).$$

Finally, if we choose an orientation of $H_{k,t}$ such that each edge from $v_{i,j}$ is oriented to $u_{i,j}$ and such that all the cliques V_t^j are oriented transitively, we notice that we need at least $t\lceil k/2 \rceil$ vertices in C.

Thus,

$$\frac{\overrightarrow{\Gamma}_{LD}(H_{k,t})}{\gamma_{LD}(H_{k,t})} \leq \frac{kt/2 + 2k\log(t+1) + t\log(k+1)}{(k-1)t}$$

and

$$\frac{\overrightarrow{\Gamma}_{LD}(H_{k,t})}{\gamma_{LD}(H_{k,t})} \geq \frac{t\lceil k/2 \rceil}{kt} \geq \frac{1}{2}.$$

When $k \to \infty$ and $t \to \infty$, $\overrightarrow{\Gamma}_{LD}(H_{k,t})/\gamma_{LD}(H_{k,t}) \to \frac{1}{2}$. □

4.3. *Lower bound with the number of vertices*

In this subsection, we consider how small $\overrightarrow{\Gamma}_{LD}(G)$ can be compared to the number of vertices. For the best orientation and the undirected case, there exist many graphs reaching the theoretical lower bound in $\Theta(\log n)$ (see Theorem 7). For the worst orientation, we did not find any graph with $\overrightarrow{\Gamma}_{LD}(G)$ of order $\log n$.

Open problem 44. Does there exist a class of graphs $\mathcal{G}$ such that for any $G \in \mathcal{G}$ on n vertices the value $\overrightarrow{\Gamma}_{LD}(G)$ is logarithmic on n?

We have three reasons to believe there is a positive answer for Open Problem 44. First, most of the other types of locating-dominating parameters can achieve logarithmic values on n. Secondly, we did not find a non-logarithmic lower bound. Thirdly, A natural class of candidates would be (Erdős-Renyi) random graphs where an unoriented locating-dominating set has indeed logarithmic size [14]. However, the worst orientation of such a graph is not easy to manipulate and then we were not able to study efficiently upper bounds on $\overrightarrow{\Gamma}_{LD}(G)$.

On the other hand, in the following we give some properties which deny the possibility for a graph class $\mathcal{G}$ to have a logarithmic lower bound on n. Together with a well-known conjecture and an open problem, if they have a positive solution, these properties mean that if $\mathcal{G}$ has a certain type of a forbidden subgraph characterization, then it does not have a logarithmic lower bound for $\overrightarrow{\Gamma}_{LD}(G)$. In the following, we discuss these ideas and give some polynomial lower bounds for $\overrightarrow{\Gamma}_{LD}(G)$ in some graph classes.

Lemma 30 gives a linear lower bound for $\overrightarrow{\Gamma}_{LD}(G)$ in n for classes of graphs which have their chromatic number bounded by a constant since $\alpha(G) \geq n/\chi(G)$ and for classes of graphs with cliques of linear size. These results can be extended to obtain bounds in $\Omega(n^{\beta})$ where β is a constant when a class of graphs $\mathcal{G}$ is χ*-bounded* by a polynomial function, that is, if there exists a polynomial function f such that $\chi(G) \leq f(\omega(G))$ holds for all $G \in \mathcal{G}$. Note that it has been asked [23] if it is true that every χ-bounded class admits a χ-bounding function that is polynomial. Moreover, Gyárfas [19] has conjectured that if the graph class $\mathcal{G}$ is F-free for some forest F, then $\mathcal{G}$ is χ-bounded.

Theorem 45. *Let $\mathcal{G}$ be a class of graphs χ-bounded by a function $f : x \mapsto x^c$ where c is a constant. Then, for any $G \in \mathcal{G}$ with n vertices, we have:*

$$\overrightarrow{\Gamma}_{LD}(G) \geq 2^{-c/(c+1)} \cdot n^{\frac{1}{c+1}}.$$

Proof. Let $G \in \mathcal{G}$. By Lemma 30, we have $\overrightarrow{\Gamma}_{LD}(G) \geq \omega(G)/2 \geq \chi(G)^{1/c}/2$ and $\overrightarrow{\Gamma}_{LD}(G) \geq \alpha(G) \geq n/\chi(G)$. Thus, $\overrightarrow{\Gamma}_{LD}(G) \geq \max\{n/\chi(G), \chi(G)^{1/c}/2\}$. This value attains its minimum when $n/\chi(G) = \chi(G)^{1/c}/2$. In other words, when $\chi(G) = (2n)^{c/(c+1)}$. This gives the claim. □

Theorem 45 applies in particular for perfect graphs for which f is the identity function. Hence, if G is a perfect graph, then

$$\overrightarrow{\Gamma}_{LD}(G) \geq \sqrt{\frac{n}{2}}. \tag{2}$$

Theorem 45 can also be used to get a lower bound, for example, for claw-free graphs. In [8], the authors have shown that if G is a connected claw-free graph with an independent set of size at least 3, then $\chi(G) \leq 2\omega(G)$. Thus, $\overrightarrow{\Gamma}_{LD}(G) \geq \sqrt{n}/2$. Similar idea works also for C_3-free graphs. In [24], the author has shown that if G is C_3-free, then $\alpha(G) \in \Omega(\sqrt{n \log n})$. Thus, also $\overrightarrow{\Gamma}_{LD}(G) \in \Omega(\sqrt{n \log n})$.

Finally, we end the chapter by giving a class of perfect graphs which shows that Bound (2) is tight within a logarithmic multiplier. We denote by $G \square H$ the cartesian product of G and H.

Theorem 46. *Let m be an integer. Then, $m \leq \overrightarrow{\Gamma}_{LD}(K_m \square K_m) \leq 3m \log(m+1)$.*

Proof. Let us denote the vertices of $G = K_m \square K_m$ by $V(G) = \{(v_i, u_j) \mid 1 \leq i, j \leq m\}$. Moreover, we have $(v_{i_1}, u_{j_1})(v_{i_2}, u_{j_2}) \in E(G)$ if $i_1 = i_2$ or $j_1 = j_2$. There are $2m$ cliques, each of size m in G and every vertex belongs to exactly two of these cliques. We have $\omega(K_m \square K_m) = \chi(K_m \square K_m) = m$. Thus, $m \leq \overrightarrow{\Gamma}_{LD}(K_m \square K_m)$ and G is perfect.

Let D be an orientation of G such that $\overrightarrow{\Gamma}_{LD}(G) = \gamma_{LD}(D)$. Similarly, as in the proof of Theorem 43, we again construct a dominating set for each clique $\{(v_i, u_j) \mid 1 \leq i \leq m\}$ where j is fixed and a 2-dominating set for each clique $\{(v_i, u_j) \mid 1 \leq j \leq m\}$ where i is fixed. Observe that, in D, each dominating set has cardinality of at most $\log(m+1)$ ([11]) and hence, each 2-dominating set has cardinality of at most $2\log(m+1)$. Since we have m dominating sets and m different 2-dominating sets, we have $\overrightarrow{\Gamma}_{LD}(K_m \square K_m) \leq 3m \log(m+1)$. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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