On Vertex-Robust Identifying Codes of Level Three

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Abstract

Assume that $G = (V, E)$ is an undirected and connected graph, and consider $C \subseteq V$. For every $v \in V$, let $I_r(v) = \{u \in C : d(u, v) \leq r\}$, where $d(u, v)$ denotes the number of edges on any shortest path between u to v in G. If all the sets $I_r(v)$ for $v \in V$ are pairwise different, and none of them is the empty set, C is called an r-identifying code. In this paper, we consider t-vertex-robust r-identifying codes of level s, that is, r-identifying codes such that they cover every vertex at least s times and the code is vertex-robust in the sense that $|I_r(u) \bigtriangleup I_r(v)| \geq 2t + 1$ for any two different vertices u and v. Vertex-robust identifying codes of different levels are examined, in particular, of level 3. We give bounds (sometimes exact values) on the density or cardinality of the codes in binary hypercubes and in some infinite grids.

Keywords: Identifying code, binary hypercube, infinite grid, vertex robustness, covering code.

1 Introduction

Assume that $G = (V, E)$ is an undirected and connected graph. We denote the *(graphic) distance* by $d(u, v)$ which is the number of edges in any shortest path between $u \in V$ and $v \in V$. Let $r \geq 1$ be an integer. For $v \in V$, the *ball* of radius r centered at v is defined by

$$B_r(v) = \{u \in V \mid d(u, v) \leq r\}.$$

If $d(x, y) \leq r$, we say that x and y r-*cover* (or *cover)* each other.

A *code* is a nonempty subset of V and its elements are called *codewords*. Let C be a code. For $x \in V$, we denote

$$I_r(x) = I_r(G, C; x) = C \cap B_r(x).$$

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The set $I_r(x)$ is called the I_r*-set* of x. In this paper, we concentrate on $r = 1$ and write $I_1(x) = I(x)$ (and call the set $I(x)$ the I*-set* of x).

A code $C \subseteq V$ is called r*-identifying* (in G) if the sets $I_r(x)$ are distinct and non-empty for all $x \in V$.

Karpovsky, Chakrabarty and Levitin [9] introduced the concept of identifying codes in 1998. An application (see [9]) of identifying codes to fault diagnosis of multiprocessor architectures is described next.

Suppose that each vertex of G contains a processor and an edge is a communication link between two processors. Assume that at most one processor can be malfunctioning. We wish to locate the malfunctioning processor, say $x \in V$, or decide that there is none. We choose a code $C \subseteq V$, i.e., a subset of processors, and a codeword is assigned the following task. Each codeword $c \in C$ checks all the processors in $B_r(c)$ and sends a single bit value "1" to the host if it detects any problems and "0" if everything is fine in the ball. If the code C is r-identifying, the host can determine the faulty vertex, say x, by knowing $I_r(x)$ which is the set of processors sending "1". Of course, we would like to use as few codewords as possible.

Identification has been widely studied in graphs such as binary Hamming spaces (i.e., binary hypercubes), the square, triangular and king grids and the hexagonal mesh (see, e.g., [1],[2],[5],[7],[8],[9],[13] and the references therein). In the papers [14],[4],[6] and [11], codes that remain identifying although I_r-sets can be corrupted are considered.

The symmetric difference $(A \setminus B) \cup (B \setminus A)$ is denoted by $A \triangle B$.

Definition 1. *[6] Let s and t be non-negative integers. A code $C \subseteq V$ is a t-vertex-robust r-identifying code of level s if*

i) $|I_r(v)| \geq s$ for all $v \in V$, and

ii) $|I_r(u) \triangle I_r(v)| \geq 2t + 1$ for all distinct vertices u and v of V.

These codes remain identifying even if some codewords can be missing from an I_r-set or some new ones added to it (together again at most t changes in one I_r-set). If C satisfies ii), we can be sure that $I_r(u) \triangle A \neq I_r(v) \triangle B$ for any subsets A and B of C with $|A| \leq t$ and $|B| \leq t$. The requirement i) determines the level of protection against a false alarm.

Let C satisfy ii) and suppose that there exists a vertex $x \in V$ such that $|I_r(x)| = s$ with $s \in \{0, 1, \ldots, t\}$. Then all the other vertices are r-covered by at least $2t - s + 1$ codewords. Indeed, if $y \in V$ is such that $|I_r(y)| < 2t - s + 1$, then $|I_r(x) \triangle I_r(y)| < s + (2t - s + 1)$, a contradiction. Adding (if the degree of the vertex x in G allows it) $2t - 2s + 1$ codewords to C guarantees that all the vertices are covered by at least $2t - s + 1$ codewords (and the condition ii) is, of course, still valid). Adding at most $2t + 1$ (when t is a constant) codewords does not increase the density (see Section 4) in the above mentioned grids. Therefore, the values $s \geq t + 1$ are of main interest.

In [4], the case $s = t + 1$ is studied, in particular, 1-vertex-robust 1-identifying codes of level 2 in the above mentioned graphs. For the case $s = 0$, see [14], and for $s = 2t + 1$ where $t \geq 2$, see [10].

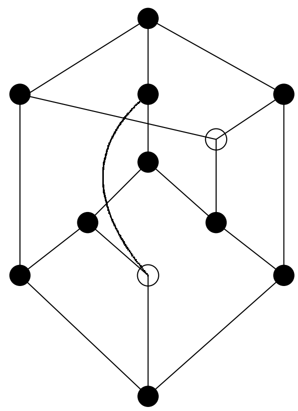

Figure 1: A 1-vertex-robust 1-identifying code of level 3 that attains the bound (1). Circles denote the vertices and the black ones are codewords.

In this paper, we assume that $t \geq 1$. In Section 2, we give bounds on t-vertex-robust 1-identifying codes of level $s \geq t+2$ in general regular graphs, and in Section 3, we concentrate on binary hypercubes and consider mainly 1-vertex-robust 1-identifying codes of level 3. Section 4 is devoted to the bounds on the density of codes in the infinite square and king grids and in the infinite hexagonal mesh.

2 A lower bound on regular graphs

Lemma 1. *Let $t \geq 1$ be an integer and $d \geq t+2$. Assume that $G=(V,E)$ is a finite d-regular graph and that C is a t-vertex-robust 1-identifying code of level $t+2$. Then*

$$|C| \geq \frac{(t+2)|V|}{d+1-\frac{t+1}{2t+3}}. \tag{1}$$

If C is a t-vertex-robust 1-identifying code of level $s \geq t+3$, then

$$|C| \geq \frac{s|V|}{d+1}. \tag{2}$$

Proof. Let C be t-vertex-robust 1-identifying of level $t+2$. We denote by C_i the set of codewords c of C for which $|I(c)|=i$. We also denote

$$C_{\geq i} = \bigcup_{j \geq i} C_j.$$

Clearly, $C_i = \emptyset$ for all $i \in \{1,2,\ldots,t+1\}$. Count the number of ordered pairs (c,c'), where $c \in C_{t+2}$, $c' \in C_{\geq t+3}$ and there is an edge between c and

c'. For every $c \in C_{t+2}$, there are exactly $t+1$ choices for c', because by ii) of Definition 1 every c' is in $C_{\geq t+3}$. For each $c' \in C_i$, $i \geq t+3$, there are at most $i-1$ choices for c. Hence

$$(t+1)|C_{t+2}| \leq \sum_{i=t+3}^{d+1} (i-1)|C_i|.$$

Using this inequality (on the second line below) we get

$$\begin{aligned}
&\sum_{i=t+2}^{d+1} i|C_i| \\
&= (t+2)|C_{t+2}| + \sum_{i=t+3}^{d+1} (i - \frac{i-1}{2t+3})|C_i| + \frac{1}{2t+3} \sum_{i=t+3}^{d+1} (i-1)|C_i| \\
&\geq (t+2+\frac{t+1}{2t+3})|C_{t+2}| + \sum_{i=t+3}^{d+1} (i - \frac{i-1}{2t+3})|C_i| \\
&\geq (t+2+\frac{t+1}{2t+3})|C|.
\end{aligned}$$

In other words, each codeword in C is in average covered at least $t+2+\frac{t+1}{2t+3}$ times. By counting the number of pairs $c \in C$, $x \in V$ where $d(x,c) \leq 1$, this leads to the bound

$$(t+2)|V| + \frac{t+1}{2t+3}|C| \leq (d+1)|C|,$$

and the claim follows.

The claim for t-vertex-robust 1-identifying codes of level $s \geq t+3$ is clear because every vertex of G is covered by at least s codewords. □

For $d = 3$, the bound (1) says that the cardinality of 1-vertex-robust 1-identifying code of level 3 is at least $5|V|/6$ and this is attained by the code in the graph of Figure 1 with twelve vertices and ten codewords. The bound (2) is also attained (infinitely many times) according to Theorem 1. A bound for the case $s = t+1$ can be found in [4].

3 Binary hypercubes

In this section, we consider binary Hamming spaces (i.e., binary hypercubes); the vertex set is denoted by F^n (the n-fold Cartesian product of the binary field F) and there exists an edge between two vertices (usually called *words*) if and only if the Hamming distance equals one. With a slight abuse of notation, the obtained graph is also denoted by F^n.

Let μ be a positive integer. A code $C \subseteq F^n$ is called a *μ-fold r-covering* if for every word $x \in F^n$ we have $|I_r(F^n, C; x)| \geq \mu$. For coverings, consult, for instance, [3, Chapter 14]. The requirement i) from Definition 1 says, for F^n, that a t-vertex-robust r-identifying code of level s must be an s-fold r-covering. The following theorem shows that for $s \geq t+3$ and $r=1$ the converse is also true in the binary hypercubes.

Theorem 1. *Let $s \geq t+3$. An s-fold 1-covering of F^n is a t-vertex-robust 1-identifying code of level s. In particular, for all integers $i \geq 0$ and $\mu_0 > 0$ such that μ_0 divides s and $s \leq 2^i \mu_0$, the smallest possible cardinality of a t-vertex-robust 1-identifying code of level s in F^n with $n = \mu_0 2^i - 1$ equals*

$$s\frac{2^n}{n+1}.$$

Proof. Let C be an s-fold 1-covering with $s \geq t+3$. Since in the binary hypercube the intersection of two balls of radius 1 consists of at most two points, we get

$$\begin{aligned} |I(u) \bigtriangleup I(v)| &= |I(u)| + |I(v)| - 2|I(u) \cap I(v)| \\ &\geq 2s - 2 \cdot 2 \\ &\geq 2t+1. \end{aligned}$$

Therefore, C is t-vertex-robust 1-identifying of level s. The cardinality now follows by combining (2) with [3, Theorem 14.2.4] which says that for all integers $i \geq 0$ and $\mu_0 > 0$ such that μ_0 divides μ and $\mu \leq 2^i \mu_0$, there exists a μ-fold 1-covering in F^n with $n = \mu_0 2^i - 1$ with the cardinality $\mu 2^n/(n+1)$. $\square$

Let us now consider the level 3 and $t = r = 1$. We have often found (see, e.g., [13]) the construction below useful in building good identifying codes in F^n. In what follows, the notation (a, u, w) with $a \in F$, $u \in F^n$ and $w \in F^n$ means a vector in F^{2n+1}, and $\pi(u)$ with $u = (u_1 u_2 \dots u_n) \in F^n$ stands for the parity check bit, that is, $\pi(u) = \sum_{i=1}^n u_i \mod 2$. The following construction gives us, from 1-vertex-robust 1-identifying codes of level 3 in F^n, codes with the same properties in F^{2n+1}, see Theorem 2 (and continuing we get infinite families of codes with the desired properties).

Construction 1. *([13],[3, p.67,381])* Let $C \subseteq F^n$ be a code. Denote

$$\mathcal{D}(C) = \{(\pi(u), u, u+v) \mid u \in F^n, v \in C\} \subseteq F^{2n+1}.$$

Let us examine the I-set of an arbitrary word $w \in F^{2n+1}$. The word can be written as $w = (a_w, u_w, u_w + v_w)$ where $a_w \in F$, $u_w \in F^n$ and $v_w \in F^n$. Noticing that the first bit of a codeword is the parity check bit of the following n bits and that the radius equals one, we can verify the following.

(i) If $\pi(u_w) = a_w$, then

$$I(F^{2n+1}, \mathcal{D}(C); w) = \{(\pi(u_w), u_w, u_w + c) \mid c \in C,\ d(c, v_w) \leq 1 \text{ in } F^n\}.$$

(ii) If $\pi(u_w) \neq a_w$ then

$$\begin{aligned} I(F^{2n+1}, \mathcal{D}(C); w) = & \\ & \{(a_w, u', u_w + v_w) \mid c \in C,\ d(c, v_w) = 1 \text{ in } F^n,\ u' + c = u_w + v_w\} \\ \cup \quad & (\{(\pi(u_w), u_w, u_w + v_w)\} \cap \mathcal{D}(C)). \end{aligned}$$

The set $\{(\pi(u_w), u_w, u_w + v_w)\} \cap \mathcal{D}(C)$ is non-empty if and only if $v_w \in C$.

It is easy to see that in both cases — here (a_i, u_i, s_i) with $a_i \in F$, $u_i \in F^n$ and $s_i \in F^n$ stands for any codeword in the I-set of w and k is the number of codeword in the I-set — we have

$$I(F^{2n+1}, \mathcal{D}(C); w) = \{(a_i, u_i, s_i) \mid 1 \leq i \leq k\} \tag{3}$$

$$\Rightarrow I(F^n, C; v_w) = \{u_i + s_i \mid 1 \leq i \leq k\}.$$

□

According to the next example, this construction does **not** work for 1-vertex-robust 1-identifying codes of level 2. However, the construction works for level 3 by Theorem 2 — and we get also good code constructions for level 2.

Example 1. Denote by $\mathbf{0}$ the all-zero word. The code $C_5 = (F^5 \setminus B_1(\mathbf{0})) \cup \{\mathbf{0}, 10000\}$ is 1-vertex-robust 1-identifying of level 2 in F^5 (it is straightforward to verify this using the definition), but $\mathcal{D}(C_5)$ is not in F^{11} because

$$I(F^{11}, \mathcal{D}(C_5); \mathbf{0}) = \{\mathbf{0}, 00000010000\}$$

and

$$I(F^{11}, \mathcal{D}(C_5); 01000010000) = \{11000010000, 00000010000\}$$

giving $|I(F^{11}, \mathcal{D}(C_5); \mathbf{0}) \bigtriangleup I(F^{11}, \mathcal{D}(C_5); 01000010000)| = 2 < 3$. □

As mentioned earlier, all 1-vertex-robust 1-identifying codes of level 3 in F^n are 3-fold 1-coverings and in the next lemma we show which 3-fold 1-coverings are suitable for us.

Lemma 2. *A 3-fold 1-covering is 1-vertex-robust 1-identifying of level 3 if and only if there does not exist a pair x and α in F^n such that $|I(F^n, C; x)| = |I(F^n, C; \alpha)| = 3$ and $|I(F^n, C; x) \cap I(F^n, C; \alpha)| = 2$.*

Proof. Let $C \subseteq F^n$ be a 3-fold 1-covering. Suppose that

$$|I(F^n, C; x) \bigtriangleup I(F^n, C; \alpha)| \leq 2$$

for some $x \in F^n$ and $\alpha \in F^n$. If x or α is covered by at least four codewords then immediately $|I(F^n, C; x) \bigtriangleup I(F^n, C; \alpha)| \leq 2$ is not possible. Therefore, it suffices to assume that $|I(F^n, C; x)| = |I(F^n, C; \alpha)| = 3$. Moreover, if $|I(F^n, C; x) \cap I(F^n, C, \alpha)| \leq 1$, then again $|I(F^n, C; x) \bigtriangleup I(F^n, C; \alpha)| \geq 3$. Because $|B_1(a) \cap B_1(b)| \leq 2$ for any distinct a and b in F^n, we conclude that x and α form a forbidden pair.

The other direction of the claim is trivial. □

Example 2. Using the previous lemma it is easy to deduce that the code $C_3 = F^3 \setminus \{\mathbf{0}\}$ is 1-vertex-robust 1-identifying of level 3. Moreover, there does not exist such a code with six or less codewords. This can be seen as follows. We can assume without loss of generality that one of the non-codewords equals $\mathbf{0}$. Because the code is a 3-fold 1-covering, there can be another non-codeword only at 111. But if 111 is a non-codeword, then $|I(100) \bigtriangleup I(111)| < 3$ and ii) of Definition 1 does not hold. $\square$

Theorem 2. *Let $C \subseteq F^n$ be* 1*-vertex-robust* 1*-identifying of level 3. Then $\mathcal{D}(C)$ is* 1*-vertex-robust* 1*-identifying of level 3 in F^{2n+1}.*

Proof. Since C is a 3-fold 1-covering, the code $\mathcal{D}(C)$ is also by (3). We apply Lemma 2. Suppose to the contrary that we have a pair $x = (a_x, u_x, u_x + v_x)$ and $\alpha = (a_\alpha, u_\alpha, u_\alpha + v_\alpha)$ in F^{2n+1} such that they both are covered by exactly three codewords and

$$|I(F^{2n+1}, \mathcal{D}(C); x) \cap I(F^{2n+1}, \mathcal{D}(C); \alpha)| = 2. \tag{4}$$

Let $w \in F^{2n+1}$. If $a_w = \pi(u_w)$ then the first $(n+1)$-bits in all of the codewords in $I(F^{2n+1}, \mathcal{D}(C); w)$ are equal. If $a_w \neq \pi(u_w)$, then the last n-bits in all of the codewords in $I(F^{2n+1}, \mathcal{D}(C); w)$ are the same.

Consequently, if $a_x = \pi(u_x)$, then, by (4), also $a_\alpha = \pi(u_\alpha)$ and $u_x = u_\alpha$. On the other hand, if $a_x \neq \pi(u_x)$, then also $a_\alpha \neq \pi(u_\alpha)$ and $u_x + v_x = u_\alpha + v_\alpha$. Because $x \neq \alpha$, this implies in both cases that $v_x \neq v_\alpha$.

By virtue of (3), we get $|I(F^n, C; v_x) \cap I(F^n, C; v_\alpha)| = 2$ and

$$|I(F^n, C; v_x)| = |I(F^n, C; v_\alpha)| = 3.$$

But then v_x and v_α constitute a pair in F^n which is forbidden by Lemma 2. This is a contradiction and the claim follows. $\square$

Denote by $L(n)$ the smallest possible cardinality of a 1-vertex-robust 1-identifying code of level 3 in F^n.

Corollary 1. *Let C be a* 1*-vertex-robust* 1*-identifying code of level 3 in F^n and let k be the real number defined by $k = |C|(n+1)/2^n$. Then any such code C gives rise to an infinite family of* 1*-vertex-robust* 1*-identifying codes of level 3 for the lengths N for which there exists a non-negative integer q such that N equals $2^q n + 2^q - 1$. Moreover, for these lengths, we have*

$$\frac{15 \cdot 2^N}{5N+3} \leq L(N) \leq k \cdot \frac{2^N}{N+1}. \tag{5}$$

Proof. We get the lower bound from (1). For the upper bound, apply the above construction repeatedly to the code C. $\square$

The lower bound in (5) shows that the coefficient $k = 3$ cannot be reached. Taking the code C_3 from Example 2 as our initial code, we get $L(N) \leq k \cdot 2^{N-r}$ for the lengths $N = 2^r - 1$ where $r \geq 2$ and $k = 3.5$.

Let $A \subseteq F^n$. We define a code C_A as follows, where $a + b$ is the componentwise sum modulo 2:

$$C_A = A + (B_2(\mathbf{0}) \setminus \{\mathbf{0}\}) = \{c \in F^n \mid c = a + b,\ a \in A, b \in B_2(\mathbf{0}) \setminus \{\mathbf{0}\}\}.$$

This code is a simple modification of [4, Theorem 12] where we now use the punctured ball $B_2(\mathbf{0}) \setminus \{\mathbf{0}\}$ instead of $B_2(\mathbf{0})$. Assume that $A_{23} \subset F^{23}$ is the binary Golay code [3, p. 286] with 2^{12} codewords and covering radius 3 (recall that the covering radius of a code $C \subseteq F^n$ is the smallest integer R such that every $x \in F^n$ is R-covered by at least one codeword of C).

It is easy to see, by Lemma 2, that the code $C_{A_{23}}$ is 1-vertex-robust 1-identifying of level 3. From $C_{A_{23}}$ we get an infinite sequence of codes for which $k = \frac{207}{64}$ (≈ 3.23).

We can immediately generalize this as follows.

Theorem 3. *If $A \subseteq F^n$, $n \geq 4$, has covering radius at most 3, then C_A is a 1-vertex-robust 1-identifying code of level 3.* □

Let $M(n)$ be the minimum cardinality of a 1-vertex-robust 1-identifying code of level 2 in F^n and $K(n, R)$ be the minimum cardinality of a binary code in F^n with covering radius R. Theorem 13 in [4] says that

$$M(n) = 3\frac{2^n}{n}(1 + g(n)) \tag{6}$$

where $g(n) \to 0$ when $n \to \infty$, provided that

$$\lim_{n\to\infty} \frac{K(n,3)}{2^n / \sum_{i=0}^{3} \binom{n}{i}} = 1.$$

Theorem 3 shows us that we can replace $M(n)$ by $L(n)$ in the asymptotic result (6) — notice that $M(n)$ is for codes of level 2 and $L(n)$ for codes of level 3.

4 Two grids and a mesh

The vertex set of the square and the king grids is $\mathbb{Z}^2$. Two vertices are adjacent in the square grid, if their Euclidean distance equals 1 (see Figure 3); two vertices are adjacent in the king grid, if their Euclidean distance equals 1 or $\sqrt{2}$ (see Figure 2).

Denote by R_n the set of vertices $(i, j) \in \mathbb{Z}^2$ with $|i| \leq n$ and $|j| \leq n$. The density of a code C in the square or the king grid is defined to be

$$D = \limsup_{n\to\infty} |C \cap R_n| / |R_n|.$$

Notice that the levels s greater than the size of the ball of radius 1 are not possible for any 1-vertex-robust 1-identifying codes.

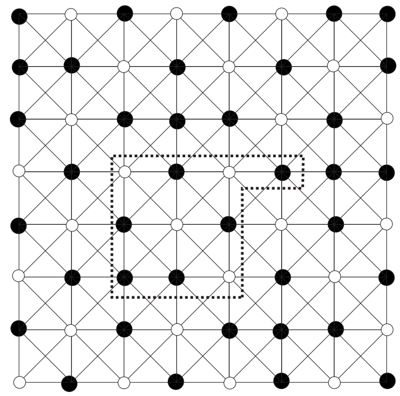

Figure 2: A code of density $\frac{3}{5}$ in the king grid.

Theorem 4. *Denote the smallest possible density of a* 1*-vertex-robust* 1*-identifying code of level* s *in the king grid by* D_s^K*. Then* $D_s^K = \frac{1}{2}$ *for* $0 \le s \le 4$ *and* $\frac{7}{12} \le D_5^K \le \frac{3}{5}$ *and* $D_s^K = \frac{s}{9}$ *for* $6 \le s \le 9$*.*

Proof. The code $C = \{(x, y) \in \mathbb{Z}^2 \mid x + y = 0 \mod 2\}$ is 1-vertex-robust 1-identifying of level up to 4 and its density is equal to $\frac{1}{2}$. For any $(x, y) \in \mathbb{Z}^2$ we know that $B_1((x, y)) \bigtriangleup B_1((x + 1, y))$ consists of six vertices. By ii) of Definition 1, there must be at least three codewords among those six. Because this is true for arbitrary (x, y) the density must be at least $\frac{1}{2}$. Hence, $D_s^K = \frac{1}{2}$ for all $0 \le s \le 4$ (the values for $s = 2$ and $s = 3$ are from [4] and [12], respectively).

Figure 2 gives a 1-vertex-robust 1-identifying code of level 5 with density $\frac{3}{5}$. Consider the set $R = B_1(a, b) \cup B_1(a+1, b)$ of twelve vertices. Either the set $\{(a-1, b-1), (a-1, b), (a-1, b+1)\}$ or $\{(a+2, b-1), (a+2, b), (a+2, b+1)\}$ contains at least two codewords by ii) of Definition 1. If the first one does (resp., the second one), then $|I(a + 1, b)| \ge 5$ (resp., $|I(a, b)| \ge 5$) shows that there are at least seven codewords in R. This gives the lower bound $D_5^K \ge \frac{7}{12}$.

We get the lower bound $\frac{s}{9}$ from i) of Definition 1 for any s. Let $s = 6$. Set $N_6 = \{(x, y) \in \mathbb{Z}^2 \mid x + y \equiv 0 \mod 3\}$. The code $C_6 = \mathbb{Z}^2 \setminus N_6$ is 1-vertex-robust 1-identifying of level 6 with density $\frac{2}{3}$. Indeed, every vertex is covered by six codewords and because every set $L_1 = \{(a, b), (a, b + 1), (a, b + 2)\}$ as well as every set $L_2 = \{(a, b), (a + 1, b), (a + 2, b)\}$ contains at most one non-codeword, we know that $|I(x) \bigtriangleup I(y)| \ge 3$, since the set $I(x) \bigtriangleup I(y)$ contains at least two disjoint sets of the type L_1 or L_2.

Set $N_7 = \{(x, y) \mid x + y \equiv 0 \mod 3 \text{ and } y \equiv 0, 1 \mod 3\}$ and $N_8 = \{(x, y) \mid x + y \equiv 0 \mod 3 \text{ and } y \equiv 0 \mod 3\}$. By noticing that $N_8 \subseteq N_7 \subseteq N_6$, it is easy to check that $C_7 = \mathbb{Z}^2 \setminus N_7$ is 1-vertex-robust 1-identifying of level 7 and $C_8 = \mathbb{Z}^2 \setminus N_8$ is of level 8. Trivially, $C = \mathbb{Z}^2$ gives the value

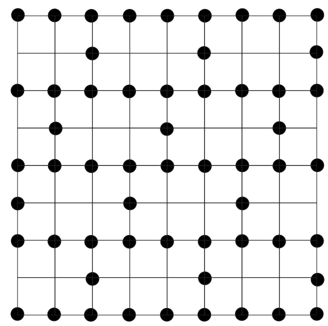

Figure 3: A code of density $\frac{2}{3}$ in the square grid.

$D_9^K = 1$. $\square$

By this theorem, we can see that the condition i) of Definition 1 becomes "dominant" after the level 5 (notice that $7/12 > 5/9$) and for small levels it is the condition ii) which is more demanding.

Theorem 5. *Denote the smallest possible density of a 1-vertex-robust 1-identifying code of level s in the square grid by D_s^S. Then $D_s^S \leq \frac{5}{8}$ for $s \in \{0, 1, 2\}$, $D_3^S = \frac{2}{3}$, $D_4^S = \frac{4}{5}$ and $D_5^S = 1$.*

Proof. The values for $s \leq 2$ follow from [4] and the value for $s = 3$ is from [12] (see Figure 3). The lower bound $\frac{4}{5}$ (by i) of Definition 1) combined with the code $\mathbb{Z}^2 \setminus \{(x, y) \mid 2x + y \equiv 0 \mod 5\}$ gives $D_4^S = \frac{4}{5}$. The case $s = 5$ is trivial. $\square$

Let us now consider the hexagonal mesh (see Figure 4).

Theorem 6. *Denote the smallest possible density of a 1-vertex-robust 1-identifying code of level s in the hexagonal mesh by D_s^H. Then $D_s^H \leq \frac{41}{50}$ for $s \in \{0, 1, 2\}$ and $D_3^H = \frac{5}{6}$ and $D_4^H = 1$.*

Proof. The values for $s \in \{0, 1, 2\}$ follow from [4, Theorem 17] and Section 1. The case $s = 4$ is trivial, so assume the level is 3. A code C is 1-vertex-robust 1-identifying of level 3 in the hexagonal mesh if and only if the distance between any two non-codewords is at least four. This is shown as follows (for an analogous proof for the level 2, see [4, Theorem 17]).

Let first C be a 1-vertex-robust 1-identifying code of level 3. If any two non-codewords are at distance one or two apart, there exists a vertex which is within distance one from both of them and its I-set contains at most two codewords — this contradicts the level 3. Let then any two non-codewords be at distance three from each other and P be any path of length three between them. The path P contains two vertices, say u and v, besides the non-codewords. Now $|I(u) \triangle I(v)| < 3$, which contradicts ii) of Definition 1.

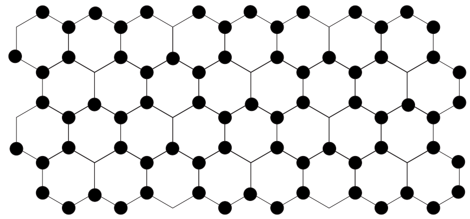

Figure 4: A code of density $\frac{5}{6}$ in the hexagonal mesh

Assume then that non-codewords are at least at distance four apart. We show that the code C is then 1-vertex-robust 1-identifying of level 3. Clearly, the condition i) of Definition 1 is fulfilled. Take then two vertices u and v. If $d(u,v) \geq 2$, we obtain $|I(u) \bigtriangleup I(v)| = |I(u)| + I(v)| - 2|I(u) \cap I(v)| \geq 4$, because $|I(u) \cap I(v)| \leq 1$. If $d(u,v) = 1$ and $|I(u) \cap I(v)| \leq 1$, then the same argument works, so assume that $d(u,v) = 1$ and $|I(u) \cap I(v)| = 2$ (the intersection cannot be larger than two). Since the distance between two non-codewords is at least four, there can be at most one non-codeword among the four neighbours of u and v. This yields $|I(u) \bigtriangleup I(v)| \geq 3$.

Now we know that a code C is 1-vertex-robust 1-identifying of level 3 if and only if the distance between any two non-codewords is at least four. Consequently, any hexagon in the graph consists of six vertices and at least five of them are in the code. Therefore, the density is at least $\frac{5}{6}$.

On the other hand, the code given in Figure 4 satisfies the requirement that any two non-codewords are at distance at least four apart. In addition, the density equals $\frac{5}{6}$. $\square$

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