Quantification of concurrence via weak measurement

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Since entanglement is not an observable *per se*, measuring its value in practice is a difficult task. Here we propose a protocol for quantifying a particular entanglement measure, namely, concurrence, of an arbitrary two-qubit pure state via a single fixed measurement setup by exploiting so-called weak measurements and the associated weak values together with the properties of the Laguerre-Gaussian modes. The virtue of our technique is that it is generally applicable for all two-qubit systems and does not involve simultaneous copies of the entangled state. We also propose an explicit optical implementation of the protocol.

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I. INTRODUCTION

In the course of the past decades, the role of entanglement has evolved into a genuine quantum resource utilized in various quantum communication and computation protocols [1–5]. This evolution has been supported by the formidable progress made on the techniques of generating entanglement in practice. Inevitable and inescapable noise, together with imperfections present in every real experiment, may, however, degrade the intended entangled state. Being able to detect and measure the entanglement content becomes important since any amount of entanglement can be harnessed in nonclassical tasks [6,7]. Although several theoretical measures have been developed for this purpose [5,8], realizing them in practice remains challenging in general. The reason is that typically these measures of entanglement contain rather involved, even unphysical, operations or are nonlinear functions of the state.

One of the most widely used measures of entanglement is the so-called concurrence [9], which in the case of two qubits in a pure state takes a particularly simple form. Despite the mathematical simplicity, the task of quantifying the value of concurrence of an *unknown* two-qubit pure state using only a single measurement setup of a fixed normalized projection-valued measure (PVM) is impossible [10]. Nevertheless, several different procedures circumventing this impossibility have been reported that exploit collective measurements done with simultaneous copies of the state [11–14] or utilize the curious relation between concurrence and two-particle interference [15]. Furthermore, measurements of concurrence that rely on relaxing the aforementioned PVM criterion have been developed [16–18].

In this study, we propose a local tomographic strategy to quantify the concurrence of *any* two-qubit pure state that takes advantage of so-called weak measurements. We also consider an experimental implementation on an optical setup that can be deployed to measure the concurrence of two polarization entangled photons using the proposed protocol. Our method is, however, universal in the sense that it works for all two-qubit systems.

The key tools of our proposal are weak measurements and the resulting weak values [19,20]. Weak measurements are (von Neumann) standard measurements [21] where the coupling strength λ between the measured system and the measuring pointer is minuscule. Consequently, the disturbance of the weak measurement to any subsequent (strong) measurement, usually called postselection, is negligible. By postselecting on a particular pure state, $|\varphi\rangle\langle\varphi|$, in the vanishing interaction strength limit $\lambda \to 0$, one can derive the weak value of the observable A as

$$_{\langle\varphi|}\langle \mathsf{A}\rangle_{\rho}^{w} := \frac{\mathrm{tr}\,[\mathsf{A}\,\rho\,|\varphi\rangle\langle\varphi|]}{\mathrm{tr}\,[\rho\,|\varphi\rangle\langle\varphi|]}, \tag{1}$$

where ρ is the preselected (mixed) state of the measured system [22]. Throughout this paper, we omit the preselection subindex whenever it is clear from the context. Weak values are intrinsically complex, which has already proved useful in characterizing the mathematically observable-independent probability space [23], several quantum paradoxes [24], the quantum state [25–31], and unobservable quantities such as the geometric phase [32–35] and the non-Hermitian operator [36]; see also the review papers [37–40]. We show that one may also take advantage of the complex feature of the weak values in assessing the amount of entanglement with a single measurement setup. This result builds upon the fact first noted in Ref. [29] that, when a Laguerre-Gaussian beam is used as the pointer state of the weak measurement, certain weak values can be interpreted as stereographical projections of the Bloch sphere onto $\mathbb{R}^2$ plane.

II. CONCURRENCE AND WEAK VALUES

Let us assume that two observers, Alice and Bob, are tasked with determining the amount of entanglement in a bipartite state ρ_{AB} by means of performing local operations. Furthermore, assume that the source generates only pure two-qubit states, that is, $\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$ for some

$$|\Psi_{AB}\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle, \tag{2}$$

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where $|0\rangle$ and $|1\rangle$ are the eigenvectors of Pauli operator σ_z, and $|ij\rangle := |i\rangle \otimes |j\rangle$ and a_{ij} $(i,j = 0,1)$ are complex numbers satisfying the normalization $\sum_{i,j=0}^{1} |a_{ij}|^2 = 1$. One of the most widely used entanglement measures in two-qubit systems is the *concurrence* C. In the case of a pure state $|\Psi_{AB}\rangle$, the concurrence $C(\Psi_{AB})$ takes the simple form [9]

$$C(\Psi_{AB})^2 = 4|a_{00}a_{11} - a_{01}a_{10}|^2 = 4\det(\rho_A) = 4\det(\rho_B), \tag{3}$$

where $\rho_{A(B)}$ is the reduced density matrix of Alice (Bob), e.g.,

$$\rho_A = \begin{pmatrix} |a_{00}|^2 + |a_{01}|^2 & a_{00}^* a_{10} + a_{01}^* a_{11} \\ a_{00}a_{10}^* + a_{01}a_{11}^* & |a_{10}|^2 + |a_{11}|^2 \end{pmatrix}. \tag{4}$$

The concurrence has a one-to-one connection to the von Neumann entropy [9]

$$\begin{aligned} E(\Psi_{AB}) &= -\mathrm{tr}\,[\rho_A \log_2(\rho_A)] = -\mathrm{tr}\,[\rho_B \log_2(\rho_B)] \\ &= -\frac{1+\sqrt{1-C^2}}{2}\log_2\left(\frac{1+\sqrt{1-C^2}}{2}\right) \\ &\quad -\frac{1-\sqrt{1-C^2}}{2}\log_2\left(\frac{1-\sqrt{1-C^2}}{2}\right) \end{aligned} \tag{5}$$

and via that to a plethora of other entanglement measures [5,8], which makes it a natural choice of figure of merit for our task.

Our main result is to reveal a mathematical relationship between the concurrence and the weak values corresponding to weak measurements of either one of the local observers. For instance, Alice's weak values of the observable $\sigma_x^A := |0\rangle\langle 1| + |1\rangle\langle 0|$, preselected on her reduced state ρ_A and postselected on either $|0\rangle$ or $|1\rangle$, read

$$\begin{aligned} {}_{\langle 0|}\langle\sigma_x^A\rangle^w &:= \frac{\mathrm{tr}\big[\sigma_x^A\,\rho_A\,|0\rangle\langle 0|\big]}{\mathrm{tr}\,[\rho_A\,|0\rangle\langle 0|]} = \frac{a_{00}a_{10}^* + a_{01}a_{11}^*}{|a_{00}|^2 + |a_{01}|^2}, \\ {}_{\langle 1|}\langle\sigma_x^A\rangle^w &:= \frac{\mathrm{tr}\big[\sigma_x^A\,\rho_A\,|1\rangle\langle 1|\big]}{\mathrm{tr}\,[\rho_A\,|1\rangle\langle 1|]} = \frac{a_{00}^*a_{10} + a_{01}^*a_{11}}{|a_{10}|^2 + |a_{11}|^2} \end{aligned} \tag{6}$$

[see Fig. 1(a)]. A weak value may not be well defined if its denominator vanishes. Physically this corresponds to receiving no signal on the measuring pointer whatsoever. We notice that either one of the above weak values being nonvanishing automatically implies that the other one is also nonzero. Therefore, whenever ${}_{\langle 0|}\langle\sigma_x^A\rangle^w \neq 0 \neq {}_{\langle 1|}\langle\sigma_x^A\rangle^w$, we may write $(|a_{00}|^2 + |a_{01}|^2)/(|a_{10}|^2 + |a_{11}|^2) = |{}_{\langle 1|}\langle\sigma_x^A\rangle^w|/|{}_{\langle 0|}\langle\sigma_x^A\rangle^w|$ and solve

$$C(\Psi_{AB})^2 = 4\det(\rho_A) = 4\big(1 - \big|{}_{\langle 0|}\langle\sigma_x^A\rangle^w\big|\big|{}_{\langle 1|}\langle\sigma_x^A\rangle^w\big|\big)\frac{\big|{}_{\langle 0|}\langle\sigma_x^A\rangle^w\big|\big|{}_{\langle 1|}\langle\sigma_x^A\rangle^w\big|}{\big(\big|{}_{\langle 0|}\langle\sigma_x^A\rangle^w\big| + \big|{}_{\langle 1|}\langle\sigma_x^A\rangle^w\big|\big)^2}, \tag{7}$$

where we have used the information ${}_{\langle 0|}\langle\sigma_x^A\rangle^w\,{}_{\langle 1|}\langle\sigma_x^A\rangle^w = |{}_{\langle 0|}\langle\sigma_x^A\rangle^w||{}_{\langle 1|}\langle\sigma_x^A\rangle^w|$. Since $|a_{00}a_{10}^* + a_{01}a_{11}^*| \leqslant |a_{00}||a_{10}| + |a_{01}||a_{11}| \leqslant 1/2$, we additionally conclude that

$$\big|{}_{\langle 0|}\langle\sigma_x^A\rangle^w\big|\big|{}_{\langle 1|}\langle\sigma_x^A\rangle^w\big| \leqslant 1. \tag{8}$$

On the other hand, one of the weak values in Eq. (6) being zero implies that the other one either also vanishes or is not well defined. Assume for example that ${}_{\langle 0|}\langle\sigma_x^A\rangle^w$ is not well defined. Then $|a_{00}|^2 + |a_{01}|^2 = 0$ implying $C(\Psi_{AB}) = 0$. Similarly, $C(\Psi_{AB}) = 0$ if ${}_{\langle 1|}\langle\sigma_x^A\rangle^w$ is not well defined. These observations can also be reproduced from Eq. (7) as limiting cases ${}_{\langle i|}\langle\sigma_x^A\rangle^w \to \infty$, $i = 0,1$, since, except for the point $(|{}_{\langle 0|}\langle\sigma_x^A\rangle^w|, |{}_{\langle 1|}\langle\sigma_x^A\rangle^w|) = (0,0)$, concurrence $C(\Psi_{AB})$ is a continuous function of $|{}_{\langle 0|}\langle\sigma_x^A\rangle^w|$ and $|{}_{\langle 1|}\langle\sigma_x^A\rangle^w|$.[1] Therefore, the concurrence, plotted in Fig. 1(b), may be determined from Eq. (7), with the exception of the singularity in the origin $(|{}_{\langle 0|}\langle\sigma_x^A\rangle^w|, |{}_{\langle 1|}\langle\sigma_x^A\rangle^w|) = (0,0)$; this case will be analyzed later separately. It is noteworthy that the protocol presented works completely locally.

We note in passing that $|{}_{\langle 0|}\langle\sigma_x^A\rangle^w| = |{}_{\langle 1|}\langle\sigma_x^A\rangle^w|$ is the only line passing through the origin on which $C(\Psi_{AB})$ attains its maximum value 1. The reduced states ρ_A corresponding to this line are those which are on the equatorial plane of the Bloch sphere in Fig. 1(a). On this line, Eq. (7) simplifies to

$$C(\Psi_{AB}) = \sqrt{1 - \big|{}_{\langle 0|}\langle\sigma_x^A\rangle^w\big|^2}. \tag{9}$$

This observation is useful in order to calibrate C as close to unity (or any other value from the interval [0,1]) as desired. Because the process is completely local, the other party (Bob) can validate the result of this "optimization" for instance via state tomography.

III. DETERMINATION OF ENTANGLEMENT WITH A FIXED MEASUREMENT SETUP

Determining $C(\Psi_{AB})$ of arbitrary $|\Psi_{AB}\rangle$ directly via measurement of only a single set of orthogonal projectors $\mathsf{P}_i = |O_i\rangle\langle O_i|$, $\sum_{i=1}^4 \mathsf{P}_i = \mathbf{1}$, where $\langle O_i|O_j\rangle = \delta_{ij}$ (Kronecker δ), is impossible [10]. In other words, one cannot quantify concurrence of all bipartite states with a single fixed measurement setup if the measured observable is a PVM. This is due to the fact that the measured probabilities $p_i = |\langle\, O_i \,|\, \Psi_{AB}\,\rangle|^2$ result in three independent real numbers, which are not in general sufficient to determine $C(\Psi_{AB})$, a nonlinear function of four complex parameters. In fact, even deciding if a completely unknown (hence possibly mixed) bipartite state is the entanglement or not requires as many resources as state tomography [41].

The relationship between Alice's weak values and the concurrence introduced in the previous section suggests that weak measurements allow one to circumvent this impossibility. To extract the real and imaginary parts of the weak

[1] Actually, in the point (0,0) concurrence is not even a function of two weak values. Namely, with a proper choice of Ψ_{AB}, $C(\Psi_{AB})$ can acquire any value from the interval [0,1], while satisfying ${}_{\langle 0|}\langle\sigma_x^A\rangle^w = 0 = {}_{\langle 1|}\langle\sigma_x^A\rangle^w$.

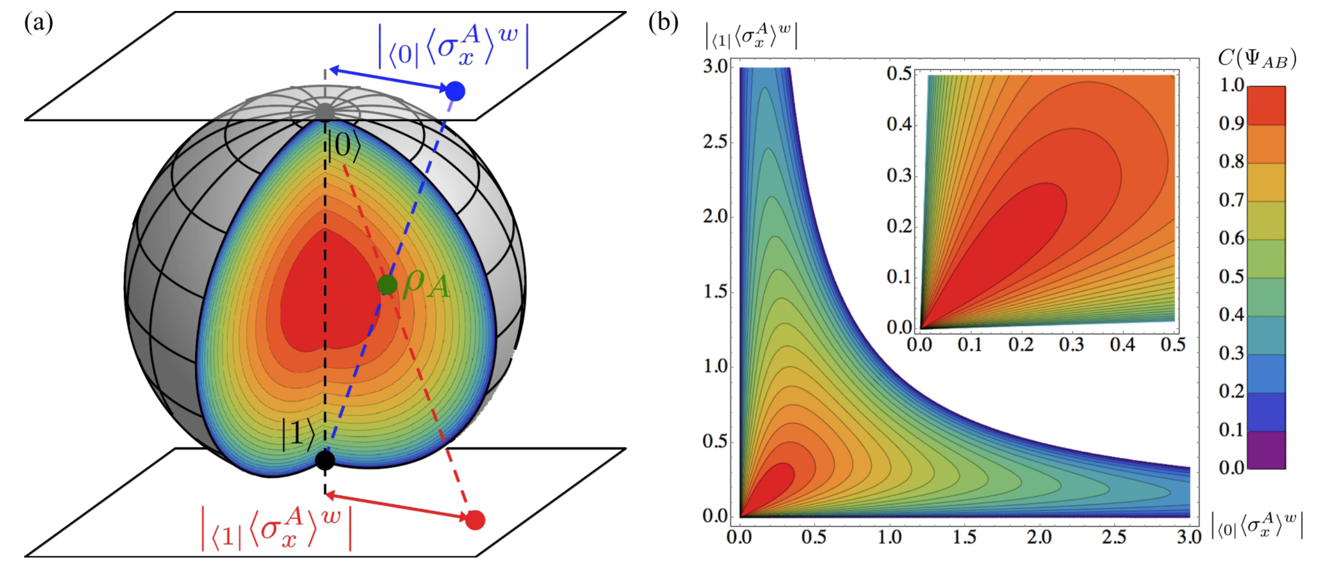

FIG. 1. (a) Stereographical representation of the weak value of the state ρ_A. Following Ref. [29], the weak measurement of σ_x on the state ρ_A, followed by postselection on $|0\rangle$ or $|1\rangle$, may be interpreted as stereographic projections of the qubit state ρ_A on the two $\mathbb{R}^2$ planes that intersect the north and south pole of the Bloch sphere. For our purposes, the absolute values $|{}_{\langle 0|}\langle\sigma_x^A\rangle^w|$ and $|{}_{\langle 1|}\langle\sigma_x^A\rangle^w|$ are particularly important because they may be used to measure the distance between ρ_A and the maximally mixed state $\frac{1}{2}\mathbf{1}_A$, which in turn is related to the amount of entanglement. (b) Concurrence $C(\Psi_{AB})$ in terms of $|{}_{\langle 0|}\langle\sigma_x^A\rangle^w|$ and $|{}_{\langle 1|}\langle\sigma_x^A\rangle^w|$. The concurrence is fully determined by these variables except for the point (0,0), which corresponds to the black dashed line in panel (a). The white region equals the canceled area $|{}_{\langle 0|}\langle\sigma_x^A\rangle^w||{}_{\langle 1|}\langle\sigma_x^A\rangle^w| > 1$.

value two complementary pointer observables are usually used [19,42–44], that is, two separate measurements have to be set up. Remarkably however, it is also possible to quantify both of these components simultaneously by using so-called Laguerre-Gaussian (LG) modes [29,45,46] as the initial pointer state due to the initial correlations [47] related to these states.

As alluded to in the previous section, the determination of entanglement fails only in problematic cases where ${}_{\langle 0|}\langle\sigma_x^A\rangle^w = 0 = {}_{\langle 1|}\langle\sigma_x^A\rangle^w$. The vanishing weak values imply that Alice's state is simplified to

$$\rho_A = \begin{pmatrix} |a_{00}|^2 + |a_{01}|^2 & 0 \\ 0 & |a_{10}|^2 + |a_{11}|^2 \end{pmatrix}. \qquad (10)$$

These cases correspond to the states on a line connecting the opposite poles $|0\rangle$ and $|1\rangle$ of the Bloch sphere [see Fig. 1(a)]. In our protocol the set of these states has only minor relevance since mathematically it has null measure (in the relevant measurable space). Accordingly, the impossibility of determining the concurrence of states with a single PVM strategy persists even if these problematic states are excluded. Nevertheless, in such instances a local measurement of the postselection probabilities can be used to reveal the amount of entanglement in the state $|\Psi_{AB}\rangle$. To this end, Alice can measure the relative intensities of the postselected states to solve the diagonal elements of ρ_A in Eq. (10). Since this measurement may be done jointly with the weak measurement protocol described above, the whole procedure of determining the entanglement content in $|\Psi_{AB}\rangle$ can be achieved with a single fixed measurement device. Moreover, the protocol uses only a single fixed PVM as the postselected measurement; as discussed above, without the preceding weak interaction such an entanglement-measuring strategy would be impossible.

The weak values ${}_{\langle 0|}\langle\sigma_x^A\rangle^w$ and ${}_{\langle 1|}\langle\sigma_x^A\rangle^w$, in addition to the intensity measurements described above, give sufficient information to determine the reduced state ρ_A [see Fig. 1(a)]. In this regard, the protocol we presented essentially relies on local tomography of the reduced state of Alice (or Bob): in the absence of classical communication between the two parties, as is the case in our protocol, this is the optimal local strategy to determine the entanglement of the two-qubit state [10]. As a consequence, we have generalized the one-qubit pure state tomography described in Ref. [29] for mixed states, thus expanding the scope of applications of the previously introduced technique. This underpins that the weak measurement setup exploiting Laguerre-Gaussian modes (an optical proposal of which is given in the next section) could be considered as a basic tool in quantum experiments involving tomography of qubits, such as the above-introduced entanglement quantification.

Being able to reconstruct the reduced state ρ_A will also enable one to connect the weak values to entanglement measures other than concurrence; see, for example, Eq. (5). In fact, the von Neumann entropy $E(\Psi_{AB}) = -\mathrm{tr}\,[\rho_{A(B)} \log_2(\rho_{A(B)})]$ in Eq. (5) is an entanglement measure of not only two qubits but also general bipartite pure states Ψ_{AB} and is independent of which one of the subsystems A or B it is calculated with respect to. Hence, our protocol can be immediately generalized to assess the amount of entanglement in scenarios where one of the two parties possesses a one-qubit system. Additionally, our method is not fully confined to pure states but can also be utilized in estimating the entanglement of the mixed bipartite states; we have left the details and proof of this fact to the Appendix. This is a highly important upside from the experimentalists' viewpoint, since the preparation of a perfectly pure bipartite state is not a realistic assumption in practice.

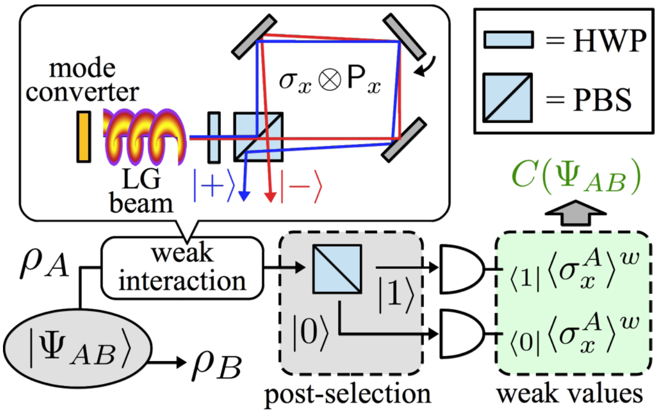

FIG. 2. Weak measurement setup for determining concurrence in the two-photon polarization state $|\Psi_{AB}\rangle$. The initial pointer state is prepared as the Laguerre-Gaussian (LG) mode using a mode converter. The weak interaction between eigenvectors of σ_x, $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, can be implemented using a polarization Sagnac interferometer (PBS, polarization beam splitter, HWP, half waveplate).

IV. PROPOSAL FOR OPTICAL EXPERIMENT

In this section we describe a possible optical setup for determining the concurrence of the polarization entangled state $|\Psi_{AB}\rangle$ of photon pairs via a weak measurement. Our proposed experimental setup is illustrated in Fig. 2. The reduced density matrix ρ_A weakly interacts with the pointer state via the interaction

$$U_\lambda = e^{-i\lambda\sigma_x \otimes \mathsf{P}_x} = \Pi_+ \otimes e^{-i\lambda \mathsf{P}_x} + \Pi_- \otimes e^{i\lambda \mathsf{P}_x}, \quad (11)$$

where P_x is the momentum operator along the x direction on the cross-sectional plane of the optical beam, λ is a small interaction strength, and $\Pi_\pm = \frac{1}{2}(\mathbf{1} \pm \sigma_x)$ are the eigenprojectors of the Pauli operator σ_x. The interaction (11) can be implemented using a polarization Sagnac interferometer and the interaction strength λ can be changed by tilting the angle of a mirror inside the interferometer (see the inset in Fig. 2).

As the initial pointer state, we choose the optical propagation mode with the two-dimensional normalized amplitude distribution $\phi_{\mathrm{i}}(x,y)$, which satisfies the paraxial wave equation [48]. After weak interaction and postselection onto $|\varphi\rangle$ $(= |0\rangle$ or $|1\rangle)$, the intensity distribution $I_{\mathrm{f}}(x,y)$ of the final pointer state becomes

$$I_{\mathrm{f}}^{\varphi}(x,y) = \sum_{j,k=\pm 1} \langle\varphi|\Pi_j \rho_A \Pi_k|\varphi\rangle \phi_{\mathrm{i}}(x - j\lambda, y) \times \phi_{\mathrm{i}}^*(x - k\lambda, y). \quad (12)$$

Assuming the "weakness" condition, $\lambda^{-1} \gg \max(1, |{}_{\langle\varphi|}\langle\sigma_x\rangle^w|)$, the interaction in Eq. (11) induces a translational shift of the pointer state with an amount proportional to the weak value ${}_{\langle\varphi|}\langle\sigma_x\rangle^w$ along the x direction [42]. Namely, under the weakness condition, Eq. (12) can be approximated as

$$I_{\mathrm{f}}^{\varphi}(x,y) = I_{\mathrm{tot}}^{\varphi} |\phi_{\mathrm{i}}(x - \lambda\, {}_{\langle\varphi|}\langle\sigma_x\rangle^w, y)|^2, \quad (13)$$

where $I_{\mathrm{tot}}^{\varphi} \equiv \int dx\, dy\, I_{\mathrm{f}}^{\varphi}(x,y) = \langle\varphi|\rho_A|\varphi\rangle$ corresponds to the total intensity of the postselected beams.

If the fundamental Gaussian beam is used for the pointer state, we can extract only the real part of the weak value from the shift in the beam average position and an alternative measurement setup with additional optical components is required to obtain the imaginary part of the weak value from the shift in the beam average momentum. A more suitable choice for the pointer state for our purpose is the (first-order) LG beam $\phi_{\mathrm{i}}(x,y) \propto (x + iy)\exp[-(x^2 + y^2)]$, which is a cylindrically symmetric solution of the paraxial wave equation [48,49]. The LG beam can be generated from a Gaussian one by using a mode converter, such as a q-plate [50] or a spatial light modulator [51]. From Eq. (13), the averaged values of the position operators Q_x and Q_y on the cross-sectional plane of the final intensity distribution are calculated as

$$\langle \mathsf{Q}_x\rangle_f = \lambda\, \mathrm{Re}[{}_{\langle\varphi|}\langle\sigma_x\rangle^w],\ \langle \mathsf{Q}_y\rangle_f = \lambda\, \mathrm{Im}[{}_{\langle\varphi|}\langle\sigma_x\rangle^w]. \quad (14)$$

Using a two-dimensional image sensor as a detector the LG pointer state therefore allows us to simultaneously visualize both the real and the imaginary part of the weak values ${}_{\langle 0|}\langle\sigma_x\rangle^w$ and ${}_{\langle 1|}\langle\sigma_x\rangle^w$ without additional optical components [29].

In the case of vanishing weak values, where Eq. (7) cannot be used, we cannot obtain any information about the entanglement from the averaged shifts of the pointer state. However, due to the aforementioned reasons these cases are physically insignificant. For the sake of completeness we nevertheless point out that measuring the total intensities $I_{\mathrm{tot}}^{\varphi} = \langle\varphi|\rho_A|\varphi\rangle$ of the two postselected beams with $|\varphi\rangle = |0\rangle, |1\rangle$ enables one to determine the diagonal elements of the state in Eq. (10). Because this can be performed jointly with measurements of the Q_x and Q_y position operators, one can determine the concurrence of the quantum state $|\Psi_{AB}\rangle$ with a single measurement setup.

Although the LG mode pointer states allow us to determine the concurrence using a single fixed PVM for postselection, there are some technical difficulties. The first problem is the mode conversion from the fundamental Gaussian mode to the LG mode. The conversion efficiency is limited by the mode converter and also by the mode coupling coefficient between the incident mode of the photon pairs and the LG mode. To increase the mode-coupling coefficient, a single-mode optical fiber is typically used for spatial mode cleaning of the photon pair beam. In this case, however, fiber coupling loss becomes a serious problem for photon-pair detection. One practical solution is photon-pair generation via four-wave mixing in the single-mode fiber [52]. Another problem is the low detection efficiency of the typical image sensor and, concurrently, the demand for a large ensemble of states needed to extract the weak values. To obtain the high-contrast two-dimensional intensity distribution, we have to generate photon pairs with high intensity using pulsed light or a high-gain imaging sensor, such as a cascade of single-photon detectors.

V. SUMMARY

We have shown how weak measurements and weak values can be used to quantify the concurrence of any two-qubit pure state. We demonstrated that the proposed protocol can be performed with a single measurement setup using a local weak interaction and a Laguerre-Gaussian mode as the pointer state. Notably, the protocol uses a single fixed PVM as for the postselection. In contrast, without the preceding weak interaction, such a measurement of concurrence is impossible [10]. We also considered a potential experimental realization for quantifying the concurrence of the polarization entangled state of photon pairs. Although the proposed implementation has some technical difficulties, such as the detection efficiency, we believe that our protocol could be practically implemented and demonstrated in the future.

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APPENDIX: ROBUSTNESS OF CONCURRENCE

Our protocol relies on the fact that for a pure state $|\Psi_{AB}\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ the concurrence is related to the reduced state $\rho_A := \mathrm{tr}_B[|\Psi_{AB}\rangle\langle\Psi_{AB}|]$ via $C(\Psi_{AB})^2 = 4\det(\rho_A)$ [see Eq. (7)]. The concurrence of a mixed two-qubit state ρ_{AB} can then be obtained from the convex roof extension $C(\rho_{AB}) = \inf_{\{p_i,\psi_i\}} \sum_i p_i C(\psi_i)$, where $p_i \geqslant 0$ satisfies $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ for some unit vectors $|\psi_i\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ [5]. In general $C(\rho_{AB})^2 \neq 4\det(\zeta_A)$, where $\zeta_A = \mathrm{tr}_B[\rho_{AB}]$. Nevertheless, in this appendix we show that $C(\rho_{AB})^2 \approx 4\det(\zeta_A)$ is a good estimate, whenever $\rho_{AB} \approx |\Psi_{AB}\rangle\langle\Psi_{AB}|$ for some unit vector $|\Psi_{AB}\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$. More precisely, we prove that for any $\varepsilon > 0$ one can find $\delta > 0$ such that $|C(\rho_{AB})^2 - 4\det(\zeta_A)| < \varepsilon$ whenever $M(\rho_{AB}) := \inf_{\Psi_{AB}} D(|\Psi_{AB}\rangle\langle\Psi_{AB}|,\rho_{AB}) < \delta$. Here D denotes the trace distance defined for arbitrary states ρ_1 and ρ_2 via $D(\rho_1,\rho_2) = \frac{1}{2}\mathrm{tr}\,[|\rho_1 - \rho_2|]$.

The quantity $M(\rho_{AB})$ is clearly a measure of "mixedness" of the state ρ_{AB}. To further enforce this terminology, it holds that

$$\begin{aligned} 0 &\leqslant \tfrac{1}{4}\big(1 - \mathrm{tr}\left[\rho_{AB}^2\right]\big) \\ &= \tfrac{1}{4}|\,\mathrm{tr}\,[(|\Psi_{AB}\rangle\langle\Psi_{AB}| - \rho_{AB})(|\Psi_{AB}\rangle\langle\Psi_{AB}| + \rho_{AB})]| \\ &\leqslant \tfrac{1}{2}\mathrm{tr}\,[|\,|\Psi_{AB}\rangle\langle\Psi_{AB}| - \rho_{AB}\,|] \\ &= D(|\Psi_{AB}\rangle\langle\Psi_{AB}|,\rho_{AB}), \end{aligned} \tag{A1}$$

for any $|\Psi_{AB}\rangle$. Hence $\frac{1}{4}(1 - \mathrm{tr}[\rho_{AB}^2]) \leqslant M(\rho_{AB})$, where $\mathrm{tr}[\rho_{AB}^2]$ is known as the purity of the state ρ_{AB}.

We begin by showing that concurrence is continuous with respect to the trace distance: in proving this, we closely follow the technique used in Ref. [53]. Let us extend C into the trace class of $\mathbb{C}^2 \otimes \mathbb{C}^2$ by defining

$$\widetilde{C}(T) := \begin{cases} \mathrm{tr}\,[|T|]C\big(\frac{|T|}{\mathrm{tr}\,[|T|]}\big), & T \neq 0, \\ 0, & T = 0. \end{cases} \tag{A2}$$

For all T the function $\widetilde{C}(T)$ can then be equivalently expressed as $\widetilde{C}(T) = \inf_{\{t_i,|\psi_i\rangle\}} \sum_i t_i C(\psi_i)$, where $t_i \geqslant 0$ satisfies $|T| = \sum_i t_i |\psi_i\rangle\langle\psi_i|$ and $||\psi_i|| = 1$ for all i. Assuming that $|T_1| \leqslant |T_2|$ it is then straightforward to see that

$$\widetilde{C}(T_1) \leqslant \widetilde{C}(T_2). \tag{A3}$$

Let ρ_1 and ρ_2 be quantum states in $\mathbb{C}^2 \otimes \mathbb{C}^2$ and define $\tau = \rho_1 - \rho_2$, so that $\rho_1 = |\tau + \rho_2| \leqslant |\tau| + \rho_2 = |\,|\tau| + \rho_2|$. For any $\varepsilon > 0$, one can find ensembles $\{t_i,|\psi_i\rangle\}$ and $\{p_j,|\varphi_j\rangle\}$ such that $|\tau| = \sum_i t_i|\psi_i\rangle\langle\psi_i|$ and $\rho_2 = \sum_j p_j|\varphi_j\rangle\langle\varphi_j|$ and satisfying $\widetilde{C}(|\tau|) \geqslant \sum_i t_i C(\psi_i) - \varepsilon/2$ and $\widetilde{C}(\rho_2) \geqslant \sum_j p_j C(\varphi_j) - \varepsilon/2$. Since $\sum_i \sum_j (t_i + p_j) = \mathrm{tr}[|\tau| + \rho_2|]$, we have

$$\begin{aligned} C(\rho_1) &= C(\tau + \rho_2) \leqslant \widetilde{C}(|\tau| + \rho_2) \\ &\leqslant \sum_i t_i|\psi_i\rangle\langle\psi_i| + \sum_j p_j|\varphi_j\rangle\langle\varphi_j|| \\ &\leqslant \widetilde{C}(|\tau|) + C(\rho_2) + \varepsilon. \end{aligned} \tag{A4}$$

Because the relation holds for arbitrary $\varepsilon > 0$, we can conclude that

$$\begin{aligned} |C(\rho_1) - C(\rho_2)| &\leqslant \widetilde{C}(|\rho_1 - \rho_2|) \\ &= \mathrm{tr}\,[|\rho_1 - \rho_2|]C\left(\frac{|\rho_1 - \rho_2|}{\mathrm{tr}\,[|\rho_1 - \rho_2|]}\right) \\ &\leqslant 2D(\rho_1,\rho_2). \end{aligned} \tag{A5}$$

On the other hand, whenever ρ_1 and ρ_2 are states in $\mathbb{C}^2 \otimes \mathbb{C}^2$, the reduced one-qubit states $\zeta_i = \mathrm{tr}_B[\rho_i]$, $i = 1,2$, satisfy

$$\begin{aligned} |\det(\zeta_1) - \det(\zeta_2)| &= \tfrac{1}{2}\left|\mathrm{tr}\left[\zeta_1^2\right] - \mathrm{tr}\left[\zeta_2^2\right]\right| \\ &= \tfrac{1}{2}|\mathrm{tr}\,[(\zeta_1 - \zeta_2)(\zeta_1 + \zeta_2)]| \\ &\leqslant 2D(\zeta_1,\zeta_2) \leqslant 2D(\rho_1,\rho_2), \end{aligned} \tag{A6}$$

where we have used the property $2\det(\zeta) = 1 - \mathrm{tr}[\zeta^2]$ that holds for all one-qubit states ζ and the data-processing inequality of trace distance $D(\rho_1,\rho_2) \geqslant D[\mathcal{E}(\rho_1),\mathcal{E}(\rho_2)]$ that holds for all completely positive trace-preserving linear maps $\mathcal{E}$ (such as the partial trace).

Using the above relations, we can easily prove our claim. Let $\varepsilon > 0$ and a unit vector $|\Psi_{AB}\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ be arbitrary and denote $\rho_A = \mathrm{tr}_B[|\Psi_{AB}\rangle\langle\Psi_{AB}|]$. We have

$$\begin{aligned} &|C(\rho_{AB})^2 - 4\det(\zeta_A)| \\ &\quad\leqslant |C(\rho_{AB})^2 - C(\Psi_{AB})^2| + |4\det(\rho_A) - 4\det(\zeta_A)| \\ &\quad\leqslant 2\,|C(\rho_{AB}) - C(\Psi_{AB})| + 4|\det(\rho_A) - \det(\zeta_A)| \\ &\quad\leqslant 12\,D(|\Psi_{AB}\rangle\langle\Psi_{AB}|,\rho_{AB}), \end{aligned} \tag{A7}$$

and consequently $|C(\rho_{AB})^2 - 4\det(\zeta_A)| \leqslant 12\,M(\rho_{AB})$ for all ρ_{AB}. Choosing $\delta = \frac{\varepsilon}{12}$ proves the claim. As a by-product we get the bounds

$$4C_- \leqslant C(\rho_{AB})^2 \leqslant 4C_+, \tag{A8}$$

where $C_\pm := \det(\zeta_A) \pm 3M(\rho_{AB})$.

In summary, we can conclude that $M(\rho_{AB}) \approx 0$ implies that ρ_{AB} is both approximately pure ($\mathrm{tr}[\rho_{AB}^2] \approx 1$) and that $C(\rho_{AB})^2 \approx 4\det(\zeta_A)$.

[1] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[2] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[3] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[4] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).
[5] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[6] L. Masanes, Phys. Rev. Lett. **96**, 150501 (2006).
[7] L. Masanes, J. Math. Phys. **49**, 022102 (2008).
[8] O. Gühne and G. Tóth, Phys. Rep. **474**, 1 (2009).
[9] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[10] J. M. G. Sancho and S. F. Huelga, Phys. Rev. A **61**, 042303 (2000).
[11] P. Horodecki, Phys. Rev. Lett. **90**, 167901 (2003).
[12] F. Mintert, A. R. R. de Carvalho, M. Kus, and A. Buchleitner, Phys. Rep. **415**, 207 (2005).
[13] S. P. Walborn, P. H. Souto Ribeiro, L. Davidovich, F. Mintert, and A. Buchleitner, Nature (London) **440**, 1022 (2006).
[14] L.-H. Zhang, Q. Yang, M. Yang, W. Song, and Z.-L. Cao, Phys. Rev. A **88**, 062342 (2013).
[15] S. M. Lee, S.-W. Ji, H.-W. Lee, and M. S. Zubairy, Phys. Rev. A **77**, 040301(R) (2008).
[16] J. Řeháček, B.-G. Englert, and D. Kaszlikowski, Phys. Rev. A **70**, 052321 (2004).
[17] A. Salles, F. de Melo, J. C. Retamal, R. L. de Matos Filho, and N. Zagury, Phys. Rev. A **74**, 060303(R) (2006).
[18] L. Zhou and Y.-B. Sheng, Entropy **17**, 4293 (2015).
[19] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
[20] Y. Aharonov, D. Z. Albert, A. Casher, and L. Vaidman, Phys. Lett. A **124**, 199 (1987).
[21] P. Busch, P. J. Lahti, and P. Mittelstaedt, *The Quantum Theory of Measurement*, 2nd ed. (Springer, Berlin, 1996).
[22] H. M. Wiseman, Phys. Rev. A **65**, 032111 (2002).
[23] A. Hosoya and Y. Shikano, J. Phys. A **43**, 385307 (2010).
[24] Y. Aharonov and D. Rohrlich, *Quantum Paradoxes* (Wiley-VCH, Weibheim, 2005).
[25] J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, and C. Bamber, Nature (London) **474**, 188 (2011).
[26] E. Haapasalo, P. Lahti, and J. Schultz, Phys. Rev. A **84**, 052107 (2011).
[27] J. S. Lundeen and C. Bamber, Phys. Rev. Lett. **108**, 070402 (2012).
[28] S. Wu, Sci. Rep. **3**, 1193 (2013).
[29] H. Kobayashi, K. Nonaka, and Y. Shikano, Phys. Rev. A **89**, 053816 (2014).
[30] M. Malik, M. Mirhosseini, M. P. J. Lavery, J. Leach, M. J. Padgett, and R. W. Boyd, Nat. Commun. **5**, 3115 (2014).
[31] E. Bolduc, G. Gariepy, and J. Leach, Nat. Commun. **7**, 10439 (2016).
[32] E. Sjöqvist, Phys. Lett. A **359**, 187 (2006).
[33] Y. Shikano and A. Hosoya, J. Phys. A **43**, 025304 (2010).
[34] H. Kobayashi, S. Tamate, T. Nakanishi, K. Sugiyama, and M. Kitano, Phys. Rev. A **81**, 012104 (2010).
[35] H. Kobayashi, S. Tamate, T. Nakanishi, K. Sugiyama, and M. Kitano, J. Phys. Soc. Jpn. **80**, 034401 (2011).
[36] A. K. Pati, U. Singh, and U. Sinha, Phys. Rev. A **92**, 052120 (2015).
[37] Y. Aharonov and L. Vaidman, in *Time in Quantum Mechanics*, edited by J. G. Muga, R. Sala Mayato, and I. L. Egusquiza (Springer, Berlin, Heidelberg, 2008), Vol. 1, pp. 399–447.
[38] Y. Shikano, in *Measurements in Quantum Mechanics*, edited by M. R. Pahlavani (InTech, Rijeka, Croatia, 2012), p. 75.
[39] J. Dressel, M. Malik, F. M. Miatto, A. N. Jordan, and R. W. Boyd, Rev. Mod. Phys. **86**, 307 (2014).
[40] A. G. Kofman, S. Ashhab, and F. Nori, Phys. Rep. **520**, 43 (2012).
[41] C. Carmeli, T. Heinosaari, A. Karlsson, J. Schultz, and A. Toigo, Phys. Rev. Lett. **116**, 230403 (2016).
[42] Y. Aharonov and L. Vaidman, Phys. Rev. A **41**, 11 (1990).
[43] R. Jozsa, Phys. Rev. A **76**, 044103 (2007).
[44] K. J. Resch and A. M. Steinberg, Phys. Rev. Lett. **92**, 130402 (2004).
[45] G. Puentes, N. Hermosa, and J. P. Torres, Phys. Rev. Lett. **109**, 040401 (2012).
[46] H. Kobayashi, G. Puentes, and Y. Shikano, Phys. Rev. A **86**, 053805 (2012).
[47] S. Kanjilal, G. Muralidhara, and D. Home, Phys. Rev. A **94**, 052110 (2016).
[48] A. E. Siegman, *Lasers* (University Science Books, Mill Valley, CA, 1986).
[49] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Phys. Rev. A **45**, 8185 (1992).
[50] L. Marrucci, C. Manzo, and D. Paparo, Phys. Rev. Lett. **96**, 163905 (2006).
[51] T. Ando, Y. Ohtake, N. Matsumoto, T. Inoue, and N. Fukuchi, Opt. Lett. **34**, 34 (2009).
[52] X. Li, P. L. Voss, J. E. Sharping, and P. Kumar, Phys. Rev. Lett. **94**, 053601 (2005).
[53] Y. Guo, J. Hou, and Y. Wang, Quantum Inf. Process. **12**, 2641 (2013).