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Tolerant Location Detection in Sensor Networks*

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Abstract

Location detection in sensor networks can be handled with so called identifying codes. For an identifying code to work properly, it is required that no sensors are malfunctioning. Previously, malfunctioning sensors have been typically coped with robust identifying codes. However, they are rather large and, hence, imply high signal interference and energy consumption. To overcome these issues, collections of disjoint identifying codes have been proposed for coping with malfunctioning sensors. However, these collections have some problems regarding detection of malfunctioning sensors and, moreover, it seems unnecessary to restrict oneself to disjoint codes. In this paper, we discuss a certain type of identifying codes, for which the detection of malfunctioning sensors is easy, and based on these codes we design a collection of codes tolerant against malfunctions. We present some results on general graphs as well as optimal constructions in rook's graphs and binary Hamming spaces.

Keywords: Identifying code; location detection; sensor network; malfunctioning sensor; covering design; Hamming space; rook's graph

1 Introduction

Sensor networks are systems designed for environmental monitoring. Various location detection systems are interesting applications regarding sensor networks. As an example of such a location detection system, consider an observer travelling in the network trying to determine her location. Here, a sensor, which can be turned on or off, can be placed in each location of the network. The sensor transmits a signal (unique to that sensor) to each location in its (closed) neighbourhood. Based on the received signals, the observer attempts to determine her location by considering the group of sensors from which signals are received. Usually, the aim is to minimize the number of sensors in the network or the number of sensors simultaneously turned on. The former setting is desirable if the sensors are expensive. However, even if the sensors are relatively cheap, minimizing the number of sensors simultaneously turned on reduces signal interference and energy consumption of the sensors (see [14, 16]). In this paper, we assume that the sensors are relatively cheap and, hence, interest is in the number of sensors simultaneously turned on. More explanation regarding location detection in sensor networks can be found in [3, 14, 16].

A sensor network can be modeled as a finite, simple and undirected graph $G = (V, E)$ as follows: the set of vertices V of the graph represent the locations of the network and the edge set E of the graph represent the neighbours of the locations. In other words, a sensor can be placed in each vertex of the graph and the sensor placed in the vertex u transmits a signal to u itself and the vertices neighbouring u. In what follows, we present some basic terminology and notation regarding graphs. The *open neighbourhood* of $u \in V$ consists of the vertices adjacent to u and it is denoted by $N(u)$. The *closed neighbourhood* of u is defined as $N[u] = \{u\} \cup N(u)$. A nonempty

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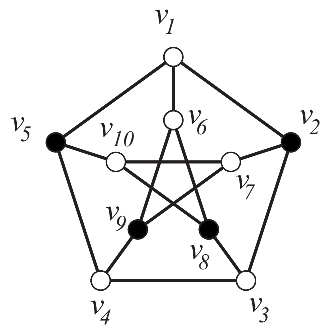

Figure 1: The black vertices form an identifying code.

subset C of V is called a *code* and the elements of the code are called *codewords*. In this paper, the code C (usually) represents the set of sensors which are simultaneously turned on. For the set of sensors from which a vertex $u \in V$ receives signals, we use the following notation:

$$I(u) = I(C; u) = N[u] \cap C.$$

We call $I(u)$ the *identifying set* of u.

Notice that if the sensors in the code C are located in such places that $I(C; u)$ are nonempty and unique for all $u \in V$, then the sought location in the network can be determined by comparing $I(C; u)$ to identifying sets of other vertices. This leads to the following definition of *identifying codes*, which were first introduced by Karpovsky *et al.* in [13]. For recent developments regarding identification, we refer to the papers [4, 6, 11]. For various other papers on the subject, see the online bibliography [15].

Definition 1. A code $C \subseteq V$ is *identifying* in G if $I(C; u) \neq \emptyset$ for all $u \in V$ and for all distinct $u, v \in V$ we have

$$I(C; u) \neq I(C; v).$$

An identifying code C in G with the smallest cardinality is called *optimal*.

The previous definition of identifying codes is illustrated in the following example.

Example 2. Consider the Petersen graph $\mathcal{P}$ illustrated in Figure 1. Let $C = \{v_2, v_5, v_8, v_9\}$ be a code in $\mathcal{P}$. By considering $I(C; u)$ for all $u \in \mathcal{P}$, it is straightforward to verify that C is an identifying code in $\mathcal{P}$. Indeed, for example, if a signal is received only from the sensor in v_5, then the sought location is v_5 as $I(C; v_5) = \{v_5\}$. Moreover, any identifying code in $\mathcal{P}$ has at least 4 codewords since using less than 4 codewords implies that at most $2^3 - 1 = 7$ different, nonempty identifying sets can be formed. Hence, C is an optimal identifying code in $\mathcal{P}$.

Remark 3. As we have seen above, identifying codes work rather well for location detection. However, they have a couple of issues with them:

- In order for an observer to determine her location after receiving the set of signaling sensors, some comparison of the identifying sets $I(u)$ has to be done. For instance, in Example 2, if a signal is received only from the sensor in v_5, then we immediately know that the observer is in $N[v_5] = \{v_1, v_4, v_5, v_{10}\}$ and then by comparing the sets $I(v_1)$, $I(v_4)$, $I(v_5)$ and $I(v_{10})$ we can deduce that the observer has to be in v_5 as $I(v_5) = \{v_5\}$.
- Identifying codes have no tolerance against malfunctioning sensors. (Here we assume that a sensor is malfunctioning if it does not transmit signals.) In particular, if a sensor fails to transmit signals to its closed neighbourhood, then a location might be incorrectly determined and, moreover, there is no indicator that a sensor is malfunctioning. For instance, in Example 2, if the sensor in v_8 is malfunctioning and the observer is in v_{10}, then a signal is received only from the sensor in v_5 and (as above) we deduce that the sought location is in v_5. Hence, an incorrect location is determined and we have no knowledge that it is wrong.

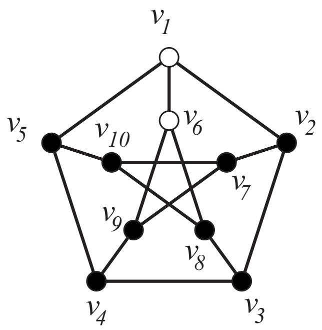

Figure 2: The black vertices form a self-identifying code.

In this paper, we address the problems of Remark 3 using self-identifying codes and their suitable collections. The self-identifying codes are introduced in Section 2. The collections of them, called *tolerant identifying collections*, are considered in Section 3, where we study the relation between our approach and robust identifying codes (cf. [7, 9, 10, 16]) as well. We also discuss the connection to collections of disjoint identifying codes presented in [14]. In Section 4, we give some existence results on tolerant identifying collections in a graph as well as present some bounds on the sizes of the collections. In Sections 5 and 6, we consider tolerant identifying collections in the rook's graphs and in the binary Hamming spaces, respectively. In both cases, we provide tolerant identifying collections of optimal self-identifying codes with the smallest number of codes.

2 Self-identifying codes

In this section, we introduce the self-identifying codes. We also discuss how these codes are related to Remark 3 and show their connection to a class of codes presented in [8].

Definition 4. A code $C \subseteq V$ is *self-identifying* in G if for all $u \in V$ we have $I(C;u) \neq \emptyset$ and

$$\bigcap_{c \in I(C;u)} N[c] = \{u\}. \tag{1}$$

A self-identifying code C is called *minimal* if $C \setminus \{u\}$ is not self-identifying for any $u \in C$. In a finite graph G, the smallest cardinality of a self-identifying code in G is denoted by $\gamma^+(G)$. Moreover, a self-identifying code C with $\gamma^+(G)$ codewords is called *optimal*.

The previous definition of self-identifying codes is illustrated in the following example.

Example 5. Consider the Petersen graph $\mathcal{P}$ illustrated in Figure 2. Let $C_1 = V \setminus \{v_1, v_6\}$. The code C_1 is self-identifying. Indeed, for every vertex $x \in V$ there are (at least) two codewords $c_1, c_2 \in I(x) \setminus \{x\}$, and because the girth of $\mathcal{P}$ is five, we obtain $N[c_1] \cap N[c_2] = \{x\}$. This implies that

$$\bigcap_{c \in I(x)} N[c] = \{x\}.$$

The code C_1 is also minimal. If we remove the vertex v_2 from C_1 (denote the code by C'), then $I(C';v_1) = \{v_5\}$ and $\cap_{c \in I(C';v_1)} N[c] = \{v_1, v_4, v_5, v_{10}\}$ which violates (1) for v_1. Similarly, no other codeword can be removed from C without violating (1). Hence the code is minimal. Furthermore, it can also be shown that the code is optimal, i.e., $\gamma^+(\mathcal{P}) = 8$.

Consider the issues discussed in Remark 3 with identifying codes in comparison to self-identifying codes. Let $C \subseteq V$ be a self-identifying code in G. Then, for location detection, no comparison to other identifying sets needs to be done since the intersection (1) immediately gives the (unique) location of the observer. For instance, in Example 5, if signals are received from the sensors in the locations v_2 and v_5, then the observer is in the unique vertex v_1 of the intersection $N[v_2] \cap N[v_5]$.

Let us then consider more closely the ability to detect malfunctioning sensors. Denote by S the set of sensors from which signals are received. If the set S is empty, then it is immediately deduced that some of the sensors are malfunctioning; indeed, a signal from some sensor should always be received. On the other hand, if $S \neq \emptyset$, then the intersection $\bigcap_{c \in S} N[c]$ is formed. Now, if the intersection contains more than one vertex, then we again know that some sensors are malfunctioning. For instance, in Example 5, if the sensor in v_7 is malfunctioning and the observer is in v_2, then signals are received from v_2 and v_3 and we can deduce that some sensors are malfunctioning since $N[v_2] \cap N[v_3] = \{v_2, v_3\}$. On the other hand, if the intersection contains exactly one vertex, then the location is correctly determined. Notice that the intersection might consists of a single vertex although some of the sensors are malfunctioning (but the obtained unique vertex is always the correct one). For instance, in Example 5, if the sensor in v_8 is malfunctioning and the observer is in v_3, then signals are received from v_2, v_3 and v_4 and we obtain that $N[v_2] \cap N[v_3] \cap N[v_4] = \{v_3\}$. Hence, the location is uniquely determined and the malfunctioning of the sensor v_8 stays unnoticed.

Previously, in [8], so called $(1, \leq 1)^+$-identifying codes have been designed for locating a single observer and detecting several ones in a sensor network. A code $C \subseteq V$ is defined to be $(1, \leq 1)^+$-*identifying* if the set $I(C; u) \setminus I(C; v) \neq \emptyset$ for all distinct $u, v \in V$. In the following theorem, it is shown that the definitions of self-identifying and $(1, \leq 1)^+$-identifying codes are equivalent.

Theorem 6. *Let $C \subseteq V$ be a code. Then the following two conditions are equivalent:*

(i) The code C is self-identifying.

(ii) $I(C; u) \setminus I(C; v) \neq \emptyset$ for all distinct $u, v \in V$.

Proof. Assume first that the code C is self-identifying, that is, (1) holds. Assume to the contrary that for some distinct vertices $u, v \in V$

$$I(C; u) \setminus I(C; v) = \emptyset.$$

This implies that $I(C; u) \subseteq I(C; v)$ and, hence, both u and v belong to the set $\cap_{c \in I(C;u)} N[c]$ which is a contradiction with (1).

Conversely, assume that the condition (ii) is valid for $C \subseteq V$. We check whether (1) is satisfied. Assume to the contrary that for some $u \in V$ the set $\cap_{c \in I(C;u)} N[c]$ contains at least two vertices u and v. But this gives that $I(C; u) \subseteq I(C; v)$ which contradicts (ii). $\square$

3 Collections of codes

We have seen above that identifying and self-identifying codes are useful for location detection in sensor networks. However, problems occur (recall Remark 3) if some of the sensors are malfunctioning. In the case of identifying codes, we have a risk of outputting an incorrect (or unknown) location. Moreover, using self-identifying codes, an imprecise location might be outputted but then we at least know that something is wrong with the current code.

Previously, in the literature, the possible malfunctioning of sensors has been handled using robust identifying codes; for different variants, see [7, 9, 10, 16]. Observe that $C \subseteq V$ is an identifying code in G if and only if $I(u) \neq \emptyset$ for all $u \in V$ and $(I(u) \setminus I(v)) \cup (I(v) \setminus I(u)) \neq \emptyset$ for all distinct $u, v \in V$. Therefore, if s sensors are malfunctioning in the network, then a code $C \subseteq V$ satisfying

$$|I(u)| \geq s + 1 \text{ and } |(I(u) \setminus I(v)) \cup (I(v) \setminus I(u))| \geq s + 1 \tag{2}$$

for all distinct $u, v \in V$ can still correctly determine the location of an observer in the network. However, albeit these codes are tolerant against s malfunctioning sensors, they are rather large and, therefore, imply signal interference and unnecessary energy consumption according to [14, 16].

For robust location detection taking into account signal interference and energy consumption, Laifenfeld and Trachtenberg [14] proposed a collection of *disjoint* identifying codes. In this setting, only the sensors corresponding to some disjoint identifying code are turned on simultaneously and the code is discarded if it is observed that some of the sensors in the code are malfunctioning. Hence, the system is tolerant against some malfunctions of the sensors. Previously, we have described that there is no built-in way to determine if an identifying code in the collection is not working properly. Therefore, in this setting, we need a separate mechanism for detecting malfunctioning sensors. Moreover, the restriction to disjoint codes also seems unnecessary.

In order to overcome the previous disadvantages, we introduce collections of self-identifying codes in the following definition.

Definition 7. Let s be a positive integer. A collection of codes $\mathcal{L} = \{C_1, C_2, \ldots, C_h\}$ is called an *s-tolerant identifying collection* in $G = (V, E)$ if

(i) $C_i \subseteq V$ is minimal self-identifying code for all $i = 1, \ldots, h$ and

(ii) for any $S \subseteq V$ of size at most s we have

$$S \cap C_i = \emptyset \quad \text{for at least one } i = 1, \ldots, h. \tag{3}$$

In the previous definition, the usage of self-identifying codes guarantees that the sought location can always be determined and a possible malfunctioning of sensors can be detected. The minimality of the self-identifying codes is motivated by the efforts to minimize signal interference and energy consumption. Moreover, the condition (3) guarantees that there always exists a code without malfunctioning vertices (or sensors), i.e., for any $S \subseteq V$ with $|S| \leq s$ there exists $C \in \mathcal{L}$ such that $S \cap C = \emptyset$. Naturally, we prefer a collection with as few codes as possible and also strive for as small self-identifying codes as possible (often using optimal codes). Notice also that we do not restrict ourselves to disjoint self-identifying codes since in this way we can cope with larger number of malfunctioning sensors (for an example, see Remark 16). Moreover, the benefits of collections of codes over single larger codes regarding signal interference and energy consumption are discussed in Remark 17.

Let $\mathcal{L}$ be an s-tolerant identifying collection in G. Then the s-tolerant identifying collection $\mathcal{L}$ is used as follows:

- A code $C \in \mathcal{L}$ is used as long as some signals from the sensors are received and the intersection

$$\bigcap_{c \in I(C;u)} N[c]$$

consists of a unique vertex, i.e., the observer can determine her location uniquely.

- If a non-unique location is outputted or no signals are received, then the code C is discarded and a new one is chosen. By (3), there exists a new correctly working code C' such that $C' \cap S = \emptyset$, where S is the set of malfunctioning sensors with at most s vertices.

The definition of s-tolerant identifying collections is illustrated in the following example.

Example 8. Let us consider the Petersen graph again. Denote the 'rotations' of the code C_1 by $C_2 = V \setminus \{v_2, v_7\}$, $C_3 = V \setminus \{v_3, v_8\}$, $C_4 = V \setminus \{v_4, v_9\}$ and $C_0 = V \setminus \{v_5, v_{10}\}$. These codes are all self-identifying and minimal (even optimal). The collection

$$\mathcal{L} = \{C_1, C_2, C_3, C_4, C_0\}$$

is 1-tolerant identifying, since the condition (3) is satisfied. Indeed, now S consists of a single vertex $S = \{v_i\}$ (or it is empty) and $S \cap C_j = \emptyset$ for j such that $j \equiv i \pmod 5$.

4 On existence of the collection

Observe that by Theorem 6 there exists a self-identifying code in a graph $G = (V, E)$ if and only if $N[x] \setminus N[y] \neq \emptyset$ for all distinct $x, y \in V$. In the following theorem, we present an existence result for the s-tolerant identifying collections in a graph.

Theorem 9. *There always exists an s-tolerant identifying collection $\mathcal{L}$ with $|\mathcal{L}| \leq \binom{|V|}{s}$ for any s in a graph $G = (V, E)$ provided that*

$$s < \min_{x \neq y} |N[x] \setminus N[y]|.$$

Moreover, if $s \geq \min_{x \neq y} |N[x] \setminus N[y]|$, then no s-tolerant identifying collection exists.

Proof. Let first $s \geq \min_{x \neq y} |N[x] \setminus N[y]|$ and the pair $a, b \in V$ be such that $|N[a] \setminus N[b]| = \min_{x \neq y} |N[x] \setminus N[y]|$. Let us choose $S = N[a] \setminus N[b]$. Now $|S| \leq s$. The condition of Theorem 6(ii) implies that any self-identifying code must contain a codeword in S. However, this means that there cannot be any self-identifying code satisfying the condition (3).

Suppose then that $s < \min_{x \neq y} |N[x] \setminus N[y]|$. Let the collection $\mathcal{L}'$ consist of all the subsets of V of size $|V| - s$. Any member of the collection $\mathcal{L}'$ forms a self-identifying code due to Theorem 6(ii) and the choice of s. In order to make a self-identifying code $C \in \mathcal{L}'$ minimal we label its codewords as $c_1, c_2, \ldots, c_{|C|}$. We start from the codeword c_1 and see if it can be removed from C and still maintain the property of self-identification. We continue this way in the given order with the other codewords. Consequently, a collection of size at most $\binom{|V|}{s}$ of minimal self-identifying codes is obtained which satisfies (3). □

Let v, k and t be positive integers such that $t \leq k \leq v$. A (v, k, t) *covering design* is a family of k-subsets, called *blocks*, chosen from a set of v elements, such that each t-subset is contained in at least one of the blocks. The minimum size of such covering design is denoted by $C(v, k, t)$. More about covering designs can be read in [2, pp. 365–373].

In the following theorem, we present a lower bound on the size of an s-tolerant identifying collection.

Theorem 10. *Let s be a positive integer. If $\mathcal{L}$ is an s-tolerant identifying collection in $G = (V, E)$, then*

$$|\mathcal{L}| \geq C(|V|, |V| - \gamma^+(G), s).$$

Proof. Let $\mathcal{L} = \{C_1, C_2, \ldots, C_{|\mathcal{L}|}\}$ be an s-tolerant identifying collection in a graph $G = (V, E)$. Let further $V = \{v_1, v_2, \ldots, v_{|V|}\}$. We define a $|\mathcal{L}| \times |V|$-matrix $M = (a_{ij})$ whose entry

$$a_{ij} = \begin{cases} 1 & \text{if } v_j \in C_i \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, |\mathcal{L}|$ and $j = 1, \ldots, |V|$. Since $\mathcal{L}$ is an s-tolerant identifying collection, we know by (3) that for any s columns of the matrix M there is a row such that the entries corresponding to the s columns are all 0's, i.e., non-codewords. This fact remains true if we change in a row any 1 to 0 in order to make the rows have the same number $|V| - \gamma^+(G)$ of 0's in them. If some of the rows coincide, we prune them away from the matrix. Consequently, the remaining rows of the altered matrix now correspond to the blocks in a $(|V|, |V| - \gamma^+(G), s)$ covering design. Consequently, we have

$$|\mathcal{L}| \geq C(|V|, |V| - \gamma^+(G), s)$$

as claimed. □

In the following example, we show that the lower bound of the previous theorem can be attained. Another case of the optimality of the bound is given later in Example 20.

Example 11. Since in the Petersen graph $\mathcal{P}$ the value $|N[x] \setminus N[y]|$ equals either two or three, there does not exist, by Theorem 9, an s-tolerant identifying collection for $s \geq 2$. For $s = 1$ the collection $\mathcal{L} = \{C_1, C_2, C_3, C_4, C_0\}$ is of optimal size due to the previous theorem, since $\gamma^+(\mathcal{P}) = 8$ and $C(10, 2, 1) \geq 5$. Indeed, we need at least 5 blocks of size 2 for covering all the elements of a set of size 10.

5 Smallest collections in the rook's graphs

Let us consider s-tolerant identifying collections in the Cartesian product $K_n \square K_m$, where K_n and K_m are complete graphs of order n and m, respectively. The graph $K_n \square K_m$ is also known as the rook's graph. Notice that the rook's graph can be viewed as a chess board with n *columns* and m *rows*, and the neighbourhood of a vertex is determined by the movement of a rook. Previously, identification and related problems in the rook's graphs have been considered, for example, in [4, 5]. In the following theorem, we give a characterization for self-identifying codes in $K_n \square K_m$ as well as determine the sizes of optimal codes. Let us denote $K_n = \{v_1, v_2, \ldots, v_n\}$ and $K_m = \{w_1, w_2, \ldots, w_m\}$. By the k-th *row* (resp. h-th *column*) of the graph $K_n \square K_m$, we mean the set of vertices $R_k = \{(v_i, w_k) \mid i = 1, \ldots, n\}$ (resp. $P_h = \{(v_h, w_j) \mid j = 1, \ldots, m\}$).

Theorem 12. *Let $n \geq 2$ and $m \geq 2$ be integers. Then a code C is self-identifying in $K_n \square K_m$ if and only if there exist at least two codewords in each row and column of the graph. Moreover, we have the following result for the size of an optimal self-identifying code in the graph:*

$$\gamma^+(K_n \square K_m) = 2 \cdot \max\{m, n\}.$$

Proof. For any vertex $x = (v_i, w_j)$ of $K_n \square K_m$, we have

$$N[x] = \{(v_a, w_b) \mid a = i \text{ or } b = j\}.$$

Let C be a self-identifying code in $K_n \square K_m$. In what follows, we show that there must be at least two codewords of C in each row and column. Suppose to the contrary that there is at most one codeword in the row R_k (the argument for the columns is analogous). Let z be the codeword if it exists and if there is no codeword in the row R_k let z be any vertex from that row. Denote $z = (v_i, w_k)$. Now all the codewords in $I(z)$ belong to the column P_i. But this implies that

$$P_i \subseteq \bigcap_{c \in I(z)} N[c]$$

which is a contradiction with (1). The fact that there are at least two codewords in every row and column yields the claim

$$\gamma^+(K_n \square K_m) \geq 2 \cdot \max\{m, n\}.$$

Now suppose that the code C is such that all the columns and rows contain at least two codewords of C. We will show that C is then self-identifying. Let $x = (v_i, w_j)$ be any vertex of $K_n \square K_m$. By the assumption, there exist two codewords c_1 and c_2 in the same row as x and, obviously, c_1 and c_2 belong to $I(x)$. Then we clearly have $N[c_1] \cap N[c_2] = R_j$. Since there are also two codewords in the same column as x, there exists $c_3 = (v_i, w_k) \in C$ such that $c_3 \neq x$, i.e., $k \neq j$. This implies that $(N[c_1] \cap N[c_2]) \cap N[c_3] = R_j \cap N[(v_i, w_k)] = \{(v_i, w_j)\} = \{x\}$. Therefore, we have

$$\bigcap_{c \in I(C;x)} N[c] = \{x\}$$

and the code C is self-identifying.

In what follows, we show that there always exists a self-identifying code in $K_n \square K_m$ with $2 \cdot \max\{m, n\}$ codewords. Without loss of generality, we may assume that $n \geq m$. First we define an auxiliary function $f : \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$, $f(i, j) = k$, where k is an integer such that $k \in \{1, 2, \ldots, m\}$ and $k \equiv j + (i - 1) \pmod{m}$. Then taking any two distinct vertices (v_1, w_{j_1}) and (v_1, w_{j_2}) from the first column of the graph, we construct a code C as follows:

$$C = \bigcup_{i=1}^{n} \{(v_i, w_{f(i,j_1)}), (v_i, w_{f(i,j_2)})\}. \tag{4}$$

In other words, the code C is formed from the first column by doing a cyclic shift (of one position) upwards to obtain the codewords in the next column. It is immediate by the construction that each column contains exactly two codewords and, hence, $|C| = 2n$. It is also easy to check that each row contains at least two codewords since $n \geq m$ and the function $f(i, j)$ obtains all the values of the target set $\{1, 2, \ldots, m\}$ for any j when i goes through the values $1, 2, \ldots, n$. Therefore, as discussed above, C is a self-identifying code. Thus,

$$\gamma^{+}(K_n \Box K_m) \leq 2 \cdot \max\{m, n\}.$$

and we have verified the claims of the theorem.

□

Considering the last paragraph of the previous proof, we can construct a self-identifying code in the rook's graph from any two codewords in the first column. Hence, we can actually form $\binom{m}{2}$ different self-identifying codes. This technique proves to be useful in the proof of the following theorem, where we determine the smallest number of codes in an s-tolerant identifying collection in the rook's graph $K_n \Box K_m$, when $1 \leq s \leq \min\{m, n\} - 2$. By Theorem 9, it is immediate that no s-tolerant identifying collection exists if $s \geq \min\{m, n\} - 1$.

Theorem 13. *Let n, m and s be integers such that $n, m \geq 2$, $n \geq m$ and $1 \leq s \leq m - 2$. Now the following statements hold:*

(i) There exists an s-tolerant identifying collection in $K_n \Box K_m$ with $C(m, m - 2, s)$ optimal *self-identifying codes.*

(ii) Any s-tolerant identifying collection has at least $C(m, m-2, s)$ minimal self-identifying codes.

Proof. (i) Consider the first column P_1 of $K_n \Box K_m$, which is of size m. Let $\mathcal{B}$ be a minimum $(m, m - 2, s)$ covering design of P_1. In other words, $\mathcal{B}$ is a collection of $C(m, m - 2, s)$ $(m - 2)$-blocks of P_1 such that every s-subset of P_1 is contained in at least one block. From each block, we construct a self-identifying code as follows: Take a block B from $\mathcal{B}$ and designate $P_1 \setminus B$ as codewords (now $|P_1 \setminus B| = 2$). Then construct a self-identifying code based on these two codewords of the first column as in (4). As observed previously, the constructed codes are optimal. Thus, we have obtained a collection $\mathcal{L}$ of optimal self-identifying codes with $C(m, m-2, s)$ members. In what follows, we show that for any $S \subseteq K_n \Box K_m$ with $|S| \leq s$ there always exists $C \in \mathcal{L}$ such that $S \cap C = \emptyset$; hence, concluding that $\mathcal{L}$ is an s-tolerant identifying collection.

In order to show the previous claim, we define an auxiliary function $f' : \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$, $f'(i, j) = k$, where k is an integer such that $k \in \{1, 2, \ldots, m\}$ and $j - (i - 1) \equiv k \pmod{m}$. Observe first that for any $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$ we have $j = f'(i, f(i, j))$, where the function f is defined as in the proof of Theorem 12. Hence, based on the construction of the codes in $\mathcal{L}$, we observe that for each $C \in \mathcal{L}$ the vertex (v_i, w_j) belongs to C if and only if $(v_1, w_{f'(i,j)})$ belongs to C. Let then $S \subseteq K_n \Box K_m$ be an arbitrary set of (malfunctioning) vertices with at most s elements. Due to the previous observation, we may without loss of generality assume that all the vertices in S belong to the first column P_1 of the graph. Since $\mathcal{B}$ is an $(m, m - 2, s)$ covering design of P_1, there exists a block $B \in \mathcal{B}$ such that $S \subseteq B$. Therefore, the self-identifying code C constructed from the block B is such that $S \cap C = \emptyset$. This concludes the first part of the proof.

(ii) Let $\mathcal{L}$ be an s-tolerant identifying collection. Suppose that $S \subseteq P_1$ is a set of s (malfunctioning) vertices. Then, as $\mathcal{L}$ is an s-tolerant identifying collection, there exists a code $C \in \mathcal{L}$ such that $S \cap C = \emptyset$, i.e., $S \subseteq P_1 \setminus C$. Form a collection of subsets of P_1 by taking $P_1 \setminus (C \cap P_1)$ for all codes $C \in \mathcal{L}$. Each subset of the collection contains at most $m-2$ vertices (as each column contains at least two codewords). By the previous observation any s-subset of P_1 is contained in some subset of the formed collection. Thus, the formed collection is almost an $(m, m-2, s)$ covering design with the exception that some of the blocks may be of size smaller than $m-2$. However, this can be easily fixed by adding some redundant vertices to each subset with less than $m-2$ vertices. Hence, we have a $(m, m-2, s)$ covering design with $|\mathcal{L}|$ blocks. Thus, it follows that $|\mathcal{L}| \geq C(m, m-2, s)$. $\square$

6 Smallest collections in the Hamming spaces

In this section, we consider s-tolerant identifying collections in binary Hamming spaces (or hypercubes) of length n. The binary Hamming space of length n is a graph with the vertex set $\mathbb{F}^n = \{0,1\}^n$ and two vertices of $\mathbb{F}^n$ are adjacent if they differ in exactly one coordinate place. Hence, the distance of $x, y \in \mathbb{F}^n$ is the number of coordinate places, where x and y differ, and the distance is denoted by $d(x,y)$. The vertices of $\mathbb{F}^n$ are also called *words*. The all-zero word of length n is denoted by $\mathbf{0} = (0,0,\ldots,0) \in \mathbb{F}^n$, and e_i denotes the word of length n, where the ith coordinate is 1 and all the other coordinates are 0. The *weight* $w(x)$ of a word $x \in \mathbb{F}^n$ is the number coordinates equal to 1, i.e., $w(x) = d(x, \mathbf{0})$.

Let C be a code in $\mathbb{F}^n$. The sum of the code C and a word $x \in \mathbb{F}^n$ is defined as follows:

$$x + C = \{x + c \mid c \in C\}.$$

We say that a code C is a *μ-fold 1-covering* in $\mathbb{F}^n$ if $|I(C;x)| = |N[x] \cap C| \geq \mu$ for all $x \in \mathbb{F}^n$. Moreover, a μ-fold 1-covering is said to be *perfect* if $|I(C;x)| = \mu$ for all $x \in \mathbb{F}^n$. It is immediate that $\mathbb{F}^n$ is a vector space under the usual addition of vectors and multiplication with scalars. We say that the code C is *linear* if C is a subspace of $\mathbb{F}^n$.

Now we are ready to present a characterization of self-identifying codes in $\mathbb{F}^n$. In [8], it has been shown that a code $C \subseteq \mathbb{F}^n$ is $(1, \leq 1)^+$-identifying (or self-identifying using our terminology) if and only if each word of $\mathbb{F}^n$ is 1-covered by at least 3 codewords of C, i.e., C is a 3-fold 1-covering. This characterization is reformulated in the following theorem.

Theorem 14. *Let $n \geq 2$ be an integer and C be a code in $\mathbb{F}^n$. Then C is a self-identifying code in $\mathbb{F}^n$ if and only if C is a 3-fold 1-covering, i.e., $|I(x)| = |N[x] \cap C| \geq 3$ for all $x \in \mathbb{F}^n$.*

In the definition of s-tolerant identifying collections, we are interested in minimal self-identifying codes and, especially, in optimal ones. In general, the optimal 3-fold 1-coverings, i.e., the smallest possible coverings, are not well-known (see, for example, [1]). However, in [1], it has been shown that there exists a *perfect* 3-fold 1-covering C in $\mathbb{F}^n$, when r is positive integer and $n = 2^{r+1} - 1$ or $n = 3 \cdot 2^r - 1$, and furthermore that no perfect 3-fold 1-coverings exist for other lengths. Observe that a perfect 3-fold 1-covering in $\mathbb{F}^n$ is an optimal self-identifying code in $\mathbb{F}^n$. In what follows, we are going to concentrate on the lengths when perfect 3-fold 1-coverings exist.

Let us first assume that $n = 2^r - 1$, where $r \geq 2$ is an integer. In what follows, we present some relevant properties of binary Hamming codes $\mathcal{H}_n \subseteq \mathbb{F}^n$. More detailed coverage of the codes $\mathcal{H}_n$ can be found, for example, in [1]. Recall that a binary Hamming code $\mathcal{H}_n \subseteq \mathbb{F}^n$ is a perfect 1-covering, i.e., a code such that $|I(x)| = |N[x] \cap \mathcal{H}_n| = 1$ for all $x \in \mathbb{F}^n$. Moreover, the binary Hamming code $\mathcal{H}_n$ is linear. Hence, $\mathcal{H}_n$ is a subgroup of $\mathbb{F}^n$ under the addition of vectors and we may consider the cosets of $\mathcal{H}_n$. Thus, a 3-fold 1-covering in $\mathbb{F}^n$ can be formed as a union of three cosets of $\mathcal{H}_n$. In the following theorem, we show that a suitable collection of such codes is an s-tolerant identifying collection — even with smallest possible number of codes. By Theorem 9, it is immediate that no s-tolerant identifying collection exists if $s \geq n-1$. Hence, in the theorem, we may assume that $1 \leq s \leq n-2$.

Theorem 15. *Let r and s be integers such that $r \geq 2$, $1 \leq s \leq n-2$ and $n = 2^r - 1$. Now the following statements hold:*

(i) There exists an s-tolerant identifying collection in $\mathbb{F}^n$ with $C(n+1, n-2, s)$ optimal *self-identifying codes.*

(ii) Any s-tolerant identifying collection has at least $C(n+1, n-2, s)$ minimal self-identifying codes.

Proof. Let $\mathcal{H}_n$ be a binary Hamming code of length $n = 2^r - 1$. Then all the cosets of $\mathcal{H}_n$ are $\mathcal{H}_n$ itself and $e_i + \mathcal{H}_n$, where $i = 1, 2, \dots n$, and they form a partition of $\mathbb{F}^n$. In the proof, we use the notation $[n+1]$ for the set $\{1, 2, \dots, n+1\}$.

(i) Let $\mathcal{B}$ be a minimum $(n+1, n-2, s)$ covering design of $[n+1]$. In other words, $\mathcal{B}$ is a collection of $C(n+1, n-2, s)$ $(n-2)$-blocks of $[n+1]$ such that every s-subset of $[n+1]$ is contained in at least one block. From each block, we construct a self-identifying code as follows. Assuming B is a block of $\mathcal{B}$, consider the set $[n+1] \setminus B$ containing three elements. Each element of $[n+1] \setminus B$ corresponds to a coset of $\mathcal{H}_n$ as follows: $n+1$ corresponds to $\mathcal{H}_n$ and each other element i corresponds to $e_i + \mathcal{H}_n$. Then the union of these three cosets forms an optimal self-identifying code in $\mathbb{F}^n$ by the observation above. Denote by $\mathcal{L}$ the collection of self-identifying codes obtained in the previous manner. In what follows, we show that for any $S \subseteq \mathbb{F}^n$ with $|S| \leq s$ there always exists $C \in \mathcal{L}$ such that $S \cap C = \emptyset$; hence, concluding that $\mathcal{L}$ is an s-tolerant identifying collection.

Let S be a subset of $\mathbb{F}^n$ with at most s elements. Observe that each element of S belongs to some coset of $\mathcal{H}_n$. Therefore, S is included in a union of at most s cosets. By the previous correspondence of a coset and an integer of $[n+1]$, these cosets correspond to a set $S' \subseteq [n+1]$ such that $|S'| \leq s$. Hence, as $\mathcal{B}$ is an $(n+1, n-2, s)$ covering design, there exists $B \in \mathcal{B}$ such that $S' \subseteq B$. Therefore, the self-identifying code $C \in \mathcal{L}$ obtained from the block B is such that $C \cap S = \emptyset$. Thus, in conclusion, $\mathcal{L}$ is an s-tolerant identifying collection with $C(n+1, n-2, s)$ codes.

(ii) Assume then that $\mathcal{L}$ is an s-tolerant identifying collection. Recall first that $N[\mathbf{0}] = \{\mathbf{0}, e_1, e_2, \dots, e_n\}$. Observe then that $|N[\mathbf{0}] \setminus C| \leq n-2$ for each $C \in \mathcal{L}$ since C is a self-identifying code and, therefore, $|C \cap N[\mathbf{0}]| \geq 3$. Let S be a subset of $N[\mathbf{0}]$ with exactly s words. Then there exists a self-identifying code $C \in \mathcal{L}$ such that $S \subseteq N[\mathbf{0}] \setminus C$. Thus, as C goes through all the self-identifying codes in the collection $\mathcal{L}$, the sets $N[\mathbf{0}] \setminus C$ form a collection $\mathcal{B}'$ such that each $S \subseteq N[\mathbf{0}]$ with s elements is contained in one of the sets in $\mathcal{B}'$. Hence, $\mathcal{B}'$ is almost an $(n+1, n-2, s)$ covering design with the exception that some of the sets/blocks might have less than $n-2$ elements. However, this can be overcome by adding some additional dummy words into the sets/blocks with less than $n-2$ elements. Thus, we have constructed an $(n+1, n-2, s)$ covering design with $|\mathcal{L}|$ blocks from the s-tolerant identifying collection $\mathcal{L}$. Therefore, the claim immediately follows. □

In the following remark, we discuss the benefits of not using only disjoint self-identifying codes.

Remark 16. Recall that previously, in [14], collections of *disjoint* identifying codes have been studied. In our model, we do not restrict ourselves to collections of disjoint codes. This has the benefit that we can handle greater values s of malfunctioning sensors. Indeed, if we consider binary hypercubes with $n = 2^r - 1$ and $r \geq 2$, then using disjoint collections of self-identifying codes the maximum number of malfunctioning sensors that can be handled is $s \leq \lfloor (n+1)/3 \rfloor$. By Theorem 15, in our model, we can handle up to $s \leq n-2$ malfunctioning sensors.

As described in Section 3, tolerance against malfunctioning sensors can be handled with collections of codes or larger codes satisfying (2). In the following remark, we discuss the benefits of collections of self-identifying codes over the single larger codes.

Remark 17. Previously, so called robust identifying codes have been designed to handle possible malfunctioning sensors (see [7, 9, 10, 16]). As stated earlier in (2), a single identifying code C is robust against s malfunctions if for all distinct $x, y \in V$ we have $|I(x)| \geq s+1$ and $|(I(x) \setminus I(y)) \cup (I(y) \setminus I(x))| \geq s+1$. This implies that, for example in $\mathbb{F}^n$, we have the following lower bound for identifying codes robust against $s = n-2$ malfunctions:

$$|C| \geq \frac{n-1}{n+1} \cdot 2^n = \left(1 - \frac{2}{n+1}\right) 2^n$$

This further means that almost all the sensors have to be on; hence, implying high signal interference and energy consumption. However, in our model, if $n = 2^r - 1$, the size of (optimal) self-identifying code is only $\gamma^+(\mathbb{F}^n) = 3 \cdot 2^n/(n+1)$. Thus, albeit we have to be ready to switch the set of operating sensors if errors occur, the number of sensor simultaneously turned on is much smaller.

Let us then assume that $n = 3 \cdot 2^r - 1$, where r is a positive integer. By [1], there exists a *linear* perfect 3-fold 1-covering C in $\mathbb{F}^n$, i.e., an optimal linear self-identifying code in $\mathbb{F}^n$. Hence, the cosets of C form a partition of $\mathbb{F}^n$. Based on these disjoint 3-fold 1-coverings, the following simple theorem is obtained.

Theorem 18. *Let r and s be positive integers such that $n = 3 \cdot 2^r - 1$ and $s \leq 2^r - 1$. Then there exists an s-tolerant identifying collection in $\mathbb{F}^n$ with $s+1$ codes. Conversely, any s-tolerant identifying collection has at least $s+1$ minimal self-identifying codes.*

Proof. Let C be a linear perfect 3-fold 1-covering in $\mathbb{F}^n$. The number of cosets of C is $(n+1)/3 = 2^r$. Hence, there exist 2^r disjoint self-identifying codes in $\mathbb{F}^n$. Since $s \leq 2^r - 1$, an s-tolerant identifying collection $\mathcal{L}$ can be formed by taking $s+1$ (disjoint) cosets of C. Indeed, for any $S \subseteq \mathbb{F}^n$ with $|S| \leq s$, there exists a self-identifying code $C' \in \mathcal{L}$ such that $C' \cap S = \emptyset$. Moreover, it is clear that no s-tolerant identifying collection with less than $s+1$ codes exists. Thus, the claim follows. $\square$

Above, we have presented constructions for s-tolerant identifying collections for some specific lengths n. In what follows, we present a couple of results by which we can construct new s-tolerant identifying collections from known ones. The construction in the first theorem is based on a direct sum (see, for example, [1]). Recall that the direct sum of codes in $\mathbb{F}^n$ is defined as follows: if n_1 and n_2 are positive integers, and $C_1 \subseteq \mathbb{F}^{n_1}$ and $C_2 \subseteq \mathbb{F}^{n_2}$ are codes, then the direct sum of C_1 and C_2 is defined as

$$C_1 \oplus C_2 = \{(x, y) \mid x \in C_1, y \in C_2\}.$$

Theorem 19. *Let $n \geq 3$ be an integer and $\mathcal{L}$ be an s-tolerant identifying collection in $\mathbb{F}^n$. Then there also exists an s-tolerant identifying collection in $\mathbb{F}^{n+1}$ of size at most $|\mathcal{L}|$.*

Proof. Let C be a self-identifying code. By Theorem 14, the code C is a 3-fold 1-covering. Therefore, it is clear that

$$C' = C \oplus \mathbb{F} = \{(x, y) \mid x \in C, y \in \mathbb{F}\} \subseteq \mathbb{F}^{n+1}$$

is also 3-fold 1-covering. Hence, by Theorem 14, the code C' is self-identifying. Let then $\mathcal{L} = \{C_1, C_2, \ldots, C_{|\mathcal{L}|}\}$ be an s-tolerant identifying collection in $\mathbb{F}^n$, where all C_i are minimal self-identifying codes. Construct then a collection $\mathcal{L}'$ of self-identifying codes in $\mathbb{F}^{n+1}$ as follows:

$$\mathcal{L}' = \{C_1 \oplus \mathbb{F}, C_2 \oplus \mathbb{F}, \ldots, C_{|\mathcal{L}|} \oplus \mathbb{F}\}.$$

Notice that some of the self-identifying codes in $\mathcal{L}'$ might not be minimal. However, we can get a *minimal* self-identifying code from any such code by deleting vertices as long as the code stays self-identifying. Thus, without loss of generality, we may assume that $\mathcal{L}'$ is a collection of minimal self-identifying codes. In what follows, we show that $\mathcal{L}'$ also meets the condition (3) of the definition.

Let S' be a subset of $\mathbb{F}^{n+1}$ with k words, where $k \leq s$. Clearly, we can write $S' = \{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}$, where $x_i \in \mathbb{F}^n$ and $y_i \in \mathbb{F}$. Considering the set $S = \{x_1, x_2, \ldots, x_k\} \subseteq \mathbb{F}^n$ (having at most k different elements), we immediately obtain that there exists a self-identifying code C in $\mathcal{L}$ such that $C \cap S = \emptyset$. Therefore, there also exists a self-identifying code $C' = C \oplus \mathbb{F}$ in $\mathcal{L}'$ such that $C' \cap S' = \emptyset$. Thus, $\mathcal{L}'$ is an s-tolerant identifying collection in $\mathbb{F}^{n+1}$ with $|\mathcal{L}|$ elements. $\square$

In the following example, we illustrate the direct sum construction of the previous theorem and the construction of Theorem 18.

Example 20. Let us construct a 3-tolerant identifying collection in $\mathbb{F}^{11}$. By [2, p. 369], the size of a minimum (8,5,3) covering design is 8, i.e., $C(8,5,3) = 8$. Hence, by Theorem 15, we get a 3-tolerant identifying collection of size 8. By applying Theorem 19 (four times), we obtain a 3-tolerant identifying collection of size eight in $\mathbb{F}^{11}$. On the other hand, using Theorem 18 with $r = 2$, we get a 3-tolerant identifying collection of size four and no smaller collection exists. Notice that the collection with 4 codes also attains the lower bound $C(2^{11}, 2^{11}-3\cdot 2^{11}/12, 3) = C(2048, 1536, 3) \geq 4$ of Theorem 10. Indeed, a $(2048, 1536, 3)$ covering design has more than 3 blocks as otherwise a 3-subset not belonging to any of these block could be formed by taking an element for each block such that the element does not belong to the block (a contradiction).

The second construction is based on the $(\pi(u), u, u + v)$-construction (used, for example, in [1]). For further considerations, let $\pi : \mathbb{F}^n \to \mathbb{F}$ be a mapping defined as follows:

$$\pi(u) = \begin{cases} 0, & \text{if } w(u) \text{ is even;} \\ 1, & \text{if } w(u) \text{ is odd.} \end{cases}$$

By [1, Theorem 14.4.3], it is known that if C is a μ-fold 1-covering in $\mathbb{F}^n$, then

$$C' = \{(\pi(u), u, u + v) \mid u \in \mathbb{F}^n, v \in C\}$$

is a μ-fold 1-covering in $\mathbb{F}^{2n+1}$ with $2^n|C|$ words. In the following theorem, we present a refinement of this result reformulated in the terminology of self-identifying codes.

Theorem 21. *Let $n \geq 2$ be an integer. If C is a minimal self-identifying code in $\mathbb{F}^n$, then*

$$C' = \{(\pi(u), u, u + v) \mid u \in \mathbb{F}^n, v \in C\} \tag{5}$$

and

$$C'' = \{(\pi(u) + 1, u, u + v) \mid u \in \mathbb{F}^n, v \in C\} \tag{6}$$

are both minimal self-identifying codes in $\mathbb{F}^{2n+1}$ with $2^n|C|$ words.

Proof. Let C be a minimal self-identifying code in $\mathbb{F}^n$. By Theorem 14, C is a 3-fold 1-covering in $\mathbb{F}^n$. Therefore, by the result above, C' is a 3-fold 1-covering in $\mathbb{F}^{2n+1}$ with $2^n|C|$ codewords. Hence, by Theorem 14, C' is a self-identifying code in $\mathbb{F}^{2n+1}$. Thus, it remains to be shown that C' is also minimal.

Let $c' = (\pi(u), u, u + c_1)$ be a codeword of C', where $c_1 \in C$. Since C is a minimal self-identifying code, there exists a word $w \in N[c_1]$ such that $I(C; w) = \{c_1, c_2, c_3\}$ for some $c_2, c_3 \in C$. Considering the word $w' = (\pi(u), u, u+w)$, we obtain that for $i = 1, 2, \ldots, n+1$ the word $w' + e_i \notin C'$ since the parity of the first bit of the word is not suitable for a codeword of C'. Therefore, as $I(C; w) = \{c_1, c_2, c_3\}$, we have $I(C'; w') = \{(\pi(u), u, u + c_1), (\pi(u), u, u + c_2), (\pi(u), u, u + c_3)\}$. Hence, if c' is removed from the code C', then the code is not self-identifying anymore. Thus, the code C' is a minimal self-identifying code.

The proof for the code C'' is similar as $C'' = e_1 + C'$. Thus, the claim follows. $\square$

In the following theorem, based on the previous result, we present a method for constructing a new $(2s+1)$-tolerant identifying collection in $\mathbb{F}^{2n+1}$ from a known s-tolerant identifying collection in $\mathbb{F}^n$. Compare this result to the direct sum construction of Theorem 19, where the number of malfunctioning sensors that can be coped with is not increased.

Theorem 22. *Let n and s be integers such that $n \geq 2$ and $s \geq 1$. Let $\mathcal{L}$ be an s-tolerant identifying collection in $\mathbb{F}^n$. Then form a collection $\mathcal{L}'$ of self-identifying codes in $\mathbb{F}^{2n+1}$ as follows: for all $C \in \mathcal{L}$ add the self-identifying codes $C' \subseteq \mathbb{F}^{2n+1}$ of (5) and $C'' \subseteq \mathbb{F}^{2n+1}$ of (6) to the collection $\mathcal{L}'$. The obtained collection $\mathcal{L}'$ is a $(2s+1)$-tolerant identifying collection in $\mathbb{F}^{2n+1}$ with $2|\mathcal{L}|$ codes.*

Proof. Notice first that all the codes $C' \subseteq \mathbb{F}^{2n+1}$ and $C'' \subseteq \mathbb{F}^{2n+1}$ of $\mathcal{L}'$ are minimal self-identifying codes by Theorem 21. Observe also that each word of $\mathbb{F}^{2n+1}$ can be written either as $(\pi(u_1), u_1, u_2)$

or $(\pi(u_1)+1, u_1, u_2)$, where u_1 and u_2 belong to $\mathbb{F}^n$. In other words, the Hamming space $\mathbb{F}^{2n+1}$ can be partitioned into two separate subsets based on whether the first bit is $\pi(u_1)$ or $\pi(u_1)+1$.

Let then S be a subset of $\mathbb{F}^{2n+1}$ with at most $2s+1$ words. By the previous observation, there are at most s words of the form $(\pi(u_1), u_1, u_2)$ or $(\pi(u_1)+1, u_1, u_2)$ in S. Without loss of generality, we may assume that there are at most s words of the form $(\pi(u_1), u_1, u_2)$. In what follows, we show that there exists a code $C' \in \mathcal{L}'$ of type (5) such that $S \cap C' = \emptyset$. Observe first that all the codewords of C' are of the form $(\pi(u_1), u_1, u_2)$ and, therefore, we may concentrate only on the at most s words of S of the form $(\pi(u_1), u_1, u_2)$, denoted by S'. By the construction of C', we notice that $(\pi(u_1), u_1, u_2) \in C'$ if and only if $u_2 = u_1 + v$, where $v \in C$. Furthermore, this is equivalent to saying that $u_1 + u_2 \in C$. Considering the at most s words of the form $u_1 + u_2$, we obtain that there exists a code $C \in \mathcal{L}$ avoiding all such words as $\mathcal{L}$ is an s-tolerant identifying collection. Therefore, the self-identifying code C' in $\mathbb{F}^{2n+1}$ corresponding to C is such that $C' \cap S' = \emptyset$ and, hence, $C' \cap S = \emptyset$. Thus, as $|\mathcal{L}'| = 2|\mathcal{L}|$, the claim immediately follows. □

7 Conclusion

In this paper, we considered a location problem in sensor networks when sensors can be malfunctioning. We noticed that malfunctioning poses challenges to usual identifying codes. One way to address this problem is to use larger codes, that is, robust identifying codes. However, as pointed out by Laifenfeld and Trachtenberg [14], one can also use collections of disjoint identifying codes to address the problem. We introduced the tolerant identifying collections which are an improvement on the collections of disjoint identifying codes. We studied the smallest tolerant identifying collections in the rook's graphs and the binary Hamming spaces. In light of Section 4, it would be interesting to find out the smallest tolerant identifying collections also in other families of graphs.

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