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A new subclass of starlike functions

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Abstract: Motivated by the Rønning-starlike class [Proceedings of the American Mathematical Society 1993; 118: 189-196], we introduce the new class $\mathcal{S}_c^*$ that includes analytic and normalized functions f, which satisfy the inequality

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{f(z)}{z}-1\right| \quad (|z|<1).$$

In this paper, we first give some examples that belong to the class $\mathcal{S}_c^*$. Also, we show that if $f \in \mathcal{S}_c^*$ then $\mathrm{Re}\{f(z)/z\} > 1/2$ in $|z|<1$ (Marx–Strohhäcker problem). Afterwards, upper and lower bounds for $|f(z)|$ are obtained where f belongs to the class $\mathcal{S}_c^*$. We also prove that if $f \in \mathcal{S}_c^*$ and $\alpha \in [0,1)$, then f is starlike of order α in the disc $|z| < (1-\alpha)/(2-\alpha)$. At the end, we estimate logarithmic coefficients, the initial coefficients, and the Fekete–Szegö problem for functions $f \in \mathcal{S}_c^*$.

Key words: Starlike, subordination, Marx–Strohhäcker problem, logarithmic coefficients, Fekete–Szegö problem

1. Introduction

Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc on the complex plane $\mathbb{C}$ and $\mathcal{H}(\Delta)$ be the class of functions f that are analytic in Δ. Also let $\mathcal{A} \subset \mathcal{H}(\Delta)$ be the class of all functions f that satisfy the standard normalization $f(0) = 0 = f'(0) - 1$. It is known that if $f \in \mathcal{A}$, then it has the following Taylor–Maclaurin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta). \tag{1.1}$$

The set of all univalent functions f in Δ is denoted by $\mathcal{U}$. If f and g belong to class $\mathcal{H}(\Delta)$, then we say that a function f is subordinate to g, written as

$$f(z) \prec g(z) \quad \text{or} \quad f \prec g,$$

if there exists a Schwarz function $w : \Delta \to \Delta$ with the following properties:

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \Delta),$$

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such that $f(z) = g(w(z))$ for all $z \in \Delta$. Notice that if $g \in \mathcal{U}$, then we have the following geometric equivalence: relation

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

Let $\alpha \in [0, 1)$. A function $f \in \mathcal{A}$ is called starlike of order α if and only if f satisfies the following inequality:

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \Delta).$$

The familiar class of the starlike functions of order α is denoted by $\mathcal{S}^*(\alpha)$. An extremal function for the class $\mathcal{S}^*(\alpha)$, namely the Koebe function of order α, is defined by:

$$k_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \quad (0 \le \alpha < 1). \tag{1.2}$$

We denote by $\mathcal{S}^* \equiv \mathcal{S}^*(0)$ the class of the starlike functions. For each $\alpha \in [0, 1)$ we have $\mathcal{S}^*(\alpha) \subset \mathcal{U}$. Also, we say that a function $f \in \mathcal{A}$ is convex of order α if and only if $zf'(z) \in \mathcal{S}^*(\alpha)$. We denote by $\mathcal{K}(\alpha)$ the class of the convex functions of order α in Δ. Also $\mathcal{K}(\alpha) \subset \mathcal{U}$ where $0 \le \alpha < 1$. The class of the convex functions in Δ is denoted by $\mathcal{K} \equiv \mathcal{K}(0)$. Analytically, $f \in \mathcal{K}(\alpha)$ if and only if:

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in \Delta).$$

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were introduced by Robertson [8]. Next, we consider the class $\mathcal{S}^*_\alpha \subset \mathcal{S}^*(\alpha)$ as follows:

$$\mathcal{S}^*_\alpha := \left\{f \in \mathcal{A} : \left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \alpha\right\}.$$

Let $\mathcal{R}(\alpha)$ denote the class of functions $f \in \mathcal{A}$ satisfying the following inequality:

$$\mathrm{Re}\left\{\frac{f(z)}{z}\right\} > \alpha \quad (z \in \Delta, 0 \le \alpha < 1).$$

It is know that $\mathcal{S}^*(1/2) \subset \mathcal{R}(1/2)$ for all $z \in \Delta$ and that the constant $1/2$ is the best possible; see [2, p. 73].

Rønning (see [10]) introduced a certain subclass of the starlike functions, denoted by S_p, consisting of all functions $f \in \mathcal{A}$ with the following property:

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \ge \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in \Delta). \tag{1.3}$$

Since $\mathrm{Re}\{\xi\} = |\xi - 1|$ describes a parabola with vertex at $\xi = 1/2$ and $(1/2, \infty)$ as symmetry axis, the functions satisfying condition (1.3) are associated with a parabolic region. Also, $S_p \subset \mathcal{S}^*(1/2)$.

Motivated by the class S_p, we introduce a new subclass of the starlike functions as follows:

Definition 1.1 *Let $f \in \mathcal{A}$. Then we say that a function f belongs to the class $\mathcal{S}^*_c$ if it satisfies the following condition:*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \ge \left|\frac{f(z)}{z} - 1\right| \quad (z \in \Delta). \tag{1.4}$$

We observe that the class $\mathcal{S}^*_c$ is a subclass of the starlike functions. It is easy to see that the identity function satisfies inequality (1.4) and thus $\mathcal{S}^*_c \neq \emptyset$. In Section 2 we give more examples that satisfy inequality (1.4).

2. Examples

First, consider the function f_γ as follows:

$$f_\gamma(z) = z + \gamma z^2 \quad (z \in \Delta). \tag{2.1}$$

We are looking for a $\gamma \in \mathbb{C}$ such that f_γ belong to the class $\mathcal{S}_c^*$. With a little calculation, (2.1) implies that

$$\frac{zf'_\gamma(z)}{f_\gamma(z)} = 1 + \frac{\gamma z}{1+\gamma z} \quad \text{and} \quad \frac{f_\gamma(z)}{z} - 1 = \gamma z \quad (z \in \Delta).$$

Now let $\gamma z = re^{i\theta}$ where $\theta \in [-\pi, \pi]$. Then

$$\operatorname{Re}\left\{\frac{zf'_\gamma(z)}{f_\gamma(z)}\right\} = \operatorname{Re}\left\{1 + \frac{\gamma z}{1+\gamma z}\right\} = 1 + \operatorname{Re}\left\{\frac{re^{i\theta}}{1+re^{i\theta}}\right\} = 1 + \frac{r(r+\cos\theta)}{1+2r\cos\theta+r^2}$$

and

$$\left|\frac{f_\gamma(z)}{z} - 1\right| = |\gamma z| = |re^{i\theta}| = r.$$

Therefore, we are looking for r_0 such that

$$h(x,r) := 1 + \frac{r(r+x)}{1+2rx+r^2} - r \geq 0 \quad (0 \leq r < r_0, \quad -1 \leq x \leq 1, \quad x := \cos\theta).$$

Since h is an increasing function with respect to $x \in [-1,1]$, we have

$$\begin{aligned} h(-1,r) = 1 + \frac{r(r-1)}{1-2r+r^2} - r \geq 0 \\ \Leftrightarrow \frac{1-3r+r^2}{1-r} \geq 0 \\ \Leftrightarrow r \in (-\infty, (3-\sqrt{5})/2] \cup [(3+\sqrt{5})/2, \infty). \end{aligned}$$

Consequently if $|\gamma| \leq (3-\sqrt{5})/2 = 0.38\ldots$, then the function (2.1) belongs to the class $\mathcal{S}_c^*$.

Next, we consider the function $\mathfrak{f}_\beta$ as follows:

$$\mathfrak{f}_\beta(z) = \frac{z}{1-\beta z} \quad (z \in \Delta). \tag{2.2}$$

We will look for some β such that $\mathfrak{f}_\beta$ belongs to the class $\mathcal{S}_c^*$. A simple calculation gives us

$$\frac{z\mathfrak{f}'_\beta(z)}{\mathfrak{f}_\beta(z)} = \frac{1}{1-\beta z} \quad \text{and} \quad \frac{\mathfrak{f}_\beta(z)}{z} - 1 = \frac{\beta z}{1-\beta z} \quad (z \in \Delta).$$

If we let $\beta z = re^{i\theta}$, where $0 \leq r < 1$ and $\theta \in [-\pi, \pi]$, then

$$\operatorname{Re}\left\{\frac{z\mathfrak{f}'_\beta(z)}{\mathfrak{f}_\beta(z)}\right\} = \operatorname{Re}\left\{\frac{1}{1-\beta z}\right\} = \frac{1-r\cos\theta}{1-2r\cos\theta+r^2}$$

and

$$\left|\frac{\mathfrak{f}_\beta(z)}{z}-1\right| = \left|\frac{\beta z}{1-\beta z}\right| = \frac{r}{\sqrt{1-2r\cos\theta+r^2}}.$$

Therefore, we are looking for r_0, such that

$$g(x,r) := \frac{1-rx}{r\sqrt{1-2rx+r^2}} \geq 1 \quad (0 \leq r < r_0, \quad -1 \leq x \leq 1, \quad x := \cos\theta).$$

It is easy to check that g attains its minimum with respect to $x \in [-1,1]$ at $x = r$, so we are looking for r_0 such that

$$g(r) := \frac{1-r^2}{r\sqrt{1-r^2}} \geq 1 \quad (0 \leq r < r_0),$$

and this gives $r_0 = \sqrt{2}/2$. Therefore, if $|\beta| \leq \sqrt{2}/2 = 0.707\ldots$ exactly, then (2.2) belongs to the class $\mathcal{S}_c^*$.

The following lemma will be useful.

Lemma 2.1 (*See [6]*) *Let $p(z)$ be an analytic function in Δ of the form*

$$p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n \quad (c_m \neq 0),$$

with $p(z) \neq 0$ in Δ. If there exists a point $z_0 \in \Delta$ such that

$$|\arg\{p(z)\}| < \frac{\pi\varphi}{2} \quad for \quad |z| < |z_0|$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi\varphi}{2}$$

for some $\varphi > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z)} = il\varphi,$$

where

$$l \geq \frac{m}{2}\left(a + \frac{1}{a}\right) \geq m \quad when \quad \arg\{p(z_0)\} = \frac{\pi\varphi}{2} \tag{2.3}$$

and

$$l \leq -\frac{m}{2}\left(a + \frac{1}{a}\right) \leq -m \quad when \quad \arg\{p(z_0)\} = -\frac{\pi\varphi}{2}, \tag{2.4}$$

where

$$\{p(z_0)\}^{1/\varphi} = \pm ia \quad and \quad a > 0.$$

In the next section, we shall investigate some geometric properties of the class $\mathcal{S}_c^*$.

3. Main results

We begin this section with the following.

Theorem 3.1 *Let the function $f \in \mathcal{A}$ belong to the class $\mathcal{S}_c^*$. Then*

$$\frac{f(z)}{z} \prec \varphi(z), \tag{3.1}$$

where

$$\varphi(z) := \frac{1}{1-z} \quad (z \in \Delta). \tag{3.2}$$

Proof Let $f \in \mathcal{A}$ be in the class $\mathcal{S}_c^*$. Define

$$p(z) := \frac{f(z)}{z} \quad (z \in \Delta). \tag{3.3}$$

Therefore p is analytic in Δ and $p(0) = 1$. From (3.3), we obtain

$$1 + \frac{zp'(z)}{p(z)} = \frac{zf'(z)}{f(z)} \quad (z \in \Delta). \tag{3.4}$$

Since $f \in \mathcal{S}_c^*$, by relation (3.4) and by definition of $\mathcal{S}_c^*$, we have

$$\begin{aligned} \mathrm{Re}\left\{1 + \frac{zp'(z)}{p(z)}\right\} &= \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \\ &\geq \left|\frac{f(z)}{z} - 1\right| = |p(z) - 1| \\ &\geq \mathrm{Re}\{1 - p(z)\}. \end{aligned}$$

The last inequality implies that

$$\mathrm{Re}\left\{p(z) + \frac{zp'(z)}{p(z)}\right\} \geq 0 \quad (z \in \Delta). \tag{3.5}$$

By making use of the subordination principle, inequality (3.5) results in

$$p(z) + \frac{zp'(z)}{p(z)} \prec \frac{1+z}{1-z}. \tag{3.6}$$

If we apply Theorem 3.3d, [5, p. 109], then from (3.6) we conclude that

$$p(z) \prec q(z) \prec \frac{1+z}{1-z},$$

where $q(z)$ is the univalent solution of the differential equation

$$q(z) + \frac{zq'(z)}{q(z)} = \frac{1+z}{1-z} \quad (z \in \Delta). \tag{3.7}$$

Also $q(z)$ is the best dominant of (3.6). A simple calculation shows that the solution of the differential equation (3.7) is equal to

$$q(z) = \left(\int_0^1 \left(\frac{1-z}{1-tz} \right)^2 dt \right)^{-1} = \frac{1}{1-z} \quad (z \in \Delta),$$

concluding the proof. Here, the proof ends. □

Marx and Strohhäcker (see [4, 12]) proved that if $f \in \mathcal{A}$, then the following implication is sharp:

$$\mathrm{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \Rightarrow \mathrm{Re}\left\{ \frac{f(z)}{z} \right\} > \frac{1}{2} \quad (z \in \Delta).$$

The same results of this kind are known as the Marx–Strohhäcker problem and they have many applications in complex dynamical systems; see [11, 13]. Following this, we obtain the Marx–Strohhäcker problem for the class $\mathcal{S}_c^*$.

Theorem 3.2 *If f given by (1.1) belongs to class $\mathcal{S}_c^*$, then*

$$\mathrm{Re}\left\{ \frac{f(z)}{z} \right\} > \frac{1}{2} \quad (z \in \Delta).$$

This means that $\mathcal{S}_c^* \subset \mathcal{R}(1/2)$.

Proof By (3.1), using the definition of subordination and from

$$\mathrm{Re}\{\varphi(z)\} = \mathrm{Re}\left\{ \frac{1}{1-z} \right\} > \frac{1}{2} \quad (z \in \Delta),$$

we get the desired result. □

Open problem. Find the largest α such that $f \in \mathcal{S}_c^*$ implies that

$$\mathrm{Re}\left\{ \frac{f(z)}{z} \right\} > \alpha \quad (z \in \Delta).$$

From Theorem 3.2 we see that $\alpha \geq 1/2$. Furthermore, function (2.2) shows that this α cannot be greater than $2 - \sqrt{2} = 0.58\ldots$.

The following theorem, called the growth theorem, gives upper and lower bounds for $|f(z)|$, where f belongs to the class $\mathcal{S}_c^*$.

Theorem 3.3 *Let $f \in \mathcal{S}_c^*$. Then we have*

$$r\varphi(-r) \leq |f(z)| \leq r\varphi(r) \quad (|z| = r < 1), \tag{3.8}$$

where $\varphi(z)$ is defined in (3.2).

Proof Let φ be given by (3.2). If $f \in \mathcal{S}_c^*$, then by Theorem 3.1 we have

$$\frac{f(z)}{z} \prec \varphi(z).$$

The last subordination relation implies that

$$\frac{f(z)}{z} \in \varphi(|z| \le r) \tag{3.9}$$

for each $r \in (0,1)$ and $|z| \le r$. Since

$$\mathrm{Re}\left\{1 + \frac{z\varphi''(z)}{\varphi'(z)}\right\} = \mathrm{Re}\left\{1 + 2\frac{z}{1-z}\right\} > 0 \quad (z \in \Delta),$$

φ is convex univalent in Δ and for each $r \in (0,1)$ the set $\varphi(|z| \le r)$ is symmetric with respect to the real axis. This leads us to the following two-sided inequality:

$$\varphi(-r) \le |\varphi(z)| \le \varphi(r), \tag{3.10}$$

where $r \in (0,1)$ and $|z| \le r$. The assertion now is obtained from (3.9) and (3.10). This is the end of the proof. □

Theorem 3.4 *Let $f \in \mathcal{S}_c^*$ and $\alpha \in [0,1)$. Then*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \alpha \quad (|z| < (1-\alpha)/(2-\alpha)).$$

Proof Let $f \in \mathcal{S}_c^*$. Then by Theorem 3.1 we have

$$\frac{f(z)}{z} \prec \frac{1}{1-z}.$$

By definition of subordination there exists a Schwarz function w such that

$$\frac{f(z)}{z} = \frac{1}{1-w(z)} \quad (z \in \Delta).$$

Clearly w is analytic in Δ with $w(0) = 0$ and

$$\log\left\{\frac{f(z)}{z}\right\} = \log\left\{\frac{1}{1-w(z)}\right\} \quad (z \in \Delta). \tag{3.11}$$

We find from the last equation, (3.11), that

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zw'(z)}{1-w(z)} \quad (z \in \Delta). \tag{3.12}$$

It is well known that $|w(z)| \le |z|$ (cf. [2]), and also, by the Schwarz–Pick lemma, for a Schwarz function w the following inequality holds:

$$|w'(z)| \le \frac{1-|w(z)|^2}{1-|z|^2} \quad (z \in \Delta). \tag{3.13}$$

Thus, by $|w(z)| \le |z|$ and (3.13), the relation (3.12) implies that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{zw'(z)}{1-w(z)}\right| \le \frac{|z||w'(z)|}{1-|w(z)|} \le \frac{|z|}{1-|z|} < 1-\alpha,$$

provided that $|z| < \frac{1-\alpha}{2-\alpha}$. This completes the proof. □

In the sequel, the following lemma (see [3]) (popularly known as Jack's lemma) will be required.

Lemma 3.5 *Let the (nonconstant) function* $\omega(z)$ *be analytic in* Δ *with* $\omega(0) = 0$. *If* $|\omega(z)|$ *attains its maximum value on the circle* $|z| = r < 1$ *at a point* $z_0 \in \Delta$, *then*

$$z_0\omega'(z_0) = k\omega(z_0),$$

where k *is a real number and* $k \geq 1$.

Theorem 3.6 *Let the function* $f \in \mathcal{A}$ *satisfy the inequality*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \quad (z \in \Delta). \tag{3.14}$$

Then $f \notin \mathcal{S}_c^*$. *This means that* $\mathcal{S}^*(1/2) \not\subset \mathcal{S}_c^*$.

Proof If the function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}_c^*$, then by the proof of Theorem 3.4 we have

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zw'(z)}{1-w(z)} \quad (z \in \Delta). \tag{3.15}$$

Suppose now that there exists a point $z_0 \in \Delta$ such that $|w(z_0)| = 1$ and $|w(z)| < 1$ when $|z| < |z_0|$. If we apply Lemma 3.5, then we have

$$z_0w'(z_0) = kw(z_0) \quad (w(z_0) = e^{it}; t \in \mathbb{R}; k \geq 1). \tag{3.16}$$

Therefore, we find from (3.15) and (3.16) that

$$\operatorname{Re}\left\{\frac{z_0f'(z_0)}{f(z_0)}\right\} = \operatorname{Re}\left\{1 + \frac{z_0w'(z_0)}{1-w(z_0)}\right\} = 1 + \operatorname{Re}\left\{\frac{kw(z_0)}{1-w(z_0)}\right\} = 1 + \operatorname{Re}\left\{\frac{ke^{it}}{1-e^{it}}\right\} = 1 - \frac{k}{2} \leq \frac{1}{2},$$

which contradicts the hypothesis (3.14). This completes the proof. □

Actually, there exists a function $f \in \mathcal{A}$, a starlike function of order $1/2$ such that $f \notin \mathcal{S}_c^*$. The functions (2.2) are starlike of order $1/2$ for every β, $|\beta| \leq 1$, while they are in $\mathcal{S}_c^*$ only for $|\beta| \leq \sqrt{2}/2$.

Remark 3.7 *Finding some* $\alpha \in [0, 1)$ *such that* $\mathcal{S}_c^* \subset \mathcal{S}^*(\alpha)$ *is an open problem. In the sequel, we will answer this problem partially. Indeed, we conjecture that* $\mathcal{S}_c^* \subset \mathcal{S}^*(\alpha)$ *when* $\alpha \in (1/2, 1)$. *For this purpose, let* $\gamma = 0.2$ *in* (2.1). *Then the function* $f_{0.2}(z) = z + 0.2z^2$ *belongs to the class* $\mathcal{S}_c^*$. *A simple calculation gives us*

$$\operatorname{Re}\left\{\frac{zf_{0.2}'(z)}{f_{0.2}(z)}\right\} = \operatorname{Re}\left\{\frac{1+0.4z}{1+0.2z}\right\} > \frac{3}{4} \quad (z \in \Delta).$$

Therefore, $f_{0.2}$ *is a starlike function of order* $3/4$. *Also, if we let* $\beta = 0.2$ *in* (2.2), *then the function* $\mathfrak{f}_{0.2}(z) = \frac{z}{1-0.2z}$ *belongs to the class* $\mathcal{S}_c^*$. *We have*

$$\operatorname{Re}\left\{\frac{z\mathfrak{f}_{0.2}'(z)}{\mathfrak{f}_{0.2}(z)}\right\} = \operatorname{Re}\left\{\frac{1}{1-0.2z}\right\} > 0.83 \quad (z \in \Delta).$$

This means that $\mathfrak{f}_{0.2} \in \mathcal{S}^*(0.83)$. *These examples show that* $\mathcal{S}_c^* \subset \mathcal{S}^*(\alpha)$ *where* $1/2 < \alpha < 1$. *On the other hand, we know that the function* k_α *is starlike of order* α $(0 \le \alpha < 1)$, *where* k_α *is defined in* (1.2). *A simple calculation of* (1.2) *gives that*

$$\frac{zk'_\alpha(z)}{k_\alpha(z)} = 1 + 2(1-\alpha)\frac{z}{1-z} \quad (z \in \Delta) \tag{3.17}$$

and

$$\left|\frac{k_\alpha(z)}{z} - 1\right| = \left|\frac{1}{(1-z)^{2(1-\alpha)}} - 1\right| \quad (z \in \Delta). \tag{3.18}$$

If k_α *belongs to the class* $\mathcal{S}_c^*$, *then from* (3.17), (3.18), *and the definition of* $\mathcal{S}_c^*$ *we have*

$$\operatorname{Re}\left\{1 + 2(1-\alpha)\frac{z}{1-z}\right\} \ge \left|\frac{1}{(1-z)^{2(1-\alpha)}} - 1\right| \quad (z \in \Delta). \tag{3.19}$$

If the last inequality holds for all $z \in \Delta$, *then it holds for* $|z| = 1$, *too. Also, for real* z *close to* 1, *we have LHS*$\to \alpha$, *while RHS* $\to \infty$. *This shows that there are no* $\alpha \ge 0$ *so that* $\mathcal{S}^*(\alpha) \subset \mathcal{S}_c^*$.

In order to estimate the logarithmic coefficients and because φ is univalent, we may rewrite Theorem 3.1 in the following form.

Theorem 3.8 *If the function* $f \in \mathcal{A}$ *belongs to the class* $\mathcal{S}_c^*$, *then*

$$\log\left\{\frac{f(z)}{z}\right\} \prec -\log\{1-z\}.$$

The logarithmic coefficients γ_n of $f \in \mathcal{A}$ are defined by

$$\log\left\{\frac{f(z)}{z}\right\} = \sum_{n=1}^{\infty} 2\gamma_n z^n \quad (z \in \Delta). \tag{3.20}$$

The sharp upper bounds for the modulus of logarithmic coefficients are known for functions in very few subclasses of $\mathcal{U}$. For functions in the class $\mathcal{S}^*$ we have the sharp inequality $|\gamma_n| \le 1/n$ where $n \ge 1$, but this is false for the full class $\mathcal{U}$, even in order of magnitude. Also, if $f \in \mathcal{S}^*(\alpha)$, then $|\gamma_n| \le (1-\alpha)/n$ where $0 \le \alpha < 1$ and $n \ge 1$. Since the estimate of the logarithmic coefficients is an important problem in the theory of univalent functions, we shall investigate this problem for the functions in the class $\mathcal{S}_c^*$.

The following lemma is due to Rogosinski [9, 2.3 Theorem X].

Lemma 3.9 *Let* $q(z) = \sum_{n=1}^{\infty} Q_n z^n$ *be analytic and univalent in* Δ *such that it maps* Δ *onto a convex domain. If* $p(z) = \sum_{n=1}^{\infty} P_n z^n$ *is analytic in* Δ *and satisfies the subordination* $p(z) \prec q(z)$, *then* $|P_n| \le |Q_1|$ *where* $n = 1, 2, \ldots$.

Theorem 3.10 *Let* $f \in \mathcal{A}$. *If* $f \in \mathcal{S}_c^*$ *and the coefficient of* $\log(f(z)/z)$ *is given by* (3.20), *then*

$$|\gamma_n| \le \frac{1}{2} \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots\}). \tag{3.21}$$

The result is sharp.

Proof Let the function $f \in \mathcal{A}$ belong to the class $\mathcal{S}_c^*$. Then, by Theorem 3.8, we have

$$\log\left\{\frac{f(z)}{z}\right\} \prec -\log\{1-z\}. \tag{3.22}$$

Replacing the Taylor–Maclaurin series on both sides of (3.22) gives

$$\sum_{n=1}^{\infty} 2\gamma_n z^n \prec \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

It is easily seen that the function $-\log\{1-z\}$ is convex univalent in Δ; therefore, by Lemma 3.9 we get the inequality (3.21). □

In the sequel, we estimate the initial coefficients of the function f of the form (1.1) belonging to the class $\mathcal{S}_c^*$. First, we recall the following lemma.

Lemma 3.11 (See [1, Lemma 1]) *If f is a Schwarz function of the form*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots,$$

then

$$|w_2 - t w_1^2| \le \begin{cases} -t, & \text{if } t \le -1; \\ 1, & \text{if } -1 \le t \le 1; \\ t, & \text{if } t \ge 1. \end{cases}$$

For $t < -1$ or $t > 1$, the equality holds if and only if $w(z) = z$ or one of its rotations. For $-1 < t < 1$, the equality holds if and only if $w(z) = z^2$ or one of its rotations. The equality holds for $t = -1$ if and only if $w(z) = z\frac{\lambda+z}{1+\lambda z}$ $(0 \le \lambda \le 1)$ or one of its rotations, while for $t = 1$, the equality holds if and only if $w(z) = -z\frac{\lambda+z}{1+\lambda z}$ $(0 \le \lambda \le 1)$ or one of its rotations.

Theorem 3.12 *Let f be of the form (1.1). If f belongs to the class $\mathcal{S}_c^*$, then*

$$|a_2| \le 1, \quad |a_3| \le 1 \quad \text{and} \quad |a_4| \le 1.$$

All inequalities are sharp.

Proof Let the function f be of the form (1.1). Since $f \in \mathcal{S}_c^*$, by Theorem 3.1 we have

$$\frac{f(z)}{z} \prec \frac{1}{1-z}.$$

By the definition of subordination there exists a Schwarz function w with $w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots$ and $|w(z)| < 1$ so that

$$\frac{f(z)}{z} = \frac{1}{1-w(z)} \quad (z \in \Delta),$$

or equivalently,

$$f(z) = \frac{z}{1-w(z)} \quad (z \in \Delta). \tag{3.23}$$

By substituting the Taylor series of f and w in (3.23) and comparing the coefficients, we obtain

$$a_2 = w_1, \quad a_3 = w_2 + w_1^2 \quad \text{and} \quad a_4 = w_3 + 2w_1w_2 + w_1^3. \tag{3.24}$$

Since $|w_1| \le 1$ (see [7, p. 128]), we get $|a_2| \le 1$. In order to estimate a_3, we apply Lemma 3.11. However, we have

$$|a_3| = |w_2 + w_1^2| = |w_2 - (-1)w_1^2| \le 1.$$

Prokhorov and Szynal in [7, Lemma 2] proved that if $(\mu, \nu) = (2, 1)$, then $|w_3 + \mu w_1 w_2 + \nu w_1^3| \le 1$. Therefore,

$$|a_4| = |w_3 + 2w_1w_2 + w_1^3| \le 1.$$

This completes the proof. □

The problem of finding sharp upper bounds for the coefficient functional $|a_3 - \mu a_2^2|$ $(\mu \in \mathbb{C})$ for different subclasses of class $\mathcal{A}$ is known as the Fekete–Szegö problem. Next, we study this problem for the class $\mathcal{S}_c^*$.

Theorem 3.13 *If $f \in \mathcal{A}$ of the form (1.1) belongs to the class $\mathcal{S}_c^*$, then for any complex number μ*

$$|a_3 - \mu a_2^2| \le \begin{cases} 1 - \mu, & \text{if } \mu \le 0; \\ 1, & \text{if } 0 \le \mu \le 2; \\ \mu - 1, & \text{if } \mu \ge 2. \end{cases}$$

The result is sharp.

Proof By use of Lemma 3.11 and (3.24), the proof is obtained. □

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