Quantum Incompatibility Witnesses

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We demonstrate that quantum incompatibility can always be detected by means of a state discrimination task with partial intermediate information. This is done by showing that only incompatible measurements allow for an efficient use of premeasurement information in order to improve the probability of guessing the correct state. Thus, the gap between the guessing probabilities with pre- and postmeasurement information is a witness of the incompatibility of a given collection of measurements. We prove that all linear incompatibility witnesses can be implemented as some state discrimination protocol according to this scheme. As an application, we characterize the joint measurability region of two noisy mutually unbiased bases.

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Introduction.—Quantum incompatibility is one of the key features that separate the quantum from the classical world [1]. It gives rise to several among the most intriguing quantum phenomena, including measurement uncertainty relations [2], contextuality [3], and nonlocality [4]. So far, however, the direct experimental verification of quantum incompatibility has been a demanding task, as the known detection methods, based on Bell experiments [5–7] and steering protocols [8–11], rely on entanglement.

In this paper, we show that quantum incompatibility can be detected by means of a state discrimination task with partial intermediate information. More precisely, we consider a scenario where Alice sends Bob a quantum system that she has prepared into a state chosen from one of n disjoint state ensembles, but she reveals to him the chosen ensemble only at a later time. Bob can then decide to perform his measurement either before or after Alice's announcement and, importantly, the achievable success probabilities can be compared. We show that Bob can benefit from prior compared to posterior measurement information and improve his probability of guessing the correct state only if his measurements are incompatible.

Looking at it from another perspective, the difference between Bob's guessing probabilities with pre- and postmeasurement information is a witness of the incompatibility of the collection of measurements he uses in the discrimination task. Since the complement set of incompatible collections of measurements is the closed and convex set of all the compatible collections of measurements, this observation sets the previous detection scheme for incompatibility within the broader framework of witnesses.

In general, a witness is any experimentally assessable linear function whose value is greater than or equal to 0 whenever the measured object does not have the investigated property, but gives a negative value at least for some object with that property. The paradigmatic example of witnesses is that of entanglement witnesses, which have become one of the main methods to detect entanglement [12,13]. Other examples include the detection of non-Gaussianity of states [14], dimensionality of correlations [15], or for the unital channels the detection of not being a random unitary channel [16]. The fact that witnesses can be applied to detect incompatibility has been recently noted in [17,18].

We prove that any incompatibility witness essentially arises as a state discrimination task with intermediate information of the type described above. By standard separation results for convex sets, this implies that all incompatible sets of measurements can be detected by performing some state discrimination where premeasurement information is strictly better than postmeasurement information. This yields a novel operational interpretation of quantum incompatibility, and provides a method to detect it in a physically feasible experiment. In particular, this proves that entanglement is not needed to reveal incompatibility.

General framework of witnesses.—We briefly recall the general setting of witnesses as this clarifies our main results on incompatibility witnesses and makes the reasoning behind them easy to follow [19].

Let $\mathcal{V}$ be a real linear space and $C \subset \mathcal{V}$ a compact convex subset that mathematically describes the objects we are interested in. This set is further divided into two disjoint

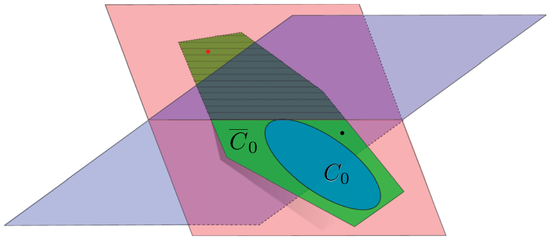

FIG. 1. Witnesses are associated with hyperplanes, and they are detection equivalent if they yield the same separation of the set C. Here, two tight equivalent witnesses detect the red point, but not the black one.

subsets C_0 and $\bar{C}_0$, with C_0 being closed and convex. We can think of C_0 and $\bar{C}_0$ as properties—either an element $x \in C$ is in C_0 or in $\bar{C}_0$. A witness of the property $\bar{C}_0$, or $\bar{C}_0$-witness, is a map $\xi: C \to \mathbb{R}$ such that

(W1) $\xi(x) \geq 0$ for all $x \in C_0$ and $\xi(x) < 0$ at least for some $x \in \bar{C}_0$;

(W2) $\xi(tx + (1-t)y) = t\xi(x) + (1-t)\xi(y)$ for all x, $y \in C$ and $t \in [0,1]$.

By condition (W2), each witness generates a hyperplane separating $\mathcal{V}$ into two half-spaces. Condition (W1) then asserts that one of the two halves entirely contains C_0, but still does not contain all of C (see Fig. 1).

We say that an element $x \in \bar{C}_0$ is detected by ξ if $\xi(x) < 0$, and we denote by $\mathcal{D}(\xi)$ the subset of all elements of $\bar{C}_0$ that are detected by ξ. Another $\bar{C}_0$-witness ξ' is called finer than ξ if $\mathcal{D}(\xi') \supseteq \mathcal{D}(\xi)$, and in this case we write $\xi \preceq \xi'$. If $\mathcal{D}(\xi') = \mathcal{D}(\xi)$, we say that ξ and ξ' are detection equivalent and denote this by $\xi \approx \xi'$ (see Fig. 1). As we typically aim to detect as many elements as possible, we favor witnesses that cannot be made any finer. A necessary condition for ξ being optimal in that sense is that ξ is tight, meaning that $\xi(x) = 0$ for some $x \in C_0$.

Any $\bar{C}_0$-witness ξ can be written in the form

$$\xi(x) = \delta - v^*(x) \quad \forall\ x \in C, \tag{1}$$

where $v^*: \mathcal{V} \to \mathbb{R}$ is a linear map and $\delta \in \mathbb{R}$ is a constant. An essential point for our later developments is that the representation (1) of a witness ξ is not unique but there is some freedom in the choice of v^* and δ. In addition, if we are only interested in the set of detected elements $\mathcal{D}(\xi)$, we have a further degree of freedom, coming from the possibility to switch from ξ to an equivalent $\bar{C}_0$-witnesses $\xi' = \alpha\xi$ for some constant $\alpha > 0$.

Detecting quantum incompatibility.—A measurement with a finite outcome set X is mathematically described as a positive operator valued measure (POVM), i.e., a map A from X to the set $\mathcal{L}_s(\mathcal{H})$ of self-adjoint linear operators on a Hilbert space $\mathcal{H}$ such that the operators $\mathsf{A}(x)$ are positive (meaning that $\langle\psi|\mathsf{A}(x)\psi\rangle \geq 0$ for all $\psi \in \mathcal{H}$) and they satisfy the normalization condition $\sum_x \mathsf{A}(x) = \mathbb{1}$.

For clarity, we limit our discussion to pairs of measurements. The treatment of finite collections of measurements is similar. Two measurements A and B, having outcome sets X and Y, respectively, are compatible if there exists a measurement M, called their joint measurement, with outcome set $X \times Y$, such that $\sum_y \mathsf{M}(x,y) = \mathsf{A}(x)$ and $\sum_x \mathsf{M}(x,y) = \mathsf{B}(y)$. Otherwise, A and B are incompatible.

By $\mathcal{O}_{X,Y}$ we denote the compact set of all pairs of measurements (A,B) with outcome sets X, Y, respectively. This set is divided into compatible pairs $\mathcal{O}_{X,Y}^{\rm com}$ and incompatible pairs $\mathcal{O}_{X,Y}^{\rm inc} \equiv \overline{\mathcal{O}_{X,Y}^{\rm com}}$. We define convex combinations in $\mathcal{O}_{X,Y}$ componentwise, and it follows that the subset $\mathcal{O}_{X,Y}^{\rm com}$ of compatible pairs is closed and convex. Hence we can consider $\mathcal{O}_{X,Y}^{\rm inc}$-witnesses; we call them incompatibility witnesses (IWs).

Discrimination scenario as an incompatibility witness.—In the standard state discrimination scenario [23–25], Alice picks a label z from a given set Z with probability $p(z)$. She encodes the label into a quantum state ϱ_z and delivers the state to Bob. Bob knows the set $\{\varrho_z\}_{z\in Z}$ of states used in the encoding. He is trying to recover the label by making a measurement on the quantum system that he has received. It is convenient to merge the *a priori* probability distribution p and the state encoding into a single map $\mathcal{E}$, given as $\mathcal{E}(z) = p(z)\varrho_z$. We call this map a state ensemble; its defining properties are that $\mathcal{E}(z)$ is positive for all z, and $\sum_z \mathrm{tr}[\mathcal{E}(z)] = 1$. The guessing probability depends on the measurement M that Bob uses, and it is given as

$$P_{\rm guess}(\mathcal{E};\mathsf{M}) = \sum_z \mathrm{tr}[\mathcal{E}(z)\mathsf{M}(z)].$$

Further, we denote

$$P_{\rm guess}(\mathcal{E}) = \max_{\mathsf{M}} P_{\rm guess}(\mathcal{E};\mathsf{M}), \tag{2}$$

where the optimization is done over all measurements with outcome set Z.

We are then considering two modifications of the standard state discrimination scenario, where partial classical information concerning the correct label is given either before or after the measurement is performed [26–29]. The form of the partial information is given as a partitioning $Z = X \cup Y$ of Z into two disjoint subsets. By conditioning the state ensemble $\mathcal{E}$ to the occurrence of a label in X or Y, we obtain new state ensembles $\mathcal{E}_X$ and $\mathcal{E}_Y$, which we call subensembles of $\mathcal{E}$; they are given as

$$\mathcal{E}_X(x) = \frac{1}{p(X)}\mathcal{E}(x), \qquad \mathcal{E}_Y(y) = \frac{1}{p(Y)}\mathcal{E}(y),$$

and their label sets are X and Y, respectively. Here we have denoted $p(X) = \sum_{z\in X} p(z)$ and $p(Y) = \sum_{z\in Y} p(z)$. We write $\hat{\mathcal{E}} = (\mathcal{E}, \{X, Y\})$ for the partitioned state ensemble, i.e., the state ensemble $\mathcal{E}$ with the partitioning of Z into disjoint subsets X and Y.

If Alice announces the correct subensemble before Bob chooses his measurement, we call the task discrimination with premeasurement information. In this case, Bob can choose a measurement A with the outcome set X to discriminate $\mathcal{E}_X$ and a measurement B with the outcome set Y to discriminate $\mathcal{E}_Y$. At each round of the experiment he measures either A or B, depending on Alice's announcement. Bob's total guessing probability is

$$P_{\text{guess}}^{\text{prior}}(\hat{\mathcal{E}};\mathsf{A},\mathsf{B}) = p(X)P_{\text{guess}}(\mathcal{E}_X;\mathsf{A}) + p(Y)P_{\text{guess}}(\mathcal{E}_Y;\mathsf{B}) \tag{3}$$

and its maximal value is

$$\begin{aligned} P_{\text{guess}}^{\text{prior}}(\hat{\mathcal{E}}) &= \max_{(\mathsf{A},\mathsf{B})\in\mathcal{O}_{X,Y}} P_{\text{guess}}^{\text{prior}}(\hat{\mathcal{E}};\mathsf{A},\mathsf{B}) \\ &= p(X)P_{\text{guess}}(\mathcal{E}_X) + p(Y)P_{\text{guess}}(\mathcal{E}_Y). \end{aligned} \tag{4}$$

In the other variant of the discrimination scenario, Alice announces the correct subensemble only after Bob has performed his measurement. Bob has to use a fixed measurement at each round but he can postprocess the obtained measurement outcome according to the additional information. We call this task discrimination with post-measurement information. It has been shown in [28] that now the maximal guessing probability, denoted as $P_{\text{guess}}^{\text{post}}(\hat{\mathcal{E}})$, is given by

$$P_{\text{guess}}^{\text{post}}(\hat{\mathcal{E}}) = \max_{(\mathsf{A},\mathsf{B})\in\mathcal{O}_{X,Y}^{\text{com}}} P_{\text{guess}}^{\text{prior}}(\hat{\mathcal{E}};\mathsf{A},\mathsf{B}). \tag{5}$$

A comparison of (4) and (5) reveals that the maximal guessing probabilities $P_{\text{guess}}^{\text{prior}}(\hat{\mathcal{E}})$ and $P_{\text{guess}}^{\text{post}}(\hat{\mathcal{E}})$ result in optimizing the same mathematical quantity, with the important difference that in the latter the optimization is restricted to compatible pairs of measurements. From this, we already conclude that if $P_{\text{guess}}^{\text{prior}}(\hat{\mathcal{E}};\mathsf{A},\mathsf{B}) > P_{\text{guess}}^{\text{post}}(\hat{\mathcal{E}})$ for some partitioned state ensemble $\hat{\mathcal{E}}$, then A and B are incompatible. This conclusion is essentially Theorem 1 of [28], stated in slightly different words. In the following, we develop this observation into a necessary and sufficient condition for incompatibility by using the framework of witnesses.

We first notice that, for a partitioned state ensemble $\hat{\mathcal{E}} = (\mathcal{E},\{X,Y\})$ with $P_{\text{guess}}^{\text{prior}}(\hat{\mathcal{E}}) > P_{\text{guess}}^{\text{post}}(\hat{\mathcal{E}})$, the function

$$\xi_{\hat{\mathcal{E}}}(\mathsf{A},\mathsf{B}) = P_{\text{guess}}^{\text{post}}(\hat{\mathcal{E}}) - P_{\text{guess}}^{\text{prior}}(\hat{\mathcal{E}};\mathsf{A},\mathsf{B}) \tag{6}$$

is a tight IW for pairs of measurements in $\mathcal{O}_{X,Y}$; we call it the incompatibility witness associated with $\hat{\mathcal{E}}$. In some cases, the exact evaluation of $P_{\text{guess}}^{\text{post}}(\hat{\mathcal{E}})$ may be a difficult task, but still by finding a number δ such that $P_{\text{guess}}^{\text{post}}(\hat{\mathcal{E}}) \leq \delta < P_{\text{guess}}^{\text{prior}}(\hat{\mathcal{E}})$ one obtains an IW by setting

$$\xi_{\hat{\mathcal{E}}}^{\delta}(\mathsf{A},\mathsf{B}) = \delta - P_{\text{guess}}^{\text{prior}}(\hat{\mathcal{E}};\mathsf{A},\mathsf{B}). \tag{7}$$

Clearly, we then have $\xi_{\hat{\mathcal{E}}}^{\delta} \preceq \xi_{\hat{\mathcal{E}}}$.

An important feature of the witnesses arising from partitioned state ensembles is that their physical implementation is straightforward. Namely, the quantities $P_{\text{guess}}(\mathcal{E}_X;\mathsf{A})$ and $P_{\text{guess}}(\mathcal{E}_Y;\mathsf{B})$ are obtained by performing standard state discrimination experiments, and $P_{\text{guess}}^{\text{prior}}(\hat{\mathcal{E}};\mathsf{A},\mathsf{B})$ is then given via (3). The constant term $P_{\text{guess}}^{\text{post}}(\hat{\mathcal{E}})$ must be calculated analytically or numerically, or at least upper bounded tightly enough. It has been shown in [28] that the calculation of $P_{\text{guess}}^{\text{post}}(\hat{\mathcal{E}})$ reduces to the evaluation of the standard guessing probability $P_{\text{guess}}(\mathcal{E}')$ of an auxiliary state ensemble $\mathcal{E}'$, and the techniques for calculating the standard guessing probability (see, e.g., [30]) are thereby applicable.

Characterization of incompatibility witnesses.—The following two theorems are the main results of this paper.

Theorem 1. For any incompatibility witness ξ, there exists a partitioned state ensemble $\hat{\mathcal{E}}$ such that the associated incompatibility witness $\xi_{\hat{\mathcal{E}}}$ is finer than ξ. Further, if ξ is tight, there exists a partitioned state ensemble $\hat{\mathcal{E}}$ such that ξ is detection equivalent to $\xi_{\hat{\mathcal{E}}}$.

In the case of IWs, the natural choice for the ambient vector space $\mathcal{V}$ containing $\mathcal{O}_{X,Y}$ is the Cartesian product $\mathcal{F}(X)\times\mathcal{F}(Y)$, where $\mathcal{F}(X)$ is the vector space of all operator valued functions $F: X\to\mathcal{L}_s(\mathcal{H})$. All linear maps on $\mathcal{F}(X)\times\mathcal{F}(Y)$ are expressible in terms of scalar products with elements $(F,G)\in\mathcal{F}(X)\times\mathcal{F}(Y)$, so that the basic representation (1) of witnesses takes the form

$$\xi(\mathsf{A},\mathsf{B}) = \delta - \sum_x \mathrm{tr}[F(x)\mathsf{A}(x)] - \sum_y \mathrm{tr}[G(y)\mathsf{B}(y)] \tag{8}$$

for all $(\mathsf{A},\mathsf{B})\in\mathcal{O}_{X,Y}$. The proof of Theorem 1 is based on the freedom in the choice of (F,G) and δ.

Proof of Theorem 1.—Starting from an IW ξ of the general form (8), we similarly define a map ξ' by choosing $F'(x) = \alpha[F(x)-\mu\mathbb{1}]$, $G'(y) = \alpha[G(y)-\mu\mathbb{1}]$ and $\delta' = \alpha(\delta - 2\mu d)$, where d is the dimension of the Hilbert space and $\alpha,\mu\in\mathbb{R}$ are constants that we determine next. A direct calculation shows that $\xi' = \alpha\xi$ on $\mathcal{O}_{X,Y}$. First, we fix the value of μ by setting

$$-\mu = \sum_{x\in X}\|F(x)\| + \sum_{y\in Y}\|G(y)\|,$$

where $\|\cdot\|$ denotes the uniform operator norm on $\mathcal{L}_s(\mathcal{H})$. With this choice, all the operators $\mathcal{E}(x) = |\alpha|[F(x)-\mu\mathbb{1}]$ and $\mathcal{E}(y) = |\alpha|[G(y)-\mu\mathbb{1}]$ are positive. Secondly, we fix the value of α by setting

$$\frac{1}{\alpha} = \sum_{x\in X}\mathrm{tr}[F(x)-\mu\mathbb{1}] + \sum_{y\in Y}\mathrm{tr}[G(y)-\mu\mathbb{1}].$$

The right-hand side of this expression is strictly positive, as otherwise $F(x) = G(y) = \mu \mathbb{1}$ for all x, y and so the original IW (8) would be constant on $\mathcal{O}_{X,Y}$, which is impossible. Thereby, $\alpha > 0$; hence, the map $\xi' = \alpha\xi$ is an IW and $\xi' \approx \xi$. Moreover, in this way we have obtained a partitioned state ensemble $\hat{\mathcal{E}} = (\mathcal{E}, \{X, Y\})$, for which the witness ξ' has the form (7): $\xi'(\mathsf{A}, \mathsf{B}) = \delta' - P^{\text{prior}}_{\text{guess}}(\hat{\mathcal{E}}; \mathsf{A}, \mathsf{B})$. Since ξ' is an IW and hence satisfies (W1), we must have $P^{\text{post}}_{\text{guess}}(\hat{\mathcal{E}}) \leq \delta' < P^{\text{prior}}_{\text{guess}}(\hat{\mathcal{E}})$. Thereby, $\xi' \preceq \xi_{\hat{\mathcal{E}}}$. If in addition ξ is tight, then $\delta' = P^{\text{post}}_{\text{guess}}(\hat{\mathcal{E}})$, and thus $\xi' = \xi_{\hat{\mathcal{E}}}$. □

An important consequence of Theorem 1 is the following novel operational interpretation for quantum incompatibility.

Theorem 2. Two measurements A and B are incompatible if and only if there exists a partitioned state ensemble $\hat{\mathcal{E}}$ such that $P^{\text{prior}}_{\text{guess}}(\hat{\mathcal{E}}; \mathsf{A}, \mathsf{B}) > P^{\text{post}}_{\text{guess}}(\hat{\mathcal{E}})$.

The probability $P^{\text{prior}}_{\text{guess}}(\hat{\mathcal{E}}; \mathsf{A}, \mathsf{B})$ is assessable by using Alice's classical information, and then performing quantum measurements only on Bob's side. Since no entangled state is shared in the state discrimination protocol, Theorem 2 provides a much more practical way to detect incompatibility than schemes based on Bell experiments or steering. In particular, as a fundamental fact, entanglement is not needed to detect incompatibility.

Proof of Theorem 2.—The "if" statement has already been observed earlier, so here we prove the "only if" part. Let us assume that $(\mathsf{A}, \mathsf{B}) \notin \mathcal{O}^{\text{com}}_{X,Y}$. Then, by the usual separation results for compact convex sets (Corollary 11.4.2 of [31]), there exist $(F, G) \in \mathcal{F}(X) \times \mathcal{F}(Y)$ and $\delta \in \mathbb{R}$ such that, defining ξ as in (8), we have $\xi(\mathsf{A}', \mathsf{B}') \geq 0$ for all $(\mathsf{A}', \mathsf{B}') \in \mathcal{O}^{\text{com}}_{X,Y}$ and $\xi(\mathsf{A}, \mathsf{B}) < 0$. By Theorem 1 there exists a partitioned state ensemble $\hat{\mathcal{E}}$ such that $\xi \preceq \xi_{\hat{\mathcal{E}}}$. It follows that $\xi_{\hat{\mathcal{E}}}(\mathsf{A}, \mathsf{B}) < 0$, i.e., $P^{\text{prior}}_{\text{guess}}(\hat{\mathcal{E}}; \mathsf{A}, \mathsf{B}) > P^{\text{post}}_{\text{guess}}(\hat{\mathcal{E}})$. □

Bounding the compatibility region by means of two mutually unbiased bases.—As we have seen, constructing an IW involves the solution of two convex optimization problems: the evaluation of the maximal guessing probabilities defined in (4) and (5). In particular, if $\hat{\mathcal{E}}$ is a partitioned state ensemble for which the two probabilities differ, whenever the maximum in the right-hand side of (5) admits an analytical computation, one can insert the resulting value of $P^{\text{post}}_{\text{guess}}(\hat{\mathcal{E}})$ into (6) and thus write the tight IW associated with $\hat{\mathcal{E}}$ in an explicit form.

Interestingly, solving the optimization problem (5) yields even more. Indeed, evaluating a constrained maximum typically requires finding some feasible points where the maximum is attained; if the optimization problem is convex, these points are necessarily located on the relative boundary of the feasible domain. In our specific case, it means that, as a byproduct of solving (5), we get points lying on the relative boundary $\partial\mathcal{O}^{\text{com}}_{X,Y}$ of the convex set $\mathcal{O}_{X,Y}$. Then, by taking convex combinations of these points, we can even have an insight into the set $\mathcal{O}^{\text{com}}_{X,Y}$ itself. We thus see that the solution of (5) has a twofold purpose: on the one hand, through the IW constructed in (6), it provides a simple method to detect the incompatibility of many measurement pairs; on the other hand, by using the resulting optimal points, some information on the set of compatible pairs can be inferred.

An interesting special case in which the optimization problems (4) and (5) admit an analytical solution is when the partitioned state ensemble $\hat{\mathcal{E}}$ is made up of two mutually unbiased bases (MUB) of the system Hilbert space $\mathcal{H}$, or, more generally, smearings of two MUB. Indeed, suppose $\{\varphi_h\}_{h\in\{1,\ldots,d\}}$ and $\{\psi_k\}_{k\in\{1,\ldots,d\}}$ is a fixed pair of MUB; then, we can use it to construct a partitioned state ensemble as follows. First, we choose $Z = \{1, \ldots, d\} \times \{\varphi, \psi\}$ as the overall label set of the ensemble

$$\mathcal{E}_{\boldsymbol{\mu}}(j, \ell) = \frac{1}{2d}\left[\mu_\ell |\ell_j\rangle\langle\ell_j| + (1 - \mu_\ell)\frac{1}{d}\mathbb{1}\right], \tag{9}$$

where $\boldsymbol{\mu} = (\mu_\varphi, \mu_\psi)$ and $\mu_\varphi, \mu_\psi \in [1/(1-d), 1]$ are real parameters. Next, we partition Z into the subsets $X = \{(1, \varphi), \ldots (d, \varphi)\}$ and $Y = \{(1, \psi), \ldots (d, \psi)\}$; here, the letters φ and ψ are just symbols, which are needed to distinguish labels in different subsets. Finally, we set $\hat{\mathcal{E}}_{\boldsymbol{\mu}} = (\mathcal{E}_{\boldsymbol{\mu}}, \{X, Y\})$.

The detailed solution to the optimization problems (4) and (5) for the partitioned state ensemble $\hat{\mathcal{E}}_{\boldsymbol{\mu}}$ is provided in Supplemental Material [19]. It turns out that the pair of measurements

$$\begin{aligned} \mathsf{A}(h, \varphi) &= \gamma_\varphi |\varphi_h\rangle\langle\varphi_h| + (1 - \gamma_\varphi)\frac{1}{d}\mathbb{1} \\ \mathsf{B}(k, \psi) &= \gamma_\psi |\psi_k\rangle\langle\psi_k| + (1 - \gamma_\psi)\frac{1}{d}\mathbb{1} \end{aligned} \tag{10}$$

is a feasible maximum point for a suitable choice of real numbers γ_φ and γ_ψ, which depend on $\boldsymbol{\mu}$. The next two theorems then follow by our earlier observations.

Theorem 3. Let $\boldsymbol{\mu} = (\mu_\varphi, \mu_\psi) \in [1/(1-d), 1] \times [1/(1-d), 1]$ with $\boldsymbol{\mu} \neq (0, 0)$. Then $P^{\text{post}}_{\text{guess}}(\hat{\mathcal{E}}_{\boldsymbol{\mu}}) < P^{\text{prior}}_{\text{guess}}(\hat{\mathcal{E}}_{\boldsymbol{\mu}})$ if and only if $\mu_\varphi\mu_\psi \neq 0$ and either $d = 2$ or $\max\{\mu_\varphi, \mu_\psi\} > 0$. In this case, the tight incompatibility witness associated with the partitioned state ensemble $\hat{\mathcal{E}}_{\boldsymbol{\mu}}$ is

$$\begin{aligned} \xi_{\hat{\mathcal{E}}_{\mu}}(\mathsf{A}, \mathsf{B}) = &\frac{1}{4}\left[\mu_\varphi + \mu_\psi + \sqrt{\mu_\varphi^2 + \mu_\psi^2 - 2\left(1 - \frac{2}{d}\right)\mu_\varphi\mu_\psi}\right] \\ &- \frac{1}{2d}\sum_{j=1}^{d}[\mu_\varphi\langle\varphi_j|\mathsf{A}(j, \varphi)\varphi_j\rangle + \mu_\psi\langle\psi_j|\mathsf{B}(j, \psi)\psi_j\rangle]. \end{aligned} \tag{11}$$

Finally, the ensembles $\hat{\mathcal{E}}_{\boldsymbol{\mu}}$ and $\hat{\mathcal{E}}_{\boldsymbol{\nu}}$ determine detection equivalent incompatibility witnesses if and only if $\boldsymbol{\nu} = \alpha\boldsymbol{\mu}$ for some $\alpha > 0$.

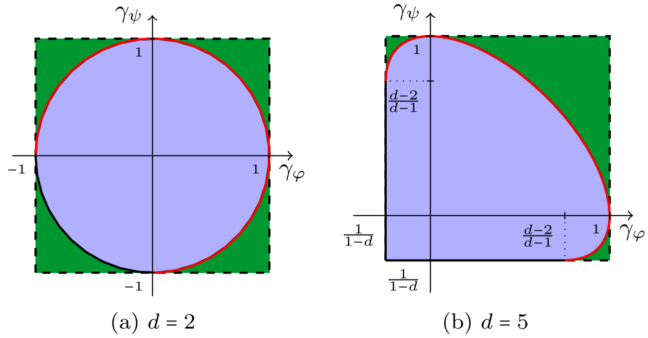

FIG. 2. The set of $\boldsymbol{\gamma} = (\gamma_\varphi, \gamma_\psi)$ for which Eq. (10) defines two measurements (green square), and the one for which these measurements are compatible (blue region) for different values of the dimension d. The red line is the curve (12). The case $d = 2$ is special, and was already treated in [35].

By the equivalence statement in the previous theorem, no generality is lost if we express the vector $\boldsymbol{\mu}$ in terms of a single real parameter θ. Consequently, also the vector $\boldsymbol{\gamma} = (\gamma_\varphi, \gamma_\psi)$ parametrizing the optimal measurements (10) becomes a function of θ. Thus, solving the optimization problem (5) for the present case actually yields a curve in the relative boundary $\partial\mathcal{O}^{\mathrm{com}}_{X,Y}$.

Theorem 4. The pair of measurements (A, B) of (10) lies on the relative boundary $\partial\mathcal{O}^{\mathrm{com}}_{X,Y}$ if

$$\boldsymbol{\gamma} = \left(\frac{d-2-d\cos(\theta+\theta_0)}{2(d-1)}, \frac{d-2-d\cos(\theta-\theta_0)}{2(d-1)}\right) \qquad (12)$$

for $\theta \in [-\theta_0, \theta_0]$ and $\theta_0 = \pi - \arctan\sqrt{d-1}$.

When $\theta = 0$, the common value of the two components of (12) is the noise robustness of the two MUB at hand; it was already derived by different methods in [32,33]. On the other hand, under the assumption that the two MUB are Fourier conjugate, the portion of the curve (12) with $\gamma_\varphi > 0$ and $\gamma_\psi > 0$ was found in [34].

The operators in (10) are positive if and only if $\boldsymbol{\gamma} \in [1/(1-d), 1] \times [1/(1-d), 1]$. Thus, all pairs of measurements of the form (10) constitute a square-shaped section of the set $\mathcal{O}_{X,Y}$. Remarkably, the lower-left vertex $(1/(1-d), 1/(1-d))$ of this square corresponds to a compatible pair of measurements if and only if $d \geq 3$; on the contrary, when $d = 2$ the relative boundary is symmetric around (0,0) [35]. Combining these considerations and Theorem 4, we can give a partial inspection of the two sets $\mathcal{O}_{X,Y}$ and $\mathcal{O}^{\mathrm{com}}_{X,Y}$, as shown in Fig. 2.

Discussion.—The framework of witnesses is an effective tool in the detection of properties described by sets with compact and convex complements. We have shown that for incompatibility of measurements, witnesses are not only a mathematical tool, but can be implemented in simple discrimination experiments. An important feature of this implementation is that it does not require entanglement.

Our characterization yields a novel operational interpretation of incompatibility: a collection of measurements is incompatible if and only if there is a state discrimination task where premeasurement information is strictly better than postmeasurement information.

Entanglement witnesses have been used not only to detect entanglement but also to quantify entanglement [36]. Further, one can drop the condition (W2) and consider nonlinear witnesses [37]. These and other modifications or generalizations will be an interesting matter of investigation in the case of incompatibility witnesses.

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Note added.—Recently we have been informed about two works that contain related results [38,39].

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