Abstract

A discrete variant of a multicriteria investment portfolio optimization problem with Savage's risk criteria is considered. One of the three problem parameter spaces is endowed with Hölder's norm, and the other two are endowed with Chebyshev's norm. The lower and upper attainable bounds on the stability radius of one Pareto optimal portfolio are obtained. We illustrate the application of our theoretical results by modeling a relevant case study.

Multicriteria investment problem with Savage's risk criteria: theoretical aspects of stability and case study

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1 Preliminaries

Many problems of making multipurpose decisions (individual or group) in management, planning and design can be formulated as multicriteria problems of continuous and/or discrete optimization. Modern financial environments require mitigation in limitations of modern portfolio theory to make portfolio choice easier in the context of long-term and goal-based investing [1]. Most of business and management decisions are being made within uncertain and risky environment that are caused by the influence of various factors such as an inadequacy of the mathematical models used by real processes, errors in measurements or rounding, and many other factors. Investment managing problems are of type problems with uncertainty in the initial data (see e.g. [2, 3, 4, 5]). Usually, any separate investment asset has higher a level of risk and less return than a portfolio of those assets and there is no reason to invest in one partic-

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ular asset. Creating a portfolio by diversifying assets and mixing variety of investments, an investor reduces the riskiness of the portfolio. Different aspects of portfolio optimization and project investment are scrutinized in literature, e.g. investments in projects [6, 7, 8, 9], grouping projects into portfolios [10], project portfolio selection [11, 12, 13], conflicting situations [14] etc. Many authors also focus on solving optimization problems with multiple criteria [15, 16].

To manage financial investments, in [17, 18] an optimization model was developed that demonstrates how an investor can minimize the level of risk with a given expected level of income. This formulation assumes the use of statistical and expert assessments of risks (financial, environmental, etc.) as initial data. It is well known that the complexity of calculating such quantities is accompanied by a large number of errors and leads to a high degree of uncertainty in the initial information. In these conditions, the question arises naturally of the plausibility of the results obtained in the solution of such problems, which leads to the necessity of carrying out post-optimal analysis. Following classical portfolio theory, the investor plots on the graph an efficient frontier depending on various pairs of risk and expected return and chooses a portfolio drawing on individual risk-return preferences. It gives an ability to construct a portfolio with the same expected return and less risk, classifying and measuring risk [19, 20]. The risk values are usually derived from historical data [21].

The model we consider is rather different from the classical models. The risk matrix is constructed for several market states related to each type of risk. Unlike the classical modern portfolio theory, where a portfolio consists of a percentage of each asset, in our model a Boolean decision vector is used to describe feasible portfolios. The problem is in finding a set of Pareto optimal portfolios with Savage's risk criteria.

The model formulation requires statistical and expert evaluations of different types of risk (e.g. financial or ecological) [22] to be specified as the initial data. To construct an efficient portfolio, the investor must be able to quantify risk and provide the necessary inputs. Usually, the collected data contain computational errors and inaccuracies. It leads to the situation when the initial data that represent risk values are inaccurate and uncertain. A number of approaches has been described in literature to treat the issue of uncertainty. For example, stability aspects for scalar problems are studied in

[23, 24, 25, 26, 27] as well as stability for multicriteria cases is analyzed in [28, 29].

One of the key points in portfolio optimization under uncertainty is an estimation of perturbation ranges for the initial data. The quantitative measure of the data perturbation level that does not violate optimality is known as the stability radius. The concept, widely presented and analyzed in the recent literature, focused on finding analytical expressions and bounds (see e.g. [30, 31, 32, 33]). Similar approaches were also developed in parallel for scheduling theory (see [34, 35, 36]). Analytic formulae are pairwise comparisons of solutions depending on selected optimality principles. The structure of global perturbations of this problem and the structure of the solution set should be taken into account. The particular definition of the stability radius concept depends on the chosen optimality principles (the given problem is multicriteria), uncertain data and a type of a distance metric used to measure the closeness in problem parameter spaces. Various types of metrics allow to consider a specificity of problem parameters perturbations. So in the case of Chebyshev's metric l_∞ the maximum changes in the initial data are taken into account only. Thus the perturbations are considered to be independent. In the case of the Manhattan metric l_1 every change in the initial data can be monitored in total. Hölder's metric l_p, $1 \leq p \leq \infty$, is the metric with the parameter and includes such extreme cases as Chebyshev's metric l_∞, the Manhattan metric l_1 and also the Euclidean metric l_2. Thus, l_p norm allows to monitor the level and type of admissible perturbations, and therefore gives to the decision maker more flexibility. For more details on the issue of using Hölder's metric in portfolio optimization we refer the reader to [37].

Along with the quantitative analysis, a qualitative approach is developed in parallel. This approach concentrates on specifying analytical conditions that will guarantee some certain pre-specified behavior of the set of optimal solutions. To highlight the ideas of this approach, it is worth to mention papers [38, 39], where the comparative analysis of five different types of stability is presented for a multicriteria integer linear programming problem. Similar results were obtained for multicriteria combinatorial problems with the bottleneck criteria [40] as well as with some other nonlinear criteria [41].

In the previous papers (see e.g. [42, 43, 44, 45]), some bounds on the stability radii were obtained in the cases where the three-dimensional problem parameters space is equipped with different combinations of l_1 and l_∞ norms. In the present paper, we obtain the lower and upper bounds on the stability radius of one Pareto optimal portfolio for the multicriteria investment problem with Savage's risk criteria where we assume that in one space an arbitrary l_p norm is defined with $1 \leq p \leq \infty$. At the same time, we measure distances with l_∞ norm in the remaining spaces. It crates a possibility to make a more detailed and customized monitoring over changes in the initial data in the framework of the different problem parameters spaces. For example, the Euclidean metric is often used to deal with risks, and l_p norm can treat the case once the decision maker needs it.

2 Basic notations and concepts

Consider a multicriteria discrete variant of portfolio optimization problem. We assume the model can be described by the following primitives listed below. Let

$N_n = \{1, 2, \dots, n\}$ be a variety of alternatives (investment assets);

N_m be a set of possible financial market states (market situations, scenarios);

N_s be a set of possible risks;

r_{ijk} be a numerical measure of economic risk of type $k \in N_s$ if investor chooses project $j \in N_n$ given the market state $i \in N_m$;

$R = [r_{ijk}] \in \mathbf{R}^{m \times n \times s}$;

$x = (x_1, x_2, \dots, x_n)^T \in \mathbf{E}^n$ be an investment portfolio, where $\mathbf{E} = \{0, 1\}$,

$$x_j = \begin{cases} 1, & \text{if investor chooses project } j, \\ 0, & \text{otherwise;} \end{cases}$$

$X \subset \mathbf{E}^n$ be a set of all admissible investment portfolios;

$\mathbf{R}^m$ be a financial market state space;

$\mathbf{R}^n$ be a portfolio space;

$\mathbf{R}^s$ be a risk space.

In our model, we assume that the risk measure is addictive, i.e. the total risk of one portfolio is a sum of risks of the projects included in the portfolio. The risk of each project can be measured, for instance, by means of the associated implementation cost.

The presence of a risk factor is integral feature of a financial market functioning. One can find information about risk measurement methods and their classification in [46]. The last trend is to quantify risks using five R: robustness, redundancy, resourcefulness, response and recovery. The natural target of any investor is to minimize different types of risks. It creates a motivation for the multicriteria analysis within risk modeling. It leads to the usage of multicriteria decision making tools [47].

Assume that the efficiency of a chosen portfolio (Boolean vector) $x \in X,\ |X| \geq 2$, is evaluated by a vector objective function

$$f(x, R) = (f_1(x, R_1), f_2(x, R_2), \dots, f_s(x, R_s)),$$

each partial objective represents Savage's minimax risk criterion [48].

$$f_k(x, R_k) = \max_{i \in N_m} R_{ik}x = \max_{i \in N_m} \sum_{j \in N_n} r_{ijk}x_j \to \min_{x \in X}, \quad k \in N_s,$$

where $R_k \in \mathbf{R}^{m \times n}$ – k-th cut $R = [r_{ijk}] \in \mathbf{R}^{m \times n \times s}$ with rows $R_{ik} = (r_{i1k}, r_{i2k}, \dots, r_{ink}) \in \mathbf{R}^n$, $i \in N_m$.

If the investor chooses Savage's risk criterion [49], then (s)he minimizes the total risk of the selected portfolio in the worst (maximum risk state) case. This approach takes place when the decision maker has most pessimistic expectations about the market.

The problem of finding Pareto optimal (efficient) portfolios is referred to as the multicriteria investment Boolean problem with Savage's risk criteria and denoted as $Z^s(R)$, $s \in \mathbf{N}$. The set of Pareto optimal portfolios is defined as follows

$$P^s(R) = \{x \in X :\ \nexists x' \in X\ \ (g(x, x', R) \geq 0_{(s)}\ \ \&\ \ g(x, x', R) \neq 0_{(s)})\},$$

where

$$g(x, x', R) = (g_1(x, x', R_1), g_2(x, x', R_2),\ \dots, g_s(x, x', R_s)),$$

$$g_k(x, x', R_k) = f_k(x, R_k) - f_k(x', R_k) = \min_{i' \in N_m} \max_{i \in N_m} (R_{ik}x - R_{i'k}x'), \quad k \in N_s,$$

$$0_{(s)} = (0, 0, \dots, 0) \in \mathbf{R}^s.$$

If $m = 1$, then the problem $Z^s(R)$ transforms into s-criteria linear Boolean programming problem:

$$Z^s_B(R): \quad Rx \to \min_{x \in X}, \tag{1}$$

where $X \subseteq \mathbf{E}^n$, $R = [r_{kj}] \in \mathbf{R}^{s\times n}$ is a matrix with rows $R_k = (r_{k1}, r_{k2}, \ldots, r_{kn}) \in \mathbf{R}^n$, $k \in N_s$. The case $m = 1$ can be interpreted as a stable market with one state only.

While solving investment problems, it is necessary to take into account an inaccuracy of initial information (statistical and expert risks evaluation errors) that are very common in real life. Under these conditions, it is highly recommended to get numerical bounds about possible changes in the initial data that preserve efficiency of the original Pareto optimal portfolio for any perturbation. Similarly to [43, 50], the number

$$\rho = \rho^s(x^0) = \begin{cases} \sup\ \Xi, & \text{if } \Xi \neq \emptyset, \\ 0, & \text{if } \Xi = \emptyset, \end{cases}$$

is called a stability radius of a Pareto optimal solution $x^0 \in P^s(R)$, where

$$\Xi = \{\varepsilon > 0 : \ \forall R' \in \Omega(\varepsilon) \quad (x^0 \in P^s(R + R'))\},$$

$$\Omega(\varepsilon) = \{R' \in \mathbf{R}^{m\times n\times s} : \ \|R'\| < \varepsilon\}.$$

Here $\Omega(\varepsilon)$ is a set of feasible perturbation matrices, $P^s(R + R')$ is a Pareto set of perturbed problem $Z^s(R+R')$, $\|R'\|$ is the norm of the matrix $R' = [r'_{ijk}]$. This norm depends on norms specified in the portfolio space $\mathbf{R}^n$, the market state space $\mathbf{R}^m$ as well as the risk space $\mathbf{R}^s$.

Further, we investigate the stability radius in three different cases depending on which of those three spaces $\mathbf{R}^n$, $\mathbf{R}^m$ or $\mathbf{R}^s$ is equipped with Hölder's l_p-norm, $1 \leq p \leq \infty$. For any dimension d and $1 \leq p \leq \infty$, Hölder's l_p norm of $a = (a_1, a_2, \ldots, a_d) \in \mathbf{R}^d$ in $\mathbf{R}^d$ is defined by the following equation

$$\|a\|_p = \begin{cases} \left(\sum_{j\in N_d} |a_j|^p\right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \\ \max\{|a_j| : \ j \in N_d\}, & \text{if } p = \infty. \end{cases}$$

It is well-known that l_p norm, defined in $\mathbf{R}^d$, induces conjugated $l_{p'}$ norm in $(\mathbf{R}^d)^*$. For p and p', the following relations hold

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad 1 < p < \infty.$$

Here as usual, we set $p' = 1$ if $p = \infty$, and $p' = \infty$ if $p = 1$. Thus, we assume that p and p' vary within the range $[1, \infty]$. We also assume $1/p = 0$ if $p = \infty$.

It is easy to see

$$\|z\|_p \|z\|_{p'} = \|z\|_1 \ \text{ for } \ z \in \{-1, 0, 1\}^n, \ p \in [1, \infty]. \tag{2}$$

For any $\alpha > 0$ and $m \in \mathbf{N}$,

$$\|(\underbrace{\alpha, ..., \alpha}_{m})\|_p = m^{1/p}\alpha. \tag{3}$$

Further, we will use classical Hölder's inequality

$$ab \leq \|a\|_p \|b\|_{p'},$$

where $a = (a_1, a_2, \ldots, a_n)^T \in \mathbf{R}^n$, $b = (b_1, b_2, \ldots, b_n)^T \in \mathbf{R}^n$.

3 Theoretical aspects of stability: main results

3.1 Case A: portfolio space R^n is endowed with l_p

We endow the portfolio space $\mathbf{R}^n$ with an arbitrary Hölder's l_p norm, $1 \leq p \leq \infty$, while in the market state space $\mathbf{R}^m$ and the risk space $\mathbf{R}^s$ we measure distances by means of l_∞. Thus, for any matrix $R = [r_{ijk}] \in \mathbf{R}^{m \times n \times s}$

$$\|R\|_{p\infty\infty} = \|(\|R_1\|_{p\infty}, \|R_2\|_{p\infty}, \ldots, \|R_s\|_{p\infty})\|_\infty = \max_{k \in N_s} \|R_k\|_{p\infty},$$

where

$$\|R_k\|_{p\infty} = \|(\|R_{1k}\|_p, \|R_{2k}\|_p, \ldots, \|R_{mk}\|_p)\|_\infty, \quad k \in N_s.$$

Obviously,

$$\|R_{ik}\|_p \leq \|R_k\|_{p\infty} \leq \|R\|_{p\infty\infty}, \quad i \in N_m, \ k \in N_s.$$

Additionally, due to Hölder's inequality, for any x, $x^0 \in X$ we get

$$R_{ik}x - R_{i'k}x^0 \geq -(\|R_{ik}\|_p \|x\|_{p'} + \|R_{i'k}\|_p \|x^0\|_{p'}) \geq$$

$$\geq -\|R_k\|_{p\infty}(\|x\|_{p'} + \|x^0\|_{p'}), \quad i, i' \in N_m,\ k \in N_s. \tag{4}$$

In this context $\rho_1 = \rho_1^s(x^0, m, p, \infty, \infty)$ denotes the stability radius of x^0. For Pareto optimal portfolio x^0 in $Z^s(R)$, we will use the following notation

$$\varphi_1 = \varphi_1^s(x^0, m, p, \infty, \infty) = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_\infty}{\|x\|_{p'} + \|x^0\|_{p'}},$$

$$\psi_1 = \psi_1^s(x^0, m, p, \infty, \infty) = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_\infty}{\|x - x^0\|_{p'}}.$$

Obviously, $\psi_1 \geq \varphi_1 \geq 0$. Here and henceforth we will use a vector $a = (a_1, a_2, \dots, a_s) \in \mathbf{R}^s$ projection operator to the nonnegative orthant:

$$[a]^+ = (a_1^+, a_2^+, \dots, a_s^+),$$

where sign $+$ means the positive projection of the vector, i.e. $a_k^+ = \max\{0, a_k\}$, $k \in N_s$.

Theorem 1. *For any m, $s \in \mathbf{N}$ and $p \in [1, \infty]$, the stability radius $\rho_1^s(x^0, m, p, \infty, \infty)$ of a Pareto optimal portfolio $x^0 \in P^s(R)$ in $Z^s(R)$ has the following upper and lower bounds*

$$\varphi_1^s(x^0, m, p, \infty, \infty) \leq \rho_1^s(x^0, m, p, \infty, \infty) \leq \psi_1^s(x^0, m, p, \infty, \infty). \tag{5}$$

Attainability of the upper and lower bounds specified in (5) when $p = \infty$ follows also from the following evident statement, which is a direct consequence of Theorem 1.

Corollary 1. *If for any investment portfolio $x \neq x^0$ the set $\{j \in N_n : \ x_j^0 = x_j = 1\}$ is empty, then for any number $m \in \mathbf{N}$ the formula*

$$\rho_1^s(x^0, m, \infty, \infty, \infty) = \varphi_1^s(x^0, m, \infty, \infty, \infty) = \psi_1^s(x^0, m, \infty, \infty, \infty) =$$

$$= \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_\infty}{\|x + x^0\|_1}$$

holds.

From Theorem 1 it also follows the corollary below.

Corollary 2. *[44] For any $m \in \mathbf{N}$, the following bounds take place*

$$\varphi_1^s(x^0, m, \infty, \infty, \infty) \leq \rho_1^s(x^0, m, \infty, \infty, \infty) \leq \psi_1^s(x^0, m, \infty, \infty, \infty).$$

The following theorem gives an evidence about the attainability of lower bound specified in Corollary 2, i.e. the lower bound (5) while $p = \infty$.

Theorem 2. *There exists a class of problems $Z^s(R)$, such that for portfolio $x^0 \in P^s(R)$ the following relations are valid*

$$0 < \rho_1^s(x^0, m, \infty, \infty, \infty) = \varphi_1^s(x^0, m, \infty, \infty, \infty) < \psi_1(x^0, m, \infty, \infty, \infty). \quad (6)$$

The following known result gives us the evidence about attainability of the upper bound on the stability radius of $x^0 \in P^s(R)$ in $Z^s(R)$ for the case $m = 1$ (see (1)). In this context $\mathbf{R}^n$ is endowed with l_p, and $\mathbf{R}^s$ is endowed with l_∞.

Theorem 3. *[50] For any $p \in [1, \infty]$ and $s \in \mathbf{N}$, the stability radius of $x^0 \in P^s(R)$ in the linear Boolean programming problem $Z_B^s(R)$, $R \in \mathbf{R}^{s \times n}$ is expressed by the formula*

$$\rho_1^s(x^0) = \min_{x \in X \setminus \{x^0\}} \frac{\|[R(x - x^0)]^+\|_\infty}{\|x - x^0\|_{p'}}.$$

3.2 Case B: market state space R^m is endowed with l_p

Now consider the case when the portfolio space $\mathbf{R}^n$ and the risk space $\mathbf{R}^s$ are endowed with l_∞, whereas the market state space $\mathbf{R}^m$ is equipped with Hölder's l_p norm, $1 \leq p \leq \infty$. Thus, the norm of the matrix is defined by

$$\|R\|_{\infty p \infty} = \|(\|R_1\|_{\infty p}, \|R_2\|_{\infty p}, \ldots, \|R_s\|_{\infty p})\|_\infty = \max_{k \in N_s} \|R_k\|_{\infty p},$$

where

$$\|R_k\|_{\infty p} = \|(\|R_{1k}\|_\infty, \|R_{2k}\|_\infty, \ldots, \|R_{mk}\|_\infty)\|_p, \;\; k \in N_s.$$

Obviously,

$$\|R_{ik}\|_\infty \leq \|R_k\|_{\infty p} \leq \|R\|_{\infty p \infty}, \;\; i \in N_m, \; k \in N_s.$$

Additionally, due to Hölder's inequality, for any x, $x^0 \in X$ we have

$$R_{ik}x - R_{i'k}x^0 \geq -(\|R_{ik}\|_\infty \|x\|_1 + \|R_{i'k}\|_\infty \|x^0\|_1) \geq$$

$$\geq -\|R_k\|_{\infty p} \|x + x^0\|_1, \quad i, i' \in N_m,\ k \in N_s. \tag{7}$$

In this context, $\rho_2 = \rho_2^s(x^0, m, \infty, p, \infty)$ is the stability radius of x^0. For a Pareto optimal portfolio x^0 in $Z^s(R)$ we use the following notations

$$\varphi_2 = \varphi_2^s(x^0, m) = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_\infty}{\|x + x^0\|_1},$$

$$\psi_2 = \psi_2^s(x^0, m) = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_\infty}{\|x - x^0\|_1}.$$

Evidently, $\psi_2 \geq \varphi_2 \geq 0$.

Theorem 4. *For any m, $s \in \mathbf{N}$ and $p \in [1, \infty]$, the stability radius $\rho_2^s(x^0, m)$ of a Pareto optimal portfolio $x^0 \in P^s(R)$ in $Z^s(R)$ has the following lower and upper bounds*

$$\varphi_2^s(x^0, m) \leq \rho_2^s(x^0, m, \infty, p, \infty) \leq m^{1/p} \psi_2^s(x^0, m).$$

The following known results confirms an attainability on the upper bound of the stability radius of $x^0 \in P^s(R)$ in $Z^s(R)$ for the case $m = 1$ (see (1)). In this context, both $\mathbf{R}^n$ and $\mathbf{R}^s$ are equipped with l_∞.

Theorem 5. *[51] For the stability radius of $x^0 \in P^s(R)$ in the Boolean linear programming problem $Z_B^s(R)$, $R \in \mathbf{R}^{s \times n}$, and $s \in \mathbf{N}$ the following analytical expression*

$$\rho_2^s(x^0) = \min_{x \in X \setminus \{x^0\}} \frac{\|[R(x - x^0)]^+\|_\infty}{\|x - x^0\|_1}$$

holds.

Since $\rho_1 = \rho_2$ while $p = \infty$, then Corollaries 2 and 3 follow directly from Theorem 4.

3.3 Case C: risk space R^s is endowed with l_p

We measure distances by means of l_∞ in the portfolio space $\mathbf{R}^n$ and the market state space $\mathbf{R}^m$. At the same time in the risk space $\mathbf{R}^s$, we use l_p, $1 \leq p \leq \infty$. In this case under the norm of the matrix R we understand the number

$$\|R\|_{\infty\infty p} = \|(\|R_1\|_{\infty\infty}, \|R_2\|_{\infty\infty}, \dots, \|R_s\|_{\infty\infty})\|_p,$$

where

$$\|R_k\|_{\infty\infty} = \|(\|R_{1k}\|_\infty, \|R_{2k}\|_\infty, \dots, \|R_{mk}\|_\infty)\|_\infty, \quad k \in N_s.$$

Obviously,

$$\|R_{ik}\|_\infty \le \|R_k\|_{\infty\infty} \le \|R\|_{\infty\infty p}, \quad i \in N_m,\ k \in N_s.$$

It is easy to check that for any portfolios x and x' the following inequalities

$$R_{ik}x - R_{i'k}x' \ge -\|R_k\|_{\infty\infty}\|x + x'\|_1, \quad i,\ i' \in N_m,\ \ k \in N_s \tag{8}$$

hold.

In this context, $\rho_3 = \rho_3^s(x^0, m, \infty, \infty, p)$ denotes the stability radius of x^0. For a Pareto optimal portfolio x^0 in $Z^s(R)$ we introduce the notation

$$\varphi_3 = \varphi_3^s(x^0, m, \infty, \infty, p) = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_p}{\|x + x^0\|_1},$$

$$\psi_3 = \psi_3^s(x^0, m, \infty, \infty, p) = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_p}{\|x - x^0\|_1}.$$

Evidently, $\psi_3 \ge \varphi_3 \ge 0$.

Theorem 6. *For any $m,\ s \in \mathbf{N}$ and $p \in [1, \infty]$, the stability radius $\rho_3^s(x^0, m, \infty, \infty, p)$ of a portfolio $x^0 \in P^s(R)$ in $Z^s(R)$ has the following lower and upper bounds*

$$\varphi_3^s(x^0, m, \infty, \infty, p) \le \rho_3^s(x^0, m, \infty, \infty, p) \le \psi_3^s(x^0, m, \infty, \infty, p).$$

The following statement gives the evidence about the attainability on the lower and upper bounds specified in Theorem 6.

Corollary 3. *If for any $x \ne x^0$ the set $\{j \in N_n : \ \ x_j^0 = x_j = 1\}$ is empty, then for any $m \in \mathbf{N}$ any $p \in [1, \infty]$ the following holds:*

$$\rho_3^s(x^0, m, \infty, \infty, p) = \varphi_3^s(x^0, m, \infty, \infty, p) =$$

$$= \psi_3^s(x^0, m, \infty, \infty, p) = \min_{x \in X \setminus \{x^0\}} \frac{\|[g(x, x^0, R)]^+\|_p}{\|x + x^0\|_1}.$$

If $m = 1$, as it was pointed out before, $Z^s(R)$ transforms into s-criteria Boolean linear programming problem $Z^s_B(R)$, $R \in \mathbf{R}^{s\times n}$ (see (1)). In this context, $\mathbf{R}^n$ is equipped with l_∞, and $\mathbf{R}^s$ is equipped with l_p, $1 \leq p \leq \infty$. The following known result illustrates the fact that the upper bound specified in Theorem 6 is right.

Theorem 7. *[50] For any $p \in [1, \infty]$ and $s \in \mathbf{N}$, the stability radius of $x^0 \in P^s(R)$ in $Z^s_B(R)$, $R \in \mathbf{R}^{s\times n}$ is expressed by the formula*

$$\rho^s_3(x^0) = \min_{x \in X\setminus\{x^0\}} \frac{\|[R(x - x^0)]^+\|_p}{\|x - x^0\|_1}.$$

4 Case Study

For evaluating possibilities of investments in particular regions we chose five economic unions: Caribbean Single Market and Economy (CSME), Eurasian Economic Union (EAEU), Mercosur, Gulf Cooperation Council (GCC), Central American Integration System (SICA). Gathering the countries, included in those unions, we formed the set of portfolios. The portfolios were evaluated using values for the global economic risk (Table 1). The risk evaluations were published in the Global Risk Report for the World Economic Forum in 2016 (http://weforum.org/risks/). It has eight different types: asset bubble, deflation, energy price shock, failure of critical infrastructure, failure of financial mechanism or institution, fiscal crises, unemployment or underemployment, unmanageable inflation.

All five portfolios are Pareto optimal and based only on the risk evaluation it is not possible to make a rational decision.

Every portfolio includes different countries. It make sense to implement Corollary 1 and Corollary 3. Following Corollary 2 for the case when $p = \infty$ and $p' = 1$ the stability radius for every Pareto optimal portfolio can be calculated by the formula that will be equal to the lower bound, described in Theorem 1. Based on Corollary 3 the stability radius can also be calculated using the formula, which is equal to the lower bound from Theorem 6 for any parameter $p \in [1, \infty]$.

In Figures 1-6 there are represented changes of the vales for the lower and upper bounds of the stability radii depending on the parameter $p \in [1, \infty]$.

Table 1: Value function for portfolios

	s							
CSME	81	63	110	102	79	161	168	61
EAEU	120	68	155	92	137	149	231	90
MERCOSUR	144	50	186	100	124	152	146	119
GCC	125	58	182	192	125	136	254	116
SICA	58	66	171	94	126	139	323	106

In the case when we evaluate the stability radius, varying the parameter p in the market state space or in the risk space, the values of the lower bounds of the stability radius for GCC are larger than the values for EAEU. Increasing p the EAEU portfolio becomes more robust than GCC. Similar situation happens when we consider MERCOSUR and SICA portfolios. Following Figures 1 and 5 SICA is more robust than MERCOSUR for the parameter $p = 1$. Selecting the parameter p close to ∞ MERCOSUR becomes the portfolio with the bigger lower bound that SICA. This kind of behavioral can be explained that for $p = 1$ it is assumed that the changes in the risk evaluations between the countries are dependent. Setting the parameter p close to ∞ we monitor only biggest changes in the initial data and we suppose that the adjustments of the risk evaluations in different countries do not have influences on risk in the other countries. As we can notice, depending on the investor assumptions through changing the parameter p it is possible to customize the monitoring of perturbations.

5 Conclusion

The investor's goal is to minimize the level of various types of risks, while portfolio development motivates the use of multicriteria environment in accordance with the mathematical and economic models. This approach makes it possible to use a variety of multicriteria decision-making tools [47, 52]. In this paper, to model various types of

Figure 1: Values for $\varphi_1^s(x^0, m, p, \infty, \infty)$

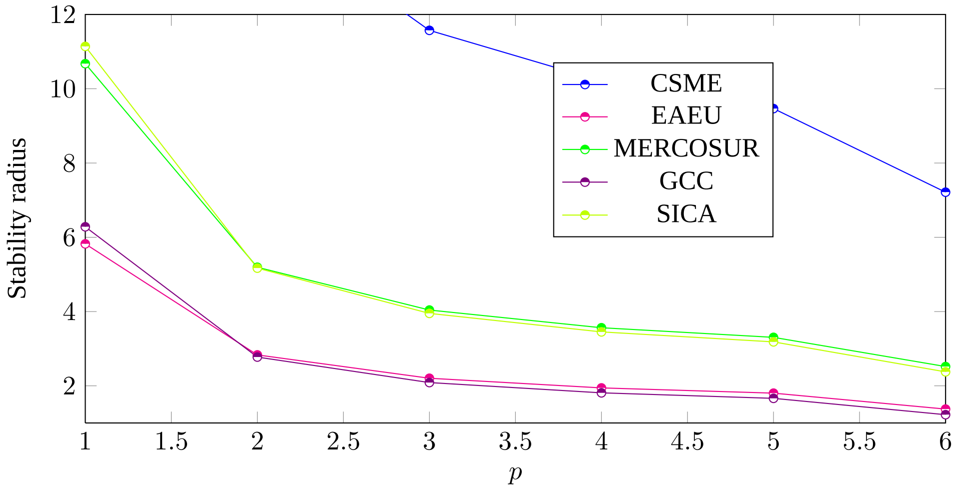


Figure 2: Values for $\psi_1^s(x^0, m, p, \infty, \infty)$

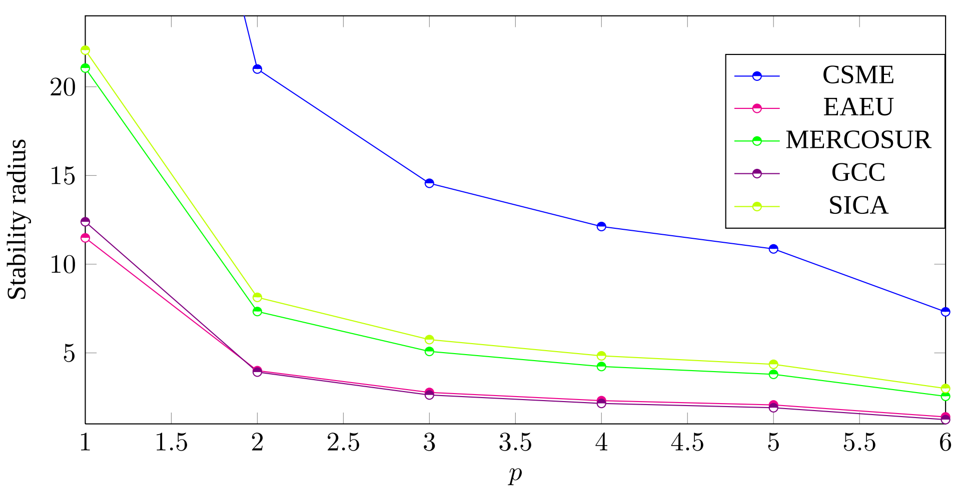
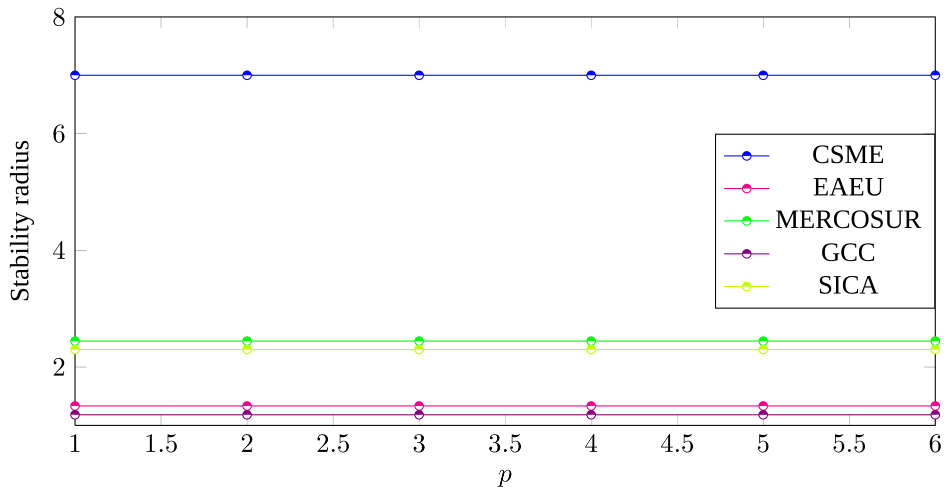


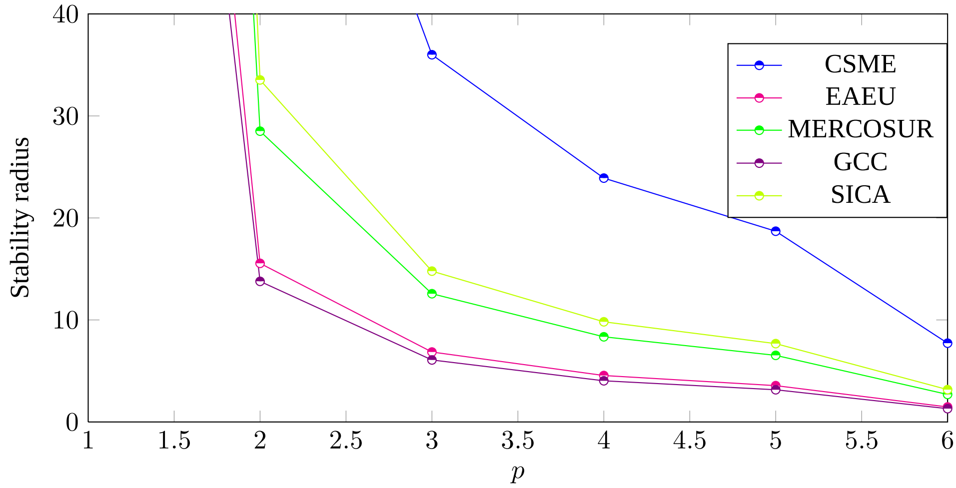
Figure 3: Values for $\varphi_2^s(x^0, m, \infty, p, \infty)$



risk, we used bottleneck partial objectives that make the investor to choose a portfolio with a minimum total level of risk in the worst scenario in the market, i.e., in a situation where the risk values are at the maximum.

Another challenge, while measuring various risks, is associated with inaccuracies of statistical observations and expert assessments. In this context, there is a need to perform post-optimal analysis for the quantitative evaluation of an extreme level of initial changes in data that do not violate the portfolio optimality. In this work, the different cases are analyzed depending on the type of metric, used in the problem parameter space. In every considered cases the lower and upper bounds for the stability radius of an effective portfolio have been specified. The straightforward application of the results to practical calculation is limited due to the enumerative structure of analytical expressions, which may require a number of comparisons growing exponentially with n and s. In the case when direct calculation is time consuming (it may happen if $n \geq 40$ and $s \geq 3$), getting the values should be calculated heuristically, for example

Figure 4: Values for $\psi_2^s(x^0, m, \infty, p, \infty)$



some multicriteria genetic algorithms can be used.

6 Disclosure

We have no potential conflict of interests.

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Figure 5: Values for $\varphi_3^s(x^0, m, \infty, \infty, p)$

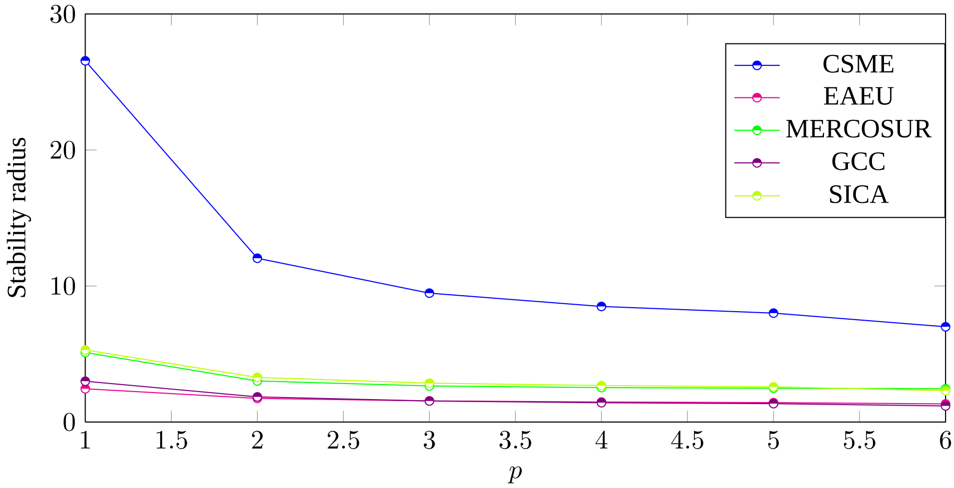
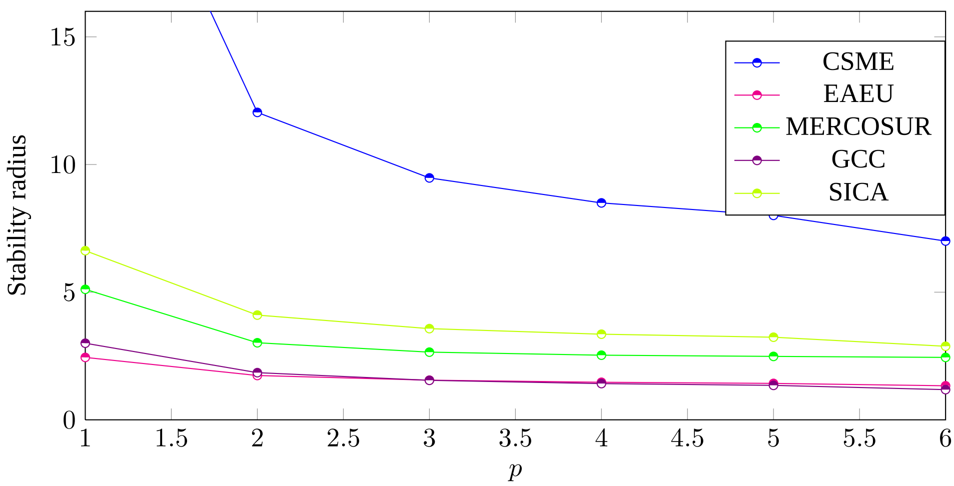


Figure 6: Values for $\psi_3^s(x^0, m, \infty, \infty, p)$



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7 Appendix (for reviewing only,not for publishing)

The following lemma can be easily proven by contradiction.

Lemma. Let $x^0 \in P^s(R)$, $\gamma > 0$. If for any portfolio $x \in X \setminus \{x^0\}$ and every perturbing matrix $R' \in \Omega(\gamma)$ there exists an index $l \in N_s$ such that $g_l(x, x^0, R_l + R'_l) > 0$. Then x^0 is Pareto optimal in any perturbed problem $Z^s(R + R')$, i.e. $x^0 \in P^s(R + R')$ as $R' \in \Omega(\gamma)$.

Proof of Theorem 1.

Proof. Let $x^0 \in P^s(R)$. First we prove $\rho_1 \geq \varphi_1$. The claim is evident if $\varphi_1 = 0$. Assume $\varphi_1 > 0$. According to the definition of φ_1, for any portfolio $x \in X \setminus \{x^0\}$ the inequality

$$\|[g(x, x^0, R)]^+\|_\infty \geq \varphi_1(\|x\|_{p'} + \|x^0\|_{p'}) \tag{9}$$

holds. Further, we are going to prove by contradiction that

$$\forall R' \in \Omega(\varphi_1) \quad \exists l \in N_s \quad (g_l(x, x^0, R'_l) > 0).$$

Suppose, there exists the perturbing matrix $R^0 \in \Omega(\varphi_1)$ with cuts R^0_k, $k \in N_s$ such that

$$g_k(x, x^0, R_k + R^0_k) \leq 0, \quad k \in N_s.$$

Then due to (4) for any $k \in N_s$, we obtain

$$0 \geq g_k(x, x^0, R_k + R^0_k) = \max_{i \in N_m}(R_{ik} + R^0_{ik})x - \max_{i \in N_m}(R_{ik} + R^0_{ik})x^0 =$$

$$= \min_{i' \in N_m} \max_{i \in N_m}(R_{ik}x - R_{i'k}x^0 + R^0_{ik}x - R^0_{i'k}x^0) \geq$$

$$\geq g_k(x, x^0, R_k) - \|R^0_k\|_{p\infty}(\|x\|_{p'} + \|x^0\|_{p'}) \geq$$

$$\geq g_k(x, x^0, R_k) - \|R^0\|_{p\infty\infty}(\|x\|_{p'} + \|x^0\|_{p'}) > g_k(x, x^0, R_k) - \varphi_1(\|x\|_{p'} + \|x^0\|_{p'}).$$

From the statements above, we deduce

$$\|[g(x, x^0, R)]^+\|_\infty < \varphi_1(\|x\|_{p'} + \|x^0\|_{p'}),$$

and it contradicts to (9). Finally, using Lemma, we get $x^0 \in P^s(R + R')$ for any $R' \in \Omega(\varphi_1)$. Hence, $\rho_1 \geq \varphi_1$.

Now we prove that $\rho_1 \leq \psi_1$. According to definition of $\psi_1 > 0$ there exists a portfolio $x^* \in X \setminus \{x^0\}$ such that

$$g_k(x^*, x^0, R_k) \leq [g_k(x^*, x^0, R_k)]^+ \leq$$

$$\leq \|[g(x^*, x^0, R)]^+\|_\infty = \psi_1 \|x^* - x^0\|_{p'}, \quad k \in N_s. \tag{10}$$

Assuming $\varepsilon > \psi_1$, consider a perturbing matrix $R^0 = [r^0_{ijk}] \in \mathbf{R}^{m \times n \times s}$ with elements

$$r^0_{ijk} = \delta \, \frac{x^0_j - x^*_j}{\|x^* - x^0\|_p}, \quad i \in N_m, \;\; j \in N_n, \;\; k \in N_s,$$

where $\psi_1 < \delta < \varepsilon$. Since in any cuts $R^0_k \in \mathbf{R}^{m \times n}$, $k \in N_s$, all the rows R^0_{ik}, $i \in N_m$, are the same (let A denotes such a row), we have

$$A = \delta \frac{(x^0 - x^*)^T}{\|x^* - x^0\|_p}. \tag{11}$$

Therefore, $\|R^0\|_{p\infty\infty} = \|R^0_k\|_{p\infty} = \|R^0_{ik}\|_p = \|A\|_p = \delta$, $i \in N_m$, $k \in N_s$, and, hence $R^0 \in \Omega(\varepsilon)$ for any $\varepsilon > \delta$. Further, due to (2) and (11), for any $p \in [1, \infty]$ the chain of equalities is true

$$A(x^* - x^0) = -\delta \frac{\|x^* - x^0\|_1}{\|x^* - x^0\|_p} = -\delta \|x^* - x^0\|_{p'}.$$

Finally, using the above equalities along with (10), we conclude that for any $k \in N_s$ the following is true

$$g_k(x^*, x^0, R_k + R^0_k) = \max_{i \in N_m} (R_{ik} + A)x^* - \max_{i \in N_m} (R_{ik} + A)x^0 =$$

$$= g_k(x^*, x^0, R_k) + A(x^* - x^0) = g_k(x^*, x^0, R_k) - \delta \|x^* - x^0\|_{p'} <$$

$$< g_k(x^*, x^0, R_k) - \psi_1 \|x^* - x^0\|_{p'} \leq 0.$$

Therefore, $x^0 \notin P^s(R + R^0)$. And hence, $\rho_1 \leq \psi_1$. Theorem 1 is proved. $\square$

Proof of Theorem 2.

Proof. Let $\varphi_1 > 0$. To fulfill the inequality $\varphi_1 < \psi_1$ it is sufficient that inequality $\|x + x^0\|_1 > \|x - x^0\|_1$ holds for any $x \in X \setminus \{x^0\}$. To prove $\rho_1 = \varphi_1$, according to Theorem 1, it is sufficient to specify a class of problems with $\rho_1 \leq \varphi_1$. So, the rest of the proof is about this. From the definition of $\varphi_1 > 0$ there exists $x^* \in X \setminus \{x^0\}$ with

$$\varphi_1 \|x^* + x^0\|_1 \geq g_k(x^*, x^0, R_k), \quad k \in N_s. \tag{12}$$

Further conclusions are true for any $k \in N_s$. Denote

$$i(x^0) = \arg\max\{R_{ik} x^0 : \; i \in N_m\},$$

$$i(x^*) = \arg\max\{R_{ik}x^* : \ i \in N_m\},$$

$$\Delta = \|x^* + x^0\|_1 - \|x^* - x^0\|_1 > 0.$$

Further assume

$$(R_{i(x^*)k} - R_{i(x^0)k})x^* > \varphi_1 \Delta, \tag{13}$$

which implies $i(x^0) \neq i(x^*)$ since $\varphi_1 \Delta > 0$. For any $\varepsilon > \varphi_1$ the elements of the cut R_k^0 in the perturbing matrix R^0 we define as follows

$$r_{ijk}^0 = \begin{cases} \delta, & \text{if } \ i = i(x^0), \ \ x_j^0 = 1, \\ \\ -\delta, & \text{otherwise,} \end{cases} \tag{14}$$

where

$$\min\left\{\varepsilon, \frac{1}{\Delta}(R_{i(x^*)k} - R_{i(x^0)k})x^*\right\} > \delta > \varphi_1. \tag{15}$$

Notice also that last inequalities are valid due to (13). Because of the specific construction of R_k^0 we have

$$R_{ik}^0 x^* = -\delta\|x^*\|_1, \quad \ i \in N_m \setminus \{i(x^0)\}, \tag{16}$$

$$R_{i(x^0)k}^0 x^0 = \delta\|x^0\|_1, \tag{17}$$

$$\|R_k^0\|_{p\infty} = \|R^0\|_{p\infty\infty} = \delta, \ \ R^0 \in \Omega(\varepsilon).$$

Additionally,

$$R_{i(x^0)k}^0 x^* = \delta(\Delta - \|x^*\|_1). \tag{18}$$

Indeed, let

$$Q_1 = \{j \in N_n : \ \ x_j^* = x_j^0 = 1\},$$

$$Q_2 = \{j \in N_n : \ \ x_j^* = 1, \ x_j^0 = 0\}.$$

Then

$$|Q_1| = \Delta/2,$$

$$|Q_2| = \|x^*\|_1 - \Delta/2,$$

$$R_{i(x^0)k}^0 x^* = \delta(|Q_1| - |Q_2|),$$

and (18) follows.

Further we prove $g_k(x^*, x^0, R_k + R_k^0) < 0$. According to (17) we have

$$f_k(x^0, R_k + R_k^0) = \max_{i \in N_m}(R_{ik} + R_{ik}^0)x^0 = f_k(x^0, R_k) + \delta\|x^0\|_1. \tag{19}$$

Now we show

$$f_k(x^*, R_k + R_k^0) = f_k(x^*, R_k) - \delta\|x^*\|_1. \tag{20}$$

Using (16), we yield

$$f_k(x^*, R_k + R_k^0) = \max\Big\{(R_{i(x^*)k} + R_{i(x^*)k}^0)x^*,\ \max_{i \neq i(x^*)}(R_{ik} + R_{ik}^0)x^*\Big\} =$$

$$= \max\Big\{(f_k(x^*, R_k) - \delta\|x^*\|_1),\ \max_{i \neq i(x^*)}(R_{ik} + R_{ik}^0)x^*\Big\}.$$

Therefore, taking into account that

$$f_k(x^*, R_k) - \delta\|x^*\|_1 \geq (R_{ik} + R_{ik}^0)x^*, \quad i \in N_m \setminus \{i(x^0), i(x^*)\},$$

in order to prove (20) it is sufficient to check the following inequality

$$f_k(x^*, R_k) - \delta\|x^*\|_1 \geq (R_{i(x^0)k} + R_{i(x^0)k}^0)x^*.$$

To do this, we use (15) and (18)

$$f_k(x^*, R_k) - \delta\|x^*\|_1 - (R_{i(x^0)k} + R_{i(x^0)k}^0)x^* =$$

$$= (R_{i(x^*)k} - R_{i(x^0)k})x^* - \delta\|x^*\|_1 - R_{i(x^0)k}^0 x^* >$$

$$> \delta(\Delta - \|x^*\|_1) - R_{i(x^0)k}^0 x^* = 0.$$

Finally, sequentially applying (19), (20), (12) and (15), for any index $k \in N_s$ we get

$$g_k(x^*, x^0, R_k + R_k^0) = g_k(x^*, x^0, R_k) - \delta\|x^* + x^0\|_1 \leq (\varphi_1 - \delta)\|x^* + x^0\|_1 < 0.$$

And hence, the formula below holds

$$\forall \varepsilon > \varphi_1 \quad \exists R^0 \in \Omega(\varepsilon) \quad (x^0 \notin P^s(R + R^0)),$$

which due to $x^0 \in P^s(R)$ produces $\rho_1 \leq \varphi_1$. Summarizing, the correctness of (6) now becomes clear. Theorem 2 is proved. $\square$

Proof of Theorem 4.

Proof. Let $x^0 \in P^s(R)$. First we prove $\rho_2 \geq \varphi_2$. It is evident if $\varphi_2 = 0$. Let $\varphi_2 > 0$. According to the definition of φ_2, for any portfolio $x \in X \setminus \{x^0\}$ the inequality

$$\|[g(x, x^0, R)]^+\|_\infty \geq \varphi_2 \|x + x^0\|_1 \tag{21}$$

is true. Further, by contradiction, we show the correctness of the formula given below

$$\forall R' \in \Omega(\varphi_2) \quad \exists l \in N_s \qquad (g_l(x, x^0, R'_l) > 0).$$

From contrary, let it be so that there exists a perturbing matrix $R^0 \in \Omega(\varphi_2)$ with cuts R^0_k, $k \in N_s$ such that

$$g_k(x, x^0, R_k + R^0_k) \leq 0, \quad k \in N_s.$$

Then according to (7) for any index $k \in N_s$ we obtain

$$0 \geq g_k(x, x^0, R_k + R^0_k) = \max_{i \in N_m} (R_{ik} + R^0_{ik})x - \max_{i \in N_m} (R_{ik} + R^0_{ik})x^0 =$$

$$= \min_{i' \in N_m} \max_{i \in N_m} (R_{ik}x - R_{i'k}x^0 + R^0_{ik}x - R^0_{i'k}x^0) \geq$$

$$\geq g_k(x, x^0, R_k) - \|R^0_k\|_{\infty p} \|x + x^0\|_1 \geq$$

$$\geq g_k(x, x^0, R_k) - \|R^0\|_{\infty p \infty} \|x + x^0\|_1 > g_k(x, x^0, R_k) - \varphi_2 \|x + x^0\|_1.$$

From the statements above, we deduce inequality

$$\|[g(x, x^0, R)]^+\|_\infty < \varphi_2 \|x + x^0\|_1,$$

which contradicts to (21). Finally, applying Lemma, we have $x^0 \in P^s(R + R')$ for every perturbing matrix $R' \in \Omega(\varphi_2)$. Hence, $\rho_2 \geq \varphi_2$.

Now we prove $\rho_2 \leq m^{1/p}\psi_2$. According to the definition $\psi_2 > 0$, there exists a portfolio $x^* \in X \setminus \{x^0\}$ such that

$$g_k(x^*, x^0, R_k) \leq [g_k(x^*, x^0, R_k)]^+ \leq$$

$$\leq \|[g(x^*, x^0, R)]^+\|_\infty = \psi_2 \|x^* - x^0\|_1, \qquad k \in N_s. \tag{22}$$

Assuming $\varepsilon > m^{1/p}\psi_2$, consider a perturbing matrix $R^0 = [r^0_{ijk}] \in \mathbf{R}^{m\times n\times s}$ whose elements are defined as follows

$$r^0_{ijk} = \delta(x^0_j - x^*_j), \quad i \in N_m, \ \ j \in N_n, \ \ k \in N_s,$$

where $\varepsilon/m^{1/p} > \delta > \psi_2$. Since all the rows R^0_{ik}, $i \in N_m$ in the cut $R^0_k \in \mathbf{R}^{m\times n}$, $k \in N_s$ are the same in the matrix $\mathbf{R}^0$, then we have (let $A \in \mathbf{R}^m$ denotes such a row)

$$A = \delta(x^0 - x^*)^T. \tag{23}$$

$$\|R^0_{ik}\|_\infty = \|A\|_\infty = \delta, \quad i \in N_m, \ k \in N_s.$$

From the equalities above and (3), we get

$$\|R^0_k\|_{\infty p} = m^{1/p}\delta, \quad k \in N_s,$$

$$\|R^0\|_{\infty p \infty} = m^{1/p}\delta \geq m^{1/p}\psi_2.$$

Thus $R^0 \in \Omega(\varepsilon)$ for any $\varepsilon > m^{1/p}\psi_2$. Further due to (23), we have

$$A(x^* - x^0) = -\delta\|x^* - x^0\|_1.$$

Finally, combining the equality above and (22), we conclude that for any $k \in N_s$ the following relations are true

$$g_k(x^*, x^0, R_k + R^0_k) = \max_{i\in N_m}(R_{ik} + A)x^* - \max_{i\in N_m}(R_{ik} + A)x^0 =$$

$$= g_k(x^*, x^0, R_k) + A(x^* - x^0) = g_k(x^*, x^0, R_k) - \delta\|x^* - x^0\|_1 <$$

$$< g_k(x^*, x^0, R_k) - \psi_2\|x^* - x^0\|_1 \leq 0.$$

Thus $x^0 \notin P^s(R + R^0)$. Hence, $\rho_2 \leq m^{1/p}\psi_2$. Theorem 4 is proved. $\square$

Proof of Theorem 6.

Proof. Let $x^0 \in P^s(R)$. First we prove $\rho_3 \geq \varphi_3$. Without loss of generality, assume $\varphi_3 > 0$ (otherwise the inequality $\rho_3 \geq \varphi_3$ is obvious). According to the definition of φ_3, for any $x \neq x^0$ the following is true

$$\|[g(x, x^0, R)]^+\|_p \geq \varphi_3\|x + x^0\|_1. \tag{24}$$

To prove the lower bound, it is necessary to show that the formula below is true

$$\forall R' \in \Omega(\varphi_3) \quad \exists l \in N_s \qquad (g_l(x, x^0, R_l + R'_l) > 0). \tag{25}$$

From contrary, assume there exists a perturbing matrix $R^0 \in \Omega(\varphi_3)$ such that

$$g_k(x, x^0, R_k + R^0_k) \leq 0, \quad k \in N_s.$$

Then using (8), we easily deduce

$$0 \geq g_k(x, x^0, R_k + R^0_k) = \min_{i' \in N_m} \max_{i \in N_m} (R_{ik}x - R_{i'k}x^0 + R^0_{ik}x - R^0_{i'k}x^0) \geq$$

$$\geq g_k(x, x^0, R_k) - \|R^0_k\|_{\infty\infty} \|x + x^0\|_1,$$

i.e.

$$g^+_k(x, x^0, R_k) \leq \|R^0_k\|_{\infty\infty} \|x + x^0\|_1, \quad k \in N_s.$$

Thus, due to $R^0 \in \Omega(\varphi_3)$ while $p \in [1, \infty]$ we have

$$\|[g(x, x^0, R)]^+\|_p \leq \|R^0\|_{\infty\infty p} \|x + x^0\|_1 < \varphi_3 \|x + x^0\|_1.$$

This contradicts to (24), and hence (25) is true. From here, according to Lemma, $x^0 \in P^s(R + R')$ for any $R' \in \Omega(\varphi_3)$. Hence, $\rho^s_3(x^0, m, \infty, \infty, p) \geq \varphi^s_3(x^0, m, \infty, \infty, p)$.

Further, we prove that $\rho_3 \leq \psi_3$ holds for any $p \in [1, \infty]$. Let $\varepsilon > \psi_3 > 0$, and the portfolio $x^* \neq x^0$ is such that

$$\|[g(x^*, x^0, R)]^+\|_p = \psi_3 \|x - x^0\|_1.$$

Then, taking into account that the norm l_p depends on the vector continuously, we take $\delta \in \mathbf{R}^s$ with positive components such that

$$\delta_k \|x^* - x^0\|_1 > g^+_k(x^*, x^0, R_k), \qquad k \in N_s, \tag{26}$$

and $\varepsilon > \|\delta\|_p > \psi_3$. Then we construct a perturbing matrix $R^0 \in \Omega(\varepsilon)$, where $\varepsilon > \|\delta\|_p$, with the cuts R^0_k, $k \in N_s$ such that for every $k \in N_s$ the inequality

$$g_k(x^*, x^0, R_k + R^0_k) < 0 \tag{27}$$

holds. Using components of the vector δ, we define the elements of any k-th cut $R_k^0 = [r_{ijk}^0] \in \mathbf{R}^{m\times n}$ of the perturbing matrix $R^0 = [r_{ijk}^0] \in \mathbf{R}^{m\times n\times s}$ using the formula

$$r_{ijk}^0 = \begin{cases} \delta_k, & \text{if } i \in N_m, \ x_j^0 \geq x_j^*, \\ -\delta_k, & \text{if } i \in N_m, \ x_j^0 < x_j^*. \end{cases}$$

Then, we have

$$\|R_k^0\|_{\infty\infty} = \delta_k, \quad k \in N_s.$$

Therefore, it is easy to see that $\|R^0\|_{\infty\infty p} = \|\delta\|_p < \varepsilon$. Additionally, all the rows R_{ik}^0 ($i \in N_m$) in the cut R_k^0, $k \in N_s$ are the same and contain components δ_k and $-\delta_k$ only. Denoting such a row as A_k, we obtain

$$A_k(x^* - x^0) = -\delta_k\|x^* - x^0\|_1, \quad k \in N_s.$$

From this for any $k \in N_s$ due to (26) we get (27) as follows:

$$g_k(x^*, x^0, R_k + R_k^0) = g_k(x^*, x^0, R_k) + A(x^* - x^0) =$$

$$= g_k(x^*, x^0, R_k) - \delta_k\|x^* - x^0\|_1 \leq$$

$$\leq g_k^+(x^*, x^0, R_k) - \delta_k\|x^* - x^0\|_1 < 0, \quad k \in N_s.$$

Thus, while $\varepsilon > \psi_3$ there exists a perturbing matrix $R^0 \in \Omega(\varepsilon)$ such that $x^0 \in P^s(R)$ is not Pareto optimal in the perturbed problem $Z^s(R + R^0)$. This implies that for any $\varepsilon > \psi_3$ we have $\rho_3 < \varepsilon$. Hence, $\rho_3 \leq \psi_3^s$, and then $p \in [1, \infty]$. Theorem 4 is proved. $\square$