On Expansivity and Pseudo-Orbit Tracing Property for Cellular Automata*

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February 17, 2020

Abstract

Ultimate expansivity extends concepts of expansivity and positive expansivity. We consider one-sided variants of ultimate expansivity and pseudo-orbit tracing property (also known as the shadowing property) for surjective one-dimensional cellular automata. We show that ultimately right (or left) expansive surjective cellular automata are chain-transitive; this improves a result by Boyle that expansive reversible cellular automata are chain-transitive. We then use this to show that left-sided pseudo-orbit tracing property and right-sided ultimate expansivity together imply pseudo-orbit tracing property for surjective cellular automata. This reproves some known results, most notably some of Nasu's. Our result improves Nasu's result by dropping an assumption of chain-recurrence, however, we remark that this improvement can also be achieved using the Poincaré recurrence theorem.

The pseudo-orbit tracing property implies that the trace subshifts of the cellular automaton are sofic shifts. We end by mentioning that among reversible cellular automata over full shifts we have examples of right expansive cellular automata with non-sofic traces, as well as examples of cellular automata with left pseudo-orbit tracing property but non-sofic traces, illustrating that neither assumption can be dropped from the theorem mentioned above.

This paper is a generalized and improved version of a conference paper presented in AUTOMATA 2018.

1 Introduction

Observing a cellular automaton (in this paper all cellular automata are one-dimensional) through a finite observation window produces a subshift which is called a trace subshift. The trace subshifts are useful in understanding the

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dynamics of cellular automata. For example, cellular automaton has the same entropy as its wide enough trace subshift [1, Lemma 1]. For (reversible) expansive cellular automata trace subshifts are even more important since every (reversible) expansive cellular automaton is conjugate to its wide enough trace subshift. Thus it is not surprising that there has been considerable effort to understand the trace subshifts of expansive cellular automata: In [2, Theorem 10] Kůrka proved that a positively expansive two-sided cellular automaton is conjugate to a one-sided subshift of finite type (henceforth called SFT's). Earlier Nasu, using textile systems, had proven that if a two-sided cellular automaton is conjugate to a one-sided SFT then it is in fact conjugate to a one-sided full shift [3, Theorem 3.12 (1)], and so overall every positively expansive two-sided cellular automaton is conjugate to a full shift. That one-sided positively expansive cellular automata are conjugate to SFT's was proven by Blanchard and Maass [4, Theorem 3.8]. However, an example by Boyle, D. Fiebig and U.-R. Fiebig shows that one-sided positively expansive cellular automata need not be conjugate to one-sided full shifts [5, Example 5.6]. In the case of reversible cellular automata, Nasu has obtained that the trace subshifts of one-sided expansive cellular automata are SFT's and some generalizations of this result [6, 7].

The complexity of the trace subshifts has strong connections to the pseudo-orbit tracing property (also known as the shadowing property): Pseudo-orbit tracing property implies that all trace subshifts are sofic, and if all trace subshifts are SFT's then the cellular automaton has the pseudo-orbit tracing property, however the converses do not hold (see [2] or [8, Section 5.7.1]). For expansive cellular automata the connection is even stronger: An expansive cellular automaton has the pseudo-orbit tracing property if and only if its trace subshifts are SFT's. Kůrka has conjectured [9, Conjecture 30] that every expansive cellular automaton is conjugated to an SFT, or equivalently, has pseudo-orbit tracing property; this conjecture remains unsolved today.

In this paper we consider one-sided variants of expansivity and pseudo-orbit tracing property. We will give a new proof of Boyle's result that reversible expansive cellular automata are chain-transitive [10, Corollary 4.3]. Our result also improves Boyle's in two regards: The cellular automaton does not need to be reversible, only surjective, and we use only one-sided variant of expansivity (extended for surjective cellular automata). We then use this to give a new proof for some known results mentioned above, most notably some of Nasu's results. We do not use textile systems but many ideas and definitions are similar to those considered in textile system theory, thus this paper can provide, to some, a more familiar setting to understand results previously only proved using textile systems.

We will in fact end up with an improved version of some of Nasu's results as chain-transitivity implies, trivially, chain-recurrence which Nasu has as an assumption. However we note that this improvement can also be achieved in a different, and more direct, way: Nasu remarks that "It is an open problem whether an onto endomorphism ϕ of a mixing SFT has ϕ-periodic points dense [...]. If the answer is affirmative, then the chain recurrence condition on ϕ in Theorem 6.3 and that on $\tilde{\phi}$ in Corollary 6.4 can be removed." [7, pp. 185],

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but there is no need to solve this longstanding open problem as denseness of recurrent points implies chain-recurrence, and this follows from the Poincaré recurrence theorem. Alternatively, the first half of the proof of our Theorem 1 (that surjectivity and right-expansivity together imply chain-transitivity) proves exactly what Nasu uses chain-recurrence to prove.

This paper is a generalized version of the conference paper [11] presented in AUTOMATA 2018. In [11] all results considered cellular automata over full shifts, in this paper we extend the results for more general shift spaces, in particular our assumptions regarding the underlying shift spaces of cellular automata are now the same as in the two main references [10, 7].

The paper is organized as follows. In Section 2 we review basic definitions of symbolic dynamics and mention some relevant known results. In Section 3 we introduce the notions of ultimate expansivity and one-sided pseudo-orbit tracing property (right- and left-POTP) for cellular automata. We also prove some simple results which provide visual interpretations for various definitions. In Section 4 we prove our improved version of Boyle's result mentioned above. In Section 5 we prove that left-POTP and ultimate right-expansivity imply together that the trace subshifts are sofic, which reproves some known results regarding traces of cellular automata. In Section 6 we remark that neither the assumption of left-POTP nor that of ultimate right-expansivity can be dropped from the theorem proved in Section 5.

2   Preliminaries

2.1   Notations

For two integers $i, j \in \mathbb{Z}$ such that $i < j$ the interval from i to j is denoted $[i, j] = \{i, i+1, \ldots, j\}$. We also denote $[i, j) = \{i, i+1, \ldots, j-1\}$ and $(i, j] = \{i+1, \ldots, j\}$. Notation $\mathbb{M}$ is used when it does not matter whether we use $\mathbb{N}$ or $\mathbb{Z}$. Composition of functions $f : X \to Y$ and $g : Y \to Z$ is written as gf and defined by $gf(x) = g(f(x))$ for all $x \in X$.

Sometimes we want to write indices as superscripts. We then write the index in parenthesis to separate them from exponents (i.e. $x^{(i)}$ denotes an element indexed by i while x^i denotes the ith power of x).

2.2   Topological Dynamics

A *(topological) dynamical system* is a pair (X, f) where X is a compact metric space and f a continuous map $X \to X$. Let (X, f) and (Y, g) be two dynamical systems. A continuous map $\phi : X \to Y$ is a *homomorphism* if $\phi f = g\phi$. If ϕ is surjective, it is a *factor map*, and (Y, g) is a *factor* of (X, f). If ϕ is injective, it is an *embedding*, and (X, f) is a *subsystem* of (Y, g). If ϕ is a bijection, it is a *conjugacy*, and (X, f) and (Y, g) are *conjugate*. Let $d : X \times X \to \mathbb{R}_+ \cup \{0\}$ be the metric considered. A sequence $(x_i)_{i \in \mathbb{Z}}$ is a *(two-way) orbit* of f if $f(x_i) = x_{i+1}$ for every $i \in \mathbb{Z}$. Let $x, y \in X$. There is an *ε-chain from x to y* if there exists

$n > 0$ and a sequence $x = x_0, x_1, \ldots, x_n = y \in X$ such that $d(f(x_i), x_{i+1}) < \varepsilon$, for all $i \in \{0, 1, \ldots, n-1\}$. Two-way infinite ε-chains are called ε-*pseudo-orbits*. The dynamical system (X, f)

- is *recurrent* if for every non-empty open set U there exists $n > 0$ such that $F^n(U) \cap U \neq \emptyset$.

- is *transitive* if for all non-empty open sets U, V there exists $n > 0$ such that $F^n(U) \cap V \neq \emptyset$.

- is *mixing* if for all non-empty open sets U, V there exists $N > 0$ such that for all $n \geq N$ it holds that $F^n(U) \cap V \neq \emptyset$.

- is *chain-recurrent* if for all $x \in X$ and $\varepsilon > 0$ there exists an ε-chain from x to x.

- is *chain-transitive* if for all $x, y \in X$ and $\varepsilon > 0$ there exists an ε-chain from x to y.

- is *chain-mixing* if for all $x, y \in X$ and $\varepsilon > 0$ there exists $N > 0$ such that for all $n \geq N$ there exists an ε-chain $x = x_0, x_1, \ldots, x_n = y$ from x to y.

- has the *pseudo-orbit tracing property (POTP)*, often also called the *shadowing property*, if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-pseudo-orbit $(x_i)_{i \in \mathbb{Z}}$ there exists an orbit $(y_i)_{i \in \mathbb{Z}}$ such that $d(x_i, y_i) < \epsilon$ for all $i \in \mathbb{Z}$.

A point $x \in X$ is f-*periodic* if there exists $n \in \mathbb{N} \setminus \{0\}$ such that $f^n(x) = x$. The set of all f-periodic points is denoted by $\mathrm{Per}_f(X)$.

2.3 Symbolic Dynamics

A finite non-empty set A is called an *alphabet*. The functions $\mathbb{M} \to A$ are called *(one-dimensional) configurations* and the set of all configurations over A is denoted by $A^{\mathbb{M}}$. For us $\mathbb{M}$ will usually be $\mathbb{Z}$. We denote $c_i = c(i)$ for $c \in A^{\mathbb{M}}$ and $i \in \mathbb{M}$. We consider $A^{\mathbb{M}}$ to be a metric space with the metric

$$d(c, e) = \begin{cases} 2^{-\min(\{|i| \,|\, c_i \neq e_i\})}, & \text{if } c \neq e \\ 0, & \text{if } c = e \end{cases}$$

for all $c, e \in A^{\mathbb{M}}$. Let $D \subseteq \mathbb{M}$ be finite and $u \in A^D$, then the set $[u] = \{c \in A^{\mathbb{M}} \mid c_D = u\}$ is called a *cylinder*. It is well-known that the topology induced by metric d is compact and that cylinders form a countable clopen (open and closed) base of this topology.

For any $n \in \mathbb{N}$ we denote by A^n the *words of length n* (i.e. finite sequences of n symbols). We also denote $A^+ = \bigcup_{n \in \mathbb{N} \setminus \{0\}} A^n$ and call any subset $L \subset A^0 \cup A^+$ a *language*. The language L is *finite* if L is finite, and *regular* if it is accepted by a finite state automaton. For a word $u \in A^n$ we denote by u^ω the configuration $c \in A^{\mathbb{N}}$ defined by $c_i = u_{i \mod n}$, and in the same way we define $^\omega u^\omega \in A^{\mathbb{Z}}$.

The *shift map* $\sigma : A^{\mathbb{M}} \to A^{\mathbb{M}}$, defined by $\sigma(c)_i = c_{i+1}$ for all $i \in \mathbb{M}$, is continuous. A subset $X \subseteq A^{\mathbb{M}}$ which is non-empty, topologically closed, and such that $\sigma^i(X) \subseteq X$ for all $i \in \mathbb{M}$, is a *shift space* or a *subshift*. The dynamical system $(A^{\mathbb{Z}}, \sigma)$ is the *full (A-)shift*. A configuration $c \in A^{\mathbb{M}}$ *avoids* $u \in A^n$ if $\sigma^i(c)_{[0,n)} \neq u$ for all $i \in \mathbb{M}$, otherwise u *appears* in c. Let $S \subseteq A^+$, and let X_S be the set of configurations that avoid S, i.e. $X_S = \{c \in A^{\mathbb{M}} \mid \forall u \in S : c \text{ avoids } u\}$. It is well-known that the given topological definition of subshift X is equivalent to saying that there exists a set of forbidden words S such that $X = X_S$. If there exists a finite set S such that $X = X_S$, then X is a *subshift of finite type (SFT)*. A factor of an SFT is a *sofic shift*. It is known that a subshift X is sofic if and only if there exists a regular language S such that $X = X_S$.

The *language of a subshift* (X, σ) is the set of words that appear in some configuration of X, and is denoted by $\mathcal{L}(X)$. We also denote $\mathcal{L}_n(X) = \mathcal{L}(X) \cap A^n$ the set of words of length n that appear in X. The subshift (X, σ) is transitive if and only if for every $u, v \in \mathcal{L}(X)$ there exists $w \in \mathcal{L}(X)$ such that $uwv \in \mathcal{L}(X)$. The subshift (X, σ) is mixing if and only if for every $u, v \in \mathcal{L}(X)$ there exists N such that for every $n \geq N$ there exists $w \in \mathcal{L}_n(X)$ such that $uwv \in \mathcal{L}(X)$. We will use the following fact about transitive SFT's.

Proposition 1 ([12, §4.5]). *Let (X, σ) be a transitive SFT. There exists $k \in \mathbb{N}$ such that (X, σ^k) is a finite union of disjoint mixing SFT's.*

The *entropy* of (X, σ) is the exponential growth rate of the number of appearing words as the length of words increases:

$$h(X, \sigma) = \lim_{n \to \infty} \frac{1}{n} \log_2(|\mathcal{L}_n(X)|).$$

We only need the following facts about entropy.

Proposition 2 ([12, Proposition 4.1.9.]). *Let X and Y be subshifts and $F : X \to Y$ a factor map. Then $h(Y, \sigma) \leq h(X, \sigma)$.*

Proposition 3 ([12, Corollary 4.4.9.]). *Let X be a transitive sofic shift and $Y \subseteq X$ a subshift. If $h(Y, \sigma) = h(X, \sigma)$, then $X = Y$.*

We refer to [12] for a more detailed introduction to symbolic dynamics.

2.4 Cellular Automata

A *cellular automaton* is a dynamical system (X, F) where $X \subseteq A^{\mathbb{M}}$ is a shift space and F is a shift-commuting map, i.e. $F\sigma = \sigma F$. In other words, cellular automata are endomorphisms of shift spaces. When $\mathbb{M} = \mathbb{N}$, the cellular automaton is called *one-sided* and when $\mathbb{M} = \mathbb{Z}$, the cellular automaton is called *two-sided*. We will sometimes refer to a cellular automaton by the function name alone, i.e. talk about a cellular automaton F. The cellular automaton (X, F) is *reversible* if there exists a cellular automaton (X, F') such that $F'F(c) = c = FF'(c)$ for all $c \in X$. Let $D = [i, j] \subset \mathbb{M}$ and let $G_{loc} : A^D \to A$ be a function. Define $G : A^{\mathbb{M}} \to A^{\mathbb{M}}$ by $G(c)_i = G_{loc}((\sigma^i(c))_D)$ for all $i \in \mathbb{M}$. Then

G is continuous and commutes with σ, so it is a cellular automaton. The set D is a *local neighborhood* of G and the function G_{loc} is a *local rule* of G. According to the Curtis-Hedlund-Lyndon Theorem all cellular automata are defined by a local rule. If G is defined by a local rule $G_{loc} : A^{[-m,a]} \to A$, where $-m \leq a$, we say that G has *anticipation* a, *memory* m and *radius* $r = \max\{m, a\}$. If $m \leq 0$ then G is *memoryless*. We overload the notation for the global function and use it also on finite words: For any $n > m + a$ and $u \in A^n$ we define in a natural way the word $G(u) \in A^{n-(m+a)}$.

For a cellular automaton (X, F) the *space-time diagrams* are the two-way infinite orbits. The set of space-time diagrams of F is denoted by

$$st(F) = \{(c^{(i)})_{i \in \mathbb{Z}} \in X^{\mathbb{Z}} \mid \forall i : F(c^{(i)}) = c^{(i+1)}\}.$$

As a pictorial presentation we consider these as coloured square lattices where rows are the points of the orbit and time advances downwards; left, right, up, and down should be understood accordingly. Notice that for us orbits, or space-time diagrams, are two-way infinite, so that we only consider points which can be seen after arbitrarily many time steps, in other words points which are in the *limit set* $\Lambda_F = \cap_{i \in \mathbb{N}} F^i(X)$ of the cellular automaton. Of course for a surjective cellular automaton $\Lambda_F = X$.

In [11] we used the balancedness property of surjective cellular automata [13] to show that ultimately right-expansive surjective cellular automata over full shifts are chain-transitive. In extending this result to cellular automata over more general shift spaces we need a more general version of the balancedness property; luckily measure theory has provided such. The following proposition is sufficient for our purposes and is directly extracted from known results.

Proposition 4. *Let X be a mixing sofic shift, (X, F) a surjective cellular automaton, and $U \subseteq X$ a clopen set. If $F(U) \subseteq U$ then $F(U) = U = F^{-1}(U)$.*

Proof. As measure theory plays no further role in this paper, we will not go into details. The following outlines known results that can be used to conclude the claim:

According to [14] X has a unique σ-invariant measure of maximal entropy μ (known as the *Parry measure*, originally presented in [15] for SFT's). From the definition one sees that $\mu(U) > 0$ for every non-empty open set $U \subseteq X$.

The *push-forward* measure of μ under F is defined by $F(\mu)(U) = \mu(F^{-1}(U))$ for any clopen set $U \subseteq X$. According to [16, Theorem 3.3.] every σ-invariant measure of maximal entropy is the push-forward measure under F of some σ-invariant measure of maximal entropy. Since μ is the unique σ-invariant measure of maximal entropy, we have that $F(\mu) = \mu$.

Now suppose that $F(U) \subseteq U$ for some clopen set $U \subseteq X$. Then $U \subseteq F^{-1}(U)$. Now we have that $\mu(F^{-1}(U) \backslash U) = \mu(F^{-1}(U)) - \mu(U) = 0$, and since $F^{-1}(U) \backslash U$ is open we get that $F^{-1}(U) \backslash U = \emptyset$. So we have that $F^{-1}(U) \subseteq U$, and thus $F^{-1}(U) = U$. Since F is surjective we also have that $F(U) = U$. $\square$

3 Ultimate Expansivity and One-Sided Pseudo-Orbit Tracing Property

3.1 Ultimate Expansivity

Expansivity of a dynamical system is a strong form of sensitivity where any small change in the state of the system is magnified above a fixed threshold under the time evolution. Here we consider the evolution of a cellular automaton in its limit set, both backward and forward in time. In other words, we are interested in the propagation of perturbations along two-way infinite orbits. If not otherwise stated the shift space X is assumed to be a subshift of $A^{\mathbb{Z}}$.

A cellular automaton (X, F) is *ultimately expansive* if there exists $\varepsilon > 0$ such that for all space-time diagrams $(c^{(i)})_{i \in \mathbb{Z}}, (e^{(i)})_{i \in \mathbb{Z}} \in st(F)$ it holds that

$$c^{(0)} \neq e^{(0)} \implies \left(\exists n \in \mathbb{Z} : d(c^{(n)}, e^{(n)}) > \varepsilon \right). \tag{1}$$

The cellular automaton is *ultimately right-expansive* if there exists $\varepsilon > 0$ such that for all space-time diagrams $(c^{(i)})_{i \in \mathbb{Z}}, (e^{(i)})_{i \in \mathbb{Z}} \in st(F)$ it holds that

$$\left(\exists i > 0 : c_i^{(0)} \neq e_i^{(0)} \right) \implies \left(\exists n \in \mathbb{Z} : d(F^n(c), F^n(e)) > \varepsilon \right). \tag{2}$$

Ultimately left-expansive is defined analogously.

In terms of multi-dimensional symbolic dynamics, the set $st(F)$ of space-time diagrams of cellular automaton F is a two-dimensional subshift. Boyle and Lind studied in [17] expansive subspaces of multidimensinonal subshifts. Ultimate expansivity means in their terminology that the vertical direction (that is, the temporal direction) is an expansive direction for subshift $st(F)$.

The terminology regarding expansivity somewhat varies, but one common practice is to call *expansive* cellular automata which are reversible and fulfill (1) and to call *positively expansive* cellular automata which are surjetive and fulfill (1) where "$\exists n \in \mathbb{Z}$" is replaced with "$\exists n \in \mathbb{N}$". Then by definition, positively expansive cellular automata and expansive cellular automata are ultimately expansive. Next examples show that ultimate expansivity covers cases which expansivity and positive expansivity do not.

Example 3.1. A cellular automaton $(A^{\mathbb{Z}}, F)$ is *nilpotent* if there exists $q \in A$ and $n \in \mathbb{N}$ such that for every $c \in A^{\mathbb{Z}}$ we have that $F^n(c) = {}^{\omega} q^{\omega}$. Then $st(F)$ is a singleton and it follows that F is ultimately expansive.

There cannot be a reversible cellular automaton that would be ultimately expansive but not expansive, as ultimate expansivity and reversibility together are equivalent to expansivity. However there are surjective ultimately expansive cellular automata which are neither expansive nor positively expansive:

Example 3.2. Let $A = \{0, 1\}$, $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be the shift map and $\mathcal{X} : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be the two-sided XOR -cellular automaton, that is the cellular automaton defined by $\mathcal{X}_{loc} : A^{[-1,1]} \to A$, $\mathcal{X}_{loc}(abc) = a \oplus c$, where $\oplus$ denotes addition modulo 2. The shift map σ is expansive, and so also ultimately expansive. The

XOR -cellular automaton cannot be expansive as it is not reversible, however it is positively expansive, and so also ultimately expansive. Consider the direct product of these, that is the cellular automaton $\sigma \times \mathcal{X} : A^{\mathbb{Z}} \times A^{\mathbb{Z}} \to A^{\mathbb{Z}} \times A^{\mathbb{Z}}$ defined by $(\sigma \times \mathcal{X})(c, e) = (\sigma(c), \mathcal{X}(e))$. This is neither expansive (not even reversible) nor positively expansive (since σ is not). However, $\sigma \times \mathcal{X}$ is ultimately expansive.

The following lemma states the well-known consequence of compactness that ultimate right-expansivity implies a deterministic local rule in the horizontal direction of space-time diagrams.

Lemma 1. *A cellular automaton (X, F) is ultimately right-expansive if and only if there exists $m, n \in \mathbb{N}$ such that for all space-time diagrams $(c^{(j)})_{j \in \mathbb{Z}}$ and $(e^{(j)})_{j \in \mathbb{Z}}$ the following holds*

$$\left(\forall j \in \{0, \ldots, 2n\} : c^{(j)}_{[0,m)} = e^{(j)}_{[0,m)} \right) \implies c^{(n)}_m = e^{(n)}_m. \tag{3}$$

Proof. For a space-time diagram $S = (s^{(i)})_{i \in \mathbb{Z}} \in st(F)$ we denote $S_{(x,y)} = s_x^{(y)}$ and similarly we define S_D for any subset $D \subseteq \mathbb{Z}^2$. Suppose the claim does not hold so that we have a sequence of pairs of space-time diagrams $(S(i), R(i))_{i \in \mathbb{N}} \in (st(F) \times st(F))^{\mathbb{N}}$ such that for every $N \in \mathbb{N}$ it holds that $S(N)_{[-N,0] \times [-N,N]} = R(N)_{[-N,0] \times [-N,N]}$ and $S(N)_{(1,0)} \neq R(N)_{(1,0)}$. This sequence has a converging subsequence $(S(i), R(i))_{i \in \mathcal{I}}$; let $(S, R) = ((s^{(i)})_{i \in \mathbb{Z}}, (r^{(i)})_{i \in \mathbb{Z}}) \in st(F) \times st(F)$ be the limit of this sequence. Now we have a counterexample to ultimate right-expansivity: For any $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that $((\sigma^{-k}(s^{(i)}))_{i \in \mathbb{Z}}, (\sigma^{-k}(r^{(i)}))_{i \in \mathbb{Z}}) \in st(F) \times st(F)$ contradicts the right-sided variant of (1). $\square$

In what follows we will consider configurations of $(A^n)^{\mathbb{Z}}$ where $n > 1$, which can lead to indexing issues. To avoid these issues we define for every $n \in \mathbb{N} \setminus \{0\}, i, j \in \{0, \ldots, n-1\}$ where $i \leq j$ projections $\pi_i : A^n \to A$ and $\pi_{[i,j]} : A^n \to A^{j-i+1}$ where $\pi_i(a_0 \cdots a_{n-1}) = a_i$ and $\pi_{[i,j]}(a_0 \cdots a_{n-1}) = a_i \cdots a_j$. We also extend these to $(A^n)^{\mathbb{Z}}$ cell-wise.

Let (X, F) be a cellular automaton with radius r. The *n-trace of F* is defined as

$$\tau_n(F) = \left\{ t \in (A^n)^{\mathbb{Z}} \mid \exists (c^{(i)})_{i \in \mathbb{Z}} \in st(F) : \forall j \in \mathbb{Z} : t_j = c^{(j)}_{[0,n)} \right\}$$

Notice that our traces are two-sided subshifts, which is in line with our definition of space-time diagrams: Traces are vertical stripes of space-time diagrams (Figure 2).

Let (X, F) be an ultimately right-expansive cellular automaton. Then the lemma above says that we have $m \in \mathbb{N}$ such that we can define a cellular automaton $(\tau_m(F), \overrightarrow{F})$ such that for every $t \in \tau_m(F)$ we have that $\overrightarrow{F}(t) \in \tau_m(F)$ is the unique configuration such that $\pi_{[1,m)}(t) = \pi_{[0,m-1)}(\overrightarrow{F}(t))$ and the last column of $\overrightarrow{F}(t)$ is the column defined by (3) (Figure 1). Then $\{(\pi_0(\overrightarrow{F}^i(t)))_{i \in \mathbb{N}} \mid t \in \tau_m(F)\}$ is the set of right halves of $st(F)$.
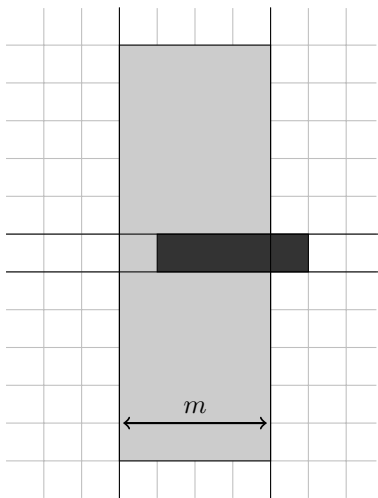
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Figure 1: An ultimately right-expansive cellular automaton defines a cellular automaton $\overrightarrow{F}$ which draws the (right halves) of the space-time diagrams. The figure illustrates how $\overrightarrow{F}$ is defined; assuming that the grid has a valid space-time diagram of F, then $\overrightarrow{F}$ maps the pattern in the light gray rectangle to the pattern in the dark gray area.

Notice that if F is surjective and ultimately expansive then (X, F) is a factor of $(\tau_m(F), \sigma)$ so $h(X, F) \leq h(\tau_m(F), \sigma)$. On the other hand, let $\psi : (A^m)^{\mathbb{Z}} \to (A^m)^{\mathbb{N}}$ be the map defined by $\psi(\cdots c_{-1} c_0 c_1 \cdots) = c_0 c_1 \cdots$. Then $(\psi(\tau_m(F)), \sigma)$ is a factor of (X, F) so $h(\psi(\tau_m(F)), \sigma) \leq h(X, F)$. Since $\mathcal{L}(\tau_m(F)) = \mathcal{L}(\psi(\tau_m(F)))$ we have that $h(\tau_m(F), \sigma) = h(\psi(\tau_m(F)), \sigma)$ and so $h(X, F) = h(\tau_m(F), \sigma)$. If (X, F) is expansive, it is conjugate to $(\tau_m(F), \sigma)$. If (X, F) is positively expansive, it is conjugate to $(\psi(\tau_m(F)), \sigma)$.

3.2 One-Sided Pseudo-Orbit Tracing Property

Next we consider pseudo-orbits for cellular automata. Let (X, F) be a cellular automaton with radius r. For every $n \in \mathbb{N} \setminus \{0\}$ we define a directed labeled multigraph $\mathcal{G}_n(F) = (V_n, E_n)$ as follows:

- The set of vertices is $V_n = \mathcal{L}_n(X)$.

- For every $u \in V_n$ and $x, y \in \mathcal{L}_r(X)$ such that $xuy \in \mathcal{L}_{n+2r}(X)$ there is a labeled edge $u \xrightarrow{xy} F(xuy)$ in E_n.

The graph $\mathcal{G}_n(F)$ defines an SFT $\mathcal{C}_n(F) \subseteq (A^n)^{\mathbb{Z}}$ where $(u, v) \in (A^n)^2$ is forbidden if there is no edge $u \longrightarrow v$. The points of $\bigcup_{n \in \mathbb{N} \setminus \{0\}} \mathcal{C}_n(F)$ are essentially the pseudo-orbits of F. From the definitions we get characterizations of chain-transitive and chain-mixing cellular automata that are more convenient for us:

Proposition 5. *A cellular automaton* (X, F) *is chain-transitive (chain-mixing) if and only if* $\mathcal{C}_n(F)$ *is transitive (mixing) for every n.*

Proof.

(X, F) is chain-transitive

$\Longleftrightarrow \forall \varepsilon > 0 : \forall c, e \in X : \exists c = c^{(0)}, \ldots, c^{(n)} = e \in X : d(F(c^{(i)}), c^{(i+1)}) < \varepsilon$

$\Longleftrightarrow \forall k \in \mathbb{N} : \forall c, e \in X : \exists c = c^{(0)}, \ldots, c^{(n)} = e \in X : F(c^{(i)})_{[-k,k]} = (c^{(i+1)})_{[-k,k]}$

$\Longleftrightarrow \forall k \in \mathbb{N} : \mathcal{G}_k(F)$ is strongly connected

$\Longleftrightarrow \forall k \in \mathbb{N} : \mathcal{C}_k(F)$ is transitive.

If (X, F) is chain-mixing $\mathcal{G}_k(F)$ is not just strongly connected, but has the property that there exists $N \in \mathbb{N}$ such that for every $u, v \in V_k$ and $l \in \mathbb{N}$ such that $l \geq N$ there exists a path of length l from u to v. Otherwise the proof is essentially the same. $\qquad\square$

For $m \in \mathbb{N} \setminus \{0\}$ and every $n, i, j \in \mathbb{N}$ such that $m + i + j = n$ we denote $_i\Sigma_j^{(m)}(F) = \pi_{[i,i+m)}(\mathcal{C}_n(F))$. We write simply $_i\Sigma_j^{(m)}$ if the cellular automaton considered is clear from the context. The subshifts $_i\Sigma_j^{(m)}$ are sofic as factors of SFT's. The elements of $_i\Sigma_j^{(m)}$ can be considered as columns of width m which can be extended with i columns to the left and with j columns to the right without introducing violations of the local rule of F. As $\mathcal{C}_n(F)$ are pseudo-orbits it is natural to call $_i\Sigma_j^{(m)}$ *pseudo-traces* (Figure 3).

Since pseudo-traces are non-empty and $_{i+1}\Sigma_j^{(m)} \subseteq {}_i\Sigma_j^{(m)}$ we have, by the finite intersection property, that $_\infty\Sigma_j^{(m)} = \bigcap_{k\in\mathbb{N}} {}_k\Sigma_j^{(m)}$ is non-empty. Since $_\infty\Sigma_j^{(m)}$ is also closed and shift-invariant it is a subshift. In similar fashion we define subshifts $_i\Sigma_\infty^{(m)} = \bigcap_{k\in\mathbb{N}} {}_i\Sigma_k^{(m)}$ and $_\infty\Sigma_\infty^{(m)} = \bigcap_{k\in\mathbb{N}} {}_k\Sigma_k^{(m)}$. The following proposition shows that $\tau_m(F) = {}_\infty\Sigma_\infty^{(m)}$ as is to be expected.

Proposition 6. *Let* (X, F) *be a cellular automaton. Then for every* $m \in \mathbb{N} \setminus \{0\}$ *it holds that* $\tau_m(F) = {}_\infty\Sigma_\infty^{(m)}$.

Proof. Let (X, F) be a cellular automaton with radius r and let $m \in \mathbb{N} \setminus \{0\}$ be arbitrary.

"$\subseteq$": If $t \in \tau_m(F)$ then any space-time diagram that contains t shows that $t \in {}_k\Sigma_k^{(m)}$ for every $k \in \mathbb{N}$.

"$\supseteq$": It is enough to show that $\mathcal{L}\big(\bigcap_{k\in\mathbb{N}} {}_k\Sigma_k^{(m)}\big) \subseteq \mathcal{L}(\tau_m(F))$. Suppose not, i.e. that there exists $u \in \mathcal{L}_n(\bigcap_{k\in\mathbb{N}} {}_k\Sigma_k^{(m)}) \setminus \mathcal{L}_n(\tau_m(F))$ for some $n \in \mathbb{N}$. Let

$U = \{(v, w) \in A^{nr} \times A^{nr} \mid vu_0w \in \mathcal{L}_{2nr+m}(X)$ and if we consider vu_0w as an

element of $A^{[-nr, m+nr)}$ then $F^i(vu_0w)_{[0,m)} = u_i$ for all $i \in \{0, \ldots, n-1\}\}$.

Since $u \in \mathcal{L}(_{nr}\Sigma_{nr}^{(m)})$ the set U is non-empty. Since $u \notin \mathcal{L}_n(\tau_m(F))$ we have that for all $(v, w) \in U$ it holds that $vu_0w \notin \mathcal{L}(\Lambda_F)$ (the language of the limit set of
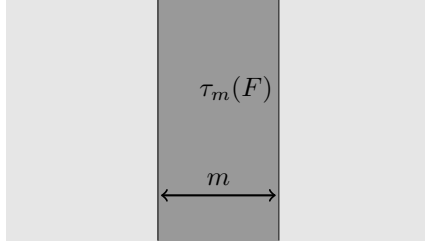
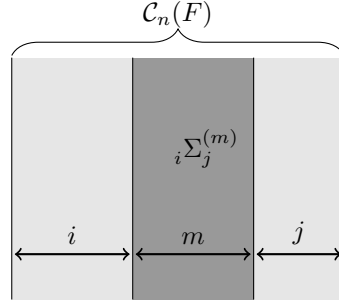Figure 2: Traces are vertical stripes of space-time diagrams.

Figure 3: Pseudo-orbits are configurations where outside of the stripes of width n we allow anything. Pseudo-traces are factors of pseudo-orbits in a natural way.

F). By compactness there exists $l \in \mathbb{N}$ such that vu_0w for every $(v,w) \in U$ is already forbidden in $F^l(X)$. But since $u \in \mathcal{L}({}_{(n+l)r}\Sigma_{(n+l)r}^{(m)})$ there has to exist $(v,w) \in U$ such that $vu_0w \in \mathcal{L}_1({}_{lr}\Sigma_{lr}^{(m+2nr)})$ so that it does appear in $F^l(X)$, thus reaching a contradiction. $\qquad\square$

Next we give a characterization of POTP for cellular automata.

Proposition 7. *Let (X,F) be a cellular automaton. The following are equivalent:*

  i. *F has POTP.*

  ii. *For every $m \in \mathbb{N} \setminus \{0\}$ there exists $n \in \mathbb{N}$ such that $\tau_m(F) = {}_n\Sigma_n^{(m)}$.*

Proof. "i. $\Rightarrow$ ii.": The POTP immediately implies that there exists $n \in \mathbb{N}$ such that the middle columns of $\mathcal{C}_{m+2n}$ are $\tau_m(F)$, i.e. that ${}_n\Sigma_n^{(m)} = \tau_m(F)$.

"ii. $\Rightarrow$ i.": If $\tau_m(F) = {}_n\Sigma_n^{(m)}$ then for pseudo-orbit $x \in \mathcal{C}_{2n+m}$ there exists an orbit $(c^{(i)})_{i \in \mathbb{Z}}$ such that $(\pi_{[n,n+m)}(c^{(i)}))_{i \in \mathbb{Z}} = \pi_{[n,n+m)}(x)$. $\qquad\square$

From this it follows that if F has POTP then $\tau_n(F)$ is sofic for every n, and that if $\tau_n(F)$ is an SFT for every n then F has POTP. These were already proved by Kůrka in [2] where also counterexamples for the converses were provided.

According to propositions 6 and 7 we have that (X,F) has POTP if and only if for every $m \in \mathbb{N} \setminus \{0\}$ there exists $n \in \mathbb{N}$ such that ${}_n\Sigma_n^{(m)} = {}_\infty\Sigma_\infty^{(m)}$. This leads to a natural definition of one-sided POTP: We say that F has the *left pseudo-orbit tracing property (left-POTP)* if for every m there exists i,j

$$
{}_i\Sigma_j^{(m)} = {}_\infty\Sigma_j^{(m)}.
$$

The *right pseudo-orbit tracing property (right-POTP)* is defined analogously. We see that for cellular automata over SFT's this definition behaves as one-sided variants are expected to:

Proposition 8. *Let X be an SFT and (X, F) a cellular automaton. Then (X, F) has POTP if and only if (X, F) has left- and right-POTP.*

Proof. "$\Rightarrow$": Immediate from propositions 6 and 7.

"$\Leftarrow$": It is enough to show that for large enough m it holds that there exists n such that $\tau_m(F) = {}_n\Sigma_n^{(m)}$. Let l be large enough so that there exists a set of forbidden words $S \subseteq A^l$ such that $X = X_S$ and let $m \geq \max\{l, 2r\}$ where r is a radius of F. Then left- and right-POTP say that we have n such that ${}_\infty\Sigma_n^{(m)} = {}_n\Sigma_n^{(m)} = {}_n\Sigma_\infty^{(m)}$. Now consider $t \in {}_n\Sigma_n^{(m)}$. It can be extended infinitely to the left without introducing violations of the local rule of F, and also to the right. If we take any valid extension to the left and glue it together with any valid extension to the right, we will have a valid space-time diagram since m was chosen large enough so that the patterns checking the validity of the space-time diagram cannot see both sides of the stripe of width m. Thus ${}_n\Sigma_n^{(m)} = \tau_m(F)$ and by Proposition 7 we are done. $\square$

The following proposition shows that memorylessness is a special case of left-POTP.

Lemma 2. *Let X be an SFT and (X, F) be a memoryless surjective cellular automaton. Then (X, F) has left-POTP.*

Proof. Let X be an SFT and $l \in \mathbb{N}$ such that there exists $S \subseteq A^l$ such that $X = X_S$. Let (X, F) be a cellular automaton with neighborhood $[0, r]$ where $r \in \mathbb{N}$. Let $m \geq \max\{l, r\}$. Take any configuration $t \in \mathcal{C}_m(F)$ and a sequence $(a_i)_{i \in \mathbb{Z}} \in A^{\mathbb{Z}}$ such that $a_i t_i \in \mathcal{L}_{m+1}(X)$ for all $i \in \mathbb{Z}$. There is no reason why $(a_i t_i)_{i \in \mathbb{Z}}$ should be in $\mathcal{C}_{m+1}(F)$, however we can construct a valid extension as follows: For every $j \in \mathbb{N}$ define a new sequence $(a_i^{(j)})_{i \in \mathbb{Z}}$ by setting $a_i^{(j)} = a_i$ for $i < -j$ and the rest of the sequence is defined recursively by

$$a_{-j}^{(j)} = a_{-j} \text{ and } a_{k+1}^{(j)} = F_{loc}(a_k^{(j)} t_k)_0 \text{ for } k \geq -j.$$

Notice that by the choice of m we have that if $x, y \in A, u, v \in A^m, w \in A^r$ such that $xu \in \mathcal{L}_{m+1}(X), uw \in \mathcal{L}_{m+r}(X)$ and $F(xu)_0 = y, F(uw) = v$, then $xuw \in \mathcal{L}_{m+r+1}(X)$ (since $m \geq l$) and $F(xuw) = yv$ (since $m \geq r$). Thus the configuration $(a_i^{(j)} t_i)_{i \in \mathbb{Z}}$ looks like a valid configuration of $\mathcal{C}_{m+1}(F)$ for all $i \geq -j$. By compactness the sequence $((a_i^{(j)})_{i \in \mathbb{Z}})_{j \in \mathbb{N}}$ has a converging subsequence, say $((a_i^{(j)})_{i \in \mathbb{Z}})_{j \in \mathcal{I}}$ where $\mathcal{I} \subseteq \mathbb{N}$ is an infinite subset. Let $(b_i)_{i \in \mathbb{Z}}$ be the limit of this subsequence. Now the configuration $(b_i t_i)_{i \in \mathbb{Z}} \in \mathcal{C}_{m+1}(F)$ shows that t can be extended to the left with one column. We can repeat the process and extend t to the left as much as we will. This shows that for every $n \in \mathbb{N}$ we have that ${}_0\Sigma_m^{(n)} = {}_\infty\Sigma_m^{(n)}$. $\square$

Now the following corollary is immediate.

Corollary 1. *Let X be an SFT. If (X, F) is memoryless and surjective, then $\tau_m(F) = \bigcap_{i \in \mathbb{N}} {}_0\Sigma_i^{(m)}$.*

Proof. From Lemma 2 it follows that for large enough n we have ${}_0\Sigma_n^{(m)} = {}_n\Sigma_n^{(m)}$, and so the claim follows from Proposition 6. □

4   Surjective Ultimately Right-Expansive Cellular Automata are Chain-Transitive

Consider a cellular automaton $(A^{\mathbb{Z}}, F)$. Let $P = \{A_i\}_{i \in \{0,\ldots,n-1\}}$ be a partition of A. We say that F *respects P at* $x \in A^{\mathbb{Z}}$ if for every $k \in \mathbb{N}$ there exists $j \in \{0, 1, \ldots, n-1\}$ such that $F^k(x) \in A_j^{\mathbb{Z}}$. Let R be the set of all points where F respects P. This is a subshift. Now we define a projection $\iota : A \to \{0, 1, \ldots, n-1\}$ by $\iota(a) = k$ if $a \in A_k$. Next we project the forward orbits of configurations in R cell-wise using ι. According to the definition of R, each row is constant after this mapping and so we can consider the projected forward orbits as a one-sided one-dimensional subshift over the alphabet $\{0, 1, \ldots, n-1\}$. We will call this subshift the *stripe shift defined by F and P* and denote it by

$$\Xi_P(F) = \{t \in \{0, \ldots, n-1\}^{\mathbb{N}} \mid \exists x \in A^{\mathbb{Z}} : \forall i \in \mathbb{N} : \forall j \in \mathbb{Z} : \iota(F^i(x)_j) = t_i\}.$$

A one-sided subshift X is called a *stripe shift* if there exists a cellular automaton $(A^{\mathbb{Z}}, F)$ and a partition P of A such that $\Xi_P(F) = X$.

In [18] we proved the following.

Lemma 3 ([18, Lemma 1]). *The binary full shift $\{0, 1\}^{\mathbb{N}}$ is not a stripe shift.*

It is natural to ask which subshifts, if any, are stripe shifts. We notice that at least every finite subshift is a stripe shift: this follows from the facts that in finite subshifts every point is eventually periodic (i.e. of form uv^ω for some finite words u, v) and that constant configurations (i.e. configurations where every cell is in the same state) respect any partition. This also says that for any cellular automaton and any partition of its state set the stripe shift defined is non-empty. Next we give a bit more interesting example of a stripe shift.

Example 4.1. The infinite firing squad cellular automaton $(B^{\mathbb{Z}}, G)$ presented in [19] has the following property: There exists $f \in B$ and $(c^{(i)})_{i \in \mathbb{Z}} \in st(G)$ such that $c^{(0)} = {}^\omega f^\omega$ and for all $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \mathbb{Z}$ we have that $c_j^{(i)} \neq f$. Now let $P = \{\{f\}, B \setminus \{f\}\}$ as the partition of B and we see that the *sunny side up* subshift $X_{\leq 1} = \{c \in \{0, 1\}^{\mathbb{N}} \mid c \text{ has at most one 1}\}$ is a stripe shift (it is obvious that $X_{\leq 1} \subseteq \Xi_P(G)$ and easy to see that $\Xi_P(G) \subseteq X_{\leq 1}$).

Using the firing squad cellular automaton and Lemma 3 we can characterize which sofic shifts are stripe shifts. Let us first prove that finite unions of stripe shifts are stripe shifts.

Proposition 9. *Let $X_1, \ldots, X_l$ be stripe shifts. Then $X = \cup_{i=1}^{l} X_i$ is a stripe shift.*

Proof. It is enough to show that the union of two stripe shifts is a stripe shift. Let X and Y be stripe shifts and assume that $X \cup Y \subseteq \{0, 1, \ldots, n-1\}^{\mathbb{N}}$ is such that every letter $0, 1, \ldots, n-1$ appears in some configuration of $X \cup Y$. Let $(A^{\mathbb{Z}}, F)$ be a cellular automaton and $\{A_i\}_{i \in \{0,1,\ldots,n-1\}}$ a collection of subsets of A such that $P_A = \{A_i \mid i \in \{0, 1, \ldots, n-1\}, A_i \neq \emptyset\}$ is a partition of A such that $\Xi_{P_A}(F) = X$. Let $(B^{\mathbb{Z}}, G)$, $\{B_i\}_{i \in \{0,1,\ldots,n-1\}}$ and P_B be defined in a similar way so that $\Xi_{P_B}(G) = Y$. We can assume that A and B are disjoint. Now let $P = \{A_i \cup B_i\}_{i \in \{0,1,\ldots,n-1\}}$, which is a partition of $A \cup B$ (A_i and B_i cannot both be empty for any $i \in \{0, 1, \ldots, n-1\}$ since all letters appear in $X \cup Y$). Our goal is to define a cellular automaton $((A \cup B)^{\mathbb{Z}}, H)$ such that $\Xi_P(H) = X \cup Y$. We can assume that the local rules of $(A^{\mathbb{Z}}, F)$ and $(B^{\mathbb{Z}}, G)$ both have neighborhood $[-r, r]$ for some $r \in \mathbb{N} \setminus \{0\}$. Let $a_0 \in A_0 \cup B_0$ and $a_1 \in A_1 \cup B_1$ be some letters. The local rule H_{loc} is a map $(A \cup B)^{[-r,r]} \to A \cup B$ defined by

$$H_{loc}(x_{-r} \cdots x_{-1} x_0 x_1 \cdots x_r) = \begin{cases} F_{loc}(x_{-r} \cdots x_r), & \text{if } x_{-r} \cdots x_r \in A^{[-r,r]} \\ G_{loc}(x_{-r} \cdots x_r), & \text{if } x_{-r} \cdots x_r \in B^{[-r,r]} \\ a_0, & \text{if } x_0 \in A \text{ and } x_{-1} \text{ or } x_1 \text{ in } B \\ a_1, & \text{otherwise} \end{cases}$$

Clearly, from the first two lines, we have that $X \cup Y \subseteq \Xi_P(H)$. On the other hand, if a configuration $c \in (A \cup B)^{\mathbb{Z}}$ contains letters from both A and B then the third and fourth lines guarantee that $H(c)$ contains a_0 and a_1 so that H does not respect P at c. Thus $\Xi_P(H) = X \cup Y$. $\square$

Now we are ready to characterize the sofic stripe shifts.

Proposition 10. *Let X be a sofic shift. If X has positive entropy then no stripe shift can contain it, and complementarily, if X has zero entropy then it is a stripe shift.*

Proof. For sofic shifts, having positive entropy is equivalent to being uncountable. Let the number of letters that appear in X be n.

Suppose that X is an uncountable sofic shift and that $(A^{\mathbb{Z}}, F)$ is a cellular automaton and $P = \{A_i\}_{i \in \{0,\ldots,n-1\}}$ a partition such that $X \subseteq \Xi_P(F)$. Then there exists $u, v \in \mathcal{L}(X)$ such that $u_0 \neq v_0$ and $\{u, v\}^{\mathbb{N}} \subseteq X$. Let $\tilde{u} = uv$ and $\tilde{v} = vu$, so that $|\tilde{u}| = |\tilde{v}|$. Of course also $\{\tilde{u}, \tilde{v}\}^{\mathbb{N}} \subseteq X$. Now let $P' = \{A'_0, A'_1\}$ be a partition of A such that P is a refinement of P' and $\iota^{-1}(u_0) \subseteq A'_0$ and $\iota^{-1}(v_0) \subseteq A'_1$ (where ι is the projection $A \to \{0, \ldots, n-1\}$ according to the partition P). Now the stripe shift defined by $F^{|\tilde{u}|}$ and P' is $\{0, 1\}^{\mathbb{N}}$ contradicting Lemma 3.

Next let X be a countable sofic shift and let $\mathcal{G}$ be a labeled directed graph such that the labels of the one-way infinite paths of $\mathcal{G}$ are the points of X.

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According to [20, Lemma 4.8.] we can assume that the connected components (connected in the sense that if the directions are erased then the component is a connected undirected graph) of $\mathcal{G}$ consist of some number of cycles $C(1), \ldots, C(k)$ and paths $P(1), \ldots, P(k-1)$ such that $P(i)$ is path from $C(i)$ to $C(i+1)$. According to Proposition 9 we can assume that there is only one connected component.

Denote the edge set of $\mathcal{G}$ by $E_{\mathcal{G}}$ and define $F_1 : E_{\mathcal{G}} \to E_{\mathcal{G}}$ so that if e has unique follower edge in $\mathcal{G}$ then $F_1(e)$ is that edge, otherwise $F_1(e)$ is the the follower edge which is on the same cycle as e (the only edges where the follower is not unique are the ones on cycles where there is a choice to either continue along the cycle or start along the path connecting to the next cycle). Similarly we define $F_2 : E_{\mathcal{G}} \to E_{\mathcal{G}}$ but F_2 does the opposite choice than F_1 in the edges where there are two possible ways to continue. Let $(B^{\mathbb{Z}}, G)$ be the firing squad cellular automaton of Example 4.1. We define a cellular automaton $((E_{\mathcal{G}} \times B \times \cdots \times B)^{\mathbb{Z}}, F)$, where we have $k-1$ copies of B, by $F(c, e^{(1)}, \ldots, e^{(k-1)}) = (c', G(e^{(1)}), \ldots, G(e^{(k-1)}))$ where

$$c_i' = \begin{cases} F_1(c_i) & \text{if } e_i^{(j)} \neq f \text{ for every } j \in \{1, 2, \ldots, k-1\} \\ F_2(c_i) & \text{if } e_i^{(j)} = f \text{ for some } j \in \{1, 2, \ldots, k-1\} \end{cases}.$$

Now X is the stripe shift defined by F and $P = \{E_x \times B \times \cdots \times B\}_{x \in \mathcal{L}_1(X)}$ where $E_x = \{a \in E_{\mathcal{G}} \mid \text{label of } a \text{ is } x\}$. $\qquad \square$

As remarked in the beginning of the proof, we could equally well formulate the above proposition with the condition of X being uncountable. We used the entropy condition instead since we have an example of an uncountable stripe shift, and on the other hand since the proof of Lemma 3 seems to suggest that positive entropy is impossible for stripe shifts in general.

While the stripe trace has certain interest in itself, we just need the above as a technical detail in the proof of the following theorem. This theorem generalizes [10, Corollary 4.3] where the cellular automaton is assumed to be reversible and expansive both to the left and to the right.

Theorem 1. *Let X be a mixing sofic shift. A surjective ultimately right-expansive cellular automaton (X, F) is chain-transitive.*

Proof. Suppose (X, F) is not chain-transitive. Then, by Proposition 5, there exists m such that $\mathcal{G}_m(F)$ is not strongly connected. We may assume that m is large enough so that we have a cellular automaton $(\tau_m(F), \overrightarrow{F}_m)$ as defined by Lemma 1. Let $\mathcal{G}_m(F)_1, \ldots, \mathcal{G}_m(F)_k$ be the strongly connected components of $\mathcal{G}_m(F)$. There has to exist a strongly connected component which has no arrows to other strongly connected components (if every connected component could be left, there would have to exist a cycle which would visit two different connected components, which is a contradiction); we may assume that $\mathcal{G}_m(F)_1$ is such. Let $V_1 \subseteq \mathcal{L}_m(X)$ be the vertex set of $\mathcal{G}_m(F)_1$ and $V_1^c = \mathcal{L}_m(X) \setminus V_1$. We denote with V_1 and V_1^c also the clopen sets of X which the vertex sets naturally define. Since V_1 has no arrows pointing outwards, we have that $F(V_1) \subseteq V_1$.

Then according to Proposition 4 we have that $F(V_1) = V_1$ and $F^{-1}(V_1) = V_1$. According to $F^{-1}(V_1) = V_1$ there are no arrows pointing from V_1^c to V_1 and so $F(V_1^c) \subseteq V_1^c$. Again, by Proposition 4, we have that $F(V_1^c) = V_1^c$.

Define a partition P of A^m as follows:

$$P_1 = V_1$$
$$P_0 = A^m \setminus V_1,$$

and let $\iota : A^m \to \{0,1\}$ be the projection defined by this partition. Of course $V_1^c \subseteq P_0$. As we saw above $\mathcal{C}_m(F) \subseteq P_0^{\mathbb{Z}} \cup P_1^{\mathbb{Z}}$. Then we also must have that $\tau_m(F) \subseteq P_0^{\mathbb{Z}} \cup P_1^{\mathbb{Z}}$. Take one vertex $u \in V_1 = P_1$ and one $v \in V_1^c \subseteq P_0$. Since X is a mixing sofic shift, we have $K \in \mathbb{N}$ and words $w_{uu}, w_{uv}, w_{vv}, w_{vu} \in A^K$ such that

$$Y = \{\cdots x_{-1} w_{x_{-1}x_0} x_0 w_{x_0 x_1} x_1 \cdots \mid x_i \in \{u,v\} \text{ for all } i \in \mathbb{Z}\} \subseteq X$$

Now extend $(\tau_m(F), \overrightarrow{F}_m)$ arbitrarily to a cellular automaton $((A^m)^{\mathbb{Z}}, \overrightarrow{F}'_m)$. But now the stripe shift defined by $\overrightarrow{F}'_m$ and P would contain an uncountable sofic shift: For $x, y \in \{u,v\}$ define $z_{x,y} \in \{0,1\}^{|u|+K-1}$ as

$$z_{x,y} = \iota\big((xw_{xy}y)_{[0,m)}\big)\iota\big((xw_{xy}y)_{[1,m+1)}\big)\cdots\iota\big((xw_{xy}y)_{[|u|+K-1,m+|u|+K)}\big),$$

then

$$\{i_0 z_{x_0,y_0} i_1 z_{x_1,y_1} \cdots \mid i_j \in \{0,1\} \text{ and } \iota(x_j) = i_j \text{ and}$$
$$\iota(y_j) = i_{j+1} \text{ for all } i,j \in \mathbb{N}\} \subseteq \Xi_P(\overrightarrow{F}'_m).$$

This contradicts Lemma 10.

$\square$

Remark 1. In Theorem 1 mixing sofic shift X cannot be replaced by a transitive sofic shift X: Take two (reversible) expansive cellular automata $(A^{\mathbb{M}}, F)$ and $(B^{\mathbb{M}}, G)$ where $\mathbb{M} = \mathbb{N}$ or $\mathbb{M} = \mathbb{Z}$ and A and B are disjoint. For convenience assume that the local neighborhood is $\{-1,0,1\} \cap \mathbb{M}$. Let $X \subseteq (A \cup B)^{\mathbb{M}}$ be a transitive SFT defined by the set of forbidden patterns $\mathcal{F} = \{xy \mid x,y \in A\} \cup \{xy \mid x,y \in B\}$. Now define a cellular automaton (X, H) by a local rule with neighborhood $\{-2,0,2\} \cap \mathbb{M}$. Within this neighborhood the local rule sees letters only from A or only from B. This local neighborhood is mapped according to F or G depending on whether the local rule sees letters from A or from B. This (X, H) is expansive as $(A^{\mathbb{M}}, F)$ and $(B^{\mathbb{M}}, G)$ are, but not chain-transitive.

Theorem 1 has the following immediate corollary.

Corollary 2. *Let X be a mixing sofic shift and let (X, F) be a surjective ultimately right-expansive cellular automaton. Then (X, F) is chain-mixing and so $_i\Sigma_j^{(m)}$ is a mixing sofic shift for every $i, j \in \mathbb{N}, m \in \mathbb{N} \setminus \{0\}$.*

Proof. Let us show that $(\mathcal{C}_n(F), \sigma^k)$ is a transitive SFT for every $k \in \mathbb{N} \setminus \{0\}$. Then it will follow from Proposition 1 that $(\mathcal{C}_n(F), \sigma)$ is a mixing SFT, and thus, by Proposition 5, (X, F) is chain-mixing. Let $(u^{(i)})_{i \in \{0,1...,k-1\}}, (v^{(i)})_{i \in \{0,1,...,k-1\}} \in \mathcal{L}_k(\mathcal{C}_n(F))$. Since also (X, F^k) is a surjective ultimately right-expansive cellular automaton, Theorem 1 and Proposition 5 imply that $\mathcal{C}_n(F^k)$ is a transitive SFT. Thus there exists an F^k-pseudo-orbit through cylinders defined by $u^{(k-1)}$ and $v^{(0)}$ in this order. Then we have an F-pseudo-orbit through the cylinders $u^{(k-1)}$ and $v^{(0)}$ (in this order) such that the number of steps from $u^{(k-1)}$ to $v^{(0)}$ is a multiple of k, and so we see that $(\mathcal{C}_n(F), \sigma^k)$ is transitive.

The pseudo-traces ${}_i\Sigma_j^{(m)}$ are mixing sofic shifts as factors of mixing SFT's. $\qquad\square$

5 Left-POTP and Ultimate Right-expansive Cellular Automata Have Sofic Traces

We can now prove that surjective ultimately right-expansive cellular automaton with left-POTP has POTP. Our proof is inspired by Taati's proof that a cellular automaton (over the full shift) which is reversible over its limit set is stable [21], i.e. reaches the limit set in finite time (here $\tau_m(F)$ corresponds to the limit set).

Theorem 2. *Let $X \subseteq A^\mathbb{Z}$ be a transitive SFT and let (X, F) be a surjective ultimately right-expansive cellular automaton with left-POTP. Then F has POTP and $\tau_m(F)$ is a sofic shift for every m. If F is memoryless, then $\tau_m(F)$ is an SFT. Further, if X is a mixing SFT, then $\tau_m(F)$ is mixing.*

Proof. By Proposition 1 there exists n such that (X, σ^n) is a finite union of disjoint mixing SFT's. As F is a surjective cellular automaton, some power of F is a cellular automaton when restricted to any of these mixing SFT's. If this power of F has POTP on each of these disjoint mixing SFT's then the original cellular automaton also has POTP. This is why it is enough to prove the claim with the additional assumption of X being a mixing SFT. This same argument has already been made in more detail in the end of Section 2 in [7] and in the proof of [10, Theorem 5.5].

We have seen that ${}_L\Sigma_L^{(m)}$ is a sofic shift for every $L, m \in \mathbb{N}$, and according to Corollary 2 it is also transitive. We will show that for large enough $L \in \mathbb{N}$ it holds that $\tau_m(F) = {}_L\Sigma_L^{(m)}$.

Let l be large enough so that left-POTP says that ${}_l\Sigma_l^{(m)} = {}_\infty\Sigma_l^{(m)}$. Let $\overrightarrow{F}_m : \tau_m(F) \to \tau_m(F)$ be the cellular automaton defined by the ultimate right-expansivity of F. Let r be a radius of F and let r' be a radius of $\overrightarrow{F}_m$. Let $\mathcal{F} = (A^m)^{2r'+1} \setminus \mathcal{L}_{2r'+1}(\tau_m(F))$, i.e. the words of length $2r' + 1$ that do not appear in $\tau_m(F)$. Now $\overrightarrow{F}_m$ can be extended to $X_\mathcal{F}$ by using the same local rule of radius r'; Let $\overrightarrow{F}$ denote the extension of $\overrightarrow{F}_m$ to $X_\mathcal{F}$. It does not necessarily hold that $\overrightarrow{F}(X_\mathcal{F}) \subseteq X_\mathcal{F}$. Notice that since F is surjective we have that $\mathcal{L}_k({}_{kr}\Sigma_{kr}^{(m)}) =$

$\mathcal{L}_k(\tau_m(F))$ for every $k \in \mathbb{N}$. In particular $_{l'}\Sigma_{l'}^{(m)} \subseteq X_{\mathcal{F}}$ for every $l' \geq (2r'+1)r$. Let $L \geq \max\{l, (2r'+1)r\}$.

Claim: $\overrightarrow{F}(_{L+1}\Sigma_{L+1}^{(m)}) = {}_L\Sigma_L^{(m)}$.

Proof: "$\subseteq$": Let $t^{(0)} \in {}_{L+1}\Sigma_{L+1}^{(m)}$. Since $t^{(0)}$ is the central stripe of some element in $\mathcal{C}_{2L+2+m}(F)$ there is a unique $t^{(1)} \in {}_{L+2}\Sigma_L^{(m)} \subseteq {}_L\Sigma_L^{(m)}$, determined by the local rule of $\overrightarrow{F}_m$, such that $t^{(0,1)} \in (A^{m+1})^{\mathbb{Z}}$ defined by

$$\pi_{[0,m)}(t^{(0,1)}) = t^{(0)} \text{ and } \pi_{[1,m]}(t^{(0,1)}) = t^{(1)}$$

is in $_{L+1}\Sigma_L^{(m+1)}$. Then it has to be that $\overrightarrow{F}(t^{(0)}) = t^{(1)}$.

"$\supseteq$": Now let $t^{(1)} \in {}_L\Sigma_L^{(m)}$. By left-POTP we have that $t^{(1)} \in {}_{L+2}\Sigma_L^{(m)}$, so that there exists $t^{(0)} \in {}_{L+1}\Sigma_{L+1}^{(m)}$ such that $\overrightarrow{F}(t^{(0)}) = t^{(1)}$. $\square$

Now we have that $_L\Sigma_L^{(m)}$ is a factor of $_{L+1}\Sigma_{L+1}^{(m)}$, so the entropy of $_L\Sigma_L^{(m)}$ is at most the entropy of $_{L+1}\Sigma_{L+1}^{(m)}$ (Proposition 2). But we also have that $_{L+1}\Sigma_{L+1}^{(m)} \subseteq {}_L\Sigma_L^{(m)}$, and so $_{L+1}\Sigma_{L+1}^{(m)}$ and $_L\Sigma_L^{(m)}$ have the same entropy. According to Proposition 3 we have that $_{L+1}\Sigma_{L+1}^{(m)} = {}_L\Sigma_L^{(m)}$ and so $_L\Sigma_L^{(m)} = {}_\infty\Sigma_\infty^{(m)}$. Now F has POTP according to Proposition 7, and POTP always implies soficness of the trace subshifts.

Next let F be memoryless, i.e. it has a local neighborhood $[0, r]$. According to Corollary 1 we now have that $_0\Sigma_k^{(m)} = \tau_m(F)$ for some k. Let $X_i^{(j)}$ denote the SFT of $(A^j)^{\mathbb{Z}}$ defined by forbidding $(A^j)^i \setminus \mathcal{L}_i(\tau_j(F))$. Let $x \in X_{2r'k+2}^{(m)}$ be arbitrary and $y \in (A^{m+k})^{\mathbb{Z}}$ be the unique configuration defined by $\pi_{[i,i+m)}(y) = \overrightarrow{F}^i(x)$ for all $i \in \{0, 1, \ldots, k\}$. Now $\pi_{[0,m+1)}(y) \in X_{2r'(k-1)+2}^{(m+1)}$ since $x = \pi_{[0,m)}(y) \in X_{2r'k+2}^{(m)}$ and $\pi_{[1,m)}(y)$ is defined using the local rule of $\overrightarrow{F}_m$. We can repeat this for k times and we get that $y \in X_2^{(m+k)}$, i.e. $y \in \mathcal{C}_{m+k}(F)$. But then, since $_0\Sigma_k^{(m)} = \tau_m(F)$, we have that $x \in \tau_m(F)$. Of course $\tau_m(F) \subseteq X_{2r'k+2}^{(m)}$ and so we are done.

Lastly, if X is mixing, then by Corollary 2 also $\tau_m(F)$ is mixing. $\square$

Remark 2. Theorem 2 implies the following:

- If $(A^{\mathbb{N}}, F)$ is surjective and positively expansive, then $\tau(F)$ is an SFT ([4, Theorem 3.3]).

- If $(A^{\mathbb{N}}, F)$ is reversible and expansive, then $\tau(F)$ is an SFT ([6, Theorem 1.3])

- If (X, F), where X is a transitive SFT, is surjective, ultimately right-expansive, memoryless, and chain-recurrent, then (X, F) has POTP ([7, Theorem 6.3 (i)], the last assumption is not actually needed, see introduction or the theorem above).

6 Left-POTP or Right-Expansive Cellular Automata Can Have Non-Sofic Traces

Let X be a transitive SFT. We have seen that surjective ultimately right-expansive cellular automaton (X, F) with left-POTP has POTP. Especially this means that right-expansive cellular automata with left-POTP have sofic traces. In [18] we gave an example of a right-expansive cellular automaton over the full shift (based on constructions in [22]) whose trace subshifts are non-sofic, which then also cannot have POTP. Here we will give a reversible cellular automaton over one-sided full shift with non-sofic traces, and so a reversible cellular automaton with left-POTP and with non-sofic traces.

Let $X \subseteq A^{\mathbb{Z}}$ be a subshift. The set of *isolated* points of X is

$$\text{Iso}(X) = \{c \in X \mid \exists n \in \mathbb{N} : [c_{-n} \cdots c_n] \cap X = \{c\}\}.$$

Lemma 4. *If $X \subseteq A^{\mathbb{Z}}$ is a sofic shift, then $|Iso(X) \cap Per_\sigma(X)| < \infty$.*

Proof. Suppose X is sofic but the intersection of its isolated and periodic points is infinite. Let $\mathcal{A}$ be a finite state automaton that recognizes $\mathcal{L}(X)$; we can assume a model where every state is accepting and every state has both incoming and outgoing edges. Let $\{c_i\}_{i \in \mathbb{N}} \subseteq \text{Iso}(X) \cap \text{Per}_\sigma(X)$ be an infinite subset such that if $i \neq j$ then for all k it holds that $\sigma^k(c_i) \neq c_j$. For every $i \in \mathbb{N}$ let u_i be the shortest word such that $c_i = {}^\omega u_i^\omega$. Since for every i any repetition of u_i is accepted, there has to exist a cycle in $\mathcal{A}$ whose labels read u_i some number of times. Let $i, j \in \mathbb{N}$ be arbitrary but different. Let $\mathcal{A}_i$ and $\mathcal{A}_j$ be cycles in $\mathcal{A}$ whose labels read u_i and u_j (resp.) some number of times. The cycles $\mathcal{A}_i$ and $\mathcal{A}_j$ must be separate in the sense that there cannot be a directed path from $\mathcal{A}_i$ to $\mathcal{A}_j$ or from $\mathcal{A}_j$ to $\mathcal{A}_i$, since otherwise there would exist a word $w \in A^+$ such that ${}^\omega u_i w u_j^\omega$ or ${}^\omega u_j w u_i^\omega$ would be in X contradicting the isolation of c_i. Having an own separate cycle for infinitely many points contradicts the finiteness of $\mathcal{A}$. $\qquad\square$

We defined traces for two-sided cellular automata but the definition is essentially the same for one-sided cellular automata: For one-sided cellular automaton (X, F) the *m-trace* of F is

$$\tau_m(F) = \{t \in (A^m)^{\mathbb{Z}} \mid \exists (c^{(i)})_{i \in \mathbb{Z}} \in st(F) : \forall i \in \mathbb{Z} : t_i = c^{(i)}_{[0,m)}\}.$$

Now we are ready to present a reversible one-sided cellular automaton with non-sofic traces.

Proposition 11. *There exists a reversible one-sided cellular automaton whose traces are non-sofic.*

Proof. Let $(A^{\mathbb{N}}, F)$ be a cellular automaton where $A = \{0, 1, 2, 3\}$ and $F_{loc} : A^{[0,1]} \to A$ is defined by $F_{loc}(ab) = \rho_b(a)$ for permutations (in cycle notation with 1-cycles included):

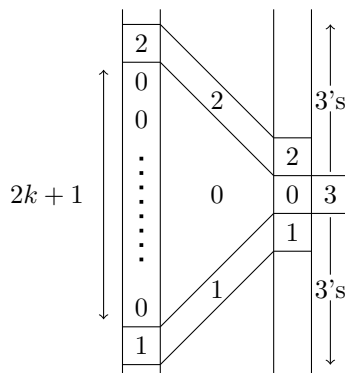$$\rho_0 = \rho_2 = (0)(12)(3), \qquad \rho_1 = \rho_3 = (012)(3).$$

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Figure 4: Word $20^{2k+1}1$ forces the word $20^{2k-1}1$ next to it. Eventually this leads to the word 201. Next to this there can only be a 3. Since 3's are fixed, the whole column is a constant 3. This uniquely determines the column containing $20^{2k+1}1$ and forces it to be periodic.

This is a reversible cellular automaton, since the permutations $\pi_0 = \pi_1 = (0)(12)(3)$ and $\pi_2 = \pi_3 = (021)(3)$ can be verified to define the inverse of F. For more on one-sided reversible cellular automata defined in this way see [23]. We will show that $\tau_1(F)$ is non-sofic, it is then easy to see that also $\tau_m(F)$ for any $m > 1$ is non-sofic. Notice that when considered visually, the words in $\mathcal{L}(\tau_1(F))$ are vertical.

First notice that 3 is always mapped to 3 so that every $c \in \tau_1(F)$ is either $^\omega 3^\omega$ or has no appearances of the letter 3. Notice also that if there is a column $^\omega 3^\omega$ in a space-time diagram of $(A^{\mathbb{N}}, F)$ then every column to the left of it has to be periodic. This can be seen, for example, by noticing that ρ_3 can be extended into a permutation $\rho_{3,n} : A^n \to A^n$ for any n by $\rho_{3,n}(u) = F(u3)$ where $u \in A^n$. Now since 3 is fixed, the two-way infinite sequences $(\rho_{3,n}^i(u))_{i \in \mathbb{Z}}$ are precisely the elements of $\tau_n(F)$ obtained by fixing 3 into the cell n. Now the order of the permutation $\rho_{3,n}$ gives an upper bound for the period of the nth column to the left of $^\omega 3^\omega$.

The following fact is not difficult to see, and is explained in detail in [18, Example 1]:

- If $20^l1 \in \mathcal{L}(\tau_1(F))$ where $l \in \mathbb{N} \setminus \{0, 1\}$ then the only word that can appear in a space-time diagram to the right of it is $20^{l-2}1$ ("to the right" here means that the words are aligned so that 2 in $20^{l-2}1$ is one step right and one step down from the 2 in 20^l1).

Next we will show that for every $k \in \mathbb{N}$ the word $20^{2k+1}1$ does appear in $\tau_1(F)$ and that it uniquely determines the (periodic) configuration it appears in (refer to Figure 4). That $20^{2k+1}1$ appears in $\tau_1(F)$ can be verified by considering the zeroth column of the space-time diagram generated by $0^{k+1}3x \in A^{\mathbb{N}}$ where $x \in A^{\mathbb{N}}$ is arbitrary; the word $20^{2k+1}1$ will appear in the zeroth column between

time steps $-k-1$ and $k+1$. Now consider a space-time diagram where the word $20^{2k+1}1$ appears in the zeroth column. By the fact mentioned above, the only word that can appear k steps to the right of $20^{2k+1}1$ is 201. The only letter which maps 0 to 1 when going forwards in time and 0 to 2 when going backwards in time is 3. Thus the next column to the right must have a 3 in it. As was reasoned above, this implies that the column containing $20^{2k+1}1$ must be periodic. Notice that the configuration that $20^{2k+1}1$ appears in was uniquely determined, which means it is isolated. Now $\tau_1(F)$ has infinitely many isolated periodic points, and thus, according to Lemma 4, it cannot be sofic.

$\square$

Since any memoryless cellular automaton over a full shift has left-POTP we have the following.

Corollary 3. *There exists a reversible cellular automaton over a full shift with left-POTP whose trace is non-sofic.*

7 Conclusion

The following list summarizes what we can say about cellular automata having various combinations of one-sided POTP and expansivity.

- If surjective (X, F), where X is an SFT, has both left- and right-POTP, then F has POTP, and so the trace of F is sofic (from definitions and Proposition 8).

- If surjective (X, F), where X is a transitive SFT, has left-POTP and is ultimately right-expansive, then F has POTP, and so the trace of F is sofic (Theorem 2).

- If reversible $(A^{\mathbb{Z}}, F)$ is right-expansive, then the trace of F is not necessarily sofic, and so F does not necessarily have POTP ([18, Proposition 7]).

- If reversible $(A^{\mathbb{Z}}, F)$ has left-POTP, then the trace of F is not necessarily sofic, and so F does not necessarily have POTP (Proposition 11).

The big open question is, still, whether expansive cellular automata have the pseudo-orbit tracing property, or equivalently whether expansive cellular automata are conjugate to SFT's (conjectured for cellular automata over full shifts by Kůrka [9, Conjecture 30.]).

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