On the Generating Function of Discrete Chebyshev Polynomials

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Abstract

We give a closed form for the generating function of the discrete Chebyshev polynomials. The closed form consists of the MacWilliams transform of Jacobi polynomials together with a binomial multiplicative factor. It turns out that the desired closed form is a solution to a special case of Heun differential equation, and that the closed form implies combinatorial identities that appear quite challenging to prove directly.

Keywords: *Orthogonal polynomials, Discrete Chebyshev polynomials, Krawtchouk polynomials, MacWilliams transform, Generating function, Heun equation.*

1 Introduction

The discrete Chebyshev polynomials belong to the rich family of *orthogonal polynomials* (see [9] for a general treatise on the orthogonal polynomials and [2] for a previous work of the authors). The inner product associated to the discrete Chebyshev polynomials is defined with a discrete weight function, and hence the vector space $\mathcal{P}_N$ of polynomials having degree at most N forms a natural reference for the orthogonal polynomials discussed in this article.

The sum and the scalar product in $\mathcal{P}_N$ are defined pointwise, and the inner product is defined as

$$\langle p, q\rangle_w = \sum_{l=0}^{N} w_l p(l) q(l). \tag{1.1}$$

The *Krawtchouk polynomials* (see [6]) $K_0^{(N)}$, $K_1^{(N)}$, ..., $K_N^{(N)}$ (of order N) are orthogonal with respect to weight function $w_l = \binom{N}{l}$ and the discrete

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Chebyshev polynomials $D_0^{(N)}$, $D_1^{(N)}$, …, $D_N^{(N)}$ of order N with respect to weight function $w_l = 1$ for each l. In addition to orthogonality, we have $\deg(K_k^{(N)}) = \deg(D_k^{(N)}) = k$ for each $k \in \{0, 1, \ldots, N\}$.

As (orthogonal) polynomials with ascending degree, the discrete Chebyshev polynomials form a basis of $\mathcal{P}_N$, and hence any polynomial p of degree at most N can be uniquely represented as

$$p = d_0 D_0^{(N)} + d_1 D_1^{(N)} + \ldots + d_N D_N^{(N)}, \tag{1.2}$$

where $d_l \in \mathbb{C}$. Coefficients d_l in (1.2) are called the *discrete Chebyshev coefficients* of p. Since the discrete Chebyshev polynomials are orthogonal with respect to constant weight function, they have the following property important in the approximation theory: With respect to norm $||p-q||^2 = \sum_{l=0}^{N}(p(l)-q(l))^2$, the best approximation of p in $\mathcal{P}_M$ can be found by simply taking $M+1$ first summands of (1.2) (see [4], for instance).

2 Preliminaries

2.1 The Discrete Chebyshev Polynomials

There are various ways to construct polynomials orthogonal with respect to inner product (1.1) with weight function $w_l = 1$ so that $\deg(D_k^{(N)}) = k$.

We choose a construction analogous to that of *Legendre polynomials* [9]. We first define the difference operator Δ by $\Delta f(x) = f(x+1) - f(x)$, the binomial coefficient by $\binom{x}{k} = \frac{1}{k!}x(x-1)\ldots(x-k+1)$, and finally

$$D_k^{(N)}(x) = (-1)^k \Delta^k \left(\binom{x}{k} \binom{x-N-1}{k} \right). \tag{2.1}$$

It is straightforward to see that polynomials D_k (here and hereafter, we omit the superscript N if there is no danger of confusion) defined above form a basis of $\mathcal{P}_N$ orthogonal with respect to inner product (1.1) with weight $w_l = 1$. Moreover, clearly $\deg(D_k) = k$, since one application of Δ decreases the degree of a polynomial by one [3].

In this article, *we consider (2.1) as the definition of discrete Chebyshev polynomials*, but it is also easy to see that the following explicit expressions hold (see [3]):

$$\begin{aligned} D_k^{(N)}(x) &= \sum_{l=0}^{k}(-1)^l \binom{k}{l}\binom{N-x}{k-l}\binom{x}{l} \\ &= \sum_{l=0}^{k}(-1)^l \binom{k+l}{k}\binom{N-l}{k-l}\binom{x}{l}. \end{aligned} \tag{2.2}$$

Also, it is rather easy to verify that the discrete Chebyshev polynomials satisfy the following recurrence relation:

$$k^2 D_k = (2k-1)D_1 D_{k-1} - (N+k)(N-k+2)D_{k-2}, \tag{2.3}$$

$D_0 = 1$, $D_1 = N - 2x$ (see [3]). The recurrence (2.3) also extends the definition of D_k to cases $k > N$.

The method of using *generating functions* is among the cornerstones of various areas of mathematics, and does not need any further introduction. We merely focus on the very simple form of the generating function of Krawtchouk polynomials (see [6]):

$$(1+t)^{N-x}(1-t)^x = \sum_{k=0}^{\infty} K_k^{(N)}(x)t^k. \tag{2.4}$$

In fact, when studying binomial distributions, it is quite natural to *define* the Krawtchouk polynomials via (2.4).

On the other hand, the quest for the generating function of the discrete Chebyshev polynomials seems to be a more complicated task. In what follows, we give a closed form for the generating function

$$\sum_{k=0}^{\infty} D_k^{(N)}(x)t^k. \tag{2.5}$$

It should be noticed, however, that some useful closed-form expressions carrying information about the discrete Chebyshev polynomials have been found before. For instance in [5] an expression

$$(1+t)^k(1+s)^{N-x}(1-st)^x \tag{2.6}$$

having the property that the coefficient of $s^k t^k$ equals to $D_k^{(N)}(x)$ is given.

2.2 A Differential Equation for Jacobi Polynomials

For a nonnegative integer n, the *Jacobi polynomial* $P_n^{(\alpha,\beta)}(x)$ is, up to the constant factor, the unique entire rational solution to the *differential equation (for Jacobi polynomials)*

$$(1-x^2)y'' + \big(\beta - \alpha - (\alpha+\beta+2)x\big)y' + n(n+\alpha+\beta+1)y = 0 \tag{2.7}$$

(see [1]).

In this article, we are interested in Jacobi polynomials with parameters $\alpha = 0$, $\beta = -(N+1)$, where $N > 0$ is a fixed integer. We also substitute x for n and t for x in equation (2.7), and denote $J_x^{(N+1)}(t) = P_x^{(0,-N-1)}(t)$.

We usually omit superscript $N+1$ and denote $J_x(t) = J_x^{(N+1)}(t)$. Then $J_x(t)$ satisfies differential equation

$$(1-t^2)J_x''(t) - (N+1-(N-1)t)J_x'(t) + x(x-N)J_x(t) = 0. \tag{2.8}$$

Recall that in this context, x is a fixed nonnegative integer. Polynomial $J_x(t)$ can be expressed as

$$J_x(t) = \frac{1}{2^x}\sum_{k=0}^{x}\binom{x}{k}\binom{x-N-1}{k}(t-1)^k(t+1)^{x-k} \tag{2.9}$$

(see [1]). Since equation (2.8) is clearly invariant under substitution $x \leftarrow N-x$, we have symmetry

$$J_{N-x}(t) = J_x(t) \tag{2.10}$$

(see [1]).

2.3 MacWilliams Transform

The *MacWilliams transform* of order x for a polynomial P is defined as

$$\widehat{P}_x(t) = (1+t)^x P\left(\frac{1-t}{1+t}\right). \tag{2.11}$$

As definition (2.11) shows, MacWilliams transform is a special case of Möbius transformation together with factor $(1+t)^x$. If the subscript x is clear by context, we may omit it. It is also straightforward to see that if x is an integer so that $\deg(P) \leq x$, then $\widehat{P}$ is again a polynomial. In this article, we will however face situations with non-integer values of x, and it is worth noticing already here that (2.11) shows that if $t > -1$, then $\widehat{P}_x(t)$ is a uniquely defined differentiable function of real variable x.

In what follows, $\widehat{J}_x(t)$ stands for the MacWilliams transform of J_x of order x. It is then straightforward to uncover a representation for $\widehat{J}_x(t)$:

$$\widehat{J}_x(t) = \widehat{(J_x)}_x(t) = \sum_{k=0}^{x}(-1)^k\binom{x}{k}\binom{x-N-1}{k}t^k. \tag{2.12}$$

The symmetry (2.10) implies straightforwardly

$$\begin{aligned}
\widehat{J}_{N-x}(t) &= \widehat{(J_{N-x})}_{N-x}(t) = (1+t)^{N-x}J_{N-x}\left(\frac{1-t}{1+t}\right) \\
&= (1+t)^{N-2x}(1+t)^x J_x\left(\frac{1-t}{1+t}\right) = (1+t)^{N-2x}\widehat{J}_x(t).
\end{aligned}$$

Equality

$$\widehat{J}_{N-x}(t) = (1+t)^{N-2x}\widehat{J}_x(t) \tag{2.13}$$

thus obtained will be important in understanding the alternative representation of the generating function introduced in Section 5.

3 Heun Equation

A differential equation for the MacWilliams transform of $J_x(t)$ can be found easily. For short, we denote $J(t) = J_x(t)$ and $\widehat{J}(t) = \widehat{J}_x(t)$ in the following lemmata.

Lemma 1. *$\widehat{J}(t)$ satisfies differential equation*

$$t(1+t)\widehat{J}''(t) + (Nt + 1 - 2t(x-1))\widehat{J}'(t) + x(x-N-1)\widehat{J}(t) = 0. \quad (3.1)$$

Proof. By computing the derivatives of $\widehat{J}(t) = (1+t)^x J(\frac{1-t}{1+t})$ we can represent $\widehat{J}(t)$, $\widehat{J}'(t)$, and $\widehat{J}''(t)$ in terms of $J(\frac{1-t}{1+t})$, $J'(\frac{1-t}{1+t})$, and $J''(\frac{1-t}{1+t})$. A direct calculation allows us also to reverse the representations to get

$$\begin{aligned}
J(\frac{1-t}{1+t}) &= (1+t)^{-x}\widehat{J}(t), && (3.2)\\
J'(\frac{1-t}{1+t}) &= \frac{1}{2}x(1+t)^{-x+1}\widehat{J}(t) - \frac{1}{2}(1+t)^{-x+2}\widehat{J}'(t), \text{ and} && (3.3)\\
J''(\frac{1-t}{1+t}) &= \frac{1}{4}x(x-1)(1+t)^{-x+2}\widehat{J}(t) \\
&- \frac{1}{2}(x-1)(1+t)^{-x+3}\widehat{J}'(t) + \frac{1}{4}(1+t)^{-x+4}\widehat{J}''(t). && (3.4)
\end{aligned}$$

Replacing t with $\frac{1-t}{1+t}$ in (2.8) and substituting (3.2)-(3.4) into (2.8) gives us the claim.

Another way to prove the lemma is to use (2.12) and verify by direct calculations that differential equation (3.1) is satisfied. □

Lemma 2. *Let $T(t)$ be defined as $T(t) = (1+t)^{N-2x}\widehat{J}(-t^2)$. Then $T(t)$ satisfies differential equation*

$$\begin{aligned}
(t^3 - t)T''(t) &+ \big(2t(N-2x) + 3t^2 - 1\big)T'(t) \\
&+ \big(N - 2x - tN(N+2)\big)T(t) = 0 && (3.5)
\end{aligned}$$

Proof. As in the previous Lemma, we can express $T(t)$, $T'(t)$, and $T''(t)$ in terms of $\widehat{J}(-t^2)$, $\widehat{J}'(-t^2)$, and $\widehat{J}''(-t^2)$, and then to reverse the representations to get

$$\begin{aligned}
\widehat{J}(-t^2) &= (1+t)^{2x-N}T(t) && (3.6)\\
\widehat{J}'(-t^2) &= \frac{1}{2t}(N-2x)(1+t)^{2x-N-1}T(t) \\
&\quad -\frac{1}{2t}(1+t)^{2x-N}T'(t) && (3.7)\\
\widehat{J}''(-t^2) &= \frac{1}{4t^3}(N-2x)(1+t)^{2x-N-2}(t(N-2x+2)+1)T(t) \\
&\quad -\frac{1}{4t^3}(2t(N-2x)+t+1)(1+t)^{2x-N-1}T'(t) \\
&\quad +\frac{1}{4t^2}(1+t)^{2x-N}T''(t) && (3.8)
\end{aligned}$$

by direct calculation. By substituting $-t^2$ for t in (3.1) and by using (3.6)-(3.8), we get differential equation (3.5) after some direct calculations. $\square$

Differential equation (3.5) can be easily rewritten in *standard natural form for the Heun differential equation*

$$\begin{aligned} & t(t-1)(t-q)y''(t) + \big(c(t-1)(t-q) + d\cdot t(t-q) \\ + \quad & (a+b+1-c-d)t(t-1)\big)y'(t) + (abt-\lambda)y(t) = 0 \end{aligned}$$

(see [8]) by taking $q=-1$, $a=-N$, $b=N+2$, $c=1$, $d=N-2x+1$, and $\lambda = 2x-N$. So, *the generalized Riemann scheme* (see [8]), describing the local characteristic properties of this equation, is as follows:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \\ 0 & 1 & -1 & \infty & ;t \\ 0 & 0 & 0 & -N & ;2x-N \\ 0 & 2x-N & N-2x & N+2 & \end{pmatrix}.$$

4 The Generating Function

By equality (2.12) function $T(t) = (1+t)^{N-2x}\widehat{J}_x(-t^2)$ can be represented as

$$T(t) = (1+t)^{N-2x}\sum_{k=0}^{x}\binom{x}{k}\binom{x-N-1}{k}t^{2k}. \tag{4.1}$$

If $t\in(-1,1)$, we should keep in mind that $\widehat{J}_x(-t^2) = (1+t^2)^x J(\frac{1+t^2}{1-t^2})$ can be straightforwardly defined for any real values of x. Hence for $t\in(-1,1)$ also $T(t) = (1+t)^{N-2x}\widehat{J}_x(-t^2)$ can be defined for an arbitrary real x, even though (4.1) is meaningful only for integer values of x (because of the summation upper bound). Another way of generalizing (4.1) even to complex values of x is to expand (4.1) straightforwardly to see that if we write

$$T(t) = \sum_{k=0}^{\infty}\tau_k(x)t^k, \tag{4.2}$$

then

$$\tau_k(x) = \sum_{0\le l\le k/2}\binom{N-2x}{k-2l}\binom{x}{l}\binom{x-N-1}{l} \tag{4.3}$$

is a polynomial of degree k. For any fixed x, $T(t)$ is an analytic function of t in the disc $|t|<1$ (we can use the principal branch of the logarithm to defined to power), and hence it has a unique Maclaurin expansion (4.2) convergent when $|t|<1$.

That (4.2) converges for $|t|<1$ can be also verified by using the ratio test, but to estimate $|\tau_{k+1}(x)/\tau_k(x)|$ as k tends to infinity is not very straightforward. On the other hand, the recurrence of the next lemma reveals that $\lim\limits_{k\to\infty}|\tau_{k+1}(x)/\tau_k(x)| = 1$.

Remark 1. Polynomials $\tau_k(x)$ for small values of k are easy to find by using (4.3). For instance, $\tau_0(x) = 1$, $\tau_1(x) = N - 2x$, and $\tau_2(x) = 3x^2 - 3Nx + \frac{1}{2}N(N-1)$.

Lemma 3. *For $k \geq 2$, polynomials $\tau_k(x)$ satisfy the recurrence relation*

$$k^2\tau_k(x) = (2k-1)(N-2x)\tau_{k-1}(x) - (N+k)(N-k+2)\tau_{k-2}(x). \quad (4.4)$$

Proof. This is a general property for a generic solution to Heun equation, see [8]. Recurrence (4.4) can be also obtained by differentiating and substituting (4.2) to Equation (3.5). □

Remark 2. From (4.4) it follows that

$$\frac{\tau_k(x)}{\tau_{k-1}(x)} = \frac{(2k-1)(N-2x)}{k^2} - \frac{(N+k)(N-k+2)}{k^2}\frac{\tau_{k-2}(x)}{\tau_{k-1}(x)},$$

which shows that $\lim_{k\to\infty} |\tau_{k+1}(x)/\tau_k(x)| = \infty$ is impossible. Since clearly $\tau_k(x)$ is a rational expression in k, the limit exists and is finite. Now

$$\frac{\tau_k(x)}{\tau_{k-1}(x)} \cdot \frac{\tau_{k-1}(x)}{\tau_{k-2}(x)} = \frac{(2k-1)(N-2x)}{k^2} \cdot \frac{\tau_{k-1}(x)}{\tau_{k-2}(x)} - \frac{(N+k)(N-k+2)}{k^2}$$

shows that $\lim\limits_{k\to\infty} |\tau_{k+1}(x)/\tau_k(x)| = 1$.

We are now ready to state the main result.

Theorem 1. *Function*

$$T_{N,x}(t) = (1+t)^{N-2x}\widehat{J}_x(-t^2) \quad (4.5)$$

is the generating function of discrete Chebyshev polynomials, i.e. $\tau_k(x) = D_k(x)$ for each $k \geq 0$.

Proof. By (2.3), the Discrete Chebyshev polynomials satisfy the same recurrence relation (4.4) as polynomials $\tau_k(x)$ do. Since the initial conditions $\tau_0(x) = D_0(x)$ and $\tau_1(x) = D_1(x)$ hold by Remark 1, we have equality $\tau_k(x) = D_k(x)$ for each k. □

Remark 3. It may be useful to compare (4.5) and (2.4). Let $I_x(u) = 1 = u^0$. Then by Formula (2.11)

$$\widehat{I}_x(t) = (1+t)^x\left(\frac{1-t}{1+t}\right)^0 = (1+t)^x,$$

so the generating function of Krawtchouk polynomials can be written as

$$(1+t)^{N-x}(1-t)^x = (1+t)^{N-2x}(1-t^2)^x = (1+t)^{N-2x}\widehat{I}_x(-t^2),$$

whereas the generating function for Discrete Chebyshev polynomials is

$$(1+t)^{N-2x}\widehat{J}_x(-t^2),$$

where

$$J_x(u) = \frac{1}{2^x}\sum_{k=0}^{x}\binom{x}{k}\binom{x-N-1}{k}(u-1)^k(u+1)^k.$$

Note also that formula (2.13) is (trivially) valid for I as well for J.

Moreover,

$$I_x(u) = 1 = \frac{1}{2^x}((1-u)+(1+u))^x = \frac{1}{2^x}\sum_{k=0}^{x}\binom{x}{k}(1-u)^k(1+u)^{x-k},$$

whereas

$$\begin{aligned} J_x(u) &= \frac{1}{2^x}\sum_{k=0}^{x}(-1)^k\binom{x-N-1}{k}\left(\binom{x}{k}(1-u)^k(1+u)^{x-k}\right) \\ &= \frac{1}{2^x}\sum_{k=0}^{x}\binom{N+k-x}{k}\binom{x}{k}(1-u)^k(1+u)^{x-k}. \end{aligned}$$

In addition, the sum or coefficients is equal to

$$\sum_{k=0}^{x}\binom{N+k-x}{k} = \sum_{k=0}^{x}\binom{(N-x)+k}{(N-x)} = \binom{N+1}{x}.$$

5 Concluding Remarks

Example 1. Expression (4.5) shows that if x is an integer at most $N/2$, then $T_{N,x}(t)$ is a polynomial in t of degree $N-2x+2x=N$. Thus we can find expressions

$$T_{N,x}(t) = \sum_{n=0}^{N} D_n^{(N)}(x)t^n$$

by simply evaluating $D_n(N)(x)$ for $n \in \{0,1,\ldots,N\}$ by using (2.3) or (4.3). For example, $N=6$ gives

$$\begin{aligned} T_{6,0}(t) &= 1+6t+15t^2+20t^3+15t^4+6t^5+t^6 = (1+t)^6 \\ T_{6,1}(t) &= 1+4t+0\cdot t^2-20t^3-35t^4-24t^5-6t^6 = (1+t)^4(1-6t^2) \\ T_{6,2}(t) &= 1+2t-9t^2-20t^3+5t^4+30t^5+15t^6 \\ &= (1+t)^2(1-10t^2+15t^4) \\ T_{6,3}(t) &= 1-12t^2+30t^4-20t^6, \end{aligned}$$

which is in full accordance with (4.5) and (2.12). For $x \in \{4,5,6\}$ the power $6-2x$ of $1+t$ in (4.5) is no longer positive, so it is not clear that $T_{6,x}(t)$ would be a polynomial anymore. But if $T_{6,x}$ is not a polynomial for $x \in \{4,5,6\}$, there would be a rather mysterious asymmetry between $x \leq 3$ and $x > 3$. Fortunately it is easy to show that $T_{N,x}(t)$ is indeed a polynomial for each $x \in \{0,1,\dots,N\}$ and the asymmetry actually vanishes via trivial equality $1-t^2 = (1+t)(1-t)$.

Theorem 2. *The generating function $T_{N,x}(t)$ can be also represented as*

$$T_{N,x}(t) = (1-t)^{2x-N}\widehat{J}_{N-x}(-t^2). \tag{5.1}$$

Proof. Equality (2.13) implies

$$\widehat{J}_{N-x}(-t^2) = (1-t^2)^{N-2x}\widehat{J}_x(-t^2) = (1-t)^{N-2x}(1+t)^{N-2x}\widehat{J}_x(-t^2),$$

and the claim follows immediately. □

Example 2 (Example 1 continued)**.** Since by Theorem 2 the expressions $T_{6,x}(t)$ are polynomials in t of degree 6, we can evaluate their values for $x \in \{4,5,6\}$ as

$$\begin{aligned}
T_{6,4}(t) &= 1-2t-9t^2+20t^3+5t^4-30t^5+15t^6 \\
&= (1-t)^2(1-10t^2+15t^4) \\
T_{6,5}(t) &= 1-4t+0\cdot t^2+20t^3-35t^4+24t^5-6t^6 = (1-t)^4(1-6t^2) \\
T_{6,6}(t) &= 1-6t+15t^2-20t^3+15t^4-6t^5+t^6 = (1-t)^6.
\end{aligned}$$

This is again in full accordance with (5.1) and (2.12).

To combine Theorems 1 and 2 into a single presentation is straightforward:

Theorem 3 (The explicit polynomial form for $x \in \{0,1,\dots,N\}$)**.** *The generating function $T_{N,x}(t)$ can be presented as a polynomial in t of degree N:*

$$T_{N,x}(t) = (1+t\cdot\operatorname{sign}(N-2x))^{|N-2x|}\widehat{J}^{(N)}_{\min\{x,N-x\}}(-t^2).$$

Remark 4. Theorem 1 implies that (2.2) and (4.3) are equal, i.e.

$$\sum_{0\leq l\leq k/2}\binom{N-2x}{k-2l}\binom{x}{l}\binom{x-N-1}{l} = \sum_{l=0}^{k}(-1)^l\binom{k}{l}\binom{N-x}{k-l}\binom{x}{l}. \tag{5.2}$$

A direct combinatorial proof of (5.2) appears challenging, for instance, the techniques of [7] appear powerless in this case. Theorem 2 implies an identity similar to (5.2).

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