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On the Interplay of Direct Topological Factorizations and Cellular Automata Dynamics on Beta-shifts

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Abstract

We consider the range of possible dynamics of cellular automata (CA) on two-sided beta-shifts S_β and its relation to direct topological factorizations. We show that any reversible CA $F : S_\beta \to S_\beta$ has an almost equicontinuous direction whenever S_β is not sofic. This has the corollary that non-sofic beta-shifts are topologically direct prime, i.e. they are not conjugate to direct topological factorizations $X \times Y$ of two nontrivial subshifts X and Y. We also give a simple criterion to determine whether $S_{n\gamma}$ is conjugate to $S_n \times S_\gamma$ for a given integer $n \geq 1$ and a given real $\gamma > 1$ when S_γ is a subshift of finite type. When S_γ is strictly sofic, we show that such a conjugacy is not possible at least when γ is a quadratic Pisot number of degree 2. We conclude by using direct factorizations to give a new proof for the classification of reversible multiplication automata on beta-shifts with integral base and ask whether nontrivial multiplication automata exist when the base is not an integer.

Keywords: Cellular automata; Beta-shifts; Sensitivity; Direct topological factorizations; Multiplication automata.

1 Introduction

Let $X \subseteq A^{\mathbb{Z}}$ be a one-dimensional subshift over a finite symbol set A. A cellular automaton (CA) is a function $F : X \to X$ defined by a local rule, and it endows the space X with translation invariant dynamics given by local interactions. It is natural to ask how the structure of the underlying subshift X affects the range of possible topological dynamics that can be achieved by CA on X. One approach to this is via the framework of directional dynamics of Sablik [29]. This framework is concerned with the possible space-time diagrams of $x \in X$ with respect to F, in which successive iterations $F^t(x)$ are drawn on consecutive rows (see Figure 1 for a typical space-time diagram of a configuration with respect to the CA which shifts each symbol by one position to the left). Information cannot cross the dashed gray line in the figure so we say that the slope of this line is an almost equicontinuous direction. On the other hand, a slope is called a sensitive direction if information can cross over every line having that slope.

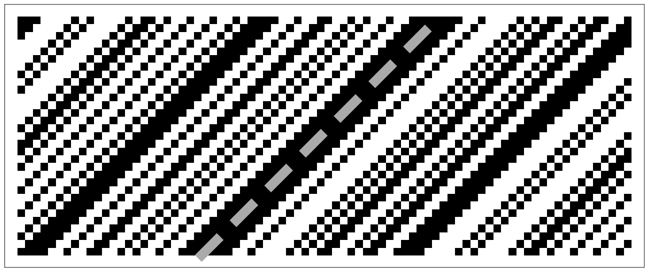

Figure 1: A space-time diagram of the binary shift map σ. White and black squares correspond to digits 0 and 1 respectively. The dashed gray line shows an almost equicontinuous direction.

It has been proven in Theorem 4.13 of [19] that every infinite transitive sofic subshift admits a reversible CA which is sensitive in all directions. On the other hand, Theorem 6.10 of [19] shows that in the class of non-sofic S-gap shifts X_S (all of them are synchronized and many have the specification property) every reversible CA has an almost equicontinuous direction. It would be interesting to extend the latter result to other natural classes of subshifts. The classical study of $\mathrm{Aut}(X)$, the group of reversible CA on X, is mostly not related to our line of inquiry. However, we highlight the result of [7] saying that $\mathrm{Aut}(X)/\langle\sigma\rangle$ is a periodic group if X is a transitive subshift that has subquadratic growth. This implies for such X that every $F \in \mathrm{Aut}(X)$ has an almost equicontinuous direction.

In this paper we consider two-sided beta-shifts, which form a naturally occurring class of mixing coded subshifts. We show in Theorem 4.4 that if S_β is a non-sofic beta-shift, then every reversible CA on S_β has an almost equicontinuous direction. As an application we use this result to show in Theorem 5.1 that non-sofic beta-shifts are topologically direct prime, i.e. they are not conjugate to direct topological factorizations $X \times Y$ of two nontrivial subshifts X and Y, thus answering a problem suggested in the presentation [25].

The proof of Theorem 5.1 relies on the observation that whenever X and Y are infinite transitive subshifts, then $X \times Y$ has a very simple reversibe CA with all directions sensitive: it just shifts information into opposite directions in the X and Y components. Therefore the problem of determining whether a given subshift is topologically direct prime is closely related to the study of directional dynamics. In Section 5 we suggest a program of studying direct topological factorizations of sofic beta-shifts and accompany this suggestion with some preliminary remarks. In Subsection 5.1 we present a characterization of integers $n \geq 1$ and reals $\gamma > 1$ such that $S_{n\gamma}$ is conjugate to $S_n \times S_\gamma$ in the case when S_γ is a subshift of finite type. Such a characterization seems more difficult to find in the case when S_γ is strictly sofic. In Subsection 5.2 we show that $S_{n\gamma}$ is not conjugate to $S_n \times S_\gamma$ for strictly sofic S_γ at least when γ is a Pisot number of degree 2.

In Section 6 we take a look at a class of CA called multiplication automata that are naturally associated to beta-shifts. The classification of all the possible multiplication automata is previously known on beta-shifts with integral base [4]. We present an alternative proof of this classification (at least in the case of reversible multiplication CA) which makes more explicit use of the topological factorizations of the underlying beta-shift. It would be interesting to extend this classification to beta-shifts S_β with nonintegral base.

This paper is an extended version of [18] published in the proceedings of DLT 2020. Subsection 5.2 and Section 6 are new. The material in Section 6 (with the exception of Example 6.8) has previously appeared in the author's doctoral dissertation [17].

2 Preliminaries

In this section we recall some preliminaries concerning symbolic dynamics and topological dynamics in general. Good references to these topics are [20, 23].

Definition 2.1. If X is a compact metrizable topological space and $T : X \to X$ is a continuous map, we say that (X, T) is a *(topological) dynamical system*.

When there is no risk of confusion, we may identify the dynamical system (X, T) with the underlying space or the underlying map, so we may say that X is a dynamical system or that T is a dynamical system.

Definition 2.2. We write $\psi : (X, T) \to (Y, S)$ whenever (X, T) and (Y, S) are dynamical systems and $\psi : X \to Y$ is a continuous map such that $\psi \circ T = S \circ \psi$ (this equality is known as the *equivariance condition*). Then we say that ψ is a *morphism*. If ψ is injective, we say that ψ is an *embedding*. If ψ is surjective, we say that ψ is a *factor map* and that (Y, S) is a factor of (X, T) (via the map ψ). If ψ is bijective, we say that ψ is a *conjugacy* and that (X, T) and (Y, S) are *conjugate* (via ψ).

A finite set A containing at least two elements (*letters*) is called an *alphabet*. In this paper the alphabet usually consists of numbers and thus for $n \in \mathbb{N}_+$ we denote $\Sigma_n = \{0, 1, \ldots, n-1\}$. The set $A^{\mathbb{Z}}$ of bi-infinite sequences (*configurations*) over A is called a *full shift*. The set $A^{\mathbb{N}}$ is the set of one-way infinite sequences over A. Formally any $x \in A^{\mathbb{Z}}$ (resp. $x \in A^{\mathbb{N}}$) is a function $\mathbb{Z} \to A$ (resp. $\mathbb{N} \to A$) and the value of x at $i \in \mathbb{Z}$ is denoted by $x[i]$. It contains finite, right-infinite and left-infinite subsequences denoted by $x[i, j] = x[i]x[i+1]\cdots x[j]$, $x[i, \infty] = x[i]x[i+1]\cdots$ and $x[-\infty, i] = \cdots x[i-1]x[i]$. Occasionally we signify the symbol at position zero in a configuration x by a dot as follows:

$$x = \cdots x[-2]x[-1]x[0].x[1]x[2]x[3]\cdots.$$

A configuration $x \in A^{\mathbb{Z}}$ or $x \in A^{\mathbb{N}}$ is *periodic* if there is a $p \in \mathbb{N}_+$ such that $x[i+p] = x[i]$ for all $i \in \mathbb{Z}$. Then we may also say that x is p-periodic or that x has period p. If x is 1-periodic, we call it a *constant configuration*. We say that x is *eventually periodic* if there are $p \in \mathbb{N}_+$ and $i_0 \in \mathbb{Z}$ such that $x[i+p] = x[i]$ holds for all $i \geq i_0$.

A *subword* of $x \in A^{\mathbb{Z}}$ is any finite sequence $x[i,j]$ where $i,j \in \mathbb{Z}$, and we interpret the sequence to be empty if $j < i$. Any finite sequence $w = w[1]w[2]\cdots w[n]$ (also the empty sequence, which is denoted by ϵ), where $w[i] \in A$, is a *word* over A. Unless we consider a word w as a subword of some configuration, we start indexing the symbols of w from 1 as we have done here. If $w \neq \epsilon$, we say that w occurs in x at position i if $x[i]\cdots x[i+n-1] = w[1]\cdots w[n]$. The concatenation of a word or a left-infinite sequence u with a word or a right-infinite sequence v is denoted by uv. A word u is a *prefix* of a word or a right-infinite sequence x if there is a word or a right-infinite sequence v such that $x = uv$. Similarly, u is a *suffix* of a word or a left-infinite sequence x if there is a word or a left-infinite sequence v such that $x = vu$. The set of all words over A is denoted by A^*, and the set of non-empty words is $A^+ = A^* \setminus \{\epsilon\}$. The set of words of length n is denoted by A^n. For a word $w \in A^*$, $|w|$ denotes its length, i.e. $|w| = n \iff w \in A^n$. For any word $w \in A^+$ we denote by ${}^\infty w$ and w^∞ the left- and right-infinite sequences obtained by infinite repetitions of the word w. We denote by $w^{\mathbb{Z}} \in A^{\mathbb{Z}}$ the configuration defined by $w^{\mathbb{Z}}[in,(i+1)n-1] = w$ (where $n = |w|$) for every $i \in \mathbb{Z}$.

Any collection of words $L \subseteq A^*$ is called a *language*. For any $S \subseteq A^{\mathbb{Z}}$ the collection of words appearing as subwords of elements of S is the language of S, denoted by $L(S)$. For $n \in \mathbb{N}$ we denote $L^n(S) = L(S) \cap A^n$. For any $L \subseteq A^*$, the *Kleene star* of L is

$$L^* = \{w_1 \cdots w_n \mid n \geq 0, w_i \in L\} \subseteq A^*,$$

i.e. L^* is the set of all finite concatenations of elements of L. If $\epsilon \notin L$, define $L^+ = L^* \setminus \{\epsilon\}$ and if $\epsilon \in L$, define $L^+ = L^*$.

To consider topological dynamics on subsets of the full shifts, the sets $A^{\mathbb{Z}}$ and $A^{\mathbb{N}}$ are endowed with the product topology (with respect to the discrete topology on A). These are compact metrizable spaces. The shift map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $\sigma(x)[i] = x[i+1]$ for $x \in A^{\mathbb{Z}}$, $i \in \mathbb{Z}$, and it is a homeomorphism. Also in the one-sided case we define $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ by $\sigma(x)[i] = x[i+1]$. Any topologically closed *nonempty* subset $X \subseteq A^{\mathbb{Z}}$ such that $\sigma(X) = X$ is called a *subshift*. A subshift X equipped with the map σ forms a dynamical system and the elements of X can also be called *points*. Any $w \in L(X) \setminus \epsilon$ and $i \in \mathbb{Z}$ determine a *cylinder* of X

$$\mathrm{Cyl}_X(w,i) = \{x \in X \mid w \text{ occurs in } x \text{ at position } i\}.$$

Definition 2.3. We say that a subshift X is *transitive* (or *irreducible* in the terminology of [23]) if for all words $u,v \in L(X)$ there is $w \in L(X)$ such that $uwv \in L(X)$. We say that X is *mixing* if for all $u,v \in L(X)$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ there is $w \in L^n(X)$ such that $uwv \in L(X)$.

Definition 2.4. Let $X \subseteq A^{\mathbb{Z}}$ and $Y \subseteq B^{\mathbb{Z}}$ be subshifts. We say that a map $F : X \to Y$ is a *sliding block code* from X to Y (with memory m and anticipation a for integers $m \leq a$) if there exists a *local rule* $f : A^{a-m+1} \to B$ such that $F(x)[i] = f(x[i+m], \ldots, x[i], \ldots, x[i+a])$. If $X = Y$, we say that F is a *cellular automaton* (CA). If we can choose m and a so that $-m = a = r \geq 0$, we say that F is a radius-r CA.

Note that both memory and anticipation can be either positive or negative. Note also that if F has memory m and anticipation a with the associated local rule $f : A^{a-m+1} \to A$, then F is also a radius-r CA for $r = \max\{|m|, |a|\}$, with possibly a different local rule $f' : A^{2r+1} \to A$.

The following observation of [12] characterizes sliding block codes as the class of structure preserving transformations between subshifts. In particular, there is a bijective sliding block code from one subshift to another if and only if the two subshifts are conjugate.

Theorem 2.5 (Curtis-Hedlund-Lyndon)**.** A map $F : X \to Y$ between subshifts X and Y is a morphism between dynamical systems (X, σ) and (Y, σ) if and only if it is a sliding block code.

Bijective CA are called *reversible*. It is known that the inverse map of a reversible CA is also a CA. We denote by $\mathrm{End}(X)$ the monoid of CA on X and by $\mathrm{Aut}(X)$ the group of reversible CA on X (the binary operation is function composition).

The notions of almost equicontinuity and sensitivity can be defined for general topological dynamical systems. We omit the topological definitions, because for cellular automata on transitive subshifts there are combinatorial characterizations for these notions using blocking words. We present these alternative characterizations below.

Definition 2.6. Let $F : X \to X$ be a radius-r CA and $w \in L(X)$. We say that w is a *blocking word* if there is an integer e with $|w| \geq e \geq r + 1$ and an integer $p \in [0, |w| - e]$ such that

$$\forall x, y \in \mathrm{Cyl}_X(w, 0), \forall n \in \mathbb{N}, F^n(x)[p, p + e - 1] = F^n(y)[p, p + e - 1].$$

The following is proved in Proposition 2.1 of [29].

Proposition 2.7. If X is a transitive subshift and $F : X \to X$ is a CA, then F is almost equicontinuous if and only if it has a blocking word.

We say that a CA on a transitive subshift is *sensitive* if it is not almost equicontinuous. The notion of sensitivity is refined by Sablik's framework of directional dynamics [29].

Definition 2.8. Let $F : X \to X$ be a cellular automaton and let $p, q \in \mathbb{Z}$ be coprime integers, $q > 0$. Then p/q is a *sensitive direction* of F if $\sigma^p \circ F^q$ is sensitive. Similarly, p/q is an *almost equicontinuous direction* of F if $\sigma^p \circ F^q$ is almost equicontinuous.

As indicated in the introduction, this definition is best understood via the *space-time diagram* of $x \in X$ with respect to F, in which successive iterations $F^t(x)$ are drawn on consecutive rows (see Figure 1 for a typical space-time diagram of a configuration with respect to the shift map). By definition $-1 = (-1)/1$ is an almost equicontinuous direction of $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ because $\sigma^{-1} \circ \sigma = \mathrm{Id}$ is almost equicontinuous. This is directly visible in the space-time diagram of Figure 1, because it looks like the space-time diagram of the identity map when it is followed along the dashed line. Note that the slope of the dashed line is equal to -1 with respect to the vertical axis extending downwards in the diagram.

The notions of subshifts of finite type (SFT) and sofic subshifts are well known and can be found in Chapters 2 and 3 of [23]. If a sofic subshift is not an SFT, we say that it is strictly sofic. Any square matrix A with nonnegative entries is an adjacency matrix of a directed graph with multiple edges. The set of all bi-infinite sequences of edges forming valid paths is an *edge SFT* (associated to A), whose alphabet is the set of edges.

Some other classes of subshifts relevant to the study of beta-shifts are the following.

Definition 2.9. Given a subshift X, we say that a word $w \in L(X)$ is *synchronizing* if

$$\forall u, v \in L(X) : uw, wv \in L(X) \implies uwv \in L(X).$$

We say that a transitive subshift X is *synchronized* if $L(X)$ contains a synchronizing word.

Definition 2.10. A language $L \subseteq A^+$ is a *code* if for all distinct $u, v \in L$ it holds that u is not a prefix of v. A subshift $X \subseteq A^{\mathbb{Z}}$ is a *coded subshift* (given by a code L) if $L(X)$ is the set of all subwords of elements of L^*.

All transitive sofic shifts are syncronized and all synchronized subshifts are coded [8].

3 Beta-shifts

We recall some preliminaries on beta-shifts from Blanchard's paper [2] and from Lothaire's book [24].

For $\xi \in \mathbb{R}$ we denote $\mathrm{Frac}\,(\xi) = \xi - \lfloor \xi \rfloor$, for example $\mathrm{Frac}\,(1.2) = 0.2$ and $\mathrm{Frac}\,(1) = 0$.

Definition 3.1. For every real number $\beta > 1$ we define a map $T_\beta : \mathbb{I} \to \mathbb{I}$, where $\mathbb{I} = [0, 1]$ and $T_\beta(\xi) = \mathrm{Frac}\,(\beta\xi)$ for every $\xi \in \mathbb{I}$.

Definition 3.2. The β-expansion of a number $\xi \in \mathbb{I}$ is the sequence $d(\xi, \beta) \in \Sigma^{\mathbb{N}}_{\lfloor \beta \rfloor + 1}$ where $d(\xi, \beta)[i] = \lfloor \beta T^i(\xi) \rfloor$ for $i \in \mathbb{N}$.

Denote $d(1, \beta) = d(\beta)$. By this convention $d(2) = 2000\ldots$ If $d(\beta)$ ends in infinitely many zeros, i.e. $d(\beta) = d_0 \cdots d_m 0^\infty$ for $d_m \neq 0$, we say that $d(\beta)$ is finite, write $d(\beta) = d_0 \cdots d_m$, and define $d^*(\beta) = (d_0 \cdots (d_m - 1))^\infty$. Otherwise we let $d^*(\beta) = d(\beta)$. Denote by D_β the set of β-expansions of numbers from $[0, 1)$. It is the set of all infinite concatenations of words from the code

$$Y_\beta = \{d_0 d_1 \cdots d_{n-1} b \mid n \in \mathbb{N}, 0 \leq b < d_n\}$$

where $d(\beta) = d_0 d_1 d_2 \ldots$. For example, $Y_2 = \{0, 1\}$. Let S_β be the coded subshift given by the code Y_β. Since S_β is coded, it also has a natural representation by a deterministic automaton (not necessarily finite) [3, 30]. These representations allow us to make pumping arguments similar to those that occur in the study of sofic shifts and regular languages.

Lemma 3.3. The subshift S_β is mixing.

Proof. Any $u, v \in L(S_\beta)$ are subwords of $u_1 \cdots u_n$ and $v_1 \cdots v_m$ respectively for some $n, m \in \mathbb{N}_+$ and $u_i, v_i \in Y_\beta$. Because the code Y_β always contains the word 0, it follows that $u_1 \cdots u_n 0^i v_1 \cdots v_m \in L(S_\beta)$ for all $i \in \mathbb{N}$ and S_β is mixing. □

The subshift S_β is sofic if and only if $d(\beta)$ is eventually periodic and it is an SFT if and only if $d(\beta)$ is finite.

There is a natural lexicographical ordering on $\Sigma_n^{\mathbb{N}}$ which we denote by $<$ and $\leq$. Using this we can alternatively characterize S_β as

$$S_\beta = \{x \in \Sigma_{\lfloor\beta\rfloor}^{\mathbb{Z}} \mid x[i, \infty] \leq d^*(\beta) \text{ for all } i \in \mathbb{Z}\}.$$

We call S_β a *beta-shift* (with base β). When $\beta > 1$ is an integer, the equality $S_\beta = \Sigma_\beta^{\mathbb{Z}}$ holds.

4 CA Dynamics on Beta-shifts

In this section we study the topological dynamics of reversible CA on beta-shifts, and more precisely the possibility of them having no almost equicontinuous directions. By Theorem 4.13 of [19] every infinite transitive sofic subshift admits a reversible CA which is sensitive in all directions, and in particular this holds for sofic beta-shifts. In this section we see that this result does not extend to the class of non-sofic beta-shifts.

We begin with a proposition showing that a CA on a non-sofic beta-shift has to "fix the expansion of one in the preimage" in some sense. For that we also need the following lemma.

Lemma 4.1. If $x \in S_\beta$ and $N \in \mathbb{N}$ is such that $x[i, \infty] \neq d(\beta)$ for all $i \geq N$, then $x[N, \infty] = w_1 w_2 w_3 \ldots$ for some $w_j \in Y_\beta$.

Proof. Because $x[N, \infty] < d(\beta)$, it follows that $x[N, \infty]$ has a prefix of the form $w_1 = d_0 d_1 \cdots d_{n-1} b \in Y_\beta$ for some $n \in \mathbb{N}$, $b < d_n$. We can write $x[N, \infty] = w_1 x_1$ for some $x_1 \in \Sigma_{\lfloor\beta\rfloor}^{\mathbb{N}}$. Because x_1 is a suffix of x, then again from our assumption it follows that $x_1 < d(\beta)$ and we can find a $w_2 \in Y_\beta$ which is a prefix of x_1. For all $j \in \mathbb{Z}$ we similarly we find $w_j \in Y_\beta$ such that $x[N, \infty] = w_1 w_2 w_3 \ldots$. □

Proposition 4.2. Let $\beta > 1$ be such that S_β is not sofic, let $F \in \mathrm{End}(S_\beta)$, let $x \in S_\beta$ be such that $x[0,\infty] = d(\beta)$ and let $y \in F^{-1}(x)$. Then there is a unique $i \in \mathbb{Z}$ such that $y[i,\infty] = d(\beta)$. Moreover, i does not depend on the choice of x or y.

Proof. Let $r \in \mathbb{N}$ be such that F is a radius-r CA.

We first claim that i does not depend on the choice of x or y when it exists. To see this, assume to the contrary that for $j \in \{1,2\}$ there exist $x_j \in S_\beta$ with $x_j[0,\infty] = d(\beta)$, $y_j \in F^{-1}(x_j)$ and $i_j \in \mathbb{Z}$ such that $i_1 < i_2$ and $y_1[i_1,\infty] = d(\beta) = y_2[i_2,\infty]$. Then in particular for $M = \max\{i_2 - i_1, i_2\}$ it holds that $y_2[M,\infty] = y_2[M - i_2 + i_2,\infty] = y_1[M - i_2 + i_1,\infty]$ and

$$\begin{aligned} &d(\beta)[M - i_2 + i_1 + r,\infty] = x_1[M - i_2 + i_1 + r,\infty] = F(y_1)[M - i_2 + i_1 + r,\infty] \\ &= F(y_2)[M + r,\infty] = x_2[M + r,\infty] = d(\beta)[M + r,\infty]. \end{aligned}$$

Then $d(\beta)$ would be eventually periodic, contradicting the assumption that S_β is not sofic.

For the other claim, let us assume that for some x and y as in the assumption of the proposition there is no $i \in \mathbb{Z}$ such that $y[i,\infty] = d(\beta)$. By Lemma 4.1 we can write $y[-r,\infty] = w_1 w_2 w_3 \cdots$ for some $w_i \in Y_\beta$.

Let r_i be such that $y[-r, r_i] = w_1 \cdots w_i$ for all $i \in \mathbb{N}$. Fix some $j, k \in \mathbb{N}$ such that $0 \leq r_j < r_k$, $|r_k - r_j| \geq 2r$ and $y[r_j - r, r_j + r] = y[r_k - r, r_k + r]$. Because x is not eventually periodic, it follows that $x[r_j + 1,\infty] \neq x[r_k + 1,\infty]$.

Assume first that $x[r_j + 1,\infty] < x[r_k + 1,\infty]$. Because S_β is coded, there is a configuration $z \in S_\beta$ such that $z[-r,\infty] = w_1 \cdots w_j w_{k+1} w_{k+2} \cdots$, i.e. this suffix can be found by removing the word $w_{j+1} \cdots w_k$ from the middle of $y[-r,\infty]$. Then $F(z) \in S_\beta$ but $F(z)[0,\infty] = x[0,r_j]x[r_k + 1,\infty] > x[0,r_j]x[r_j + 1,\infty] = d(\beta)$ contradicting one of the characterizations of S_β.

Assume then alternatively that $x[r_j + 1,\infty] > x[r_k + 1,\infty]$. Because S_β is coded, there is a configuration $z \in S_\beta$ such that

$$z[-r,\infty] = w_1 \cdots w_j (w_{j+1} \cdots w_k)(w_{j+1} \cdots w_k) w_{k+1} w_{k+2} \cdots,$$

i.e. this suffix can be found by repeating the occurrence of the word $w_{j+1} \cdots w_k$ in the middle of $y[-r,\infty]$. Then $F(z) \in S_\beta$ but

$$\begin{aligned} &F(z)[0,\infty] = x[0,r_j]x[r_j + 1, r_k]x[r_j + 1, r_k]x[r_k + 1,\infty] \\ &= x[0,r_j]x[r_j + 1, r_k]x[r_j + 1,\infty] > x[0,r_j]x[r_j + 1, r_k]x[r_k + 1,\infty] = d(\beta) \end{aligned}$$

contradicting again the characterization of S_β. □

To apply the previous proposition for a non-sofic shift S_β and $F \in \mathrm{End}(S_\beta)$, there must exist at least some $x, y \in S_\beta$ such that $x[0,\infty] = d(\beta)$ and $y \in F^{-1}(x)$. This happens at least when F is surjective, in which case we take the number $i \in \mathbb{Z}$ of the previous proposition and say that the intrinsic shift of F is equal to i. If the intrinsic shift of F is equal to 0, we say that F is shiftless.

In the class of non-synchronized beta-shifts we get a very strong result on surjective CA: they are all shift maps.

Theorem 4.3. If S_β is not synchronized, then all surjective CA in $\mathrm{End}(S_\beta)$ are powers of the shift map.

Proof. Let $F \in \mathrm{End}(S_\beta)$ be an arbitrary surjective CA and let $r \in \mathbb{N}$ be some radius of F. We may assume without loss of generality (by composing F with a suitable power of the shift if necessary) that F is shiftless. We prove that $F = \mathrm{Id}$.

Assume to the contrary that $F \neq \mathrm{Id}$, so there is $x \in S_\beta$ such that $F(x)[0] \neq x[0]$. Let $e = {}^\infty 0.d(\beta)$ and let $z \in F^{-1}(e)$ be arbitrary, so in particular $z[0,\infty] = d(\beta)$ by Proposition 4.2. Since S_β is not synchronized, it follows that every word of $L(S_\beta)$ occurs in $d(\beta)$ (as explained by Kwietniak in [21], attributed to Bertrand-Mathis [1]). In particular it is possible to choose $i \geq r+1$ such that $\sigma^i(z)[-r,r] = x[-r,r]$ and $F(x)[0] = F(\sigma^i(z))[0] = \sigma^i(z)[0] = x[0]$, a contradiction. $\square$

Next we consider only reversible CA. They do not have to be shift maps in the class of general non-sofic beta-shifts, and in fact the group $\mathrm{Aut}(X)$ contains a copy of the free product of all finite groups whenever X is an infinite synchronized subshift by Theorem 2.17 of [9]. Nevertheless $\mathrm{Aut}(S_\beta)$ is constrained in the sense of directional dynamics.

Theorem 4.4. If S_β is not sofic and $F \in \mathrm{Aut}(S_\beta)$ is shiftless then F admits a blocking word. In particular all elements of $\mathrm{Aut}(S_\beta)$ have an almost equicontinuous direction.

Proof. Let $r \in \mathbb{N}_+$ be a radius of both F and F^{-1}. Since $d(\beta)$ is not eventually periodic, it is easy to see (and is one formulation of the Morse-Hedlund theorem, see e.g. Theorem 7.3 of [27]) that there is a word $u \in \Sigma_{\lfloor\beta\rfloor}^{3r}$ and symbols $a < b$ such that both ua and ub are subwords of $d(\beta)$. Let $p = p'ub$ ($p, p' \in L(S_\beta)$) be some prefix of $d(\beta)$ ending in ub. We claim that p is blocking. More precisely we will show that if $x \in S_\beta$ is such that $x[0,|p|-1] = p$ then $F^t(x)[0,|p|-2] = p'u$ for all $t \in \mathbb{N}$.

Assume to the contrary that $t \in \mathbb{N}$ is the minimal number for which we have $F^t(x)[0,|p|-2] \neq p'u$. We can find $w, v, v' \in L(S_\beta)$ and $c, d \in \Sigma_{\lfloor\beta\rfloor}$ ($c < d$) so that $u = wdv$, $|w| \geq 2r$ (recall that $|u| = 3r$) and $F^t(x)[0,|p|-2] = p'wcv'$. Indeed, F^{-1} is shiftless because F is, and therefore the prefix $p'w$ still remains unchanged in $F^t(x)[0,\infty]$.

Now we note that x could have been chosen so that some of its suffixes is equal to 0^∞ and in particular under this choice no suffix of $F^t(x)$ is equal to $d(\beta)$. Therefore, by Lemma 4.1 we can represent $F^t(x)[0,\infty] = w_1 w_2 w_3 \dots$ where $w_i \in Y_\beta$ for all $i \in \mathbb{N}$ and in fact $w_1 = p'wc$.

Now let $q = q'ua$ ($q, q' \in L(S_\beta)$) be some prefix of $d(\beta)$ ending in ua. Then also $q'wd$ is a prefix of $d(\beta)$ and thus $q'wc \in Y_\beta$. Because S_β is a coded subshift, there is a configuration $y \in S_\beta$ such that $y[0,\infty] = (q'wc)w_2w_3\dots$. For such y it holds that $F^{-t}(y) \in S_\beta$ but $F^{-t}(y)[0,\infty] = q'(ub)x[|p|,\infty] > d(\beta)$ contradicting the characterization of S_β. $\square$

5 Topological Direct Primeness of Beta-shifts

We recall the terminology of Meyerovitch [26]. A *direct topological factorization* of a subshift X is a subshift $X_1 \times \cdots \times X_n$ which is conjugate to X and where each X_i is a subshift. We also say that each subshift X_i is a *direct factor* of X. The subshift X is *topologically direct prime* if it does not admit a non-trivial direct factorization, i.e. if every direct factorization contains one copy of X and the other X_i in the factorization contain just one point.

Non-sofic beta-shifts turn out to be examples of topologically direct prime dynamical systems. This is an application of Theorem 4.4.

Theorem 5.1. If S_β is a non-sofic beta-shift then it is topologically direct prime.

Proof. Assume to the contrary that there is a topological conjugacy $\phi : S_\beta \to X \times Y$ where X and Y are non-trivial direct factors of S_β. The subshifts X and Y are mixing because they are factors of the subshift S_β which is mixing by Lemma 3.3, and in particular both of them are infinite, because a mixing finite subshift can only contain one point.

Define a reversible CA $F : X \times Y \to X \times Y$ by $F(x, y) = (\sigma(x), \sigma^{-1}(y))$ for all $x \in X$, $y \in Y$. Because X and Y are infinite, it follows that F has no almost equicontinuous directions, i.e. $\sigma^r \circ F^s$ is sensitive for all coprime r and s such that $s > 0$. Then define $G = \phi^{-1} \circ F \circ \phi : S_\beta \to S_\beta$. The map G is a reversible CA on S_β and furthermore (S_β, G) and $(X \times Y, F)$ are conjugate via the map ϕ. By Theorem 4.4 the CA G has an almost equicontinuous direction, so we can fix coprime r and s such that $s > 0$ for which $\sigma^r \circ G^s$ is almost equicontinuous. But $\sigma^r \circ G^s$ is conjugate to $\sigma^r \circ F^s$ via the map ϕ, so $\sigma^r \circ F^s$ is also almost equicontinuous, a contradiction. □

In general determining whether a given subshift is topologically direct prime or not seems to be a difficult problem. Lind gives sufficient conditions in [22] for SFTs based on their entropies: for example any mixing SFT with entropy $\log p$ for a prime number p is topologically direct prime. The paper [26] contains results on multidimensional full shifts, multidimensional 3-colored chessboard shifts and p-Dyck shifts with p a prime number.

In the class of beta-shifts the question of topological direct primeness remains open in a countable number of cases.

Problem 5.2. Characterize the topologically direct prime sofic beta-shifts.

Example 5.3. If $n > 1$ is an integer, then $S_n = \Sigma_n^{\mathbb{Z}}$ is topologically direct prime if and only if n is a prime number. Namely, if $n = pq$ for integers $p, q \geq 2$, then S_n is easily seen to be conjugate to $S_p \times S_q$ via a coordinatewise bijective symbol map (just map $(a_1, a_0) \in \Sigma_p \times \Sigma_q$ to $a_1 q + a_0 \in \Sigma_n$). The case when $n = p$ is a prime number follows by Lind's result because the entropy of $\Sigma_p^{\mathbb{Z}}$ is $\log p$.

In this example the existence of a direct factorization is characterized by the existence of direct factorization into beta-shifts with integral base. Therefore, considering the following problem might be a good point to start with Problem 5.2.

Problem 5.4. Characterize the numbers $n, \gamma > 1$ such that n is an integer and $S_{n\gamma}$ is conjugate to $S_n \times S_\gamma$.

5.1 SFT beta-shifts

In this subsection, we consider Problem 5.4 in the case when S_γ is an SFT. We start with a definition and a lemma stated in [2].

Definition 5.5. Let $n > 1$ be an integer and $w \in \Sigma_n^*$. We say that w is *lexicographically greater than all its shifts* if $w0^\infty > \sigma^i(w0^\infty)$ for every $i > 0$.

Lemma 5.6 (Blanchard, [2])**.** S_β is an SFT if and only if $\beta > 1$ is the unique positive solution of some equation $x^d = a_{d-1}x^{d-1} + \cdots + a_0$ where $d \geq 1$, $a_{d-1}, a_0 \geq 1$ and $a_i \in \mathbb{N}$ such that $a_{d-1} \cdots a_0$ is lexicographically greater than all its shifts. Then $d(\beta) = a_{d-1} \cdots a_0$.

Proof. The polynomial equation is equivalent to $1 = a_{d-1}x^{-1} + \cdots + a_0 x^{-d}$, which clearly has a unique positive solution. If β satisfies such an equation then $d(\beta) = a_{d-1} \cdots a_0$ and S_β is an SFT. On the other hand, if S_β is an SFT, then $d(\beta)$ takes the form of a word $a_{d-1} \cdots a_0$ which is lexicographically greater than all its shifts and β satisfies $1 = a_{d-1}x^{-1} + \cdots + a_0 x^{-d}$. $\square$

We also make the following simple observation.

Lemma 5.7. If the equation $x^d = a_{d-1}x^{d-1} + \cdots + a_0$ has roots $\gamma_1, \gamma_2, \ldots, \gamma_d$ (listed with multiplicity), then $y^d = na_{d-1}y^{d-1} + \cdots + n^d a_0$ has roots $n\gamma_1, n\gamma_2, \ldots, n\gamma_d$ (listed with multiplicity).

Proof. By multiplying both sides of $x^d = a_{d-1}x^{d-1} + \cdots + a_0$ by n^d and by substituting $y = nx$ we see that the roots of $y^d = na_{d-1}y^{d-1} + \cdots + n^d a_0$ are $n\gamma_i$. Because multiplying $\prod_i (x - \gamma_i) = 0$ by n^d yields $\prod_i (y - n\gamma_i) = 0$, we also see that the multiplicities of the roots γ_i and $n\gamma_i$ are the same in their respective equations.

$\square$

For the following we also recall some facts on zeta functions. The zeta function $\zeta_X(t)$ of a subshift X is a formal power series that encodes information about the number of periodic configurations in X and it is a conjugacy invariant of X (for precise definitions see Section 6.4 of [23]). Every SFT X is conjugate to an edge SFT associated to a square matrix A. Let I be an index set and let $\{\mu_i \in \mathbb{C} \setminus \{0\} \mid i \in I\}$ be the collection of non-zero eigenvalues of A with multiplicities: it is called the non-zero spectrum of X. It is known that then $\zeta_X(t) = \prod_{i \in I}(1 - \mu_i t)^{-1}$. The number of p-periodic configurations in X is equal to $\sum_{i \in I} \mu_i^p$ for $p \in \mathbb{N}_+$. If Y is also an SFT with $\zeta_Y(t) = \prod_{j \in J}(1 - \nu_j t)^{-1}$, then the zeta function of $X \times Y$ is $\zeta_X(t) \otimes \zeta_Y(t) = \prod_{i \in I, j \in J}(1 - \mu_i \nu_j t)^{-1}$ [22].

Theorem 5.8. Let S_γ be an SFT with γ the unique positive solution of some equation $x^d = a_{d-1}x^{d-1} + \cdots + a_0$ where $d \geq 1$, $a_{d-1}, a_0 \geq 1$ and $a_i \geq 0$ such that $a_{d-1}\cdots a_0$ is lexicographically greater than all its shifts. If $n \geq 2$ is an integer such that also $(na_{d-1})\cdots(n^d a_0)$ is lexicographically greater than all its shifts, then $S_{n\gamma}$ is topologically conjugate to $S_n \times S_\gamma$. The converse also holds: if $(na_{d-1})\cdots(n^d a_0)$ is not lexicographically greater than all its shifts, then either $S_{n\gamma}$ is not an SFT or $S_{n\gamma}$ and $S_n \times S_\gamma$ have different zeta functions. In particular they are not conjugate.

Proof. We have $d(\gamma) = a_{d-1}\cdots a_0$. The roots of $x^d = a_{d-1}x^{d-1} + \cdots + a_0$ are $\gamma_1 = \gamma, \gamma_2, \ldots, \gamma_d$ (listed with multiplicity). By Lemma 5.7 the roots of $y^d = na_{d-1}y^{d-1} + \cdots + n^d a_0$ are $n\gamma_i$ (listed with multiplicity) and $n\gamma$ is the unique positive solution. If $(na_{d-1})\cdots(n^d a_0)$ is lexicographically greater than all its shifts, then $S_{n\gamma}$ is an SFT with $d(n\gamma) = na_{d-1}\cdots n^d a_0$. By Propositions 4.5 and 4.7 of [13] the shifts S_γ and $S_{n\gamma}$ are conjugate to the edge shifts X_C and X_B respectively given by the matrices

$$C = \begin{pmatrix} a_{d-1} & 1 & 0 & \cdots & 0 \\ a_{d-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad B = \begin{pmatrix} na_{d-1} & 1 & 0 & \cdots & 0 \\ n^2 a_{d-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ n^d a_0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

They are also the companion matrices of the polynomials $x^d - a_{d-1}x^{d-1} - \cdots - a_0$ and $y^d - na_{d-1}y^{d-1} - \cdots - n^d a_0$, so the eigenvalues are the roots of these polynomials and the zeta functions of S_γ and $S_{n\gamma}$ are

$$\zeta_{X_C}(t) = \prod_i (1 - \gamma_i t)^{-1} \quad \text{and} \quad \zeta_{X_B}(t) = \prod_i (1 - n\gamma_i t)^{-1}.$$

In any case $\zeta_{S_n} = (1 - nt)^{-1}$, so the zeta function of $X = S_n \times S_\gamma$ is $\zeta_X(t) = \prod_i (1 - n\gamma_i t)^{-1}$, which is equal to ζ_{X_B}.

We will construct a conjugacy between $S_n \times X_C$ and X_B. We will choose the labels of the edges in X_C and X_B as in Figures 2 and 3. The labels in the figures range according to $0 \leq i_j < n$ and $0 \leq k_j < a_{d-j}$ for $1 \leq j \leq d$.

The labeling has been chosen in a way that suggests the correct choice of the reversible sliding block code $\phi : S_n \times X_C \to X_B$. For any $x \in S_n \times X_C$ we make the usual identification $x = (x_1, x_2)$ where $x_1 \in S_n$, $x_2 \in X_C$ and we denote by $\pi_1(x) = x_1$, $\pi_2(x) = x_2$ the projecting maps. Then ϕ is defined by

$$\phi(x)[i] = \begin{cases} *_j \text{ when } \pi_2(x)[i] = *_j, \\ (i_1, i_2, \ldots, i_j, k_j) \text{ when } \pi_2(x)[i] = (j, k_j) \\ \qquad \text{and } \pi_1(x)[i-(j-1), i] = i_1 i_2 \cdots i_j \end{cases}$$

More concretely, any configuration $s \in S_n \times X_C$ can be represented as a bi-infinite concatenation of words $x = \cdots w_{-2}w_{-1}w_0w_1w_2\cdots$ such that for each $k \in \mathbb{Z}$ the word $\pi_2(w_k)$ is of the form $*_1 *_2 \cdots *_{j-1}\,(j, k_j)$ for some $1 \leq j \leq d$

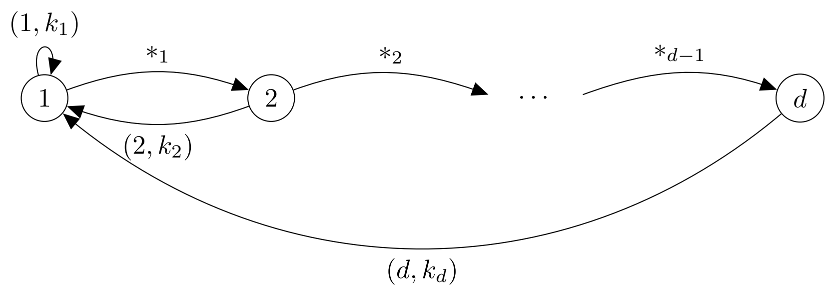

Figure 2: The choice of labels for the graph of X_C.

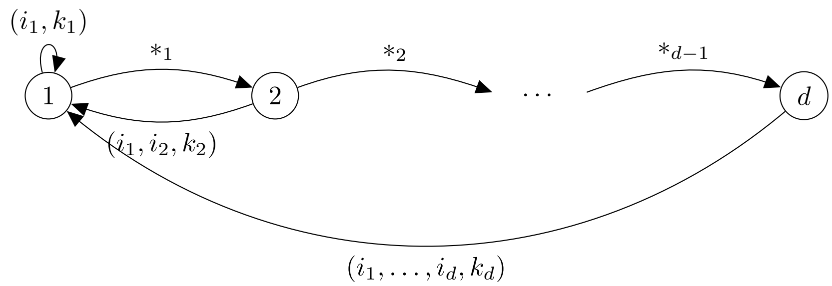

Figure 3: The choice of labels for the graph of X_B.

and $0 \leq k_j \leq n$. If $\pi_1(w_k) = i_1 i_2 \cdots i_j$ and w_k occurs in x at position $i-(j-1)$, then $\phi(x)[i-(j-1), i] = *_1 *_2 \cdots *_{j-1} (i_1, i_2, \cdots, i_j, k_j)$. Essentially the map ϕ just deposits all the information in $\pi_1(w_k)$ to the far right in $\phi(x)[i-(j-1), i]$. Clearly this map is reversible.

For the converse, assume that the word $(na_{d-1}) \cdots (n^d a_0)$ is not lexicographically greater than all its shifts and that $S_{n\gamma}$ is an SFT. Then $n\gamma$ is the unique positive solution of some equation $x^e = b_{e-1}x^{e-1} + \cdots + b_0$ where $e \geq 1$, $b_{e-1}, b_0 \geq 1$ and $b_i \geq 0$ such that $b_{e-1} \cdots b_0$ is lexicographically greater than all its shifts. As above, $S_{n\gamma}$ is conjugate to an edge shift Y given by a matrix with eigenvalues $\beta_1, \beta_2, \ldots, \beta_e$ which are also the roots of the corresponding polynomial. By our assumption the polynomials $x^e - b_{e-1}x^{e-1} - \cdots - b_0$ and $y^d - na_{d-1}y^{d-1} - \cdots - n^d a_0$ are different, so they have different sets of roots (with multiplicities taken into account) and

$$\zeta_Y(t) = \prod_j (1 - \beta_i t)^{-1} \neq \prod_i (1 - n\gamma_i t)^{-1} = \zeta_X(t),$$

because $\mathbb{C}[t]$ is a unique factorization domain. $\square$

We conclude this subsection with an example concerning an SFT beta-shift $S_{\beta_1\beta_2}$ where the assumption of either β_1 or β_2 being an integer is dropped.

Example 5.9. A beta-shift S_{γ^2} can be topologically direct prime even if S_γ and S_{γ^2} are SFTs (and then in particular S_{γ^2} is not conjugate to $S_\gamma \times S_\gamma$). Denote by γ the unique positive root of $x^3 - x^2 - x - 1$. By Lemma 5.6 we have $d(\gamma) = 111$ and in particular S_γ is an SFT. Denote $\beta = \gamma^2$. Its minimal polynomial is $x^3 - 3x^2 - x - 1$ and by Lemma 5.6 $d(\beta) = 311$, so S_β is an SFT and it is conjugate to the edge SFT given by the matrix $A = \left(\begin{smallmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right)$. It has three distinct eigenvalues $\beta_0 = \beta$, β_1 and β_2.

We claim that S_β is topologically direct prime. To see this, assume to the contrary that S_β is topologically conjugate to $X \times Y$ where X and Y are nontrivial direct factors for S_β. Since S_β is an SFT and mixing by Lemma 3.3, it follows from Proposition 6 of [22] that X and Y are mixing SFTs and in particular they are infinite. The zeta functions of X and Y are of the form

$$\zeta_X(t) = \prod_i (1 - \mu_i t)^{-1} \quad \text{and} \quad \zeta_Y(t) = \prod_j (1 - \nu_j t)^{-1}$$

for some $\mu_i, \nu_j \in \mathbb{C} \setminus \{0\}$. The zeta function of S_β is

$$\zeta_{S_\beta}(t) = (1 - \beta t)^{-1}(1 - \beta_1 t)^{-1}(1 - \beta_2 t)^{-1} = \prod_{i,j} (1 - \mu_i \nu_j t)^{-1}.$$

Because $\mathbb{C}[t]$ is a unique factorization domain and because X and Y are nontrivial SFTs, we may assume without loss of generality that $\zeta_X(t) = (1 - \mu t)^{-1}$ and $\zeta_Y(t) = (1 - \nu_1 t)^{-1}(1 - \nu_2 t)^{-1}(1 - \nu_3 t)^{-1}$ for some $\mu, \nu_1, \nu_2, \nu_3 \in \mathbb{C} \setminus \{0\}$. The quantities μ and $\nu_1 + \nu_2 + \nu_3$ are the numbers of 1-periodic points of X and Y respectively and thus the number of 1-periodic points of S_β is equal to $\mu(\nu_1 + \nu_2 + \nu_3) = 3$ where μ and $\nu_1 + \nu_2 + \nu_3$ are nonnegative integers. In particular $\mu \in \{1, 3\}$.

Assume first that $\mu = 1$. Therefore X has the same zeta function as the full shift over the one letter alphabet and X has just one periodic point. As a mixing SFT X has periodic points dense so X only contains one point, contradicting the nontriviality of X.

Assume then that $\mu = 3$. Therefore X has the same zeta function as $\Sigma_3^{\mathbb{Z}}$ and X has precisely 3^n n-periodic points for all $n \in \mathbb{N}_+$. In particular the number of 2-periodic points of S_β is divisible by $3^2 = 9$. On the other hand the number of 2-periodic points of S_β is equal to $\mathrm{Tr}(A^2) = 11$, a contradiction.

5.2 Some strictly sofic beta-shifts

Problem 5.4 remains open in the case when S_γ is strictly sofic. In this subsection we make some progress in the strictly sofic case.

We begin with a proposition that relates the fractional part of γ to the possibility of a conjugacy between $S_{n\gamma}$ and $S_n \times S_\gamma$.

Proposition 5.10. Let $\gamma > 1$ and let $n > 1$ be an integer. If $\mathrm{Frac}(\gamma) \geq \frac{1}{n}$, then $S_{n\gamma}$ has more constant configurations than $S_n \times S_\gamma$. In particular, $S_{n\gamma}$ is not conjugate to $S_n \times S_\gamma$.

Proof. We note that

$$\lfloor n\gamma \rfloor = \lfloor n(\lfloor \gamma \rfloor + \mathrm{Frac}\,(\gamma)) \rfloor \geq \lfloor n(\lfloor \gamma \rfloor + \frac{1}{n}) \rfloor = n\lfloor \gamma \rfloor + 1 > n\lfloor \gamma \rfloor.$$

The left hand side is the number of constant configurations of $S_{n\gamma}$ and the right hand side is the number of constant configurations of $S_n \times S_\gamma$. □

The main result of this subsection is stated in terms of algebraic properties of γ. We first present the relevant definitions.

Definition 5.11. A number α is an algebraic integer of degree n if it is a root of a monic polynomial $f(x)$ with integer coefficients that is irreducible over $\mathbb{Q}$ (the minimal polynomial of α). The other roots of $f(x)$ are called conjugates of α. An algebraic integer $\alpha > 1$ is a *Perron number* if all its conjugates have absolute value strictly less than α and it is a *Pisot number* if all its conjugates have absolute value strictly less than one.

It is known that if β is a Pisot number then S_β is sofic, and if S_β is sofic then β is a Perron number. A full characterization of sofic beta-shifts S_β based on the algebraic properties of β is not known.

Lemma 5.12 (Lothaire, [10], Remark 7.2.23)**.** If $\beta > 1$ has a real conjugate strictly greater than 1, then S_β is not sofic.

Lemma 5.13 (Frougny and Solomyak, [10])**.** The only Pisot numbers of degree 2 are the maximal positive roots of the following polynomials with integer coefficients:

$$\begin{aligned} &x^2 - ax - b \quad \text{with} \quad a \geq b \geq 1 \quad \text{and} \\ &x^2 - ax - b \quad \text{with} \quad a \geq 3 \quad \text{and} \quad -a + 2 \leq b \leq -1. \end{aligned}$$

Lemma 5.14. Let $\gamma = \gamma_1 > 1$ be a Pisot number of degree 2 with a conjugate γ_2. If S_γ is strictly sofic, then $\gamma_2 > 0$.

Proof. Let $x^2 - ax - b$ be the minimal polynomial of γ. It is not possible that $a \geq b \geq 1$, because then S_γ would be an SFT by Lemma 5.6. Then by Lemma 5.13 we have that $a \geq 3$ and $b \leq -1$. Because $\gamma_1\gamma_2 = -b \geq 1$ and $\gamma_1 > 1$, it follows that $\gamma_2 > 0$. □

Theorem 5.15. Let $\gamma = \gamma_1 > 1$ be a Pisot number of degree 2 with conjugate γ_2 and let $n > 1$ be an integer. If S_γ is strictly sofic, then one of the following hold.

- If the fractional part of γ_1 is greater than $1/2$, then $S_{n\gamma}$ has more constant configurations than $S_n \times S_\gamma$.
- If the fractional part of γ_2 is greater than $1/2$, then $S_{n\gamma}$ is not sofic.

In particular, $S_{n\gamma}$ is not conjugate to $S_n \times S_\gamma$ for any choice of n.

Proof. Let $\gamma_1 = \gamma$ and let γ_2 be the conjugate of γ_1. Since S_γ is strictly sofic, then $\gamma_2 > 0$ by Lemma 5.14. If the minimal polynomial of γ is of the form $x^2 - ax - b$ for $a, b \in \mathbb{Z}$, then $a = \gamma_1 + \gamma_2$ is an integer and precisely one of the numbers γ and γ_2 has fractional part greater than 1/2. If $\mathrm{Frac}(\gamma) \geq \frac{1}{2}$, the first bullet point follows from Proposition 5.10.

For the second case we note that the conjugate of $n\gamma_1$ is equal to $n\gamma_2$, because $n\gamma_i$ $(i \in \{1, 2\})$ are roots of the polynomial $x^2 - (na)x - n^2 b$ by Lemma 5.7. Because the fractional part of the positive number γ_2 is greater than 1/2, it follows in particular that $n\gamma_2 > 1$. Thus $n\gamma$ is a Perron number with a real conjugate > 1, so by Lemma 5.12 the subshift $S_{n\gamma}$ is not sofic. $\square$

6 Factorizations and Multiplication Automata

Further motivation to study direct factorizations of beta-shifts into other beta-shifts comes from the so-called multiplication automata, which have been previously studied and applied in e.g. [4, 5, 11, 15, 16, 28]. For any digit set Σ_n, when $x \in \Sigma_n^{\mathbb{Z}}$ is such that $x[i] = 0$ for all $i < N$ for some integer N, we define

$$\mathrm{real}_\beta(x) = \sum_{i=-\infty}^{\infty} x[-i]\beta^i.$$

Then, for $\alpha > 0$ and $\beta > 1$, we denote by $\Pi_{\alpha,\beta} : S_\beta \to S_\beta$ the cellular automaton such that $\mathrm{real}_\beta(\Pi_{\alpha,\beta}(x)) = \alpha\,\mathrm{real}_\beta(x)$ for every configuration $x \in S_\beta$ such that $x[i] = 0$ for all sufficiently small i, whenever such an automaton exists. We say that $\Pi_{\alpha,\beta}$ multiplies by α in base β.

In the case when $\beta = n$ is an integer and $n = pq$ for integers $p, q \geq 2$, it is known that the CA $\Pi_{p,n}$ exists. Indeed, as shown in [15], it is a memory 0 and anticipation 1 sliding block code with the local rule $g_{p,n} : \Sigma_n \times \Sigma_n \to \Sigma_n$ defined as follows. Digits $a, b \in \Sigma_n$ are represented as $a = a_1 q + a_0$ and $b = b_1 q + b_0$, where $a_0, b_0 \in \Sigma_q$ and $a_1, b_1 \in \Sigma_p$: such representations always exist and they are unique. Then

$$g_{p,n}(a, b) = g_{p,n}(a_1 q + a_0, b_1 q + b_0) = a_0 p + b_1.$$

The construction of $\Pi_{p,n}$ makes use of a direct topological factorization of S_n into $S_p \times S_q$ in an essential way, which we elaborate in the following.

Definition 6.1. A *partial shift* on a subshift $X_1 \times X_2$ is a CA $\tau : X_1 \times X_2 \to X_1 \times X_2$ defined by $\tau(x) = (\sigma(x_1), x_2)$ for all $x = (x_1, x_2)$ where $x_1 \in X_1$ and $x_2 \in X_2$. More generally, if $\psi : X \to X_1 \times X_2$ is a conjugacy, we call $\psi^{-1} \circ \tau \circ \psi : X \to X$ a partial shift on the subshift X.

Definition 6.2. A CA $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a *partitioned CA* if $F = \rho \circ \tau$ where $\rho : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a coordinatewise symbol permutation and τ is a partial shift on $A^{\mathbb{Z}}$.

On the alphabets Σ_n consisting of digits we define a collection of canonical partial shifts. For any $p \in \mathbb{N}$ dividing n let $q \in \mathbb{N}$ such that $pq = n$. Then we can define the bijection $\pi : \Sigma_n \to \Sigma_p \times \Sigma_q$ by $\pi(a) = (a_1, a_0)$ where $a = a_1 q + a_0$ is the unique way to write $a \in \Sigma_n$ so that $a_1 \in \Sigma_p$ and $a_0 \in \Sigma_q$. If τ is defined on $(\Sigma_p \times \Sigma_q)^{\mathbb{Z}}$ by $\tau(x_1, x_2) = (\sigma(x_1), x_2)$ and if π is extended naturally to the set $\Sigma_n^{\mathbb{Z}}$, we say that the map $\tau_p = \pi^{-1} \circ \tau \circ \pi : \Sigma_n^{\mathbb{Z}} \to \Sigma_n^{\mathbb{Z}}$ is *the canonical p-shift over n.*

It is now easily seen, as noted in [15], that under this definition $\Pi_{p,n} : \Sigma_n^{\mathbb{Z}} \to \Sigma_n^{\mathbb{Z}}$ is a partitioned CA when p is a factor of n. Namely, if $q \in \mathbb{N}$ is such that $pq = n$ and if $\rho : \Sigma_{pq} \to \Sigma_{pq}$ is the map $\rho(a_1 q + a_0) = a_0 p + a_1$, we see by comparison to the definition of the local rule $g_{p,n}$ that $\Pi_{p,n} = \rho \circ \tau_p$.

Since $\Pi_{p,n}$ is a partitioned CA, it is reversible and has an inverse $\Pi_{1/p,n}$. By composing these types of automata for different choices of p we see that $\Pi_{p/q,n}$ exists whenever p and q are products of prime factors of n.

Multiplication automata $\Pi_{\alpha,n}$ do not exist for all $\alpha > 0$. Consider for example multiplication by 3 in base 10 and assume that the hypothetical CA $\Pi_{3,10}$ had radius $r \geq 1$. If $\xi_1 = 0.333 \cdots 33 \in \mathbb{R}_+$ and $\xi_2 = 0.333 \cdots 34 \in \mathbb{R}_+$ are numbers with r consecutive occurrences of the digit 3 in their base-10 representations, then $3 \cdot \xi_1 < 1$ and $1 < 3 \cdot \xi_2 < 2$, so the base-10 representations of $3 \cdot \xi_1$ and $3 \cdot \xi_2$ differ to the left of the decimal point, contradicting the assumption that the radius of $\Pi_{3,10}$ is r. This is the main idea behind the proof of the following theorem from [4].

Theorem 6.3 (Blanchard, Host, Maass [4])**.** The automaton $\Pi_{\alpha,n}$ exists precisely when $\alpha = p/q$ where p and q are products of prime factors of n.

We can give an interesting alternative proof of Theorem 6.3 when we restrict our attention to reversible CA. Our proof uses the natural factorizations of S_n into products $S_p \times S_q$ for integral n, p and q. Indeed, any such factorizations give rise to possible rates of information flow that a given reversible CA can exhibit, and it turns out that reversible CA multiplying by different numbers must transfer information at different rates.

To measure the rate of information flow we will use the group homomorphism $\delta : \mathrm{Aut}(S_n) \to \mathbb{R}_+$ defined in [14]. For $F \in \mathrm{Aut}(S_n)$ let $r > 0$ be a radius of both F and F^{-1}. The set of *left stairs* of F is

$$L_F = \{(x[-r, r-1], F(x)[0, 2r-1]) \mid x \in \Sigma_n^{\mathbb{Z}}\}$$

(the reason for calling these "stairs" is apparent by Figure 4) and we then define $\delta(F) = |L_F|/n^{3r}$. The non-trivial facts that $\delta(F)$ does not depend on the choice of r and that δ is indeed a group homomorphism are shown in [14]. This homomorphism is an instance of a more general construction known as the *dimension representation* which can be defined on $\mathrm{Aut}(X)$ for any SFT X, see. e.g. Section 6 in [6].

We recall some basic properties of the map δ noted in [14]. It is easy to verify that $\delta(\sigma) = n$ for the shift $\sigma : S_n \to S_n$ and $\delta(\tau_p) = p$ for the p-shift over n. Therefore it is natural to think of the map δ as measuring the rate of

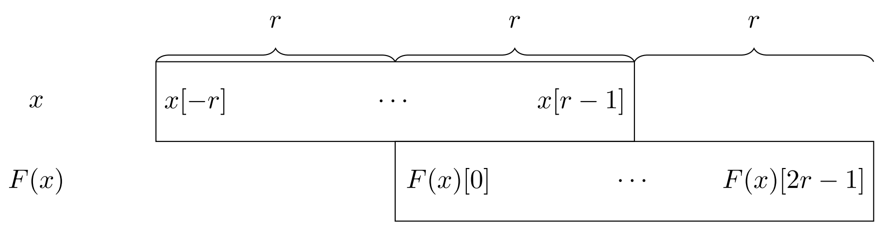

Figure 4: A left stair found in a space-time diagram.

information flow to the left that the given reversible CA gives rise to. If F has finite order, i.e. $F^t = \mathrm{Id}$ for some $t \in \mathbb{N}_+$, then $\delta(F) = 1$: this is due to the fact that 1 is the only finite order element in the multiplicative group $\mathbb{R}_+$. Also by [14] the set $\mathrm{Im}(\delta)$ is equal to the multiplicative subgroup of $\mathbb{R}_+$ generated by $p_1, \ldots, p_k$ where p_i are the prime factors of n.

Lemma 6.4. Let $p_1, \ldots, p_k$ be the prime factors of n. For any $\alpha \in \mathrm{Im}(\delta)$ there is a multiplication automaton $\Pi_{\alpha,n}$ which is some product of multiplication automata $\Pi_{p_i,n}$ and their inverses and which satisfies $\delta(\Pi_{\alpha,n}) = \alpha$.

Proof. Since $\mathrm{Im}(\delta)$ is generated by the prime factors of n, it is sufficient to prove the lemma for $\alpha = p$ which is a prime factor of n. We have seen that $\Pi_{p,n} = \rho \circ \tau_p$ where ρ is a symbol permutation and in particular it has finite order. Since δ is a homomorphism, it follows that $\delta(\Pi_{p,n}) = \delta(\rho)\delta(\tau_p) = 1 \cdot p = p$. □

Lemma 6.5. If $\Pi_{\alpha,n}$ is in the kernel of $\delta : \mathrm{Aut}(S_n) \to \mathbb{R}_+$, then $\alpha = 1$.

Proof. Assume to the contrary that $\alpha \neq 1$. We may assume without loss of generality that $\alpha > 1$ (by considering $\Pi_{\alpha,n}^{-1}$ instead of $\Pi_{\alpha,n}$ if necessary). Let r be a common radius of $\Pi_{\alpha,n}$ and its inverse. By our assumption $\delta(\Pi_{\alpha,n}) = 1$, so by the definition of δ the set $L = L_F$ should contain n^{3r} elements. We will find a contradiction by concretely enumerating the left stairs.

Let $u_i \in \Sigma_n^r$ $(0 \leq i < n^r)$ be an enumeration of the elements of Σ_n^r and let $v_j \in \Sigma_n^{2r}$ $(0 \leq j < n^{2r})$ be an enumeration of the elements of Σ_n^{2r}. For all such i, j define $y_{i,j} \in \Sigma_n^{\mathbb{Z}}$ by $y_{i,j}[-r, 2r-1] = u_i v_j$, $y_{i,j}[k] = 0$ for $k \notin [-r, 2r-1]$. Let $(w_{i,j}, v_j)$ be the left stair derived from the configuration $x_{i,j} = \Pi_{\alpha,n}^{-1}(y_{i,j})$ (note that $y_{i,j}[0, 2r-1] = v_j$ by the definition of $y_{i,j}$), so $w_{i,j} = x_{i,j}[-r, r-1]$. This is depicted in Figure 5.

We first show that all the left stairs of the form $(w_{i,j}, v_j)$ $(0 \leq i < n^r$, $0 \leq j < n^{2r})$ are distinct. Let i, i', j, j' be such that $(w_{i,j}, v_j) = (w_{i',j'}, v_{j'})$. From $v_j = v_{j'}$ it follows that $j = j'$ and it remains to show that $i = i'$. Since $\Pi_{\alpha,n}^{-1}$ has radius r, from $v_j = v_{j'}$ it follows that $x_{i,j}[r, \infty] = x_{i',j}[r, \infty]$. Since $\alpha > 1$, from $y_{i,j}[-\infty, -r-1] = y_{i',j}[-\infty, -r-1] = {}^{\infty}0$ it follows that $x_{i,j}[-\infty, -r-1] = x_{i',j}[-\infty, -r-1] = {}^{\infty}0$. Combining these observations with $w_{i,j} = w_{i',j}$ it follows that $x_{i,j} = x_{i',j}$. Applying $\Pi_{\alpha,n}$ to this equality we get $y_{i,j} = y_{i',j}$, so in particular $u_i = u_{i'}$ and $i = i'$.

We show that $w_{i,j} \neq (n-1)^{2r}$ for all choices of i and j. Assume to the contrary that $w_{i,j} = (n-1)^{2r}$ for some i, j. Consider the configuration x' with $x'[-\infty, -r-1] = {}^{\infty}0$ and $x'[-r, \infty] = (n-1)^{\infty}$. Since $\alpha > 1$ and $\mathrm{real}_n(x') = n^{r+1}$, it follows that in the configuration $y' = \Pi_{\alpha,n}(x')$ we have $y'[k] \neq 0$ for some $k < -r$. Since $x_{i,j}[-\infty, r-1] = x'[-\infty, r-1]$ and $\Pi_{\alpha,n}$ has radius r, it follows that $y_{i,j}[-\infty, -1] = y'[-\infty, -1]$ and in particular $y_{i,j}[k] \neq 0$ for some $k < -r$. This contradicts the definition of $y_{i,j}$.

There exists a left stair of the form (w, v) where $w = (n-1)^{2r}$, and by the previous paragraph it is different from all the left stairs of the form $(w_{i,j}, v_j)$ $(0 \leq i < n^r,\ 0 \leq j < n^{2r})$. It follows that $|L| \geq n^{3r} + 1$, contradicting the assumption $\delta(\Pi_{\alpha,n}) = 1$. $\square$

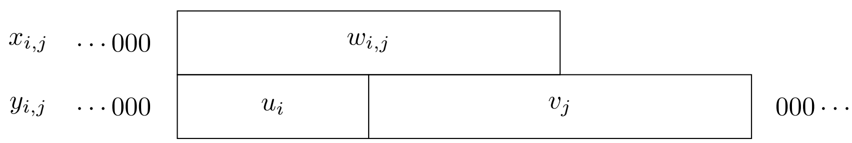

Figure 5: Left stairs of the form $(w_{i,j}, v_j)$.

Theorem 6.6. All the reversible multiplication automata over S_n are precisely of the form $\Pi_{p/q,n}$ where p and q are products of prime factors of n.

Proof. Earlier we noted that $\Pi_{p/q,n}$ exists whenever p and q are products of prime factors of n. To see the other direction, let $\Pi_{\alpha',n}$ be an arbitrary reversible multiplication automaton where $\alpha' \in \mathbb{R}_+$. By Lemma 6.4 there is $\alpha = p/q$ where p and q are products of prime factors of n such that $\delta(\Pi_{\alpha,n}) = \delta(\Pi_{\alpha',n})$. Therefore $\delta(\Pi_{\alpha/\alpha',n}) = 1$ and $\alpha' = \alpha$ by Lemma 6.5. $\square$

It would be interesting to generalize this classification to beta-shifts with nonintegral base β, or to just find examples of $\Pi_{\alpha,\beta}$ that are not shift maps.

Problem 6.7. Can $\Pi_{n,n\gamma}$ exist for an integer $n > 1$ and a nonintegral $\gamma > 0$ when there exists a conjugacy between $S_{n\gamma}$ and $S_n \times S_\gamma$?

The following example shows that this is not always possible.

Example 6.8. Let $\gamma = 1 + \sqrt{2}$ and $\beta = 2\gamma$. Since $\gamma^2 = 2\gamma + 1$, we have $\beta^2 = 4\beta + 4$ by Lemma 5.7 and $d(\beta) = 44$ by Lemma 5.6. We can use Theorem 5.8 to see that S_β is topologically conjugate to $S_2 \times S_\gamma$. However, it turns out that the CA $\Pi_{2,\beta}$ does not exist, because otherwise for $x = \ldots 000.222\ldots$ it holds that $\mathrm{real}_\beta(\Pi_{2,\beta}(x)) = 2\,\mathrm{real}_\beta(x) = \mathrm{real}_\beta(\ldots 000.444\ldots)$. Since $d(\beta) = 44$, it holds that $\mathrm{real}_\beta(\ldots 000.444\ldots) = \mathrm{real}_\beta(\ldots 001.010101\ldots)$. Because $y = \ldots 001.010101 \cdots \in S_\beta$, it follows that $\Pi_{2,\beta}(x) = y$. This is not possible because as a CA $\Pi_{2,\beta}$ has to map constant configurations to constant configurations.

Acknowledgments

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