

Optimal Bounds on Codes for Location in Circulant Graphs

Ville Junnila* and Tero Laihonen

Department of Mathematics and Statistics
University of Turku, FI-20014 Turku, Finland
viljun@utu.fi and terolai@utu.fi

Gabrielle Paris†

LIRIS, University of Lyon, France
claudia.paris-sierra@univ-lyon1.fr

Abstract

Identifying and locating-dominating codes have been studied widely in circulant graphs of type $C_n(1, 2, 3, \dots, r)$ over the recent years. In 2013, Ghebleh and Niepel studied locating-dominating and identifying codes in the circulant graphs $C_n(1, d)$ for $d = 3$ and proposed as an open question the case of $d > 3$. In this paper we study identifying, locating-dominating and self-identifying codes in the graphs $C_n(1, d)$, $C_n(1, d - 1, d)$ and $C_n(1, d - 1, d, d + 1)$. We give a new method to study lower bounds for these three codes in the circulant graphs using suitable grids. Moreover, we show that these bounds are attained for infinitely many parameters n and d . In addition, new approaches are provided which give the exact values for the optimal self-identifying codes in $C_n(1, 3)$ and $C_n(1, 4)$.

Keywords: Identifying code; locating-dominating code; circulant graph; square grid; triangular grid; king grid

1 Introduction

Let us first introduce some notations and concepts. In this paper, a graph $G = (V, E)$ with the vertex set V and the edge set E is simple, undirected and connected. The (*graphic*) *distance*, denoted by $d_G(u, v) = d(u, v)$, is the number of edges in any shortest path between the vertices u and v . We define the *open neighbourhood* $N(G; u) = \{v \in V \mid d(u, v) = 1\}$ and the *closed neighbourhood* $N[G; u] = \{v \in V \mid d(u, v) \leq 1\}$. A graph is *r-regular* provided that $|N(G; u)| = r$ for all $u \in V$. Suppose that n and d_1, d_2, \dots, d_k are positive integers such that $d_i \leq n/2$ for all $i = 1, \dots, k$. We define the *circulant graph* as follows: the vertex set is $\mathbb{Z}_n = \{0, 1, \dots, n - 1\}$ and the open neighbourhood of a vertex $u \in \mathbb{Z}_n$ is

$$N(u) = \{u - d_1, u - d_2, \dots, u - d_k\} \cup \{u + d_1, u + d_2, \dots, u + d_k\},$$

where the calculations are done modulo n .

A nonempty subset of vertices $C \subseteq V$ is called a *code* and its elements are *codewords*. We define the *I-set* or *identifier* of a vertex u with respect to a code C by

$$I(G, C; u) = N[G; u] \cap C.$$

If the graph or the code is clear from the context, we write $I(G, C; u) = I(G; u) = I(C; u) = I(u)$. A vertex u is *covered* or *dominated* by C if $I(G, C; u)$ is nonempty. A code C is *dominating* if

*Corresponding author

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every vertex $u \in V$ is dominated by C . A code C is *identifying* in G if C is dominating and for all distinct vertices $u, v \in V$ we have

$$I(C; u) \neq I(C; v).$$

The smallest possible cardinality of an identifying code in a finite graph G is denoted by $\gamma^{ID}(G)$. An identifying code with cardinality $\gamma^{ID}(G)$ is called *optimal*. The definition of identifying codes is due to Karpovsky *et al.* [17], and the original motivation for studying such codes comes from fault diagnosis in multiprocessor systems. If a code C is dominating and we have for distinct non-codewords $u, v \in V \setminus C$ that

$$I(C; u) \neq I(C; v)$$

then the code is called *locating-dominating* in G . This concept was introduced by Slater [19, 21, 22] and the original motivation for locating-dominating codes was based on fire and intruder alarm systems. The smallest possible cardinality of a locating-dominating code in a finite graph G is denoted by $\gamma^{LD}(G)$. A locating-dominating code with $\gamma^{LD}(G)$ codewords is called *optimal*.

In this paper, we focus on studying identifying and locating-dominating codes (as well as self-identifying codes which are defined later) in the circulant graphs. A circulant graph $C_n(1, 2, \dots, r)$ ($r \in \mathbb{Z}, r \geq 1$) can also be viewed as a power graph of a cycle of length n . In these circulant graphs $C_n(1, 2, \dots, r)$, identifying and locating-dominating codes have been studied previously in [2, 4, 7, 9, 13, 18, 20, 24]. Recently, in [8], Ghebleh and Niepel studied identification and location-domination in $C_n(1, 3)$. They obtained the following results for $n \geq 9$:

$$\lceil 4n/11 \rceil \leq \gamma^{ID}(C_n(1, 3)) \leq \lceil 4n/11 \rceil + 1 \quad \text{and} \quad \lceil n/3 \rceil \leq \gamma^{LD}(C_n(1, 3)) \leq \lceil n/3 \rceil + 1.$$

Moreover, they showed that in most cases the given lower bounds are actually the exact values of $\gamma^{ID}(C_n(1, 3))$ and $\gamma^{LD}(C_n(1, 3))$ and conjectured that in the rest of the cases the lower bound could be increased by one (attaining the given constructions). These conjectures have been solved in our paper [16]. They also stated as an open question what happens in the graphs $C_n(1, d)$ with d being greater than 3 and mentioned that the methods used in their paper seem to be non-applicable. In this paper, we present a new approach to determine $\gamma^{ID}(C_n(1, d))$ and $\gamma^{LD}(C_n(1, d))$ with $d \geq 3$. The new approach is based on the observation that identification and locating-domination in the circulant graphs $C_n(1, d)$ have connections to identifying and locating-dominating codes in the infinite square grid. In particular, we can take advantage of the known lower bounds for identifying and locating-dominating codes in the square grid and derive lower bounds for the circulant graphs $C_n(1, d)$. Moreover, there exist similar connections and results between the circulant graphs $C_n(1, d-1, d)$ and $C_n(1, d-1, d, d+1)$ and the infinite triangular grid and king grid, respectively. In Section 2, these connections as well as the needed definitions and known results regarding the grids are discussed, and we also present the lower bounds for the circulant graphs obtained from the grids. Then, in Section 3, we present constructions of identifying and locating-dominating codes for the circulant graphs. In particular, we obtain infinite families of circulant graphs with optimal identifying codes as well as families with optimal locating-dominating codes.

In addition to considering identification and location-domination, we also study self-identifying codes, which overcome some issues of the regular identifying codes described in the following. Indeed, if C is an identifying code in a graph $G = (V, E)$, then we can locate one irregularity (for example, a fire or an intruder) in G as all the I -sets are distinct. However, if there are more than one irregularity in G , then we can mislocate the irregularity (see [12]), since we could have $I(C; u) = I(C; v_1) \cup I(C; v_2)$ for some vertices $u, v_1, v_2 \in V$, and more disturbingly not even notice that something is wrong. Thus, to locate one irregularity and detect multiple ones, the following definition of self-identifying codes have been introduced in [12] (although in the paper the code is called 1^+ -identifying).

Definition 1. A code $C \subseteq V$ is *self-identifying* in $G = (V, E)$ if for all distinct $u, v \in V$ we have

$$I(C; u) \setminus I(C; v) \neq \emptyset.$$

	square grid \mathcal{S}	triangular grid \mathcal{T}	king grid \mathcal{K}
LD	3/10 [23]	13/57 [10]	1/5 [11]
ID	7/20 [1, 5]	1/4 [17]	2/9 [3, 6]
self-ID	1/2 [12]	1/2 [12]	1/3 [12]

Table 1: The densities of optimal identifying (ID), locating-dominating (LD) and self-identifying (self-ID) codes in the square \mathcal{S} , triangular \mathcal{T} and king grids \mathcal{K} are listed in the table. Next to each density you can find the reference to the result.

In a finite graph G , a self-identifying code with the smallest cardinality is called optimal, and the number of codewords in an optimal self-identifying code in G is denoted by $\gamma^{SID}(G)$.

The self-identifying codes have also been discussed in [14, 15]. In those papers, it has been shown that C is a self-identifying code in G if and only if for all $u \in V$ we have $I(C; u) \neq \emptyset$ and

$$\bigcap_{c \in I(C; u)} N[c] = \{u\}. \quad (1)$$

Therefore, the sought vertex can be determined only using its I -set; compare this to regular identifying codes where the I -set has to be compared to other I -sets in order to locate a vertex. In Sections 2 and 4, we present results for self-identifying codes in the circulant graphs; especially, we focus on results in the graphs $C_n(1, d)$, $C_n(1, d - 1, d)$ and $C_n(1, d - 1, d, d + 1)$.

2 Infinite grids and circulant graphs

In this section, we first recall some preliminary definitions and results regarding infinite square, triangular and king grids and then present the connections between circulant graphs and grids. Let us first present the definitions of the grids. In all the grids, the vertex set is $V = \mathbb{Z}^2$. The edges of the *square grid* \mathcal{S} are defined in such a way that the closed neighbourhood of $u = (x, y) \in \mathbb{Z}^2$ is

$$N[\mathcal{S}; u] = \{(x', y') \in \mathbb{Z}^2 \mid |x - x'| + |y - y'| \leq 1\}.$$

The edges of the *triangular grid* \mathcal{T} is defined in such a way that the closed neighbourhood of $u = (x, y) \in \mathbb{Z}^2$ is $N[\mathcal{T}; u] = N[\mathcal{S}; u] \cup \{(x + 1, y + 1), (x - 1, y - 1)\}$. The edges of the *king grid* \mathcal{K} is defined in such a way that the closed neighbourhood of $u = (x, y) \in \mathbb{Z}^2$ is $N[\mathcal{K}; u] = N[\mathcal{T}; u] \cup \{(x - 1, y + 1), (x + 1, y - 1)\}$. For comparing the sizes of codes, we need a way to measure them in the infinite grids. For this purpose, we first denote

$$Q_m = \{(x, y) \in \mathbb{Z}^2 \mid |x| \leq m, |y| \leq m\},$$

where m is a positive integer. The *density* of a code $C \subseteq \mathbb{Z}^2$ is then defined as

$$D(C) = \limsup_{m \rightarrow \infty} \frac{|C \cap Q_m|}{|Q_m|}.$$

For a finite nonempty set $S \subseteq V$ in a graph $G = (V, E)$, the (local) *density* of a code $C \subseteq V$ in S is defined as $|S \cap C|/|S|$.

Analogously to finite graphs, an identifying, locating-dominating and self-identifying code with the smallest density in the square, triangular or king grid is called *optimal*. The densities of optimal codes on these grids have been intensively studied and all the exact values are known. The optimal densities can be found in Table 1 together with the references to the papers, where the results have been presented.

In the following theorem, we present the connection between identifying, locating-dominating and self-identifying codes in the square grid and the circulant graphs $C_n(1, d)$.

Theorem 2. *Let n , d and k be positive integers such that $d \geq 2$. If C is an identifying code in $C_n(1, d)$ with k codewords, then there exists an identifying code in the infinite square grid \mathcal{S} with density k/n . Analogous results also hold for locating-dominating and self-identifying codes.*

Proof. Let $G = C_n(1, d)$ be a circulant graph and C an identifying code in it. We will use the following correspondence of the vertex $x = (i, j) \in \mathbb{Z}^2$ in the square grid with the vertex $i + j \cdot d$ in $C_n(1, d)$ where $i + j \cdot d$ is calculated modulo n (throughout the paper). Namely, the closed neighbourhood of x is $N[\mathcal{S}; x] = \{(i, j), (i-1, j), (i+1, j), (i, j-1), (i, j+1)\}$ and the corresponding set in $C_n(1, d)$ is $\{i + j \cdot d, i-1 + j \cdot d, i+1 + j \cdot d, i + (j-1) \cdot d, i + (j+1) \cdot d\} = N[C_n(1, d); i + j \cdot d]$, that is, the neighbourhood of $i + j \cdot d$, (see Figure 1).

We define the following code in the square grid

$$C_{\mathcal{S}} = \{(i, j) \in \mathbb{Z}^2 \mid i + j \cdot d \in C\}.$$

In other words, the code $C_{\mathcal{S}}$ is obtained by repeating the C along the first line (i.e., x -axis) and then do the same for other lines with a shift d as illustrated in Figure 1. In what follows, we show that $C_{\mathcal{S}}$ is an identifying code in \mathcal{S} .

Suppose there exist two distinct vertices $x = (i, j) \in \mathbb{Z}^2$ and $y = (i, j) + (a, b) \in \mathbb{Z}^2$, ($a, b \in \mathbb{Z}$), in the square grid such that $I(\mathcal{S}; x) = I(\mathcal{S}; y)$. As C is a dominating set, so is $C_{\mathcal{S}}$ and the sets $I(\mathcal{S}; x)$ and $I(\mathcal{S}; y)$ are nonempty. Consequently, it suffices to consider the cases where the distance between x and y is at most two in \mathcal{S} . Without loss of generality, we can assume further that the for the second coordinate $0 \leq b$ (if this is not the case, just switch the roles of x and y). Consequently, $(a, b) \in S = \{(1, 0), (-1, 0), (0, 1), (1, 1), (-1, 1), (2, 0), (-2, 0), (0, 2)\}$. In the circulant graph $C_n(1, d)$, the property $I(\mathcal{S}; x) = I(\mathcal{S}; y)$ implies that $I(C_n(1, d); i + j \cdot d) = I(C_n(1, d); (i+a) + (j+b) \cdot d)$. Because C is identifying, this yields that $i + j \cdot d \equiv (i+a) + (j+b) \cdot d \pmod{n}$. Hence $a + b \cdot d \equiv 0 \pmod{n}$. Notice that the choices for a and b are restricted by S . This shows that $I(\mathcal{S}; x) \neq I(\mathcal{S}; y)$ if $n > 2d$ (recall that $d \leq n/2$, i.e., $n \geq 2d$) and we are done. If $n = 2d$, then suffices to check the case $y = (i, j) + (0, 2)$. Although in this case the sets $I(i + j \cdot d)$ and $I((i+0) + (j+2) \cdot d)$ are the same in the circulant graph, it is easy to check that the sets $I(\mathcal{S}; x)$ and $I(\mathcal{S}; y)$ are not. Indeed, notice that now $N[\mathcal{S}; x] \cap N[\mathcal{S}; y] = \{(i, j+1)\}$. If $I(\mathcal{S}; x) = I(\mathcal{S}; y)$, the only codeword in $I(\mathcal{S}; y)$ can be $x + (0, 1) = (i, j+1)$. However, in that case there would be also a codeword in $y + (0, 1)$ due to the structure of $C_{\mathcal{S}}$ and thus $I(\mathcal{S}; x) \neq I(\mathcal{S}; y)$.

For the locating-dominating codes the proof is analogous — just notice that a non-codeword $x = (i, j) \in \mathbb{Z}^2$ in \mathcal{S} corresponds to a non-codeword $i + j \cdot d$ in $C_n(1, d)$.

Suppose then that C is self-identifying. We will show that $I(\mathcal{S}; x) \setminus I(\mathcal{S}; y) \neq \emptyset$ for all distinct vertices $x = (i, j) \in \mathbb{Z}^2$ and $y = x + (k, h) \in \mathbb{Z}^2$, ($k, h \in \mathbb{Z}$). Since $C_{\mathcal{S}}$ is dominating, the claim is clear if $d_{\mathcal{S}}(x, y) \geq 3$.

Suppose then that $d_{\mathcal{S}}(x, y) = 2$. Denote for any $z = (z_1, z_2) \in \mathbb{Z}^2$ the set $P(z; a, b) = \{z, z + (a, 0), z + (0, b)\}$ where $a, b \in \{-1, 1\}$. Let us first observe that $P(z; a, b)$ always contains a codeword of $C_{\mathcal{S}}$. This follows since in the circulant graph the set $I(C_n(1, d); z_1 + z_2 \cdot d) \setminus I(C_n(1, d); z_1 - a + (z_2 - b) \cdot d)$ contains a codeword of C due to the fact that C is self-identifying. Notice that $z_1 + z_2 \cdot d$ and $z_1 - a + (z_2 - b) \cdot d$ are different vertices in $C_n(1, d)$ as $n \geq 2d$. If $y = x + (-1, -1)$ (resp. $y = x + (-2, 0)$), then $N[\mathcal{S}; x] \setminus N[\mathcal{S}; y]$ equals $P(x; 1, 1)$ (resp. contains $P(x; 1, 1)$). Thus, $I(\mathcal{S}; x) \setminus I(\mathcal{S}; y) \neq \emptyset$. Similarly, it is easy to check that for all x and y such that $d_{\mathcal{S}}(x, y) = 2$, the set $N[\mathcal{S}; x] \setminus N[\mathcal{S}; y]$ contains $P(x; a, b)$ for suitable $a, b \in \{-1, 1\}$.

Let $d_{\mathcal{S}}(x, y) = 1$. Similarly as above we can show (looking now at the vertices $i + j \cdot d$ and $(i+k) + (j+h) \cdot d$ in the circulant graph) that the set $N[\mathcal{S}; x] \setminus N[\mathcal{S}; y]$ always contains a codeword of $C_{\mathcal{S}}$. □

The previous theorem (together with the results presented in Table 1) immediately imply the following corollary, which gives lower bounds for the optimal sizes of identifying, locating-dominating and self-identifying codes in the circulant graphs $C_n(1, d)$. Later, in Sections 3 and 4, we show that the lower bounds can be attained for certain circulant graphs.

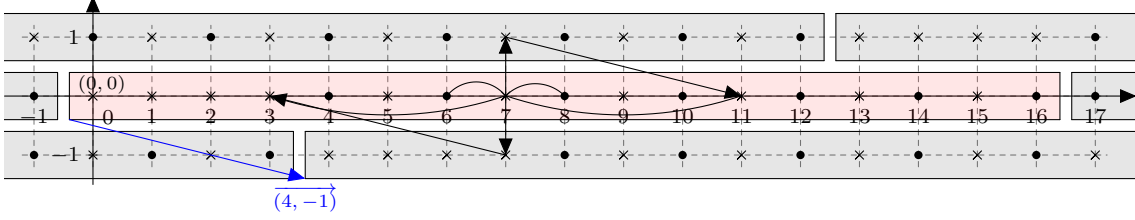


Figure 1: The code $C = \{0, 1, 2, 3, 5, 7, 9, 11, 13, 15\}$ of $C_{17}(1, 4)$ on the 2-dimensional square grid. The crosses mark the codewords and dots the non-codewords.

Corollary 3. *Let n and d be positive integers such that $d \geq 2$ and $G = C_n(1, d)$. Then we have*

$$\gamma^{LD}(G) \geq \left\lfloor \frac{3n}{10} \right\rfloor, \gamma^{ID}(G) \geq \left\lfloor \frac{7n}{20} \right\rfloor \text{ and } \gamma^{SID}(G) \geq \left\lfloor \frac{n}{2} \right\rfloor.$$

In the following theorem, we present the connection between identifying, locating-dominating and self-identifying codes in the triangular grid and the circulant graphs $C_n(1, d-1, d)$.

Theorem 4. *Let n , d and k be positive integers such that $d \geq 3$. If C is an identifying code in $C_n(1, d-1, d)$ with k codewords, then there exists an identifying code in the infinite triangular grid \mathcal{T} with density k/n . Analogous results also hold for locating-dominating and self-identifying codes.*

Proof. We take the advantage of the correspondence of a vertex $x = (i, j)$ in the triangular grid \mathcal{T} and the vertex $i + j \cdot (d-1) \pmod{n}$ in the circulant graph $C_n(1, d-1, d)$. Now the set $N[\mathcal{T}; x] = \{(i+1, j+1), (i, j+1), (i-1, j), (i, j), (i+1, j), (i, j-1), (i-1, j-1)\}$ corresponds to the set $N[i + j \cdot (d-1)]$ in the circulant graph. Let C be an identifying code in $C_n(1, d-1, d)$. The code $C_{\mathcal{T}} = \{(i, j) \in \mathbb{Z}^2 \mid i + j \cdot (d-1) \in C\}$ can be shown to be identifying in \mathcal{T} using similar arguments as in Theorem 2 and the claim follows for identifying codes. Analogous reasoning gives that if C is locating-dominating, then $C_{\mathcal{T}}$ is also locating-dominating. The case of self-identifying codes is even easier than in the proof Theorem 2, since it is enough, as discussed in [12], to check that there is a codeword of $C_{\mathcal{T}}$ in the set $N[C_{\mathcal{T}}; x] \setminus N[C_{\mathcal{T}}; y]$ for vertices such that $d(x, y) = 1$ (other cases follow from this). \square

In the following corollary, we present lower bounds for the circulant graphs $C_n(1, d-1, d)$. In Sections 3 and 4, we show that the lower bounds can be attained with locating-dominating and self-identifying codes and that there exists an infinite family of identifying codes approaching the lower bound.

Corollary 5. *Let n and d be positive integers such that $d \geq 3$ and $G = C_n(1, d-1, d)$. Then we have*

$$\gamma^{LD}(G) \geq \left\lfloor \frac{13n}{57} \right\rfloor, \gamma^{ID}(G) \geq \left\lfloor \frac{n}{4} \right\rfloor \text{ and } \gamma^{SID}(G) \geq \left\lfloor \frac{n}{2} \right\rfloor.$$

In the following theorem, we present the connection between identifying, locating-dominating and self-identifying codes in the king grid and the circulant graphs $C_n(1, d-1, d, d+1)$.

Theorem 6. *Let n , d and k be positive integers such that $d \geq 3$. If C is an identifying code in $C_n(1, d-1, d, d+1)$ with k codewords, then there exists an identifying code in the infinite king grid \mathcal{K} with density k/n . Analogous results also hold for locating-dominating and self-identifying codes.*

Proof. This goes similarly as in Theorem 2 using the correspondence of a vertex (i, j) in the king grid \mathcal{K} and the vertex $i + j \cdot d \pmod{n}$ in the circulant graph $C_n(1, d-1, d, d+1)$. The case of self-identifying codes is again easier than in Theorem 2, since it suffices, as discussed in [12], to check the situation for $d(x, y) = 1$ (as other cases follow). \square

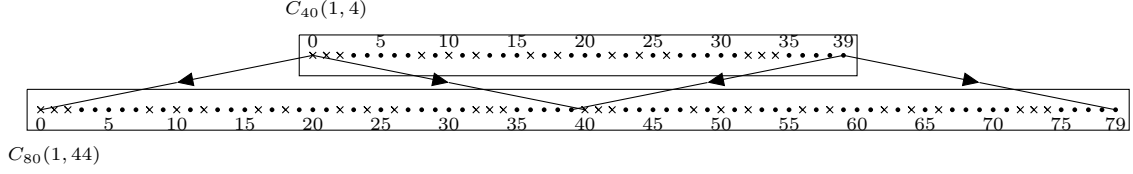


Figure 2: Optimal identifying codes for $C_{40}(1, 4)$ and $C_{80}(1, 44)$. The crosses denote the codewords and dots are non-codewords.

In the following corollary, we present lower bounds for the circulant graphs $C_n(1, d-1, d, d+1)$. In Sections 3 and 4, we show that the lower bounds can be attained with locating-dominating and self-identifying codes and that there exists an infinite family of identifying codes approaching the lower bound.

Corollary 7. *Let n and d be positive integers such that $d \geq 3$ and $G = C_n(1, d-1, d, d+1)$. Then we have*

$$\gamma^{LD}(G) \geq \left\lceil \frac{n}{5} \right\rceil, \quad \gamma^{ID}(G) \geq \left\lceil \frac{2n}{9} \right\rceil \quad \text{and} \quad \gamma^{SID}(G) \geq \left\lceil \frac{n}{3} \right\rceil.$$

3 Identifying and locating-dominating codes in circulant graphs

In this section we give optimal constructions for the following types of circulant graphs: $C_n(1, d)$, $C_n(1, d-1, d)$ and $C_n(1, d-1, d, d+1)$.

3.1 On graphs $C_n(1, d)$

In the next theorem, we will give constructions which attain the bounds in Corollary 3 for identifying and locating-dominating codes.

Theorem 8. *Let n and d be positive integers such that $n \geq 2d$.*

- (i) *If $n \equiv 0 \pmod{40}$ and $d \equiv 4 \pmod{40}$, then we have $\gamma^{ID}(C_n(1, d)) = \frac{7n}{20}$.*
- (ii) *If $n \equiv 0 \pmod{20}$ and $d \equiv 6 \pmod{20}$, then we have $\gamma^{ID}(C_n(1, d)) = \frac{7n}{20}$.*
- (iii) *If $n \equiv 0 \pmod{20}$ and $d \equiv 5 \pmod{20}$, then we have $\gamma^{LD}(C_n(1, d)) = \frac{3n}{10}$.*

Proof. (i) Let $n \equiv 0 \pmod{40}$ and $d \equiv 4 \pmod{40}$. Define

$$B_1 = \{0, 1, 2, 8, 10, 12, 16, 18, 22, 24, 26, 32, 33, 34\}$$

and

$$D_1 = \{u \in \mathbb{Z}_n \mid u \equiv b \pmod{40} \text{ for some } b \in B_1\}.$$

The codes B_1 in $C_{40}(1, 4)$ and D_1 in $C_n(1, d)$, where $n = 80$ and $d = 44$ are illustrated in Figure 2.

It is straightforward to verify that B_1 is an identifying code in $C_{40}(1, 4)$. In what follows, we prove that D_1 is an identifying code in $C_n(1, d)$ by showing that all the I -sets $I(C_n(1, d), D_1; x)$ are nonempty and unique. Observe first that by the construction of D_1 we obtain for all $x \in \mathbb{Z}_n$ that

$$I(C_n(1, d), D_1; x) \equiv I(C_{40}(1, 4), B_1; x_0) \pmod{40},$$

where x_0 is an integer such that $x \equiv x_0 \pmod{40}$ and $0 \leq x_0 \leq 39$. Therefore, the I -sets $I(C_n(1, d), D_1; x)$ are nonempty for all $x \in \mathbb{Z}_n$. Let x and y be distinct vertices of \mathbb{Z}_n . Assume first that $x \not\equiv y \pmod{40}$. Let then x' and y' be integers such that $x \equiv x' \pmod{40}$, $y \equiv y' \pmod{40}$, $0 \leq x' \leq 39$ and $0 \leq y' \leq 39$. Therefore, by the previous observation, if $I(C_n(1, d), D_1; x) =$

$I(C_n(1, d), D_1; y)$, then $I(C_{40}(1, 4); B_1, x') = I(C_{40}(1, 4); B_1, y')$ and we have a contradiction as B_1 is an identifying code in $C_{40}(1, 4)$. Hence, we may assume that $x \equiv y \pmod{40}$. Let us then show that $N[C_n(1, d); x] \cap N[C_n(1, d); y] = \emptyset$. Suppose to the contrary that there exist $x, y \in \mathbb{Z}_n$ such that $x + j = y + j'$ for some $j, j' \in \{-d, -1, 0, 1, d\}$. Since $x \equiv y \pmod{40}$, we obtain that $j \equiv j' \pmod{40}$. This further implies that $j = j'$ and $x = y$ (a contradiction). Therefore, as each vertex of \mathbb{Z}_n is covered by a codeword of D_1 , we have $I(C_n(1, d), D_1; x) \neq I(C_n(1, d), D_1; y)$. Thus, D_1 is an identifying code in $C_n(1, d)$.

(ii) Let $n \equiv 0 \pmod{20}$ and $d \equiv 6 \pmod{20}$. Define $B_2 = \{0, 2, 8, 9, 11, 12, 18\}$ and

$$D_2 = \{u \in \mathbb{Z}_n \mid u \equiv b \pmod{20} \text{ for some } b \in B_2\}.$$

It is straightforward to verify that B_2 is an identifying code in $C_{20}(1, 6)$. Then, using similar arguments as in the case (i), we can prove that D_2 is an identifying code in $C_n(1, d)$.

(iii) Let $n \equiv 0 \pmod{20}$ and $d \equiv 5 \pmod{20}$. Define $B_3 = \{0, 4, 7, 11, 14, 17\}$ and

$$D_3 = \{u \in \mathbb{Z}_n \mid u \equiv b \pmod{20} \text{ for some } b \in B_3\}.$$

It is straightforward to verify that B_3 is a locating-dominating code in $C_{20}(1, 5)$. Then, using similar arguments as in the case (i) (although now x and y are assumed to be non-codewords), we can prove that D_3 is a locating-dominating code in $C_n(1, d)$. \square

3.2 On graphs $C_n(1, d - 1, d)$

The next theorem gives optimal constructions on locating-dominating codes in $C_n(1, d - 1, d)$. In addition, we provide an infinite sequence of identifying codes approaching the lower bound for identifying codes in Corollary 5. Moreover, it will be shown in Corollary 18 (see also Theorem 17) that we cannot attain the lower bound by any identifying code.

Theorem 9. (i) For all the parameters n and d such that $d \equiv 8 \pmod{57}$, $d \geq 8$, $n \geq 2d$ and $n \equiv 0 \pmod{57}$, we have $\gamma^{LD}(C_n(1, d - 1, d)) = \frac{13n}{57}$.

(ii) We have an infinite sequence of identifying codes in the circulant graphs $C_n(1, d - 1, d)$ such that their density tends to $1/4$.

Proof. (i) Let $d \equiv 8 \pmod{57}$, $d \geq 8$, $n \geq 2d$ and $n \equiv 0 \pmod{57}$. We denote

$$B = \{0, 2, 4, 6, 15, 18, 27, 29, 31, 33, 43, 45, 47\}.$$

Let further

$$C = \{v \in \mathbb{Z}_n \mid v \equiv b \pmod{57} \text{ for some } b \in B\}.$$

It is straightforward to check that B is a locating-dominating code in $C_{57}(1, d - 1, d)$ for $d = 8$. Next we will show that C is locating-dominating in $C_n(1, d - 1, d)$. Let us first show that $I(x) = I(y)$ for $x \not\equiv y \pmod{57}$ and $x, y \notin C$. Denote $x' = x \pmod{57}$ and $y' = y \pmod{57}$ where $0 \leq x' \leq 56$ and $0 \leq y' \leq 56$. If $I(x) = I(y)$, then it follows that the codewords in $I(x)$ and in $I(y)$ would be equal modulo 57. However, that is not possible, since $I(B; x') \neq I(B; y')$ for distinct $x', y' \notin B$. Therefore, it suffices to consider $I(x) = I(y)$ for $x \equiv y \pmod{57}$, $x \neq y$ and $x, y \notin C$. Let $j \in \{-d, -d + 1, -1, 0, 1, d - 1, d\}$ and $x + j \in I(x)$. Consequently, $x + j = y + j'$ for some $j' \in \{-d, -d + 1, -1, 0, 1, d - 1, d\}$. Since $x \equiv y \pmod{57}$, we get $j = j'$ giving $x = y$. Hence C is locating-dominating and it attains the lower bound in Corollary 5.

(ii) Let $d \geq 6$ be even and $n = 6d$. Denote $S = \{j \mid 0 \leq j \leq d, j \equiv 0 \pmod{2}\}$. We define

$$C_d = \{v \in \mathbb{Z}_n \mid v \equiv b \pmod{2d} \text{ for some } b \in S\}.$$

The code C_d has cardinality $3(d/2 + 1)$. Thus $\lim_{d \rightarrow \infty} |C_d|/n = 1/4$.

We will show that C_d is identifying in $C_n(1, d - 1, d)$. If $x \equiv s \pmod{2d}$ with $d \leq s \leq 2d - 1$ and x is odd, then $\{x - d + 1, x + d - 1\} \subseteq I(x)$. Since $N[x - d + 1] \cap N[x + d - 1] = \{x\}$, it

follows that $I(x) \neq I(y)$ for any $x \neq y$. If $x \equiv s \pmod{2d}$ where x is even and $d \leq s \leq 2d - 1$ or $s = 0$, then $\{x - d, x + d\} \subseteq I(x)$. Since $N[x - d] \cap N[x + d] = \{x\}$, the $I(x)$ is distinguished from other $I(y)$'s. Suppose then that $x \equiv s \pmod{2d}$ with $1 \leq s \leq d - 1$ and x is odd. Now $\{x - 1, x + 1\} \subseteq I(x)$ and again $I(x)$ is unique among I -sets. If $x \equiv s \pmod{2d}$ with $1 \leq s \leq d - 1$ and x is even (so s is even), then $I(x) = \{x\}$. It follows that C_d is identifying. \square

3.3 On graphs $C_n(1, d - 1, d, d + 1)$

In the following theorem, we give optimal locating-dominating codes in the circulant graph $C_n(1, d - 1, d, d + 1)$. Furthermore, we give an infinite sequence of identifying codes approaching the lower bound in Corollary 7.

Theorem 10. (i) For $d \equiv 8 \pmod{10}$, $d \geq 8$, $n \geq 4d + 6$ and $n \equiv 0 \pmod{10}$, we have $\gamma^{LD}(C_n(1, d - 1, d, d + 1)) = \frac{n}{5}$.

(ii) We have an infinite sequence of identifying codes in the circulant graphs $C_n(1, d - 1, d, d + 1)$ such that their density tends to $2/9$.

Proof. (i) Let $d \equiv 8 \pmod{10}$, $n \geq 4d + 6$ and $n \equiv 0 \pmod{10}$. Next we will verify that the code

$$C' = \{v \in \mathbb{Z}_n \mid v \equiv 0 \pmod{10} \text{ or } v \equiv 4 \pmod{10}\}$$

is locating-dominating in $C_n(1, d - 1, d, d + 1)$. Notice that the size of C' attains the lower bound in Corollary 7. Since $d \equiv 8 \pmod{10}$, then we get the following I -sets depending on the value of non-codewords x modulo 10

$x \pmod{10}$	$I(x)$	$I(x) \pmod{10}$
1	$\{x - 1, x - d + 1, x + d + 1\}$	0, 4, 0
2	$\{x - d, x + d\}$	4, 0
3	$\{x + 1, x - d - 1, x + d - 1\}$	4, 4, 0
5	$\{x - 1, x + d + 1\}$	4, 4
6	$\{x + d\}$	4
7	$\{x - d + 1, x + d - 1\}$	0, 4
8	$\{x - d\}$	0
9	$\{x + 1, x - d - 1\}$	0, 0.

Let $x \neq y$. Notice we only need to show for non-codewords x and y that $I(x) \neq I(y)$.

Assume first $x \not\equiv y \pmod{10}$. Due to the differences (see the table above) in the residue classes (0 and 4) of the codewords in the I -sets, we can easily see that $I(x) \neq I(y)$ unless $x \equiv 2 \pmod{10}$ and $y \equiv 7 \pmod{10}$. Therefore, let us consider this case. Now $I(x) = \{x - d, x + d\}$ and $I(y) = \{y - d + 1, y + d - 1\}$. Due to the residue classes modulo 10 in the I -sets, we must have $x - d \equiv y + d - 1 \pmod{n}$ and $x + d \equiv y - d + 1 \pmod{n}$. This implies that $2x \equiv 2y \pmod{n}$. If n is odd, we immediately have $x = y$ (in \mathbb{Z}_n). If n is even, we also have $x = y$ due to the fact that $n \geq 4d + 6$.

Assume then that $x \equiv y \pmod{10}$. Now it easily follows that $I(x) \neq I(y)$, since the codewords in the set $I(y)$ are the codewords in the set $I(x)$ shifted (to the right or left) along \mathbb{Z}_n .

(ii) The proof is somewhat technical and postponed to the Appendix. \square

4 Self-identifying codes in circulant graphs

In the next theorem, we will show that there exist infinite families of circulant graphs in which the lower bounds of Corollaries 3, 5 and 7 can be reached.

Theorem 11. Let d be an integer such that $d \geq 4$.

(i) If d is even, $n \geq 4d + 1$ and $n \equiv 0 \pmod{2}$, then we have $\gamma^{SID}(C_n(1, d)) = \frac{n}{2}$.

(ii) If $n \geq 4d + 1$ and $n \equiv 0 \pmod{2}$, then we have $\gamma^{SID}(C_n(1, d - 1, d)) = \frac{n}{2}$.

(iii) If $d \equiv 1 \pmod{3}$, $n \geq 4d + 5$ and $n \equiv 0 \pmod{3}$, then $\gamma^{SID}(C_n(1, d - 1, d, d + 1)) = \frac{n}{3}$.

Proof. (i) We show that the code

$$C = \{v \in \mathbb{Z}_n \mid v \equiv 0 \pmod{2}\}$$

is self-identifying in the circulant graph $C_n(1, d)$. If $x \equiv 0 \pmod{2}$, then $I(x) = \{x - d, x, x + d\}$ and otherwise $I(x) = \{x - 1, x + 1\}$. Since $n \geq 4d + 1$, we get that $N[x - d] \cap N[x + d] = \{x\}$ and $N[x - 1] \cap N[x + 1] = \{x\}$. Consequently, the condition for self-identification, namely, $\bigcap_{c \in I(x)} N[c] = \{x\}$, is satisfied. As $\frac{n}{2}$ is the lower bound, we showed that $\gamma^{SID}(C_n(1, d)) = \frac{n}{2}$.

(ii) Let $d \geq 4$, $n \geq 4d + 1$ and n be even. The code

$$C = \{v \in \mathbb{Z}_n \mid v \equiv 0 \pmod{2}\}$$

is self-identifying in $C_n(1, d - 1, d)$ as will be seen next. If d is even (resp. odd) and $x \equiv 0 \pmod{2}$, then $\{x - d, x + d\} \subseteq I(x)$ (resp. $\{x - d + 1, x + d - 1\} \subseteq I(x)$). Hence in both cases $\bigcap_{c \in I(x)} N[c] = \{x\}$. If d is even (resp. odd) and $x \equiv 1 \pmod{2}$, then $\{x - d + 1, x + d - 1\} \subseteq I(x)$ (resp. $\{x - d, x + d\} \subseteq I(x)$). Consequently, again $\bigcap_{c \in I(x)} N[c] = \{x\}$. Therefore, C is self-identifying. As $\frac{n}{2}$ is the lower bound, we showed that $\gamma^{SID}(C_n(1, d - 1, d)) = \frac{n}{2}$.

(iii) Let

$$C = \{v \in \mathbb{Z}_n \mid v \equiv 0 \pmod{3}\}.$$

We verify next that C is self-identifying in $C_n(1, d - 1, d, d + 1)$. If $x \equiv 0 \pmod{3}$, we have $I(x) = \{x, x - d + 1, x + d - 1\}$ since $d \equiv 1 \pmod{3}$. If $x \equiv 1 \pmod{3}$ (resp. $x \equiv 2 \pmod{3}$), then $I(x) = \{x - 1, x - d, x + d + 1\}$ (resp. $I(x) = \{x + 1, x - d - 1, x + d\}$). Now in each case, the intersection $\bigcap_{c \in I(x)} N[c] = \{x\}$ due to the fact that $n \geq 4d + 5$. Hence C is self-identifying. \square

In what follows, we give the optimal cardinalities of self-identifying codes in $C_n(1, 3)$ and $C_n(1, 4)$ (for n odd). In these cases, the optimal cardinalities do *not* attain the $n/2$ lower bound of Corollary 3, and for this purpose, we introduce new methods for increasing the lower bounds. In the following proposition, we present some results which are useful in the upcoming proofs.

Proposition 12. *Let n and $d_1 < d_2$ be integers such that $4d_2 - 1 < n$. If K is a self-identifying code in $C_n(d_1, d_2)$, then the following statements hold:*

(i) *For all $x \in K$, we have $|I(x)| > 2$.*

(ii) *For all $x \notin K$, there exists $1 \leq i \leq 2$ such that $\{x - d_i, x + d_i\} \subseteq I(x)$.*

(iii) *If $d_1 = 1$, $d_2 = 3$ and $|I(x)| = 2$, then we have $I(x) = \{x - 3, x + 3\}$ for all $x \notin K$.*

Proof. Let x be a vertex in the code. Assume it has only two vertices: itself and y . Then $I(y)$ contains the same two vertices. Hence, $I(x)$ contains at least three vertices.

Let x then be a non-codeword. Assume that $I(x)$ does not contain the claimed subset. Then, without loss of generality, we can assume that either $I(x) = \{x - d_1, x + d_2\}$ or $I(x) = \{x + d_1, x + d_2\}$. Suppose first that $I(x) = \{x - d_1, x + d_2\}$. For $y = x - d_1 + d_2$ we have $I(y) = \{y - d_2, y - d_1, y, y + d_1, y + d_2\} \cap K$. Now $y - d_2 = x - d_1$ and $y + d_1 = x + d_2$ are both in $I(x)$ giving $I(x) \subseteq I(y)$. Thus, K is not self-identifying. Assume then that $I(x) = \{x + d_1, x + d_2\}$. Now if $y = x + d_1 + d_2$, then as above $I(x) \subseteq I(y)$ and we are done. For $d_1 = 1$ and $d_2 = 3$ we cannot have $I(x) = \{x - 1, x + 1\}$ since $N[x - 1] \cap N[x + 1] = \{x, x - 2, x + 2\}$. \square

In the following theorem, we present the sizes of optimal self-identifying codes in $C_n(1, 3)$ for all integers $n > 11$. In particular, we show that any self-identifying code in $C_n(1, 3)$ has at least $\lceil 4n/7 \rceil$ codewords and that there exists a construction attaining this lower bound when $n \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{7}$. In the cases $n \equiv 3 \pmod{7}$ and $n \equiv 5 \pmod{7}$, we increase the lower bound by one using a novel technique and present constructions meeting this improved lower bound.

Theorem 13. *The optimal cardinalities of self-identifying codes in $C_n(1, 3)$ for $n > 11$ are as follows:*

$$\gamma^{SID}(C_n(1, 3)) = \begin{cases} 4k & \text{if } n = 7k \\ 4k + 1 & \text{if } n = 7k + 1 \\ 4k + 2 & \text{if } n = 7k + 2 \\ 4k + 3 & \text{if } n \in \{7k + 3, 7k + 4\} \\ 4k + 4 & \text{if } n \in \{7k + 5, 7k + 6\} \end{cases}.$$

Proof. Let n be an integer such that $n > 11$. Observe first that we have the following characterization for self-identifying codes in $C_n(1, 3)$:

- A code K in $C_n(1, 3)$ is self-identifying if and only if $|I(K; c)| \geq 3$ for all $c \in K$ and $\{u - 3, u + 3\} \subseteq I(K; u)$ for all $u \in \mathbb{Z}_n \setminus K$.

Indeed, if K is a self-identifying code in $C_n(1, 3)$, then the given conditions are met by the previous proposition. On the other hand, if K satisfies the conditions, then it is straightforward to verify that K is a self-identifying code by the characterization (1).

Let K be a self-identifying code in $C_n(1, 3)$. In what follows, we study more closely what happens if there exists consecutive non-codewords in K :

- If there are four or more non-consecutive non-codewords, then the first one, say u , contradicts with the previous characterization as $u + 3$ does not belong to K .
- If there are exactly three consecutive non-codewords, say $\{0, 1, 2\}$ (and thus $n - 1$ and 3 are in the code), then $\{n - 4, n - 3, n - 2, 4, 5, 6\}$ are all codewords (by the characterization). Let $P3$ be the pattern with 3 consecutive non-codewords followed by four consecutive codewords (see Figure 3).
- If there are exactly two consecutive non-codewords, say $\{0, 1\}$, then $\{n - 3, n - 2, n - 1, 2, 3, 4\}$ are in the code. Let $P2$ be the pattern with two consecutive non-codewords followed by three consecutive codewords as in Figure 3.
- Suppose then that there is only one consecutive codeword, say non-codeword 0 (and $n - 1$ and 1 are in the code). If $2 \in K$, then we get the pattern $P1a$ with one non-codeword followed by two codewords. On the other hand, if $2 \notin K$, then we obtain (by the characterization) the pattern $P1b$ with five consecutive vertices with only the first and the third one being non-codewords.

Notice that the smallest density among the patterns is the one with three consecutive non-codewords followed by four codewords, i.e., the density of the codewords in the patter is $4/7$.

Due to the obtained patterns, we may conclude that there exists in the graph two consecutive codewords followed by a non-codeword. Without loss of generality, it can be assumed that $n - 2, n - 1 \in K$ and $0 \notin K$. Furthermore, there exists a vertex x_1 such that the set $s_1 = \{0, 1, \dots, x_1\}$ is one of the patterns $P3, P2, P1a$ or $P1b$. Hence $x_1 - 1$ and x_1 are codewords and we can do the same thing with the next non-codeword vertex x_2 (notice that x_2 may be different from $x_1 + 1$). Let x_3 be such that $s_2 = \{x_2, x_2 + 1, \dots, x_3\}$ is one of the patterns. We can go on to the right and define all the sets s_1, \dots, s_r that correspond to the patterns. Note that the vertices that are not in these sets are all codewords. This partition the graph in patterns with maybe some codewords separating them. Notice also that the last pattern s_r do not intersect the first one s_1 . For each of these sets s_i let d_i be its density and n_i the number of vertices. The density of K can then be estimated

$$d \geq \frac{1}{n} \left(\sum_{1 \leq i \leq r} d_i n_i + n - \sum_{1 \leq i \leq r} n_i \right) \geq \frac{1}{n} \left(\sum_{1 \leq i \leq r} \frac{4}{7} n_i + n - \sum_{1 \leq i \leq r} n_i \right) = \frac{4}{7}$$

This implies that the self-identifying code K has at least $\lceil 4n/7 \rceil$ codewords. The proof now divides into the following cases depending on the remainder of n when divided by 7:

$P3$	•	•	•	×	×	×	×
	0	1	2	3	4	5	6
$P2$	•	•	×	×	×		
	0	1	2	3	4		
$P1a$	•	×	×				
	0	1	2				
$P1b$	•	×	•	×	×		
	0	1	2	3	4		

Figure 3: The patterns for $C_n(1, 3)$. The crosses denote the codewords.

- If $n = 7k$, then the code has at least $\lceil \frac{4}{7}n \rceil$ codewords, that is, $4k$. The code $K_1 = \{i + 7j \mid 0 \leq i \leq 3, 0 \leq j \leq k - 1\}$ is self-identifying (see the case $C_{14}(1, 3)$ in Figure 4). Indeed, for every vertex $v \notin K$, we have $\{v - 3, v + 3\} \subseteq I(v)$. Furthermore, for every vertex $v \in K$, we have $|I(K_1; v)| \geq 3$. Thus, according to the characterization, the code K_1 is self-identifying in $C_n(1, 3)$.

For $n = 7k + 1, 7k + 2$ and $n = 7k + 4$, one can analogously check that the codes $K_1 \cup \{7k\}$, $K_1 \cup \{7k, 7k + 1\}$ and $K_1 \cup \{7k - 1, 7k, 7k + 1\}$ are self-identifying in $C_n(1, 3)$ attaining the lower bound $\lceil \frac{4}{7}n \rceil$, respectively (see $C_{15}(1, 3)$, $C_{16}(1, 3)$ and $C_{18}(1, 3)$ in Figure 4, respectively).

For the length $n = 7k + 6$, we obtain that $K_7 = \{i + 7j \mid 0 \leq i \leq 3, 0 \leq j \leq k\}$ is an optimal self-identifying $C_n(1, 3)$ (see the case $C_{13}(1, 3)$ in Figure 4).

- Suppose $n = 7k + 3$. We will first show that now a self-identifying code has at least $4k + 3$ codewords. Every self-identifying code on $C_{7k+3}(1, 3)$ needs at least $\lceil \frac{4}{7}n \rceil = 4k + 2$ codewords. Assume that there is a self-identifying code K on $C_n(1, 3)$ with $4k + 2$ codewords. Recall that the density of codewords in the patterns is at least $3/5$ unless the pattern is $P3$. If there are at most $k - 2$ patterns of $P3$, then $|K| \geq \frac{4}{7}(7(k - 2)) + \frac{3}{5}(n - 7(k - 2)) = 4k + \frac{11}{5} > 4k + 2$. Consequently, there must be either k or $k - 1$ patterns of $P3$. Suppose first that there are k of them. This implies that there are three vertices outside of them (not necessarily consecutive). Recall that if we have a pattern $P3$ starting from a vertex u , then the vertices $u - 1, u - 2, u - 3$ and $u - 4$ are all codewords. Therefore, as we have only three vertices outside of patterns $P3$, they all have to be codewords. Suppose then that there are $k - 1$ patterns $P3$. Now there are 10 vertices not in these patterns. If a vertex u starts a pattern $P3$ such that $u - 1$ is not part of a pattern $P3$ (indeed, such pattern has to exist), then $u - 1$ is a codeword (as above) and does not belong to any pattern since none of the patterns other than $P3$ ends with four consecutive codewords. Therefore, we obtain that $7(k - 1)$ vertices belongs to some pattern $P3$, one codeword does not belong to any pattern and the rest 9 of the vertices belong to patterns other than $P3$ (or not to any pattern). Thus, we obtain that $|K| \geq \frac{4}{7}(7(k - 1)) + 1 + \frac{3}{5}(n - 7(k - 1) - 1) = 4k + \frac{12}{5} > 4k + 2$. Hence, there is no self-identifying code with $4k + 2$ codewords and the size of the code is at least $4k + 3$. By the same argument as above, we can show that the code $K_4 = \{i + 7j \mid 0 \leq i \leq 3, 0 \leq j \leq k - 1\} \cup \{7k, 7k + 1, 7k + 2\}$ works.
- If $n = 7k + 5$, then we show next that the code has at least $4k + 4$ codewords. It needs at least $4k + 3$ codewords. Let us use the sets s_i of the patterns again. If there is at most $k - 1$ patterns $P3$, then $|K| \geq \frac{4}{7}(7(k - 1)) + \frac{3}{5}(n - 7(k - 1)) = 4k + \frac{16}{5} > 4k + 3$. Therefore, there must be k patterns of $P3$ and five vertices outside them (not necessarily consecutive). Suppose first that these five vertices are not consecutive. Then they all must be codewords since four consecutive vertices left to any pattern $P3$ are codewords. Suppose then that the five vertices are consecutive. This implies (with the same argument) that four of them has to be codewords. Thus, in both cases, at least four of the five vertices are codewords. Hence, we have $|K| \geq 4k + 4$. As above, it is straightforward to verify that $K_6 = \{i + 7j \mid 0 \leq i \leq 3, 0 \leq j \leq k\}$ is an optimal self-identifying code with $4k + 4$ vertices.

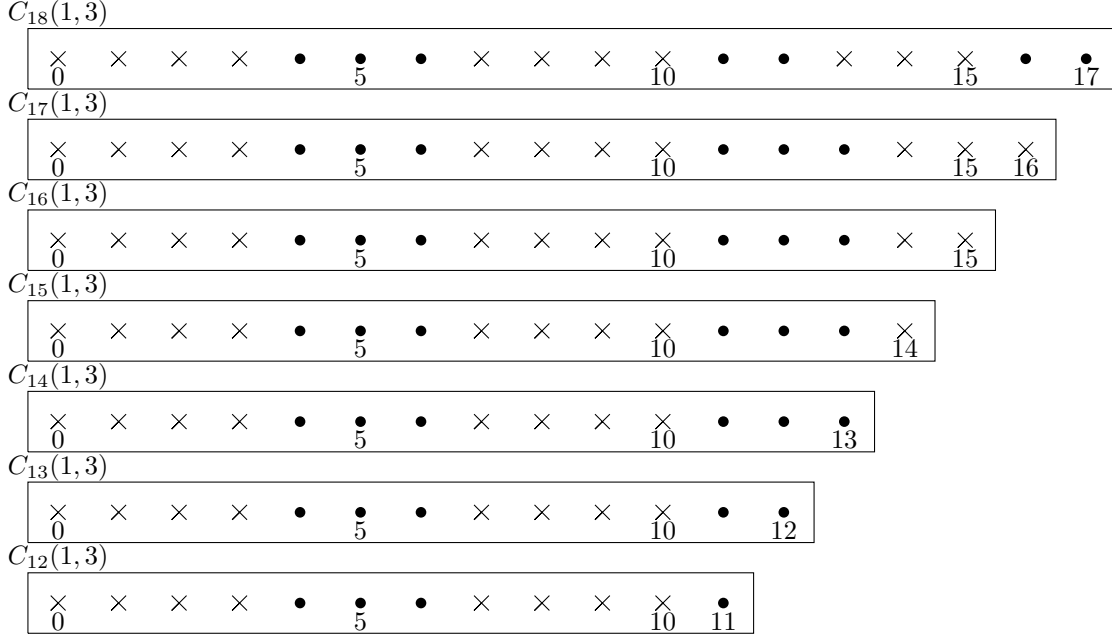


Figure 4: Optimal self-identifying codes for $C_n(1,3)$ for $n \in \{12, 13, 14, 15, 16, 17, 18\}$.

□

In the following theorem, we consider self-identifying codes in $C_n(1,4)$, when n is odd. Recall that the cardinality of an optimal self-identifying code in $C_n(1,4)$ is $\lceil n/2 \rceil$ for even n by Theorem 11. In particular, we show that the lower bound $\lceil n/2 \rceil$ of Corollary 3 can be increased by one for odd n .

Theorem 14. *If k is an integer such that $k > 5$, then we have*

$$\gamma^{SID}(C_{2k+1}(1,4)) = k + 2.$$

Proof. Let k and n be integers such that $k > 5$ and $n = 2k + 1$. Furthermore, for the lower bound, let K be a self-identifying code in $C_n(1,4)$. By Corollary 3, we immediately know that $|K| \geq \lceil n/2 \rceil = k + 1$. For the claim, we need to further show that $|K| = k + 1$ is not possible.

Suppose first that for each $u \notin K$ we have $u - 1 \in K$ and $u + 1 \in K$, i.e., there does not exist consecutive non-codewords in the graph. If now $|K| = k + 1$, then (without loss of generality) we can assume that the codewords are on the even vertices, i.e., $K = \{0, 2, \dots, 2k\}$. However, this implies a contradiction since $I(K; 2) = \{2, 6\} \subseteq I(K; 6)$. Thus, we may assume that there exist consecutive non-codewords in the graph.

Recall that we have $|I(K; c)| \geq 3$ for all $c \in K$ and $|I(K; u)| \geq 2$ for all $u \notin K$ (by Proposition 12). We say that a vertex $u \in \mathbb{Z}_n$ is *excessively covered* if $u \in K$ and $|I(K; u)| \geq 4$, or $u \notin K$ and $|I(K; u)| \geq 3$. In what follows, we first show that there exist at least three vertices that are excessively covered. Then, based on the observation, we prove that $|K| \geq k + 2$. The proof now divides into the following cases depending on how many consecutive non-codewords there exist:

- Suppose first that there exist five or more consecutive non-codewords. If u is the first one of these non-codewords, then a contradiction with Proposition 12(ii) follows as $u + 1 \notin K$ and $u + 4 \notin K$.
- Suppose then that there are exactly four consecutive non-codewords, say $u, u+1, u+2, u+3 \notin K$ and $u - 1, u + 4 \in K$. By Proposition 12, we obtain that $u - 4, u - 3, u - 2 \in K$ and



Figure 5: Optimal self-identifying code on $C_{17}(1,4)$.

$u + 5, u + 6, u + 7 \in K$. Hence, u and $u + 3$ are excessively covered since they are non-codewords with at least three neighbouring codewords. Furthermore, $u + 8$ is a codeword since the codeword $u + 4$ has to be covered by at least three codewords. Now, if $u + 9 \in K$, then the codeword $u + 5$ is excessively covered since $|I(K; u + 5)| \geq 4$. On the other hand, if $u + 9 \notin K$, then $u + 9$ is excessively covered since $u + 5$ and $u + 8$ belong to K as well as at least one of the vertices $u + 10$ and $u + 13$.

- Suppose that there are exactly three consecutive non-codewords, say $u, u + 1, u + 2 \notin K$ and $u - 1, u + 3 \in K$. As above, we deduce that $u - 4, u - 3, u - 2 \in K$ and $u + 4, u + 5, u + 6 \in K$. Similar to the previous case, we immediately obtain that u and $u + 2$ are excessively covered. If $u + 7 \in K$, then $u + 3$ is excessively covered since $|I(K; u + 3)| \geq 4$. On the other hand, if $u + 7 \notin K$, then $u + 7$ is excessively covered (as the vertex $u + 9$ in the previous case). Thus, we have three excessively covered vertices.
- Suppose that there are exactly two consecutive non-codewords, say $u, u + 1 \notin K$ and $u - 1, u + 2 \in K$. As above, we first obtain that $u - 4, u - 3 \in K$ and $u + 4, u + 5 \in K$. Using similar arguments as earlier, we immediately obtain that u and $u + 1$ are excessively covered. Furthermore, since the codewords $u - 1, u + 2$ and $u + 4$ all belong to $I(K; u + 3)$, the vertex $u + 3$ is excessively covered regardless whether it is a codeword or a non-codeword. Thus, we have three excessively covered vertices.

As stated earlier, we have $|I(K; c)| \geq 3$ for all $c \in K$ and $|I(K; u)| \geq 2$ for all $u \notin K$. In addition, we have shown that at least three vertices are excessively covered, i.e., covered more than what is required here. Therefore, by counting in two ways the pairs $c \in K$ and $u \in \mathbb{Z}_n$ such that $u \in N[c]$, we obtain the following inequality:

$$5|K| \geq 3|K| + 2(n - |K|) + 3 \Leftrightarrow |K| \geq \left\lceil \frac{2n + 3}{4} \right\rceil = k + 2.$$

Thus, in conclusion, we have shown that $|K| \geq k + 2$.

For the construction attaining the lower bound, we denote $K_1 = \{0, 2\} \cup \{i \in \mathbb{Z}_n \mid i \text{ is odd}\}$. The code K_1 is illustrated in Figure 5 (when $n = 17$). Clearly, K_1 contains $k + 2$ codewords. Furthermore, it is self-identifying in $C_n(1, 4)$. Indeed, for $v \in \{4, \dots, n - 1\}$, we have $I(v) = \{v - 4, v, v + 4\}$ if $v \in K_1$, and $I(v)$ contains $\{v - 1, v + 1\}$ if $v \notin K_1$. It is also straightforward to verify that the codewords in $I(v)$ intersect uniquely in v for $v = 0, 1, 2, 3$. Hence, K_1 is an optimal self-identifying code. \square

In the following theorem, we give optimal self-identifying codes for $C_n(1, n/2)$ for n even. Let $G = (V, E)$ be a graph and $K \subseteq V$. The *minimum distance* of a code K is defined via

$$d_{\min}(K) = \min_{x, y \in K, x \neq y} d_G(x, y).$$

We call $K \subseteq V$ a *1-error correcting code*, if $d_{\min}(K) \geq 3$. If K is 1-error correcting, then $I(G, K; x) = \{x\}$ for all $x \in K$ and $N(G; x) \cap N(G; y) = \emptyset$ for all distinct $x, y \in K$. Moreover, if G is r -regular, then a 1-error correcting code must satisfy the *sphere packing bound*: $|K| \leq |V|/(r+1)$.

Theorem 15. *Let $k \geq 5$. The optimal cardinality of self-identifying code in $C_{2k}(1, k)$ is as follows:*

$$\gamma^{SID}(C_{2k}(1, k)) = \begin{cases} \left\lceil \frac{4k}{3} \right\rceil & \text{if } k \equiv 0, 1 \pmod{3} \\ \left\lceil \frac{4k}{3} \right\rceil + 1 & \text{if } k \equiv 2 \pmod{3} \end{cases}$$

Proof. Let $k \geq 5$ and $n = 2k$. We study self-identifying codes in the graph $C_n(1, k)$. For all $x \in V = \{0, \dots, n-1\}$ the closed neighbourhood of x is $N[x] = \{x-k, x-1, x, x+1, x+k\} = \{x-k, x-1, x, x+1\}$ as $x-k \equiv x+k \pmod n$. Let K be a self-identifying code. Now it is easy to see that the I -set $I(x)$ contains $\{x-1, x+1\}$ for all $x \notin K$. Therefore, non-codewords are always surrounded by codewords (in the cycle $C_n(1)$). Furthermore, a codeword $v \in K$ cannot be surrounded by two non-codewords (in $C_n(1)$). Indeed it would imply that $I(v) = \{v, v+k\} \subseteq I(v+k)$. Hence the minimum distance $d_{\min}(V \setminus K) \geq 3$ and the set of non-codewords forms a 1-error correcting code in the cycle $C_n(1)$. Consequently, by the sphere packing bound $|V \setminus K| \leq n/3$. This observation yields $|K| \geq \lceil \frac{2}{3}n \rceil$, which gives the claimed lower bound for $k \equiv 0, 1 \pmod 3$. The constructions attaining the bound are given next:

- Let $k \equiv 0 \pmod 3$. The code $K_1 = \{v \in \{0, \dots, n-1\} \mid v \not\equiv 2 \pmod 3\}$ is self-identifying. Indeed, if $x \equiv 1 \pmod 3$, then $\{x-1, x+1\} \subseteq I(x)$ and their intersection $N[x-1] \cap N[x+1]$ contains only x . If $x \equiv 0, 2 \pmod 3$, then $\{x, x-k\} \subseteq I(x)$ and $I(x)$ also contains a third codeword and the intersection of them equals x .
- Let $k \equiv 1 \pmod 3$. If $S_2 = \{v \in \{0, \dots, k-1\} \mid v \not\equiv 2 \pmod 3\}$, then the code $K_2 = S_2 \cup \{s+k \mid s \in S_2\}$ is self-identifying. Indeed, every non-codeword v we have $\{v-1, v+1\} \subseteq I(v)$ and we are done. For every codeword v the $I(v)$ contains either $\{v, v-1, v+k\}$ or $\{v, v+1, v+k\}$ and we are done (see Figure 6).

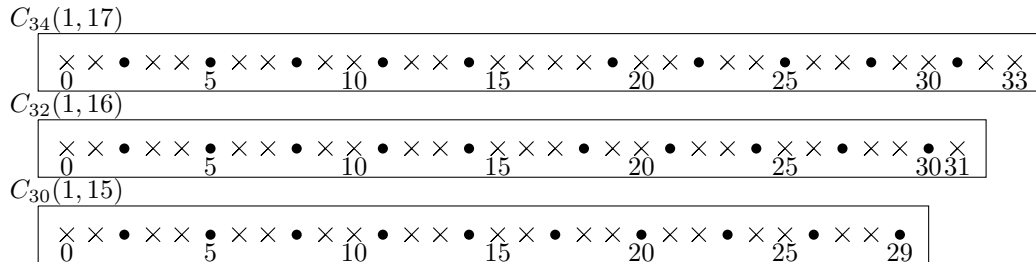


Figure 6: Examples of optimal self-identifying codes for $k = 15, 16, 17$.

It suffices to consider the case $k \equiv 2 \pmod 3$. We show that the cardinality of K must be greater than $\lceil \frac{2}{3}n \rceil = (2n+1)/3$. Suppose to the contrary that $|K| = (2n+1)/3$. Then there are $(n-1)/3$ non-codewords. Since $V \setminus K$ is 1-error correcting of cardinality $(n-1)/3$, there is exactly one vertex $y \in V$ outside the disjoint sets $N[C_n(1); v]$ for $v \in V \setminus K$. In other words, once y (clearly, $y \in K$) is given, then we know the code K without ambiguity. Without loss of generality, let $y = 0$ and thus the code is $K = \{v \in \mathbb{Z}_n \mid v \not\equiv 2 \pmod 3\}$. However, with this code we have $N[C_n(1, k); 3] = \{2, 3, 4, 3+k\}$ but 2 and $3+k$ are not in the code. Thus we have $\{3, 4\} = I(C_n(1, k); 3) \subseteq I(C_n(1, k); 4) = \{3, 4, 4+k\}$ and K cannot be self-identifying. We conclude that every self-identifying code in $C_n(1, k)$ needs at least $\lceil \frac{2}{3}n \rceil + 1$ codewords.

Denote $S_3 = \{v \in \{0, \dots, k-1\} \mid v \not\equiv 2 \pmod 3\}$. The construction attaining the bound $\lceil \frac{2}{3}n \rceil + 1$ is $K_3 = S_3 \cup \{s+k \mid s \in S_3\}$. Similarly, as above (in the case K_2) we can show that K_3 is self-identifying. \square

5 On attaining some lower bounds

Let us first introduce two basic result on identifying and self-identifying codes, which we need in Theorem 17. The first bound considering identifying codes is well-known (see [17]), but we add the proof for completeness.

Theorem 16. *Let k be an integer such that $k \geq 2$ and $G = (V, E)$ be a finite k -regular graph.*

(i) We have the following lower bound for the cardinality of an optimal identifying code:

$$\gamma^{ID}(G) \geq \left\lceil \frac{2|V|}{k+2} \right\rceil.$$

Moreover, there exists an identifying code C in G such that $|C| = 2|V|/(k+2)$ if and only if there exist exactly $|C|$ vertices $u \in V$ such that $|I(C; u)| = 1$ and for all other vertices $v \in V$ we have $|I(C; v)| = 2$.

(ii) We have the following lower bound for the cardinality of an optimal self-identifying code:

$$\gamma^{SID}(G) \geq \left\lceil \frac{2|V|}{k} \right\rceil.$$

Moreover, there exists a self-identifying code C in G such that $|C| = 2|V|/k$ if and only if $|I(C; u)| = 3$ for all $u \in C$ and $|I(C; v)| = 2$ for all $v \in V \setminus C$.

Proof. (i) Let first C be an identifying code in G . Observe then that there are at most $|C|$ I -sets with exactly one codeword since the code C is identifying. Hence, all the other $|V| - |C|$ I -sets have at least two codewords. Therefore, by counting in two ways the pairs $c \in C$ and $u \in V$ such that $u \in N[c]$, we obtain the following inequality:

$$|C|(k+1) \geq |C| + 2(|V| - |C|) \Leftrightarrow |C| \geq \frac{2|V|}{k+2}.$$

Moreover, by the previous observations, $|C| = 2|V|/(k+2)$ if and only if there exist exactly $|C|$ vertices $u \in V$ such that $|I(C; u)| = 1$ and for all other vertices $v \in V$ we have $|I(C; v)| = 2$.

(ii) Let C be a self-identifying code in G . Observe then that $N(u) \cap C$ has at least two codewords of C for all $u \in V$ since the code C is self-identifying. Hence, for each $u \in C$ we have $|I(C; u)| \geq 3$ and for each $u \in V \setminus C$ we have $|I(C; u)| \geq 2$. Therefore, by a similar double counting argument as above, we obtain: $|C| \geq 2|V|/k$. In addition, we get $|C| = 2|V|/k$ if and only if $|I(C; u)| = 3$ for all $u \in C$ and $|I(C; v)| = 2$ for all $v \in V \setminus C$. \square

The previous theorem gives lower bounds for circulant graphs as they are also regular. In the following theorem, we show that the exact bounds (above) cannot be attained in circulant graphs for identifying and self-identifying codes.

Theorem 17. Let n, r and d_2, \dots, d_r be integers such that $r \geq 3$ and $1 < d_2 < \dots < d_r \leq n/2$.

(i) Then there does not exist any identifying code C in $C_n(1, d_2, \dots, d_r)$ such that $|C| = n/(r+1)$.

(ii) Then there does not exist any self-identifying code C in $C_n(1, d_2, \dots, d_r)$ such that $|C| = n/r$.

Proof. (i) Let C be an identifying code in $C_n(1, d_2, \dots, d_r)$ such that $|C| = n/(r+1)$. By Theorem 16, it is possible if and only if there are exactly $|C|$ vertices $x_1, \dots, x_{|C|}$ such that $|I(x_i)| = 1$ and the rest of the vertices have I -sets with exactly two vertices. If there exists a vertex of C , say u , such that $|I(C; u)| = 2$, then we have $I(C; u) = \{u, v\}$ and $I(C; u) = I(C; v)$ as all I -sets have at most two codewords (a contradiction). Hence, the codewords of C are the vertices $x_1, \dots, x_{|C|}$. Therefore, in particular, we have $I(C; x_1) = \{x_1\}$ implying that $x_1 + 1 \notin C$ and $|I(C; x_1 + 1)| = 2$. If $I(C; x_1 + 1) = \{x_1, x_1 + 1 \pm d_i\}$, then the vertex $v = x_1 \pm d_i$ contains $\{x_1, x_1 + 1 \pm d_i\}$ in its I -set. Therefore, a contradiction follows as above. Hence, it has to be that $I(C; x_1 + 1) = \{x_1, x_1 + 2\}$. Now, because $x_1 + 2 \in C$, we have $I(C; x_1 + 2) = \{x_1 + 2\}$. Then, using similar arguments as above, we obtain that $x_1 + 3 \notin C$ and $I(C; x_1 + 3) = \{x_1 + 2, x_1 + 4\}$. Thus, by continuing this process, we obtain that every other vertex of $C_n(1, d_2, \dots, d_r)$ is a codeword. Clearly, this leads to a contradiction with the chosen cardinality of C . Thus, we conclude that $\gamma^{ID}(C_n(1, d_2, \dots, d_r)) > n/(r+1)$.

(ii) Let then C be a self-identifying code in $C_n(1, d_2, \dots, d_r)$ such that $|C| = n/r$. By Theorem 16, it is possible if and only if for each $u \in \mathbb{Z}_n$ we have $|N(u) \cap C| = 2$, i.e., u has exactly

two codewords of C in its open neighbourhood. Clearly, there exists a vertex $x \in \mathbb{Z}_n \setminus C$ such that $x - 1 \in C$. Using similar arguments as in the case of identifying codes, we obtain that $I(C; x) = \{x-1, x+1\}$. We continue to the right and use the same argument for each non-codeword that comes along. Hence, for any non-codeword $y \in \mathbb{Z}_n \setminus C$, we have $I(C; y) = \{y-1, y+1\}$. This further implies that $|C| \geq n/2$ which is a contradiction with the chosen cardinality of C , since $r \geq 3$. Thus, we conclude that $\gamma^{SID}(C_n(1, d_2, \dots, d_r)) > n/r$. \square

In the case of identifying codes, we immediately obtain the following corollary.

Corollary 18. *If n and d are positive integers such that $d \geq 4$ and $d \leq n/2$, then we have*

$$\gamma^{ID}(C_n(1, d-1, d)) > \frac{n}{4}.$$

Thus, the bound announced in Corollary 5 is never reached, but it is best possible since there is a sequence of codes on the circulant graphs $C_n(1, d-1, d)$ tending to this bound as proved in Theorem 9.

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6 Appendix

The proof of the Theorem 10(ii):

Let $d \geq 15$, $d \equiv 3 \pmod{6}$ and $n = 3d-9$. Notice that $n \equiv 0 \pmod{6}$. We divide the vertices of the circulant graph into three sections denoted by $A_1 = \{0, 1, 2, \dots, d-1\}$, $A_2 = \{d, d+1, \dots, 2d-1\}$ and $A_3 = \{0, 1, \dots, n-1\} \setminus (A_1 \cup A_2)$. We will first consider the code

$$C_d = \{v \mid v \in (A_1 \cup A_3), v \equiv 5 \pmod{6}\} \cup \{v \mid v \in A_2, v \equiv 0, 4 \pmod{6}\}.$$

Using this code we can construct (by adding later two more codewords) an identifying code in $C_n(1, d-1, d, d+1)$. The ratio $|C_d|/n$ tends to $2/9$ as d tends to infinity. First we exclude some ‘borderline’ vertices from the three sections and denote $A'_1 = A_1 \setminus \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, d-1\}$, $A'_2 = A_2 \setminus \{d, 2d-1\}$ and $A'_3 = A_3 \setminus \{2d\}$. We consider the borderline vertices later. It is straightforward to check that the I -sets with regard to the code C_d are as follows for $x \in A'_1 \cup A'_2 \cup A'_3$:

$x \in A'_1$	$I(x)$	$d(c_1, c_2)$	$I(x) \pmod{6}$
$\equiv 0 \pmod{6}$	$\{x-1, x+d+1\}$	$d+2$	4, 5
1	$\{x-d+1, x+d\}$	$d-8$	4, 5
2	$\{x-d, x+d-1, x+d+1\}$		0, 4, 5
3	$\{x-d-1, x+d\}$	$d-10$	0, 5
4	$\{x+1, x+d-1\}$	$d-2$	0, 5
5	$\{x\}$		
$x \in A'_2$	$I(x)$	$d(c_1, c_2)$	$I(x) \pmod{6}$
$\equiv 0 \pmod{6}$	$\{x\}$		
1	$\{x-d+1, x-1, x+d+1\}$		0, 5, 5
2	$\{x-d, x+d\}$	$2d$	
3	$\{x-d-1, x+1, x+d-1\}$		4, 5, 5
4	$\{x\}$		
5	$\{x-1, x+1\}$	2	

$x \in A'_3$	$I(x)$	$d(c_1, c_2)$	$I(x) \bmod 6$
$\equiv 0 \pmod 6$	$\{x - d + 1, x - 1\}$	$d - 2$	4, 5
1	$\{x - d, x + d + 1\}$	$d - 10$	4, 5
2	$\{x - d - 1, x - d + 1, x + d\}$		0, 4, 5
3	$\{x - d, x + d - 1\}$	$d - 8$	0, 5
4	$\{x - d - 1, x + 1\}$	$d + 2$	0, 5
5	$\{x\}$		

Let us compare these I -sets (that is, when $x \in A'_1 \cup A'_2 \cup A'_3$). Clearly, the I -sets of size one are distinguished. Consider then the I -sets of size two. In the tables above, one can find the distances $c_1 - c_2$ of the codewords in $I(x)$ with $c_1 > c_2$. If the distance is different, the I -sets cannot be the same. For those, which have the same distance, the $c_1 \pmod 6$ and $c_2 \pmod 6$ are different as shown in the table, and the I -sets again cannot be the same. Let us study the I -sets of size three then. According to the tables, the codewords in the I -sets are different modulo 6 unless $x \in A'_1$ where $x \equiv 2 \pmod 6$ and $y \in A'_3$ where $y \equiv 2 \pmod 6$. However, now $I(y)$ has distance 2 between its two largest codewords, but $I(x)$ has corresponding distance $d - 10$. Consequently, $I(x) \neq I(y)$.

For the rest of the vertices (i.e., the borderline vertices $x \notin A'_1 \cup A'_2 \cup A'_3$) we get the following I -sets: $I(0) = \{d + 1, 2d - 8, 3d - 10\}$, $I(1) = \{d + 1, 2d - 8\}$, $I(2) = \{d + 1, d + 3, 2d - 8, 2d - 6\}$, $I(3) = \{d + 3, 2d - 6\}$, $I(4) = \{5, d + 3, 2d - 6\}$, $I(5) = \{5\}$, $I(6) = \{5, d + 7, 2d - 2\}$, $I(7) = \{d + 7, 2d - 2\}$, $I(8) = \{d + 7, d + 9, 2d - 2\}$, $I(9) = \{d + 9\}$, $I(d - 1) = \{2d - 2, 3d - 10\}$, $I(d) = \{d + 1, 3d - 10\}$, $I(2d - 1) = \{2d - 2\}$ and $I(2d) = \{d + 1\}$. It is straightforward to check (considering sizes of I -sets, codewords modulo 6 in I -sets and their distances) that we have exactly the following non-distinguished I -sets: $I(9) = I(d + 9)$, $I(d - 1) = I(d - 2)$, $I(d + 1) = I(2d)$ and $I(2d - 2) = I(2d - 1)$. We add two more codewords, namely, 0 and $2d$ to the code C_d to avoid these same I -sets. Denote $C'_d = C_d \cup \{0, 2d\}$. We should bear in mind that if $I(C_d; x) \neq I(C_d; y)$, then also $I(C'_d; x) \neq I(C'_d; y)$. Now we have (with respect to C'_d) that $2d \in I(9) \setminus I(d + 9)$, $0 \in I(d - 1) \setminus I(d - 2)$, $2d \in I(2d - 1) \setminus I(2d - 2)$ and $0 \in I(d + 1) \setminus I(2d)$. Therefore, C'_d is an identifying code and the proof is completed.

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